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**STRATEGIC CAPITAL BUDGETING: ASSET
REPLACEMENT UNDER UNCERTAINTY**

By Grzegorz Pawlina and Peter M. Kort

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Strategic Capital Budgeting: Asset Replacement under Uncertainty^α

Grzegorz Pawlina^γ and Peter M. Kort^z

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Abstract

We consider a firm's decision to replace an existing production technology with a new, more cost-efficient one. Kulatilaka and Perotti [1998, Management Science] find that, in a two-period model, increased product market uncertainty could encourage the firm to invest strategically in the new technology. This paper extends their framework to a continuous-time model which adds flexibility in timing of the investment decision. This flexibility introduces an option value of waiting which increases with uncertainty. In contrast with the two-period model, despite the existence of the strategic option of becoming a market leader due to a lower marginal cost, more uncertainty always increases the expected time to invest. Furthermore, it is shown that under increased uncertainty the probability that the firm finds it optimal to invest within a given time period always decreases for time periods longer than the optimal time to invest in a deterministic case. For smaller time periods there are contrary effects so that the overall impact of increased uncertainty on the probability of investing is in this case ambiguous.

Keywords: investment under uncertainty, strategic option exercise, preemption games

JEL classification: C61, D81, G31

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^γCorresponding author, Department of Econometrics and Operations Research, and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, the Netherlands, email: g.pawlina@kub.nl, phone: + 31 13 4663178, fax: + 31 13 4663066.

^zDepartment of Econometrics and Operations Research, and CentER, Tilburg University, the Netherlands.

1 Introduction

The purpose of this paper is to investigate the firm's asset replacement decision under product market uncertainty and imperfect competition in a fully dynamic framework. The need for such an analysis emerges from the fact that the existing literature provides in general only mixed conclusions concerning the impact of uncertainty on capital budgeting decisions in the presence of strategic interactions among the firms. Since the understanding of this relationship is not only relevant for the corporate planners but also plays a significant role for policy makers, we attempt to (at least partially) solve the existing puzzle.

Modern theory of investment under uncertainty (cf. McDonald and Siegel [9], Dixit and Pindyck [2], Ch. 2) predicts that, under either perfect competition or when the firm is a monopolist, the firm will wait longer with investing if uncertainty is higher. This results from the fact that investment is irreversible and the firm has an option to postpone it until some uncertainty is resolved. However, if (i) more than one firm hold the investment opportunity, and (ii) the firm's investment decision directly influences payoffs of other firm(s), the impact of uncertainty on the investment is twofold. First, increasing uncertainty enhances the value of the option to wait. Second, the value of an early strategic investment (made in order to achieve the first mover advantage) can significantly increase as well.

As already mentioned at the beginning, there exists no unique answer to the question concerning the direction of the investment-uncertainty relationship. Huisman and Kort [6] prove that in a continuous-time duopoly model with product uncertainty (cf. Grenadier [4] and Smets [12]) the effect on the optimal investment threshold of the change in option value of waiting is always stronger than the impact of strategic interactions. This implies a negative relationship between uncertainty of the firm profit flow and investment. On the contrary, Kulatilaka and Perotti [8] find that product market uncertainty may, in some cases, stimulate investment. The latter authors analyze a two-period setting in which (one of the) duopolistic firms can invest in a cost-reducing technology. The payoff from investment is convex in the size of the demand since an increase of demand has a more-than-proportional effect on the realized duopolistic profits (firms are responding to higher demand by increasing both output and price). Based on Jensen's inequality Kulatilaka and Perotti [8] conclude that higher volatility of the product market can accelerate investment.

In this paper we transform the approach of Kulatilaka and Perotti [8], who analyze a capital budgeting decision under product market uncertainty, to continuous time. Consequently, the fully dynamic framework allows us for incorporating the option to postpone the investment. We show that, despite the strategic effect encouraging earlier replacement of the old technology by the first mover (leader), the demand level triggering the investment as well as its expected timing increase with uncertainty for both firms. Furthermore, the probability of such an investment within a given time interval always decreases with uncertainty for time intervals longer than the time to invest in a deterministic case. For shorter intervals contrary effects arise, which implies that the

overall impact of increased uncertainty on the probability of investment in a given time interval is ambiguous (see also Sarkar [10]).

The model is presented in Section 2. In Section 3 only one firm is able to invest, while investment for both firms is allowed in Section 4. Section 5 considers a new market model. In each of the Sections 3-5 the effects of uncertainty on the various investment thresholds are determined. Section 6 examines how these results can be translated into conclusions with respect to investment timing. Section 7 concludes.

2 Framework of the Model

We consider a profit-maximizing risk-neutral firm operating in a duopoly, in which, in line with basic microeconomic theory (as well as with Kulatilaka and Perotti [8]), the following inverse demand function holds¹

$$p_t = A_t - \alpha Q_t; \quad (1)$$

where p_t is the price of a non-durable good/service and can be interpreted as the instantaneous cash flow per unit sold, A_t is a measure of the size of the demand and Q_t is the total amount of the good supplied to the market at a given instant. We introduce the following formulation for the uncertainty in demand

$$dA_t = \beta A_t dt + \gamma A_t dw_t; \quad (2)$$

where β is the instantaneous drift parameter, γ is the instantaneous standard deviation, dt is the time increment and dw_t is the Wiener increment. The firm is competing with its symmetric rival in quantities (a la Cournot).² The constant marginal cost of supplying a unit of the good to the product market is K : As in Kulatilaka and Perotti [8], a new, cost-efficient technology exists that reduces the marginal cost from K to k . In order to acquire the asset representing the cost-efficient technology, the firm has to bear an irreversible cost I : I can be interpreted as a present value of the expenditure associated with installing the new asset at the time of switching the production into the more cost-efficient mode net of the present value of the selling price of the asset representing the old technology.

The associated profits of the firm i (the other firm is denoted by j) are

¹Alternatively, we could replace the assumption of the firm being risk neutral by the replicating portfolio argument.

²Quantity competition yields the same output as a two-stage game in which the capacities are chosen first and, subsequently, the firms are competing in prices (see Tirole [13], p. 216).

as follows

$$\mathcal{V}_t^{00} = \frac{1}{9} (A_t - K)^2; \quad (3)$$

$$\mathcal{V}_t^{10} = \frac{1}{9} (A_t + K - 2k)^2; \quad (4)$$

$$\mathcal{V}_t^{01} = \frac{1}{9} (A_t - 2K + k)^2; \text{ and} \quad (5)$$

$$\mathcal{V}_t^{11} = \frac{1}{9} (A_t - k)^2; \quad (6)$$

where superscript 1 (0) in \mathcal{V}_t^{ij} indicates which firm invested (did not invest) in the cost-reducing technology.

Consequently, our task is to determine the optimal timing of investment in the asset representing the cost-efficient technology. Let us consider the value of the firm before it has invested and denote it by F . Using the dynamic programming methodology (see Dixit and Pindyck [2]) we arrive at the following Bellman equation

$$rF = \frac{1}{2} \mathcal{V}_t^2 A_t^2 F^{00} + \mathcal{V}_t^1 A_t F^0 + \mathcal{V}_t^B; \quad (7)$$

where \mathcal{V}_t^B denotes the instantaneous profit flow before the firm has invested and r is an instantaneous interest rate. If the firm invests as first, \mathcal{V}_t^B is equal to \mathcal{V}_t^{00} (see (3)). If the other firm has already invested, \mathcal{V}_t^B equals \mathcal{V}_t^{01} (cf. (5)). Solving the differential equation for $\mathcal{V}_t^B = \mathcal{V}_t^{00}$ gives³

$$F = CA_t^{-\beta} + \frac{1}{9} \frac{A_t^2}{r - \beta} + \frac{2}{9} \frac{KA_t}{r - \beta} + \frac{K^2}{9r}; \quad (8)$$

where C is a constant and β is the positive root of the following equation⁴

$$\frac{1}{2} \mathcal{V}_t^2 (\beta - 1) + \mathcal{V}_t^1 - r = 0; \quad (9)$$

3 One Firm Monopolizing the Investment Opportunity

Consider first the case in which only one firm has the opportunity of replacing the existing technology with the cost-efficient one. From (4) and Ito's lemma it is obtained that the value of the firm after the investment equals

$$\begin{aligned} V^N(A_t) &= E_t \int_t^{\infty} \frac{1}{9} (A_s + K - 2k)^2 e^{i r(s-t)} ds \\ &= \frac{1}{9} \frac{A_t^2}{r - \beta} + \frac{2}{9} \frac{(K - 2k) A_t}{r - \beta} + \frac{(K - 2k)^2}{9r}; \end{aligned} \quad (10)$$

³The case $\mathcal{V}_t^B = \mathcal{V}_t^{01}$ corresponding to the follower's adoption is considered in Section 4.

⁴Note that the boundary condition $F(0) = 0$ implies that the negative root of (9) can be ignored.

To derive the optimal investment threshold we apply the value-matching and smooth-pasting conditions (see Dixit and Pindyck [2]) to (7) and (10), which leads to

$$CA_t^- = \frac{1}{9} \frac{4(K_i - k)A_t}{r_i} + \frac{4k(K_i - k)}{r} \quad (11)$$

$$-CA_t^{-1} = \frac{4(K_i - k)}{9r_i} \quad (12)$$

Consequently, we obtain the optimal investment threshold

$$A^N = \frac{I + \frac{4k(K_i - k)}{9r}}{\frac{4}{9}(K_i - k)} (r_i) \quad (13)$$

and the optimal timing of investment

$$T^N = \inf \{t | A_t \geq A^N\} \quad (14)$$

We consider the case where the investment cost, I , and the drift rate, μ , satisfy $I > \frac{4k(K_i - k)}{9r}$ and $\mu < r$. Unless the first inequality holds, the firm always invests at the initial point of time.⁵ Violating the second condition leads to the situation when it is never optimal to exercise the replacement option. Note that the optimal threshold (13) is increasing in uncertainty and in the wedge $r_i - \mu$.⁶

Now, it is possible to express the value of the firm in terms of known parameters (for derivation, see Appendix)

$$V^N(A_t) = \begin{cases} \frac{1}{9} \frac{4(K_i - k)A_t}{r_i} + \frac{4k(K_i - k)}{r} + \frac{A_t^2}{2r_i} & \text{if } A_t \leq A^N; \\ \frac{1}{9} \frac{4(K_i - k)A_t}{r_i} + \frac{4k(K_i - k)}{r} + \frac{A_t^2}{2r_i} + I & \text{if } A_t > A^N; \end{cases} \quad (15)$$

Before the investment is undertaken, the value of the firm consists of two components: the present value of cash flows from the assets in place (first row in

⁵The boundary solution is equivalent to the situation in which the firm invests at $t = 0$. If $I < \frac{4k(K_i - k)}{9r}$ then the instantaneous gain from investment will always be greater than the related cost and the firm will adopt the new technology irrespective of the realization of the stochastic random variable. This may be seen upon analyzing the Bellman equation describing the dynamics of the value of the firm before and after adopting new technology.

Before adopting we have

$$rF = \frac{1}{2} \mu^2 A_t^2 F'' + \mu A_t F' + \frac{1}{9} (A_t - K)^2;$$

whereas after making the investment we obtain

$$rF = \frac{1}{2} \mu^2 A_t^2 F'' + \mu A_t F' + \frac{1}{9} (A_t + K - 2k)^2 + I;$$

where I is interpreted as an instantaneous perpetuity equivalent to the investment cost I .

Unless $I > \frac{4}{9}k(K_i - k)$; the RHS of the second equation is always larger than of the first one. This implies reaching a boundary solution, i.e. investing immediately in optimum, irrespective from the current realization of the process A_t :

⁶Increasing wedge $r_i - \mu$ has also an indirect effect via increasing A^N but that effect is dominated.

(15)) and the option to purchase a cost-reducing technology (second row). After incurring the investment cost, the value of the firm consists of the cash flows based on the more cost-efficient technology (last row).

The impact of uncertainty on the optimal investment threshold of the firm having the exclusive investment opportunity can be calculated by directly differentiating (13) with respect to σ^2 . Consequently, we obtain that

$$\frac{\partial A^N}{\partial (\sigma^2)} = i \frac{1}{(-i-1)^2} \frac{1 + \frac{4}{9} \frac{k(K_i - k)}{r}}{\frac{4}{9} (K_i - k)} (r - i - \theta) \frac{\partial}{\partial (\sigma^2)} > 0; \quad (16)$$

since $\frac{\partial}{\partial (\sigma^2)} < 0$ (see Dixit and Pindyck [2], p. 143). Therefore, if only one firm in a duopoly has the opportunity to invest in a cost-reducing technology, the uncertainty always increases the level of market demand required to undertake the investment.

4 Two Firms Having the Possibility to Invest

In this section we relax the assumption that only one firm has the investment opportunity. In Section 4.1 we establish the payoffs in case the firm replaces the technology as second (follower), first (leader) and at the same time as the competitor. The equilibria are presented in Section 4.2, while Section 4.3 investigates the effects of uncertainty on the investment thresholds.

4.1 Payoffs

4.1.1 Follower

Define t_i to be the moment of time at which the leader invests. At time t_i the value of the follower firm is

$$V^F(A_t) = E_t \int_{t_i}^{T^F} \frac{1}{9} (A_s - 2K + k)^2 e^{i r (s_i - t)} ds + E_t e^{i r (T^F - t)} \int_{T^F}^{\infty} \frac{1}{9} (A_s - k)^2 e^{i r (s_i - T^F)} ds; \quad (17)$$

where

$$T^F = \inf \{t | A_t \geq A^F\}; \quad (18)$$

A^F is defined as

$$A^F = \frac{-i-1}{i-1} \frac{1 + \frac{4K}{9r} (K_i - k)}{\frac{4}{9} (K_i - k)} (r - i - \theta); \quad (19)$$

where $I > \frac{4K(K_i k)}{9r}$: Analogous to (15), the value of the follower at time t , V^F , can now be expressed as

$$V^F(A_t) = \begin{cases} \frac{1}{9} \frac{A_t^2}{r_i 2^{2i} 3^{4i}} + \frac{2(2K_i k)A_t}{r_i} + \frac{(2K_i k)^2}{r} & \text{if } A_t \leq A^F; \\ \frac{1}{9} \frac{4(K_i k)A^F}{3 r_i} + \frac{4K(K_i k)}{r} + \frac{A_t}{A^F} & \text{if } A_t > A^F; \end{cases} \quad (20)$$

The interpretation is similar to the case where only one firm has the investment opportunity. The first row of (20) is the present value of profits when the other firm has a cost advantage, and the second row corresponds to the value of the option to invest in the new technology. The last row is the present value of cash flows generated with the use of the more efficient technology minus the replacement cost.

4.1.2 Leader

Following a similar reasoning as in the previous section, we present the payoffs of the firm that invests as first. Consequently, the value function of the leader evaluated after the replacement of the existing technology, i.e. at time t , is

$$V^L(A_t) = E_t \int_t^{T^F} \frac{1}{9} (A_s + K_i - 2k)^2 e^{i r(s-t)} ds + \int_{T^F}^{\infty} \frac{1}{9} (A_s - k)^2 e^{i r(s-t)} ds; \quad (21)$$

This can be rewritten into

$$V^L(A_t) = \begin{cases} \frac{1}{9} \frac{A_t^2}{r_i 2^{2i} 3^{4i}} + \frac{2(K_i - 2k)A_t}{r_i} + \frac{(K_i - 2k)^2}{r} & \text{if } A_t \leq A^F; \\ \frac{1}{9} \frac{2(K_i k)A^F}{3 r_i} + \frac{(K_i - 2k)^2 k^2}{r} + \frac{A_t}{A^F} & \text{if } A_t > A^F; \end{cases} \quad (22)$$

4.1.3 Simultaneous Replacement

The value function in case of the simultaneous replacement, V^J , is

$$V^J(A_t) = E_t \int_t^{T^a} \frac{1}{9} (A_s - K)^2 e^{i r(s-t)} ds + \int_{T^a}^{\infty} \frac{1}{9} (A_s - k)^2 e^{i r(s-t)} dt + e^{i r(T^a-t)}; \quad (23)$$

where

$$T^a = \inf(t | A_t \geq A^a) \quad (24)$$

for some $A^s > A_0$, and A_0 denotes the realization of the process at $t = 0$: The value of the investment opportunity when the replacement is simultaneous can therefore be expressed as

$$V^J(A_t; A^s) = \frac{1}{3} \frac{A_t^2}{r_i} + \frac{2K_i A_t}{r_i} + \frac{K_i^2}{r} \quad \text{if } A_t \leq A^s; \\ + \frac{1}{9} \frac{2(K_i - k)A^s}{r_i} + \frac{K_i^2 - k^2}{r} \quad \text{if } A_t > A^s; \quad (25)$$

The optimal timing of simultaneous replacement is

$$T^S = \inf \{ t | A_t = A^s \} \quad (26)$$

where

$$A^s = \frac{1 + \frac{1}{9} \frac{K_i^2 - k^2}{r}}{1 - \frac{2}{9} \frac{K_i - k}{r}} (r_i); \quad (27)$$

and $1 > \frac{2K_i(K_i - k)}{9r}$: Consequently, the value of the firm in the simultaneous replacement case when the investment is optimal can be denoted as

$$V^S(A_t) = V^J(A_t; A^s); \quad (28)$$

4.2 Equilibria

Since both firms are ex ante identical, it seems natural to consider symmetric exercise strategies and assume the endogeneity of the firms' roles, i.e. that it is not determined beforehand which firm will get the leader role. There are two types of equilibria that can occur under this choice of strategies.

4.2.1 Sequential Replacement

The first type is a sequential replacement equilibrium where one firm is the leader and the other one is the follower. Figure 1 depicts the payoffs associated with the sequential equilibrium. Let us define A^P to be the smallest root of

$$V^L(A_t) = V^F(A_t); \quad (29)$$

Since on the interval $[A^P; A^F]$ the payoff of the leader is higher than the payoff of the follower (cf. Figure 1), each of the two firms will have an incentive to be the leader. In the search for equilibrium we reason backwards. At A^F the firms are indifferent between being the leader and the follower. However, an instant before, say at $A^F - \epsilon$, the payoff from being the leader is higher than the payoff of the follower. Therefore (without loss of generality) firm i has an incentive to invest there. Firm j anticipates this and would invest at $A^F - 2\epsilon$: Repeating this reasoning we reach an equilibrium in which one of the firms invests at A^P and the other waits until demand exceeds A^F .

Note that if both firms invest at A^P with probability one, they end up with the low payoff $V^J(A^P; A^P)$: At $A_t = A^P$ demand is too low for the investment to be profitable for both firms. Therefore, the firms use mixed strategies in which the expected payoff is equal to the payoff of the follower (let us recall that the roles of the firms are not predetermined). The firms are identical, so that they both have equal probability of becoming leader or follower. In Huisman and Kort [7] it is shown that the probability of a firm to become leader, P^L (or follower, P^F) equals

$$P^L = P^F = \frac{1}{2} \frac{p(A_t)}{p(A_t)}; \quad (30)$$

where

$$p(A_t) = \frac{V^L(A_t) - V^F(A_t)}{V^L(A_t) - V^J(A_t)}; \quad (31)$$

Consequently, the probability of joint investment leading to the low payoff $V^J(A_t)$ is $\frac{p(A_t)}{2 \cdot p(A_t)}$:

For $A_t < A^P$ the leader payoff curve lies below the follower curve what implies that it is optimal for both firms to refrain from investment. For $A_t = A^P$; the leader and the follower values are equal. Therefore (30) and (31) yield the probabilities of being the leader (or follower) equal to $\frac{1}{2}$: The leader invests at the moment that $A_t = A^P$, which is the smallest solution of $V^L(A_t) = V^F(A_t)$, and the follower waits until A^F is reached. If the stochastic process starts at $A_0 > A^P$; then the sequential replacement equilibrium entails the leader investing immediately (at A_0) and the follower waiting until A^F . In this case, according to (30) and (31), $p(A_0) > 0$ since the payoff of the leader exceeds the payoff of the follower. This makes the probability of making a "mistake" and investing jointly become positive.⁷

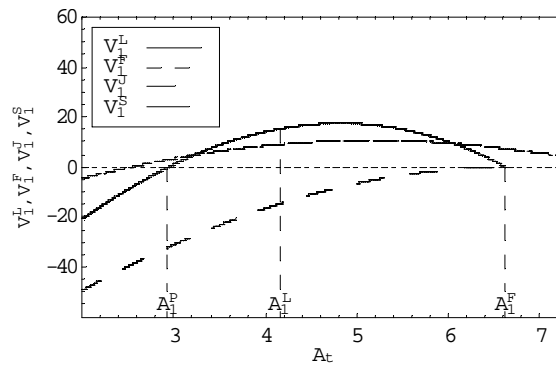


Figure 1. The differences between the value functions of the leader, V^L , optimal simultaneous replacement, V^S , early simultaneous replacement, V^J , and the value

⁷A basic reference on continuous-time preemption games is Fudenberg and Tirole [3].

function of the follower, V^F , for the set of parameter values: $K = 2$; $k = 0$; $r = 0:05$; $\theta = 0:015$; $\beta = 0:1$; and $I = 60$: For $A_0 < A^P$ the set of input parameters results in a sequential replacement at A^P (leader) and A^F (follower).

4.2.2 Simultaneous Replacement

The second type of equilibrium is the simultaneous replacement equilibrium. In such a case, the firms invest with probability 1 at the same point in time defined already by A^S (see (27)). No firm has an incentive to deviate from this equilibrium since at A^S the payoff of this strategy exceeds all other payoffs.⁸ A graphical illustration of the simultaneous equilibrium is depicted in Figure 2 below.

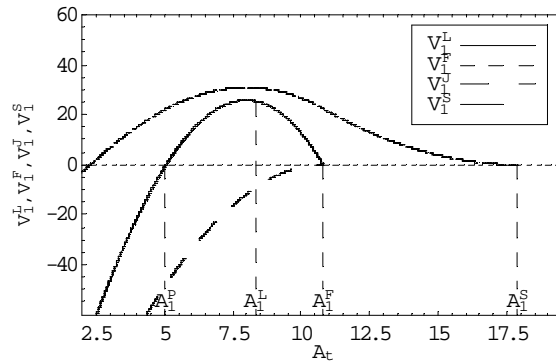


Figure 2. The differences between the value functions of the leader, V^L , optimal simultaneous replacement, V^S , early simultaneous replacement, V^J , and the value function of the follower, V^F , for the set of parameter values: $K = 2$; $k = 0$; $r = 0:05$; $\theta = 0:015$; $\beta = 0:1$; and $I = 120$: The set of input parameters results in the optimality of a simultaneous replacement at A^S :

The occurrence of a particular type of equilibrium is determined by the relative payoffs that depend on the value of the model parameters. The sequential equilibrium occurs when

$$9A_t \geq \frac{I}{K}; A^F \text{ such that } V^L(A_t) > V^S(A_t); \quad (32)$$

i.e. when for some A_t it is more profitable to become a leader than to replace simultaneously. Otherwise simultaneous replacement is the Pareto-dominant

⁸Of course, the payoffs resulting from the sequential replacement equilibrium in 4.2.1 may be lower than those associated with the optimal joint investment. However, occurrence of the sequential replacement equilibrium is due to the fact that for the same values of A_t the leader's payoff exceeds the value from the joint replacement strategy. It is the lack of coordination among the firms (with possible transfer of excess value) that leads to ex post Pareto-inefficient outcomes. In case of simultaneous replacement equilibrium the payoff of the leader never exceeds the payoff from joint optimal investment and therefore the preemption equilibrium, while still exists, is Pareto-dominated.

equilibrium. Proposition 1 says that firms replace sequentially if the investment cost is sufficiently low.

Proposition 1 A unique I^* exists such that $I > I^*$ simultaneous replacement is the Pareto-dominant equilibrium.

Proof. See Appendix. ■

4.3 Uncertainty and Investment Thresholds

First, we investigate the impact of volatility on the optimal investment thresholds of the follower and for simultaneous replacement. In these cases (see (19) and (27)) the optimal thresholds, A^{opt} , can be expressed as

$$A^{opt} = \frac{1}{i-1} f(I; K; k; r; \sigma): \quad (33)$$

It is straightforward to derive that

$$\frac{\partial A^{opt}}{\partial \sigma^2} = i \frac{1}{(i-1)^2} f(I; K; k; r; \sigma) \frac{\partial}{\partial \sigma^2} > 0; \quad (34)$$

i.e. that the optimal investment thresholds of the follower and for simultaneous replacement increase in uncertainty.

Now, we investigate the impact of volatility on the optimal investment threshold of the leader. In the remaining part of the analysis the marginal cost k is set to zero in order to simplify calculations.⁹ We already know that the entry threshold of the leader is determined by the point A^P , which is the smallest root of $\pi(A_t)$: Consequently, we calculate the derivative of $\pi(A_t)$ with respect to the market uncertainty. The change of (29) resulting from a marginal increase in σ^2 can be decomposed as follows

$$\frac{d\pi(A_t)}{d\sigma^2} = \frac{\partial \pi(A_t)}{\partial \sigma^2} + \frac{\partial \pi(A_t)}{\partial A^F} \frac{dA^F}{d\sigma^2} \quad (35)$$

The derivative $\frac{\partial \pi(A_t)}{\partial \sigma^2}$ measures directly the influence of uncertainty on the net benefit of being the leader. The product $\frac{\partial \pi(A_t)}{\partial A^F} \frac{dA^F}{d\sigma^2}$ reflects the impact on the net benefit of being the leader of the fact that the follower's investment threshold increases with uncertainty.

It is easy to show that

$$\frac{\partial \pi(A_t)}{\partial \sigma^2} < 0; \quad (36)$$

$$\frac{\partial \pi(A_t)}{\partial A^F} \frac{dA^F}{d\sigma^2} > 0; \quad (37)$$

⁹For the majority of e.g. intangible/information products this is a fairly good approximation (cf. Shapiro and Varian [11]).

Therefore, at first sight, the joint impact of both effects is ambiguous. (36) represents the simple value of waiting argument: if uncertainty is large, it is more valuable to wait for new information before undertaking the investment (Dixit and Pindyck [2]). As we have just seen, this also holds for the follower. The implication for the leader of the follower investing later is that the leader has a cost advantage for a longer time. This makes an earlier investment of the leader more beneficial. This effect is captured by (37), which can thus be interpreted as an increment in the strategic value of becoming the leader vs. the follower resulting from the delay in the follower's entry. Obviously, the latter effect is not present in the monopolistic/perfectly competitive markets, where the impact of uncertainty is unambiguous.

However, it is possible to show that the direct effect captured by (36) dominates, irrespective of the values of the input parameters.

Proposition 2 When uncertainty in the product market increases, the threshold of the leader increases, too.

Proof. See Appendix. ■

An example of the resulting leader investment thresholds is presented in Table 1. It is shown how the leader investment threshold is affected by uncertainty and the unit production cost of the old technology.

$\frac{3}{4}$	$K = 0:3$	$K = 0:8$	$K = 1:06$	$K = 1:3$	$K = 1:8$	$K = 3$
0:05	21:40	8:17	6:33	5:33	4:17	3:24
0:10	23:61	9:13	7:08	5:94	4:65	3:59
0:15	27:02	10:44	8:09	6:79	5:30	4:07
0:20	31:19	12:05	9:32	7:82	6:10	4:67
0:25	36:09	13:93	10:78	9:04	7:04	5:38
0:30	41:73	16:10	12:45	10:44	8:12	6:19
0:35	48:11	18:56	14:35	12:03	9:35	7:12
0:40	55:27	21:32	16:48	13:82	10:73	8:16
0:45	63:22	24:38	18:84	15:80	12:27	9:32
0:50	71:96	27:75	21:45	17:97	13:96	10:59

Table 1. The optimal leader investment threshold for the set of parameter values: $r = 0:05$; $\sigma = 0:015$; $k = 0$ and $I = 60$:

From Proposition 2 and Table 1 it can be concluded that, unlike in Kulatilaka and Perotti [8], the optimal investment threshold of the leader responds to both volatility and the gain from investment (via K) in a qualitatively similar way as a non-strategic threshold, i.e. it increases with uncertainty. The reason for this result is the following. First, in our model we introduced the possibility to delay the investment. Increased uncertainty could raise the profitability of investment but this holds even more for the value of the option to wait. Second,

Kulatilaka and Perotti [8] explain that in their case uncertainty could be beneficial for investment because of the convex shape of the net gain function, where the gains arise due to the cost reducing investment. Then, while performing a mean preserving spread, downside losses are more than compensated by upside gains. In the continuous time model, however, the net gain function is always linear. If the leader invests, the profit flow $\frac{1}{4}^{00}$ is replaced by the profit flow $\frac{1}{4}^{10}$; and it is clear from (3) and (4) that $\frac{1}{4}^{10}$; $\frac{1}{4}^{00}$ is linear in the stochastic variable A_t . The same holds for the follower investment ($\frac{1}{4}^{11}$; $\frac{1}{4}^{01}$ linear) and simultaneous investment (linearity of $\frac{1}{4}^{11}$; $\frac{1}{4}^{00}$). To see whether the convexity argument could also work here, we consider new market entry in the next section. Then the net gain flows of both the leader and the follower are convex in A_t :

5 New Market Entry

Two firms have an option to invest in a production asset that would enable them to operate in a new market where there is no incumbent. The new market assumption implies, in contrast with Sections 3-4, that the firms can only start realizing profits after incurring a sunk cost I : It still holds that demand follows the stochastic process (2). The marginal cost of a unit of output after launching production is set to $k = 0$:

First, we calculate the optimal threshold of the follower in the new market. After, by now, familiar steps it is obtained that

$$A^{FN} = 3 \frac{S}{r_i - 2\theta_i - \frac{3}{4}\sigma^2} I \quad (38)$$

It is straightforward to show that

$$\frac{\partial A^{FN}}{\partial (\frac{3}{4}\sigma^2)} > 0: \quad (39)$$

Moreover, for a relatively high degree of uncertainty, i.e. for $\frac{3}{4}\sigma^2 > r_i - 2\theta_i$ (what corresponds to $\bar{\sigma} > 2(1; 2)$), the follower will never invest since beyond this level the value of the option to invest always exceeds the net present value of investing so that it is optimal to never exercise the option.

Define t_i to be the moment of investment of the leader. The value of the follower at t_i is equal to

$$V_t^{FN} = \begin{cases} < \frac{8\mu}{r_i - 2\theta_i - \frac{3}{4}\sigma^2} I \left(\frac{A_t}{A^{FN}} \right)^{\frac{1}{\mu}} & \text{if } A_t < A^{FN}; \\ \frac{1}{9} \frac{A_t^2}{r_i - 2\theta_i - \frac{3}{4}\sigma^2} I & \text{if } A_t > A^{FN}; \end{cases} \quad (40)$$

The value of the leader at $t = t_i$ can be expressed as

$$V_t^{LN} = \begin{cases} \frac{1}{4} \frac{A_t^2}{r_i 2^{\otimes} i^{\frac{3}{2}}} & \text{if } A_t \leq A^{FN}; \\ \frac{5}{36} \frac{A_t^2}{r_i 2^{\otimes} i^{\frac{3}{2}}} \frac{A_t}{A^{FN}} & \text{if } A_t > A^{FN}; \end{cases} \quad (41)$$

The optimal threshold of the leader is the smallest solution of the following equation

$$V^{LN}(A_t) - V^{FN}(A_t) = \frac{1}{4} \frac{A_t^2}{r_i 2^{\otimes} i^{\frac{3}{2}}} - \frac{1}{4} \frac{A_t^2}{r_i 2^{\otimes} i^{\frac{3}{2}}} \frac{A_t}{A^{FN}} = 0: \quad (42)$$

The impact of uncertainty on the optimal investment threshold of the leader is not straightforward. Similar as in the model with an already existing market, there are two effects: the effect of the waiting option and of the strategic option. Let us denote $V^{LN}(A_t) - V^{FN}(A_t)$ by $\mathfrak{N}(A_t)$: We have

$$\frac{d\mathfrak{N}(A_t)}{d(\frac{3}{4}^2)} = \frac{\partial \mathfrak{N}(A_t)}{\partial (\frac{3}{4}^2)} + \frac{\partial \mathfrak{N}(A_t)}{\partial A^{FN}} \frac{dA^{FN}}{d(\frac{3}{4}^2)} + \frac{\partial \mathfrak{N}(A_t)}{\partial \sigma} + \frac{\partial \mathfrak{N}(A_t)}{\partial A^{FN}} \frac{dA^{FN}}{d\sigma} \frac{\partial \sigma}{\partial (\frac{3}{4}^2)}: \quad (43)$$

The derivative (43) consists of four components. The first and the second reflect the direct impact of the product market volatility on the waiting option and the strategic option, respectively. As in the previous section, the last two components correspond to the impact of uncertainty on the waiting and strategic options via parameter σ . The presence of the components reflecting the direct impact of volatility is a consequence of the convexity of the payoff (profit) in the underlying process (demand). This feature results in $\frac{3}{4}^2$ directly entering the expectation of the cumulative discounted future profits via a discount rate, $r_i 2^{\otimes} i^{\frac{3}{2}}$, corresponding to this component of the profit stream that is proportional to the square of the underlying stochastic variable A_t .

Since the analysis of the signs of the components of (43) evaluated at the leader's preemption point, A^{PN} , provides very little insight into the sign of the whole derivative, we substitute the functional forms of $V^{LN}(A_t)$ and $V^{FN}(A_t)$ into $\mathfrak{N}(A_t)$ and calculate the derivative explicitly (see Appendix). After doing so, the following result is obtained.

Proposition 3 The optimal investment threshold of the leader increases in uncertainty in the case of new market entry.

Proof. See Appendix. ■

The conclusion is that also in the case of new market entry uncertainty increases the investment thresholds. The implication is that the convex payoff argument by Kulatilaka and Perotti [8] is outweighed by the increased option value of waiting.

6 Uncertainty and Investment Timing

The aim of this paper is to analyze the impact of uncertainty and strategic interactions on the timing of the optimal exercise of the option to invest. By now we analyzed the impact of uncertainty and strategic interactions on the optimal investment threshold of the firm. Although thresholds and timing have a lot to do with each other, it cannot be concluded in general that the relation between the two is monotonic (cf. Sarkar [10]). In this section we investigate the relationship between uncertainty, optimal threshold, expected timing of asset replacement and the probability with which the threshold is reached within a time interval of a given length.

First, let us observe that the expectation of the first passage time equals¹⁰

$$E_t [T^a] = \frac{1}{\sigma^2} \frac{1}{\frac{1}{2}\sigma^2} \ln \frac{A^a(\frac{1}{2}\sigma^2)}{A_t}; \quad (44)$$

where $A^a(\frac{1}{2}\sigma^2)$ denotes the optimal investment threshold as a function of uncertainty. We note that expectation (44) tends to infinity for $\frac{1}{2}\sigma^2 \rightarrow 2\sigma^2$ and does not exist for $\frac{1}{2}\sigma^2 > 2\sigma^2$.¹¹ Consequently, for $\frac{1}{2}\sigma^2 < 2\sigma^2$ we have

$$\frac{\partial E_t [T^a]}{\partial (\frac{1}{2}\sigma^2)} = \frac{1}{2\sigma^2} \frac{1}{\frac{1}{2}\sigma^2} \ln \frac{A^a(\frac{1}{2}\sigma^2)}{A_t} + \frac{1}{\sigma^2} \frac{1}{\frac{1}{2}\sigma^2} \frac{\frac{dA^a}{d(\frac{1}{2}\sigma^2)}}{A^a(\frac{1}{2}\sigma^2)} > 0; \quad (45)$$

The expected time of investment increases in uncertainty due to two effects. First, for any given threshold, the expected first passage time is increasing in uncertainty (cf. the first component of the RHS of (45)). Second, for a fixed level of uncertainty, an increase in the optimal investment threshold leads to an increase in the expected time to reach (cf. second component of RHS of (45)). Based on (45) it can be concluded that whenever the threshold goes up due to more uncertainty, it also holds that the expected time to invest increases.

An alternative approach to measure the impact of uncertainty on the timing of investment is to look at the probability with which the threshold is reached within a time interval of a given length, say δ . After substituting

¹⁰For a derivation of the probability distribution of the first passage time see Harrison [5] for a formal exposition and Dixit [1] for a more heuristic approach.

¹¹Increasing $\frac{1}{2}\sigma^2$ beyond $2\sigma^2$ implies that the probabilities of surviving without reaching the threshold before a given time do not fall sufficiently fast for longer hitting times (moreover, the probability that the process will reach the barrier in infinity is still positive). Since the expectation is the sum of the product of the first passage times and their probabilities, an insufficient decay in the survival probabilities (without reaching the threshold) results in the divergence of the expectation.

$y = \ln \frac{A^a}{A_t}$ in the formula (8.11) in Harrison [5] and rearranging, we obtain

$$P(T < \zeta) = \Phi\left(\frac{\bar{A} \ln \frac{A^a}{A_t} + i^{\otimes} i^{\frac{1}{2} \frac{3}{4} \zeta}}{\frac{3}{4} \zeta}\right) + \frac{\mu A^a \Gamma\left(\frac{2}{3} i^{\frac{2}{3}} + 1\right)}{A_t} \Phi\left(\frac{\bar{A} \ln \frac{A^a}{A_t} + i^{\otimes} i^{\frac{1}{2} \frac{3}{4} \zeta}}{\frac{3}{4} \zeta}\right); \quad (46)$$

where T denotes the time to reach the threshold and $\Phi(\cdot)$ is the standard normal cumulative density function. As already pointed out in Sarkar [10], the derivative $\frac{\partial P(T < \zeta)}{\partial \zeta}$ does not have an unambiguous sign and it can thus be shown that, in general, uncertainty can affect the probability of reaching the threshold by a given time in both directions.

First, we illustrate the relationship between the first passage time, volatility and related probabilities for the follower threshold since it is unaffected by the strategic considerations. Subsequently, we present results of simulations related to the threshold of the leader. In this part we use the model of Section 4 but the results for the new market entry model are qualitatively similar.

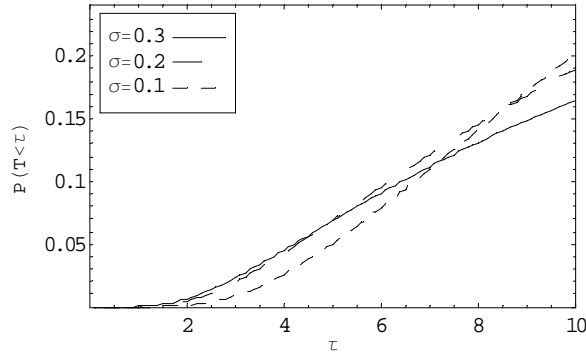


Figure 3. The cumulative probability of reaching the optimal follower investment threshold as a function of time for the set of parameter values: $A_t = 4$; $r = 0.05$; $\otimes = 0.015$; $\frac{3}{4} = 0.1$; 0.2 and 0.3 ; $K = 3$; $k = 0$ and $I = 60$:

From Figure 3 it can be concluded that the probability of reaching the follower threshold always increases with the time interval which is of course trivial. Furthermore, it can be seen that growing uncertainty raises the probability of reaching the threshold for low ζ , while the opposite is true for high ζ : This observation results from the fact that in the absence of uncertainty the optimal investment trigger is reached at a specified point in time with probability 1. Increasing volatility spreads the probability mass around this point what leads to an increased cumulative chance of reaching the trigger at points in time situated to the left of this specified point in time, while the reverse is true for the point situated to the right.

The relationship between uncertainty and probabilities of reaching the optimal follower investment threshold is depicted in Figure 4.

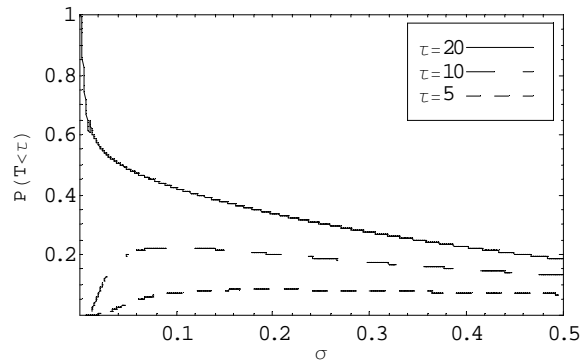


Figure 4. The cumulative probability of reaching the optimal follower investment threshold as a function of demand uncertainty for the set of parameter values: $A_t = 4$; $r = 0.05$; $\theta = 0.015$; $\zeta = 5, 10$ and 20 ; $K = 3$; $k = 0$ and $I = 60$:

Based on Figure 4 it can be concluded that the form of relationship between uncertainty and the probability of reaching the threshold depends on the time to reach. For high values of the time to reach, the cumulative probability of reaching the threshold decreases in volatility since the probability mass of the first passage time density function moves to the right. For low values of ζ the probability of reaching the threshold first increases due to a spread in the probability mass. However, since the density function is skewed, the spread occurs asymmetrically and for high volatilities the cumulative probability of reaching the threshold becomes smaller again.

The results of simulations concerning the relationship between uncertainty, the first passage time and the probabilities of reaching the leader threshold are presented in Table 2 below.

$\frac{3}{4}$	$\zeta = 1$	$\zeta = 2$	$\zeta = 5$	$\zeta = 10$	$\zeta = 15$	$\zeta = 20$
0:05	0:06	2:39	24:11	54:32	71:17	80:97
0:10	0:61	5:93	26:79	47:94	59:70	67:24
0:15	0:70	5:85	23:98	41:70	51:59	58:03
0:20	0:62	5:14	21:00	36:47	45:10	50:72
0:25	0:53	4:48	18:51	32:19	39:77	44:66
0:30	0:46	3:97	16:50	28:66	35:31	39:55
0:35	0:42	3:59	14:88	25:71	31:53	35:17
0:40	0:39	3:30	13:57	23:22	28:29	31:39
0:45	0:37	3:09	12:47	21:08	25:48	28:09
0:50	0:36	2:93	11:55	19:23	23:02	25:21

Table 2. The cumulative probability (in percentages) of reaching the optimal leader investment threshold as a function of demand uncertainty for the set parameter values: $A_t = 2$; $r = 0.05$; $\theta = 0.015$; $k = 0$, $K = 3$ and $I = 60$: The optimal timing of the replacement in a deterministic case equals $\zeta = 9:36$.

The probability of investment of the leader, despite the presence of strategic effects, responds to changes in uncertainty and time to reach in a similar way as the corresponding probabilities of the follower. For low λ 's an increase in the probability of investing is faster than for high λ 's. Moreover, for high λ 's the probability of investing is always decreasing in uncertainty, while for low λ 's the probability behaves in a non-monotonic way. These latter observations are confirmed by the following proposition, which results from developing further the observation made by Sarkar [10].

Proposition 4 Define

$$\lambda^* = \frac{1}{\sigma} \ln \frac{A^*}{A_t}; \quad (47)$$

as the point in time at which the investment threshold A^* is reached in a deterministic case. Then it holds that for $\lambda < \lambda^*$ the probability of reaching the investment threshold A^* before λ increases in uncertainty at a relatively low level of uncertainty and decreases for its relatively high level, whereas for $\lambda > \lambda^*$ the probability of reaching the optimal threshold before λ always decreases in uncertainty.

Proof. See Appendix. ■

We conclude that when under increased uncertainty the threshold increases, this implies that the probability that the firm invests within a given amount of time decreases when this amount of time is sufficiently large. However, when this amount of time is sufficiently low there are two contradictory effects. On the one hand, the investment probability goes up because higher volatility enhances the chance of reaching a particular threshold early. On the other hand, this probability eventually goes down with uncertainty since the first passage time density function becomes more skewed to the right when uncertainty increases.

7 Conclusions

The purpose of this paper was to analyze the effects of uncertainty on the decision of the duopolistic firm to exercise the option to replace an existing production asset with a new one, corresponding to a more cost-efficient technology. In comparison to past contributions to the strategic real options literature, our model allows for flexibility concerning the optimal timing of the investment and, at the same time, incorporates the convexity of the profits in the stochastic process associated with market uncertainty. We found that, irrespective of the value of input parameters, the direct effect of uncertainty (related to the waiting option) on the investment threshold of the leader is larger than the indirect effect (strategic option) resulting from the delay in the follower's entry. Furthermore, it was found that the expected investment timing increases with uncertainty.

This result supports the view that uncertainty delays investment, even in the presence of strategic interactions combined with a convex profit function.

Our paper contradicts Kulatilaka and Perotti [8], who found that, under strong strategic advantage, increased uncertainty encourages investment in growth options. The reason for this contradiction is that in the two-period model investment is a now-or-never decision, while in our continuous-time model it is possible to delay investment. The latter feature implies that there exists an option value of waiting, which increases with uncertainty. Despite the fact that the value of the project, when undertaken immediately, can rise with uncertainty, the effect of the higher value of the option to wait dominates.

Finally, we look at the probability of investing within a certain time interval. Here, the point in time in which the firm invests optimally in a deterministic case plays a crucial role. If we take a time interval that contains this point in time, then the probability of investing within this interval decreases with uncertainty. However, if the time interval is that short that the optimal investment time corresponding to the deterministic case lies outside this interval, then the investment probability goes up with uncertainty when uncertainty is low while it goes down otherwise.

8 Appendix

Derivation of (15). The value of the firm at time $t < T^N$ is equal to the sum of discounted stream of cash flow generated by using the old technology (what corresponds to the instantaneous profit $\frac{1}{9}A_t^2$) and the discounted stream of cash flow obtained after adopting the new technology at T^N (corresponding to $\frac{1}{9}A_t^2$). Consequently, we are able to write

$$V^N(A_t) = E_t \int_t^{T^N} \frac{1}{9} (A_s - K)^2 e^{i r(s-t)} ds + E_t e^{i r(T^N-t)} \int_{T^N}^{\infty} \frac{1}{9} (A_s + K - 2k)^2 e^{i r(s-T^N)} ds \quad (48)$$

After applying Ito's lemma, working out the expectations, and observing that

$$E_t e^{i r(T^N-t)} = \frac{\mu A_t}{A^N} \quad (49)$$

we obtain

$$= \frac{1}{9} \frac{\mu}{r} \frac{A_t^2}{r_i^2} + \frac{2KA_t}{r_i} + \frac{K^2}{r} + \frac{\mu A_t}{A^N} + \frac{1}{9} \frac{A_t^2}{r_i^2} + \frac{2(K - 2k)A_t}{r_i} + \frac{(K - 2k)^2}{r} + \frac{\mu A_t}{A^N} \quad (50)$$

This directly leads to (15). ■

Derivation of (38). At the optimal threshold of the follower the value of the option to wait equals the NPV of the project. The latter can be expressed by

$$F(A_t) = \int_{T^{FN}}^{\infty} \frac{1}{9} A_s^2 e^{i r (s_i - T^{FN})} ds_i \quad (51)$$

where T^{FN} denotes the point of time the investment is made. The value-matching and smooth-pasting conditions corresponding to the new market entry are therefore as follows

$$CA_t^- = \frac{1}{9 r i} \frac{A_t^2}{2 i^{3/2}} \quad (52)$$

$$-CA_t^{-1} = \frac{2}{9 r i} \frac{A_t}{2 i^{3/2}} \quad (53)$$

Solving the system of equations yields (38). ■

Proof of Proposition 1. First, let us de...

$$^3(A_t) = V^S(A_t) - V^L(A_t) \quad (54)$$

After substitution we get

$$^3(A_t) = i \frac{4 K A_t}{9 r i} + l + \frac{\frac{1}{i-1} l + \frac{K^2}{9r}}{(A^S)^-} + \frac{\frac{1}{2-i} l + \frac{4K^2}{9r} + \frac{K^2}{9r}}{(A^F)^-} A A_t^- \quad (55)$$

for $A_t \in [K; A^F]$: From (32) it follows that if the minimum of $^3(A_t)$ on the interval $[K; A^F]$ is smaller than zero, a sequential equilibrium occurs.¹² Otherwise, the firms enter simultaneously. The existence of a negative minimum of $^3(A_t)$ depends, as mentioned above, on the value of the input parameters. The minimum of $^3(A_t)$ occurs for

$$A = \frac{\frac{1}{i-1} l + \frac{K^2}{9r}}{\frac{1}{i-1} l + \frac{K^2}{9r} + \frac{1}{2-i} l + \frac{4K^2}{9r} + \frac{K^2}{9r}} \quad (56)$$

It is sufficient to show that

$$\frac{d^3(A_t)}{dl} \Big|_{A_t=A} = \frac{\partial^3(A_t)}{\partial l} + \frac{\partial^3(A_t)}{\partial A_t} \frac{dA}{dl} > 0 \quad (57)$$

From (55) we derive that

$$\frac{d^3(A_t)}{dl} = 1 + i \frac{\mu}{A^S} \frac{A_t}{i} - \frac{1}{2-i} \frac{l + \frac{K^2}{9r}}{l + \frac{4K^2}{9r}} \frac{\mu}{A^F} \frac{A_t}{i} \quad (58)$$

¹²Strictly speaking, the equilibrium with sequential entry still exists in this case but is Pareto-dominated by the simultaneous entry equilibrium (cf. Fudenberg and Tirole [3]).

Subsequently, we substitute for A_t in (58) the expression (56) for A . Although an analytical proof is not possible, numerically we are able to show that $\frac{d^3 \pi(A_t)}{d\mu^3} \Big|_{A_t=A}$ is positive for $\mu \in [1; 1)$; $\theta \in (0; 1)$; $r \in (0; 1)$; $K \in \mathbb{R}_{++}$ and $l \in (\frac{4}{9} \frac{K^2}{r}; 1)$.¹³

■ Proof of Proposition 2. A substitution for (36) and (37) in the first factor of (35) yields

$$\frac{d\pi(A_t)}{d\mu} = \frac{\frac{\mu}{i-1} \left[\frac{3}{2} l + \frac{2}{3} \frac{K^2}{r} \right] - \frac{K^2}{3r} \mu \left[l \ln \frac{A^F}{A_t} + \frac{1}{i-1} \left[\frac{1}{2} l + \frac{1}{3} \frac{K^2}{r} \right] \right]}{\frac{A^F}{A_t}}; \quad (59)$$

Since the investment threshold of the leader is equal to A^P , and A^P is the smallest root of the concave function $\pi(A_t)$; we know that

$$\frac{d\pi(A_t)}{dA_t} \Big|_{A_t=A^P} > 0; \quad (60)$$

Consequently, from the envelope theorem we conclude that it is sufficient to show that

$$\frac{d\pi(A_t)}{d\mu} \Big|_{A_t=A^P} > 0 \quad (61)$$

to conclude that the investment threshold of the leader is increasing in uncertainty (decreasing in μ). Moreover, we know from (59) that $\frac{d\pi(A_t)}{d\mu}$ changes its sign only once and the corresponding realization of A_t to the zero value of the derivative is

$$A = A^F e^{-\frac{\frac{1}{2} l + \frac{1}{3} \frac{K^2}{r}}{(\frac{1}{2} l + \frac{1}{3} \frac{K^2}{r}) + \frac{K^2}{3r} + 1}}; \quad (62)$$

Therefore

$$\frac{d\pi(A_t)}{d\mu} > 0 \iff A_t < A; \quad (63)$$

Consequently, $\frac{d\pi(A)}{d\mu} > 0$ would imply that $A > A^P$ and $\frac{d\pi(A_t)}{d\mu} \Big|_{A_t=A^P} > 0$: In order to prove that $\frac{d\pi(A)}{d\mu} > 0$, we plug (62) into (29) to obtain

$$\begin{aligned} \frac{d\pi(A)}{d\mu} &= \frac{\mu}{i-1} \left[\frac{3}{2} l + \frac{2}{3} \frac{K^2}{r} \right] e^{-\frac{\frac{1}{2} l + \frac{1}{3} \frac{K^2}{r}}{(\frac{1}{2} l + \frac{1}{3} \frac{K^2}{r}) + \frac{K^2}{3r} + 1}} - \frac{K^2}{3r} \mu \left[l \ln \frac{A^F}{A} + \frac{1}{i-1} \left[\frac{1}{2} l + \frac{1}{3} \frac{K^2}{r} \right] \right] e^{-\frac{\frac{1}{2} l + \frac{1}{3} \frac{K^2}{r}}{(\frac{1}{2} l + \frac{1}{3} \frac{K^2}{r}) + \frac{K^2}{3r} + 1}}; \quad (64) \end{aligned}$$

¹³A package Mathematica 3.0 is used in the proof of Propositions 1 - 3. The code is available from the authors.

An analytical proof is again not possible but numerically it can be shown that $\frac{d\bar{A}}{d(\frac{3}{4})} > 0$ is positive for $\beta \in [1; 2)$; $\rho \in \mathbb{R}$; $r \in (\rho; 1)$; $K \in \mathbb{R}_{++}$ and $L \geq \frac{4}{9} \frac{K^2}{r}$; $1 \leq \dots$ ■

Proof of Proposition 3. First we substitute the functional forms of the value functions into (42) and rewrite the derivative (43) as

$$\frac{d\bar{A}^N(A_t)}{d(\frac{3}{4})} = \frac{1}{4} \frac{A_t^2}{(r - \frac{1}{2} - \frac{\rho}{4})^2} \left[\frac{5}{4} \frac{1}{2} \frac{\mu}{\rho} \frac{A_t}{A^{FN}} \right] + \frac{8}{5} \frac{\mu}{\rho} \frac{A_t}{2(r - \frac{1}{2} - \frac{\rho}{4})} + \ln \frac{\mu}{\rho} \frac{A_t}{A^{FN}} \frac{\rho}{\rho(\frac{3}{4})} \quad (65)$$

Since an explicit solution of A^{PN} cannot be derived, we proceed as follows. First, we take a particular point $\bar{A} > A^{PN}$: Second, we show that $\frac{d\bar{A}^N(A_t)}{d(\frac{3}{4})}$ is negative for all $A_t \in (\underline{A}; \bar{A})$; where \underline{A} is a realization of A_t such that $\underline{A} < A^{PN}$. Let us define

$$\bar{A} = \frac{1}{\frac{1}{2} - \frac{\rho}{4}} \left[(r - \frac{1}{2} - \frac{\rho}{4}) \right] \quad (66)$$

First, we show that $\bar{A}^N > 0$ what would imply that $\bar{A} > A^{PN}$. After substituting (66) into (42) we obtain

$$\bar{A}^N = \frac{2L}{\frac{1}{2} - \frac{\rho}{4}} \left[\frac{1}{8} + 1 + \frac{\mu}{3} \right] = \frac{2L}{\frac{1}{2} - \frac{\rho}{4}} \hat{A}(\bar{A}) \quad (67)$$

Since $\beta > 2$ (recall that for $\beta \geq 2$ no firm ever invests), we know that $\frac{2L}{\frac{1}{2} - \frac{\rho}{4}}$ is always positive. Therefore we are interested only in the sign of $\hat{A}(\bar{A})$: For $\beta \neq 2$ we obtain

$$\lim_{\beta \rightarrow 2} \hat{A}(\bar{A}) = 0 \quad (68)$$

Then we establish that

$$\frac{\partial \hat{A}(\bar{A})}{\partial \beta} = \frac{\mu}{3} \frac{\rho}{\rho} - \frac{\mu}{8} + \frac{\mu}{8} + 1 + \ln \frac{\mu}{3} > 0 \quad (69)$$

for $\beta \in (2; 1)$. This implies that \bar{A}^N is positive which implies that $\bar{A} > A^{PN}$. Furthermore, we prove that (65) changes signs twice, i.e. it is positive for $A_t \in (0; \underline{A}) \cup (\bar{A}; A^{FN})$, where \bar{A} is some realization of A_t such that $\bar{A} > \bar{A}$, and negative otherwise. First, we express (65) as

$$\frac{d\bar{A}^N(A_t)}{d(\frac{3}{4})} = A_t^2 \left[K A_t^{-1} + L A_t^{-2} \right] \ln \frac{\mu}{\rho} \frac{A_t}{A^{FN}} + M \quad (70)$$

where

$$K = i^{-\frac{5}{4}} \frac{I}{2} A^{FN} \zeta_i^{-\bar{A}} \frac{\bar{A}}{\theta(\frac{3}{4}^2)} + \frac{-i - \frac{8}{5} \zeta_i^{-\bar{A}}}{2(r_i 2^{\theta} i^{\frac{3}{4}^2})}; \quad (71)$$

$$L = i^{-\frac{5}{4}} \frac{I}{2} A^{FN} \zeta_i^{-\bar{A}} \mu^{-\frac{8}{5}} \frac{\theta}{\theta(\frac{3}{4}^2)} > 0; \text{ and} \quad (72)$$

$$M = \frac{\frac{1}{4}}{(r_i 2^{\theta} i^{\frac{3}{4}^2})^2} > 0; \quad (73)$$

From (70) - (73) we know that¹⁴

$$\lim_{A_t \neq 0} K A_t^{-i^2} + L A_t^{-i^2} \ln \frac{A_t}{A^{FN}} \mu^{\theta} + M = M; \text{ and} \quad (74)$$

$$\lim_{A_t \rightarrow 1} K A_t^{-i^2} + L A_t^{-i^2} \ln \frac{A_t}{A^{FN}} \mu^{\theta} + M = 1; \quad (75)$$

Moreover

$$\begin{aligned} & \frac{\theta}{A_t} \mu^{\theta} K A_t^{-i^2} + L A_t^{-i^2} \ln \frac{A_t}{A^{FN}} \mu^{\theta} + M \\ &= A_t^{-i^3} (-i^2) K + (-i^2) L \ln \frac{A_t}{A^{FN}} \mu^{\theta} + L \end{aligned}$$

what implies that there exists only one optimum of $\frac{d\bar{A}^N(A_t)}{d(\frac{3}{4}^2)}$ that is different from zero. This result, combined with (74) and (75), implies that $\frac{d\bar{A}^N(A_t)}{d(\frac{3}{4}^2)}$ is negative at most in only one interval. Substituting \bar{A} into (65) yields

$$\begin{aligned} \frac{d\bar{A}^N(A_t)}{d(\frac{3}{4}^2)} \Big|_{A_t=\bar{A}} &= \frac{-\frac{2I}{i^2}}{r_i 2^{\theta} i^{\frac{3}{4}^2}} i^{-\frac{5}{4}} \frac{\mu^{\theta}}{3} \\ & \leq \frac{\theta}{\theta(\frac{3}{4}^2)} + \mu^{-\frac{8}{5}} \frac{\theta \mu^{\theta}}{2(r_i 2^{\theta} i^{\frac{3}{4}^2})} + \ln \frac{\mu^{\theta}}{3} \frac{\theta}{\theta(\frac{3}{4}^2)}. \end{aligned} \quad (76)$$

Numerically it can be shown that $\frac{d\bar{A}^N(A_t)}{d(\frac{3}{4}^2)} \Big|_{A_t=\bar{A}}$ is negative for $\theta \in [2; 1)$; $\theta \in \mathbb{R}$; $r \in (\theta; 1)$ and $I \in (0; 1)$. Therefore the only remaining part of the proof is to show that $\bar{A} < A^{PN}$ for any vector of input parameters. Since the explicit analytical forms of \bar{A} and A^{PN} do not exist, we use a numerical procedure. For any given vector of input parameters (from the domains as in the preceding part of the proof), we calculate the difference $A^{PN} - \bar{A}$ and show that it is positive. Given that $\frac{d\bar{A}^N(A_t)}{d(\frac{3}{4}^2)} \Big|_{A_t \in (\bar{A}; \bar{A})} < 0$; $A^{PN} \in (\bar{A}; \bar{A})$ and $\bar{A}^N i^{-\frac{5}{4}} \zeta_i^{-\bar{A}} > 0$, we conclude that $\frac{dA^{PN}}{d(\frac{3}{4}^2)} > 0$; i.e. the investment threshold of the leader increases in uncertainty. ■

¹⁴The result (74) has been derived using the de l'Hôpital rule and observing that A_t^{2i} explodes in the neighborhood of zero faster than $\ln A_t$:

Proof of Proposition 4. First, we show that \tilde{t}^a is the time to reach the investment threshold A^a in a deterministic case. After observing that $x = \sigma t$ is a solution to $dx = \sigma dt$ with an initial condition $x_0 = 0$; and substituting $x^a = \ln \frac{A^a}{A_t}$, we obtain

$$\ln \frac{A^a}{A_t} = \sigma \tilde{t}^a; \quad (77)$$

so \tilde{t}^a is the time to reach the threshold A^a . Now, we consider a density function $f(\tilde{t}; \sigma; \sigma^2)$. For a moment we assume that $\tilde{t}^a = \tilde{t}^a$ irrespective from σ : Then increasing σ is equivalent to performing a mean preserving spread. Consequently in such a case,

$$\frac{\partial}{\partial \sigma} \int_0^{\tilde{t}^a} f(s) ds \quad (\tilde{t} < \tilde{t}^a) < 0; \quad (78)$$

When $f(\tilde{t}; \sigma; \sigma^2)$ is the density function of the first passage time for a geometric Brownian motion, $E[\tilde{t}]$ is increasing in σ (cf. (45)) when A^a is increasing in σ too. Therefore, there is another effect contributing to the sign of derivative $\frac{\partial}{\partial \sigma} \int_0^{\tilde{t}^a} f(s) ds$. For $\tilde{t} > \tilde{t}^a$, an increase in uncertainty not only reduces the probability mass to the left from \tilde{t} via the mean preserving spread but also because of the mean itself moving to the right. Therefore the effect of uncertainty on the probability of investing is in this region unambiguous and negative. For $\tilde{t} < \tilde{t}^a$ the probability of investing by given \tilde{t} decreases to zero. The latter conclusion is true since from (46) it is obtained that

$$\begin{aligned} \lim_{\sigma \rightarrow 1} P(T < \tilde{t}) &= \lim_{\sigma \rightarrow 1} \int_0^{\tilde{t}^a} \frac{1}{\sigma} \frac{1}{\tilde{t}} \frac{1}{2\sigma^2} \tilde{t}^a \\ &+ \lim_{\sigma \rightarrow 1} \int_{\tilde{t}^a}^{\tilde{t}} \frac{1}{\sigma} \frac{1}{\tilde{t}} \frac{1}{2\sigma^2} \tilde{t}^a \frac{\mu A^a}{A_t} \frac{2\sigma}{\sigma^2} \tilde{t}^a \\ &= \lim_{\sigma \rightarrow 1} \int_0^{\tilde{t}^a} \frac{1}{\sigma} \frac{1}{\tilde{t}} \frac{1}{2\sigma^2} \tilde{t}^a \frac{\mu A^a}{A_t} = 0 \end{aligned} \quad (79)$$

for

$$\lim_{\sigma \rightarrow 1} A^a = 1; \quad (80)$$

For $\tilde{t} < \tilde{t}^a$, the two effects work in the opposite direction. As in the previous case, the mean $E[\tilde{t}]$ is increasing in uncertainty. Without a change in the volatility, an increase in the mean would then decrease the probability of investing. However, increasing uncertainty results in more probability mass being now present in the left tail of $f(\tilde{t})$. Therefore, the total effect of increasing uncertainty is in this region ambiguous. However, we are able to conclude that the probability of investing at a given \tilde{t} behaves in a certain non-monotonic way. For $\sigma = 0$; there is no probability mass on the interval $[0; \tilde{t})$. Therefore an

increase in uncertainty leads initially to the increased probability of investment. For relatively large $\frac{K}{A}$ the effect of moving the mean of the distribution to the right starts to dominate and the probability of investment falls. For $\frac{K}{A} \rightarrow 1$ the probability of investing by given λ decreases to zero.

Finally, we show that all the thresholds increase in uncertainty monotonically and unboundedly. We already know (from Proposition 2 and 3) that the optimal investment thresholds increase in uncertainty monotonically. So now we only have to prove that the thresholds grow in uncertainty unboundedly. For the thresholds of the follower and in case of joint replacement it is easy to observe that $\frac{K}{A} \rightarrow 1$ tends to infinity when $\frac{K}{A} \rightarrow 1$.¹⁵ The investment threshold of the leader requires slightly more attention.¹⁶ We already know that the leader invests as soon as the stochastic variable reaches the smallest root of the following equation (cf. (29))

$$0 = \frac{2}{3} \frac{KA_t}{r_i} - \frac{K^2}{3r} + \frac{1}{i} \left(\frac{2}{3} \frac{KA^F}{r_i} - \frac{K^2}{3r} + \frac{1}{A^F} \right) \quad (81)$$

After substituting (19) and rearranging, we obtain

$$0 = \frac{\frac{4}{9}KA_t}{-\frac{1}{i-1} + \frac{4K^2}{9r} + (r_i)} + \frac{\frac{K^2}{3r} + \frac{1}{A_t}}{\frac{2}{3} \frac{K}{r_i} - \frac{1}{i-1} + \frac{4K^2}{9r} + (r_i)} \quad (82)$$

It holds that

$$\lim_{\frac{K}{A} \rightarrow 1} \left(\frac{\frac{4}{9}KA_t}{-\frac{1}{i-1} + \frac{4K^2}{9r} + (r_i)} + \frac{\frac{K^2}{3r} + \frac{1}{A_t}}{\frac{2}{3} \frac{K}{r_i} - \frac{1}{i-1} + \frac{4K^2}{9r} + (r_i)} \right) = 0 \quad (83)$$

and

$$\lim_{\frac{K}{A} \rightarrow 1} \frac{\frac{4}{9}KA_t}{-\frac{1}{i-1} + \frac{4K^2}{9r} + (r_i)} < 0 \quad (84)$$

Now, we are looking for the solution of

$$0 = m(x) x_i^{-n}; \text{ where } 8x \in \mathbb{R}^{++}; m(x); n > 0; \quad (85)$$

¹⁵For new market model the similar conclusion can be drawn after the substitution of parameters in the original geometric Brownian motion.

¹⁶The unboundedness of the leader threshold in the new market entry can be proven in a similar way as in the presented case of technology adoption.

such that $m(x)$ is tending to zero from above (this is guaranteed by (84)) for $8x \in \mathbb{R}^{++}$ when the uncertainty is increasing. Consequently, any solution (so the smallest one as well) of (85) is tending to infinity. This is equivalent to

$$\lim_{\#1} A^P = 1 : \quad (86)$$

what completes the proof. ■

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