COMPUTING INTEGRAL SOLUTIONS OF COMPLEMENTARITY PROBLEMS

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Abstract

In this paper an algorithm is proposed to find an integral solution of (nonlinear) complementarity problems. The algorithm starts with a nonnegative integral point and generates a unique sequence of adjacent integral simplices of varying dimension. Conditions are stated under which the algorithm terminates with a simplex one of whose vertices is an integral solution of the complementarity problem under consideration.

1   Introduction

The complementarity problem is to find a point $x^* \in \mathbb{R}^n$ such that

$$x^* \geq 0, \;\; f(x^*) \geq 0, \;\; \text{and} \;\; x^* \cdot f(x^*) = 0,$$

where $\mathbb{R}^n$ is the n-dimensional Euclidean space and f is a given mapping from $\mathbb{R}^n$ into itself. Specifically, if f is affine, the problem is called *the linear complementarity problem.* In general, for an arbitrary function f, the problem is called *the nonlinear complementarity problem.* If the solution of the complementarity problem is required to be integral or if the function f is defined on the integer lattice $\mathbb{Z}^n$ of $\mathbb{R}^n$ instead on $\mathbb{R}^n$, then the problem is called *the discrete complementarity problem*, denoted by $\text{DCP}(f)$. There is by now a voluminous literature on the complementarity problem, see Lemke (1965), Cottle (1966), Eaves (1971), Karamardian (1972), Moré (1974a, b), Kojima (1975), van der Laan and Talman (1987) among many others. For comprehensive surveys on the subject, see for example Kojima et al. (1991), Cottle et al. (1992), or Facchinei and Pang (2003).

As the literature indicates, much work has been done to obtain results for the existence and computation of solutions to the linear and nonlinear complementarity problems, but far less is known for the discrete complementarity problem, primarily because few methods currently exist to tackle problems of this nature. In fact, it has been only recently that researchers have begun to ask under what condition a complementarity problem has an integral solution. Chandrasekaran et al. (1998) and Cunningham and Geelen (1998) have independently studied discrete linear complementarity problems and provided sufficient conditions for the existence of an integral solution. Very recently, Yang (2004a, b) has investigated discrete nonlinear complementarity problems and established several existence theorems via a nonconstructive and topological method.

The major objective of this paper is to propose a finite and systematic algorithm for finding an integral solution of the nonlinear complementarity problem. We will adopt the 2n-ray simplicial algorithm introduced by van der Laan and Talman (1981) to compute an integral solution of the nonlinear complementarity problem. We show that the algorithm will find an integral solution of the problem within a finite number of steps under two different conditions. The two conditions are similar to those given in Yang (2004a, b)

and might be viewed as discrete versions of conditions given by Moré (1974a, b) and van der Laan and Talman (1987) for the continuous case. Thus, by this algorithmic approach we also give a constructive and combinatorial proof for two existence theorems given in Yang (2004a, b). The algorithm works as follows. First, we describe a triangulation of $\mathbb{R}^n_+$ so that the set of vertices of the triangulation is equal to $\mathbb{Z}^n_+$ and the mesh size of every simplex is equal to one according to the maximum norm. Then, starting with some given point in $\mathbb{Z}^n_+$, the algorithm generates a finite sequence of adjacent simplices of varying dimension and terminates under any of the two conditions with a simplex of which one of the vertices is an integral solution of the complementarity problem. We wish to emphasize that our algorithm always finds an exact solution within a finite number of steps. This fact is in contrast to the simplicial algorithms for the computation of fixed (or zero) points of arbitrary continuous or u.h.c. mappings, which typically find an approximate solution, see for instance Todd (1976), Allgower and Georg (1990), or Yang (1999) for comprehensive treatments on simplicial fixed point algorithms developed at various stages.

In Section 2 we present basic concepts and two existence theorems for the discrete complementarity problem. In Section 3 we propose a finite algorithm to find a solution, yielding a combinatorial and constructive proof for the two existence theorems.

2 Existence results

We first give some general notation. Let n be any positive integer and $N = \{1, 2, \cdots, n\}$. The notions $\mathbb{R}^n$ and $\mathbb{Z}^n$ denote the n-dimensional Euclidean space and the set of all integral points in $\mathbb{R}^n$, respectively. Given $x, y \in \mathbb{R}^n$, $x \cdot y$ stands for the inner product of x and y. For $i \in N$, $e(i)$ denotes the ith unit vector of $\mathbb{R}^n$ for $i \in N$ and $e(-i) = -e(i)$. Given a set $D \subset \mathbb{R}^n$, $\mathrm{co}(D)$ and $\mathrm{bd}(D)$ denote the convex hull of D and the (relative) boundary of D, respectively.

Given an integer $t \geq 0$, if $x^1, \cdots, x^{t+1}$ in $\mathbb{R}^n$ are affinely independent, the convex hull of these points will be called *a simplex* or *a t-simplex*. The points $x^1, \cdots, x^{t+1}$ are called the vertices of the simplex. The convex hull of any subset of the vertices of a simplex is called *a face* of the simplex. A face of a simplex is called *a facet* if the number of its

vertices is exactly one less than the number of vertices of the simplex. A simplex is said to be *integral* if all its vertices are integral. Given an m-dimensional convex set D in $\mathbb{R}^n$, a collection $\mathcal{T}$ of m-dimensional simplices is a *triangulation or simplicial subdivision* of the set D, if (i) D is the union of all simplices in $\mathcal{T}$, (ii) the intersection of any two simplices of $\mathcal{T}$ is either the empty set or a common face of both, and (iii) any neighborhood of a point in D only meets a finite number of simplices of $\mathcal{T}$. A specific triangulation on $\mathbb{R}^n$, the so-called K'-triangulation, was proposed by Todd (1978) for arbitrary mesh size and arbitrary center point in $\mathbb{R}^n$.

In this paper we apply the K'-triangulation restricted to $\mathbb{R}^n_+$, with mesh size equal to 1 and center point equal to some specific point v in $\mathbf{Z}^n_+$. This triangulation, denoted by $K'(v)$, is the collection of n-dimensional simplices $\sigma(x^1, s, \pi)$ with vertices $x^1, x^2, \cdots, x^{n+1}$ in $\mathbb{R}^n_+$ such that (i) $\pi = (\pi(1), \cdots, \pi(n))$ is a permutation of the elements in $(1, \cdots, n)$, (ii) $s \in \{-1, 1\}^n$ with $s_i = -1$ if $x^1_i < v_i$ and $s_i = 1$ if $x^1_i > v_i$, and (iii) $x^1 \in \mathbf{Z}^n_+$ and $x^{i+1} = x^i + s_{\pi(i)}e(\pi(i))$ for $i = 1, \cdots, n$. Notice that all vertices of any simplex of the $K'(v)$-triangulation are integral points.

Two integral points x and y in $\mathbf{Z}^n$ are said to be *cell connected* if $\max_{h \in N} |x_h - y_h| \leq 1$, i.e., their distance is less than or equal to one according to the maximum norm. Observe that for given $v \in \mathbf{Z}^n_+$ any two vertices of a simplex of the $K'(v)$-triangulation are cell connected. We now have the following definition.

Definition 2.1   *A function $f : \mathbf{Z}^n_+ \to \mathbb{R}^n$ is direction preserving if for any two cell connected points x and y in $\mathbf{Z}^n_+$ it holds that $f_h(x)f_h(y) \geq 0$ for all $h \in N$.*

The class of direction preserving functions is due to Iimura (2003), see also Iimura, Murota and Tamura (2004) for a correction of Iimura's (2003) discrete fixed point theorem. Observe that the property of direction preserving prevents coordinatewisely that the function jumps from a positive value to a negative value within one cell. In this sense it replaces continuity for functions on $\mathbb{R}^n$. We state two existence theorems for direction preserving functions. The first theorem due to Yang (2004a) can be seen as a discrete analogue of the existence theorem given by Moré (1974a, b) for the continuous case. Yang (2004a) proves his result via a nonconstructive and topological method. It is known that Moré's condition is quite general in the sense that many other conditions are special cases

of it. In fact, the condition in the theorem below for the discrete case is slightly more general.

Theorem 2.2   Let $f : \mathbb{Z}_+^n \to \mathbb{R}^n$ be a direction preserving mapping. If there exist a vector $w \in \mathbb{Z}_+^n$ and for every $h \in N$ an integer $u_h > w_h$ such that for every $x \in \mathbb{Z}_+^n$ with $x_h = u_h$ for some $h \in N$ it holds that $\max_{i \in N} \ (x_i - w_i) f_i(x) > 0$, then the DCP($f$) has a solution.

Since the theorem holds for any w, it is also true when w is the origin. This yields the following corollary.

Corollary 2.3   Let $f : \mathbb{Z}_+^n \to \mathbb{R}^n$ be a direction preserving mapping. If there exists a strictly positive integral vector u such that for every every $x \in \mathbb{Z}_+^n$ with $x_h = u_h$ for some $h \in N$ it holds that $f_i(x) > 0$ for some $i \in N$ satisfying $x_i > 0$, then the DCP(f) has a solution.

It is interesting to note that this corollary implies the following result for the existence of a unique solution of DCP(f); see also Yang (2004a).

Corollary 2.4   Let $f : \mathbb{Z}_+^n \to \mathbb{R}^n$ be a direction preserving function. If there is some $c > 0$ such that for all $x, y \in \mathbb{Z}_+^n$

$$\max_{i \in N} \ (x_i - y_i)[f_i(x) - f_i(y)] \geq c \|x - y\|^2,$$

then the DCP(f) has a unique solution, where $\| \cdot \|$ is any norm.

The second main theorem can be seen as a generalization of a discrete version of an existence result given by van der Laan and Talman (1987) for the continuous case and contains an existence result given in Yang (2004a, b) as a special case.

Theorem 2.5   Let $f : \mathbb{Z}_+^n \to \mathbb{R}^n$ be a direction preserving function. Suppose there exists a strictly positive integral vector u satisfying that for every $x \in \mathbb{Z}^n$, $x_i = u_i$ implies $f_i(x) > \min_{h \in N} \ f_h(x)$ if $\min_{h \in N} \ f_h(x) < 0$ and $f_i(x) \geq 0$ otherwise. Then the DCP(f) has a solution.

3   The algorithm

In this section we adapt the 2n-ray simplicial algorithm of van der Laan and Talman (1981) to find a solution within a finite number steps of the DCP(f) under each one of the two conditions stated in the Theorems 2.2 and 2.5. In this way we also provide constructive and combinatorial proofs for the two theorems.

The 2n-ray algorithm is a variable dimension simplicial algorithm, originally introduced to approximate a fixed or zero point of a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$. To adapt the algorithm, we introduce an integer labeling rule which assigns to every nonnegative integral point an integer label out of the set $-N \cup N \cup \{0\}$.

Definition 3.1   *The integer labeling rule* $l : \mathbb{Z}_+^n \to -N \cup N \cup \{0\}$ *is specified as follows. For any* $x \in \mathbb{Z}_+^n$, *let*

$$\mu(x) = \max\{-\min_{h \in N} f_h(x), \max_{\{h \mid x_h > 0\}} f_h(x)\}.$$

Then $l(x) = 0$ *if* $\mu(x) \leq 0$. *If* $\mu(x) > 0$, *then* $l(x) = \min\{j \mid f_j(x) = -\mu(x)\}$ *when such an index* j *exists. Otherwise* $l(x) = -\min\{j \mid f_j(x) = \mu(x) \text{ and } x_j > 0\}$.

Observe that $\mu(x) < 0$ implies that $x = 0$ and $f(x) > 0$, i.e., $\mu(x) \geq 0$ unless x is the origin and all components of f at the origin are positive. In this case the origin is a solution to the DCP(f). If $f(0)$ contains at least one nonpositive component, then $\mu(0) \geq 0$. Also at any point $x \neq 0$, $\mu(x) \geq 0$. A point x solves the DCP(f) if and only if $l(x) = 0$. At any point x with $\mu(x) > 0$ we have that $l(x) \neq 0$ is well defined with $l(x) \neq -j$ when $x_j = 0$.

The algorithm begins with a given point v in $\mathbb{Z}_+^n$. In the case of Theorem 2.2, v is chosen to be w, whereas in the case of Theorem 2.5, v can be any vector in $\mathbb{Z}_+^n$ satisfying $v_h < u_h$ for every $h \in N$. In case $l(v) = 0$, then v solves DCP(f) and the algorithm terminates immediately with a solution. In the following we suppose that v does not solve the problem and therefore $l(v) \neq 0$. Now, let $\mathcal{I}$ be the family of nonempty subsets S of $-N \cup N$ such that $S \cap -S = \emptyset$ and $v_j = 0$ implies $-j \notin S$. For $T \in \mathcal{I}$, define

$$A(T) = \{x \in \mathbb{R}_+^n \mid \quad x_i \geq v_i \text{ for } i \in T; \quad x_i \leq v_i \text{ for } -i \in T;$$
$$\text{and } x_i = v_i \text{ otherwise}\}.$$

5

For $T \in \mathcal{I}$, the set $A(T)$ is a t-dimensional set, where $t = |T|$. The $K'(v)$-triangulation of $\mathbb{R}_+^n$ induces a triangulation of every set $A(T)$, $T \in \mathcal{I}$. More precisely, for $T \in \mathcal{I}$, $A(T)$ is subdivided into t-simplices $\sigma(x^1, \pi)$ with vertices $x^1, \cdots, x^{t+1}$ in $\mathbb{R}^n$, where $t = |T|$, $x^1 \in A(T) \cap \mathbb{Z}^n$, π is a permutation of the t elements of T, and $x^{i+1} = x^i + e(\pi(i))$, $i = 1, \cdots, t$.

Given a t-simplex σ in $A(T)$, $T \in \mathcal{I}$, a facet τ of σ is called T-*complete* if

$$\{l(x) \mid x \text{ is a vertex of } \tau\} = T.$$

In other words, the t vertices of a T-complete facet of a simplex in $A(T)$ are all differently labeled by the t integers in the set T. Since it is supposed that $l(v) \neq 0$, in particular we have that $\{l(v)\}$ is an element of $\mathcal{I}$ and $\{v\}$ is an $\{l(v)\}$-complete 0-simplex on the boundary of the 1-dimensional set $A(\{l(v)\})$ and therefore is a facet of precisely one 1-simplex in $A(\{l(v)\})$. In the sequel we show by several lemmas that, starting with the integral point v and $T = \{l(v)\}$, the algorithm generates in $A(T)$, for varying $T \in \mathcal{I}$, a sequence of t-simplices, where $t = |T|$, with T-complete common facets and ends within a finite number of steps with a simplex having one of its vertices which is labeled with zero and thus solves the problem. Note that every t-simplex in $A(T)$ is a face of a simplex of the $K'(v)$-triangulation.

First observe that for any given $T \in \mathcal{I}$, with $t = |T|$, since any t-simplex in $A(T)$ has at most two T-complete facets and any facet of a t-simplex in $A(T)$ either is a facet of exactly one other t-simplex in $A(T)$ or lies on the boundary of $A(T)$, the collection of t-simplices in $A(T)$ having at least one T-complete facet is either empty or consists of sequences of adjacent t-simplices in $A(T)$ with T-complete common facets. Each such sequence in $A(T)$ is either unbounded (and then has one or no end simplex) or bounded. In the latter case it is either a loop (with no end simplices) or it has two end simplices. An end simplex of a sequence in $A(T)$ of adjacent t-simplices with T-complete common facets is either (i) a t-simplex having exactly one vertex carrying a label not in the set T, or (ii) a t-simplex having a T-complete facet lying on the boundary of the set $A(T)$.

In case (i), let σ be an end simplex having a vertex x carrying a label h not in T. Then it must hold that either (a) $h = 0$ or (b) $-h \in T$ or (c) $h \neq 0$ and $-h \notin T$. In subcase (a) the vertex x is a solution to $\mathrm{DCP}(f)$. We now show that subcase (b) cannot occur if the

6

function is direction preserving. In fact, none of the simplices of the $K'(v)$-triangulation carries labels k and $-k$ together and therefore also not any of the simplices in $A(T)$.

Lemma 3.2   *If $f : \mathbb{Z}_+^n \to \mathbb{R}^n$ is a direction preserving function, there exists no simplex of the $K'(v)$-triangulation, whose vertices carry the labels k and $-k$ for some $k \in N$.*

Proof: Let σ be a simplex of the $K'(v)$-triangulation, whose vertices carry labels k and $-k$ for some $k \in N$. Then for the vertex x with label k, we have $f_k(x) < 0$ and for the vertex y with label $-k$, we have $f_k(y) > 0$. This contradicts direction preservation, because x and y are cell connected.   □

With respect to subcase (c) we have the following lemma.

Lemma 3.3   *For some $T \in \mathcal{I}$, let x be a vertex of a t-simplex σ in $A(T)$ having a T-complete facet with $l(x) = h \neq 0$. When both $h \notin T$ and $-h \notin T$, then $T \cup \{h\} \in \mathcal{I}$ and σ is on the boundary of $A(T \cup \{h\})$ and a $(T \cup \{h\})$-complete facet of exactly one $(t+1)$-simplex in $A(T \cup \{h\})$.*

Proof: Suppose that $T \cup \{h\}$ is not an element of $\mathcal{I}$. By definition $T \cup \{h\}$ is not an element of $\mathcal{I}$ if and only if both $h < 0$ and $v_{|h|} = 0$. Suppose $v_{|h|} = 0$. Since all vertices of σ lie in $A(T)$ and both h and $-h$ are not in T, we then have that $y_{|h|} = 0$ for any vertex y of σ. However, by the labelling rule this excludes that any vertex y has label $-|h|$, contradicting that both $h < 0$ and $v_{|h|} = 0$. Therefore, $T \cup \{h\}$ is an element of $\mathcal{I}$. Since σ is a t-simplex in $A(T)$ and $A(T)$ is a subset of the boundary of the $(t+1)$-dimensional set $A(T \cup \{h\})$, the t-simplex σ is on the boundary of $A(T \cup \{h\})$ and is therefore a facet of precisely one $(t+1)$-simplex in $A(T \cup \{h\})$.   □

The two Lemmas 3.2 and 3.3 imply that if σ is an end simplex of a sequence of adjacent t-simplices in $A(T)$ with T-complete facets such that a vertex x of σ carries a label h not in T, then either subcase (a) occurs and x solves the problem or subcase (c) occurs and there is a unique $(t+1)$-simplex σ' in $A(T \cup \{h\})$, having σ as a $(T \cup \{h\})$-complete facet on the boundary of $A(T \cup \{h\})$. This simplex σ' is therefore an end simplex of a sequence of adjacent $(t+1)$-simplices in $A(T \cup \{h\})$ with $(T \cup \{h\})$-complete common facets.

Next, consider case (ii) and suppose that for some $T \in \mathcal{I}$ the simplex σ is an end simplex of a sequence of adjacent t-simplices in $A(T)$ with common T-complete facets and that σ is a t-simplex having a T-complete facet τ on the boundary of $A(T)$. Clearly, the subsets of the boundary of $A(T)$ are determined by the points x in $A(T)$ at which either (a) $x_{|k|} = 0$ for some $k \in T$, $k < 0$, or (b) $x_{|k|} = v_{|k|}$ for some $k \in T$. Notice that $v_{|k|} > 0$ if $k < 0$. The next lemma shows that τ cannot be in a subset of $A(T)$ satisfying subcase (a), i.e., a T-complete facet in $A(T)$ cannot lie in the hyperplane $\{x \in \mathbb{R}^n \mid x_{|k|} = 0\}$ when $-|k| \in T$.

Lemma 3.4   *If $k < 0$ for some $k \in T$, then there exists no T-complete facet τ of a t-simplex in $A(T)$ for some $T \in \mathcal{I}$ lying in the hyperplane $\{x \in \mathbb{R}^n \mid x_{|k|} = 0\}$.*

Proof: Let τ be T-complete and $k \in T$. Then there is a vertex y of τ carrying label k. If $k < 0$, it follows from the labeling rule $f_{|k|}(y) > 0$ and $y_{|k|} > 0$ and so τ does not lie in the hyperplane $\{x \in \mathbb{R}^n \mid x_{|k|} = 0\}$. $\square$

The lemma implies that if, for some $T \in \mathcal{I}$, τ is a T-complete facet of a simplex in $A(T)$ and lies on the boundary of $A(T)$, then it lies in a subset of the boundary satisfying subcase (b). Then either $T = \{k\}$ for some $k \in N$ and τ is the 0-dimensional simplex $\{v\}$, or for some $k \in T$, τ is a T-complete $(t-1)$-simplex in $A(T \setminus \{k\})$ having a unique $(T \setminus \{k\})$-complete facet. In the latter case τ is an end simplex of a sequence of adjacent $(t-1)$-simplices in $A(T \setminus \{k\})$ with $(T \setminus \{k\})$-complete common facets.

Summarizing the results above we have that any end simplex of a sequence of adjacent simplices in $A(T)$ with T-complete common facets, not being the zero-dimensional simplex $\{v\}$ or a simplex having a vertex with label 0, either has one facet or is a facet of exactly one simplex that is an end simplex of a sequence of adjacent t'-simplices in $A(T')$ with T'-complete common facets, where either $T' = T \cup \{h\}$ for some unique $h \notin T$ and $-h \notin T$, or $T' = T \setminus \{k\}$ for some unique $k \in T$. So, the sequences of adjacent t-simplices in $A(T)$ with T-complete common facets over all different $T \in \mathcal{I}$ can therefore be linked to form sequences of adjacent simplices of variable dimension such that for any two adjacent simplices in the sequence it holds that for some $T \in \mathcal{I}$ either one of the two simplices is a t-simplex in $A(T)$ while the other one is a T-complete facet of it lying on the boundary of

8

$A(T)$, or both are t-simplices in $A(T)$ and share a T-complete facet. Doing so, we obtain a collection of such sequences. Any sequence in this collection is either an unbounded sequence with one or no end simplices or is a bounded sequence. In the latter case it is either a loop (with no end simplices or it is a sequence with two end simplices. Exactly one sequence has the starting point $\{v\}$ of the algorithm as facet of one of its end simplices. Each other end simplex has a vertex carrying label 0 and therefore solving the discrete complementarity problem.

Starting with the zero-dimensional simplex $\{v\}$, the 2n-ray algorithm generates the path of simplices of the sequence having $\{v\}$ as facet of one of its end simplices. We now prove that this path is bounded and therefore has another end simplex carrying label 0, so that the algorithm finds a solution within a finite number of steps. The next two lemmas show that under the conditions of Theorem 2.2 and Theorem 2.5 respectively, no T-complete facet of a simplex in $A(T)$ lies on the hyperplane $\{x \in \mathbb{R}^n \mid x_k = u_k\}$ for any $k \in N$.

Lemma 3.5   *For the $K'(w)$ triangulation of $\mathbb{R}^n_+$, under the conditions of Theorem 2.2, there exists no T-complete facet τ of a simplex in $A(T)$, $T \in \mathcal{I}$, lying in the hyperplane $\{x \in \mathbb{R}^n \mid x_k = u_k\}$ for some $k \in N$.*

Proof: Suppose τ is a T-complete facet in $A(T)$ lying in $\{x \in \mathbb{R}^n \mid x_k = u_k\}$ for some $k \in N$. Take any vertex x of τ. When $x_h > w_h$ for some $h \in N$, we must have $h \in T$. Then there exists a vertex y of τ with label h and so $f_h(y) < 0$. Since f is direction preserving, this implies $f_h(x) \leq 0$ due to the fact that x and y are cell connected. So, $(x_h - w_h)f_h(x) \leq 0$.

When $x_h < w_h$ for some $h \in N$, we must have $-h \in T$ and hence there exists a vertex y of τ with label $-h$, implying that $f_h(y) > 0$ and $y_h > 0$. This implies $f_h(x) \geq 0$ due to the fact that f is direction preserving and x and y are cell connected. Again it follows that $(x_h - w_h)f_h(x) \leq 0$.

Finally, if $x_h = w_h$ for some $h \in N$, then we have $(x_h - w_h)f_h(x) = 0$. In conclusion, for the vertex x of τ we have

$$\max_{h \in N} \ (x_h - w_h)f_h(x) \leq 0,$$

9

yielding a contradiction to the condition that $\max_{h \in N} \; (x_h - w_h) f_h(x) > 0$. □

Lemma 3.6   *For the $K'(v)$ triangulation of $\mathbb{R}_+^n$ with center point $v \in \mathbb{Z}_+^n$ satisfying $v_h < u_h$ for all $h \in N$, under the conditions of Theorem 2.5, there exists no T-complete facet τ in $A(T)$, $T \in \mathcal{I}$, lying on the hyperplane $\{x \in \mathbb{R}^n \mid x_k = u_k\}$ for some $k \in N$.*

Proof: Suppose τ is a T-complete facet in $A(T)$ lying in $\{x \in \mathbb{R}^n \mid x_k = u_k\}$ for some $k \in N$. Since $u_k > v_k$, we must have $k \in T$. Take any vertex x of τ. Since $x_k = u_k$ it follows from the condition in Theorem 2.5 that $f_k(x) > \min_{h \in N} \; f_h(x)$ if $\min_{h \in N} \; f_h(x) < 0$ and $f_k(x) \geq 0$ otherwise. From this it follows from the labelling rule that x cannot carry label k, contradicting τ is T-complete. □

The Lemmas 3.2-3.6 imply that under the conditions of either of the two theorems the sequence of simplices generated by the 2n-ray algorithm ends with a vertex having label 0 within a finite number of steps. To conclude this, suppose the generated sequence starting from the end simplex $\{v\}$ is unbounded. Then, by construction of the $K'(v)$-triangulation and since the simplices are adjacent, there exists a simplex σ in the sequence such that σ is a t-simplex in $A(T)$ for some $T \in \mathcal{I}$ and σ has a T-complete facet τ in the hyperplane $\{x \in \mathbb{R}^n \mid x_k = u_k\}$ for some $k \in N$. However, this contradicts the results of the Lemmas 3.5 and 3.6. Hence, the sequence of simplices must stay within the set $C = \{x \in \mathbb{R}^n \mid 0 \leq x_j \leq u_j, \; j \in N\}$. Since the number of simplices of the $K'(v)$-triangulation within this set is finite, the sequence therefore contains only a finite number of simplices and it cannot be a loop, because it has $\{v\}$ as one of its end points. Hence the sequence must have another end simplex in C. Since each other end simplex than $\{v\}$ has a vertex carrying label 0, the algorithm finds an exact solution to the discrete complementarity problem within a finite number of steps. This is summarized in the following propositions, yielding combinatorial and constructive proofs for the Theorems 2.2 and 2.5 resepctively.

Proposition 3.7   *Let $f : \mathbb{Z}_+^n \to \mathbb{R}^n$ be a direction preserving function satifying the condition of Theorem 2.2. Starting with the point w, the algorithm finds a solution of the discrete complementarity problem within the set C in a finite number of steps.*

Proposition 3.8 *Let $f : \mathbb{Z}_+^n \rightarrow \mathbb{R}^n$ be a direction preserving function so that the condition of Theorem 2.5 is satisfied. Starting with any point $v \in \mathbb{Z}_+^n$ satisfying $v_j < u_j$ for all $j \in N$, the algorithm finds a solution of the discrete complementarity problem within the set C in a finite number of steps.*

As a final remark we like to stress that the direction preserving condition is only used in the Lemmas 3.2 and 3.5, but not in the others, also not in Lemma 3.6. Lemma 3.2, saying that for some h, a T-complete simplex cannot carry both labels h and $-h$, implies that any end simplex not being $\{v\}$ must have a vertex with label 0. The proof of Lemma 3.5 shows that under Moré's condition also the direction preserving property is needed to prevent the sequence from being unbounded; by contrast the proof of Lemma 3.6, showing boundedness under the van der Laan and Talman condition, does not need the direction preserving property.

References

[1] Allgower, E.L., Georg, K., 1990. *Numerical Continuation Methods: An Introduction*, Springer, Berlin.

[2] Chandrasekaran, R., Kabadi, S. N., Sridhar, R., 1998. Integer solution for linear complementarity problem. *Mathematics of Operations Research* **23**, 390–402.

[3] Cottle, R.W., 1966. Nonlinear programs with positively bounded Jacobians. *SIAM Journal on Applied Mathematics* **14**, 147–158.

[4] Cottle, R.W., Pang, J.-S., Stone, R.E., 1992. *The Linear Complementarity Problem*, Academic Press, New York.

[5] Cunningham, W., Geelen, J., 1998. Integral solutions of linear complementarity problem. *Mathematics of Operations Research* **23**, 61–68.

[6] Eaves, B.C., 1971. On the basic theorem of complementarity. *Mathematical Programming* **1**, 68–75.

[7] Facchinei, F., Pang, J.-S., 2003. *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vol. I, Vol. II, Springer, New York.

[8] Iimura, T., 2003. A discrete fixed point theorem and its applications, *Journal of Mathematical Economics* **39**, 725–742.

[9] Iimura, T., Murota, K., Tamura, A., 2004. Discrete fixed point theorem reconsidered, Preprint No. 1449, Research Institute for Mathematical Sciences, Kyoto University, Kyoto.

[10] Karamardian, S., 1972. The complementarity problem. *Mathematical Programming* **2**, 107–129.

[11] Kojima, M., 1975. A unification of the existence theorems of the nonlinear complementarity problem. *Mathematical Programming* **9**, 257–277.

[12] Kojima, M., Megiddo, N., Noma, T., Yoshise, A., 1991. *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*, Springer-Verlag, Berlin.

[13] Laan van der, G., Talman, A.J.J., 1981. A class of simplicial restart fixed point algorithms without an extra dimension. *Mathematical Programming* **20**, 33–48.

[14] Laan van der, G., Talman, A.J.J., 1987. Simplicial approximation of solutions to the nonlinear complementarity problem. *Mathematical Programming* **38**, 1–15.

[15] Lemke, C.E., 1965. Bimatrix equilibrium points and mathematical programming. *Management Science* **11**, 681–689.

[16] Moré, J.J., 1974a. Coercivity conditions in nonlinear complementarity problem. *SIAM Review* **17**, 1–16.

[17] Moré, J.J., 1974b. Classes of functions and feasibility conditions in nonlinear complementarity problem. *Mathematical Programming* **6**, 327–338.

[18] Todd, M.J., 1976. *Computation of Fixed Points and Applications*, Springer-Verlag, Berlin.

[19] Todd, M.J., 1978. Improving the convergence of fixed point algorithms, *Mathematical Programming Study* **7**, 151–179.

[20] Yang, Z., 1999. *Computing Equilibria and Fixed Points*, Kluwer, Boston.

[21] Yang, Z., 2004a. Discrete nonlinear complementarity problems. FBA Working Paper No. 205, Yokohama National University, Yokohama.

[22] Yang, Z., 2004b. Discrete fixed point analysis and its applications. FBA Working Paper No. 210, Yokohama National University, Yokohama.