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Finite project life and uncertainty effects on investment

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Abstract

This paper revisits the important result of the real options approach to investment under uncertainty, which states that increased uncertainty raises the value of waiting and thus decelerates investment. Typically in this literature projects are assumed to be perpetual. However, in today's economy firms face a fast-changing technology environment, implying that investment projects are usually considered to have a finite life. The present paper studies investment projects with finite project life, and we find that, in contrast with the existing theory, investments may be accelerated by increased uncertainty. It is shown that this particularly happens when uncertainty is limited and project life is short.

JEL classification: D92; E22; G31

Keywords: Investment; Uncertainty; Finite project length

1 Introduction

The standard theory of the real options approach to investment, as clearly explained in Dixit and Pindyck (1994)¹, states that uncertainty in combination with irreversibility creates a value of the option to wait with undertaking capital investments. Over time more information becomes available, which enables the decision maker to make better investment decisions at a later date.

As an illustration consider the following basic real options problem (Dixit and Pindyck (1994, Ch. 6)), where a sunk investment cost, I , has to be incurred in return for a project whose value, $V(Q)$, is a function of stochastic revenue flow Q per time unit. Obviously, $V(Q)$ is increasing in Q . The simple net present value rule is to invest whenever Q exceeds Q^0 , where $V(Q^0) = I$. However, the optimal rule prescribes that investment is only optimal when $V(Q)$ is at least as large as a threshold value $V(Q^*)$ that exceeds I . The difference between $V(Q^*)$ and I is caused by the value of waiting. It is shown that Q^* , and thus the value of waiting, goes up with uncertainty. For this reason the general prediction of the real options literature is that a higher level of uncertainty will have a negative effect on investment.

In this paper we revisit the conclusion that "a higher level of uncertainty will have a negative effect on investment". To do so we adopt the standard framework with contingent claims valuation of the investment opportunity and change it in one aspect: where the vast majority of the real options literature assumes projects to be perpetual, we allow for the project life to be finite.² Clearly, most, if not all, real-life investment projects have a finite life. This is especially true in today's knowledge economy, in which innovations limit the economic lifetime of technologies. Our main result is that the investment threshold decreases with uncertainty in case uncertainty is limited and the project life is short. So, changing the project life from infinite to finite can imply a negative relationship between uncertainty and the value of waiting, which reverses the basic real options result.

To be more precise, an increase in uncertainty affects the investment threshold in three different ways. The first effect is the *discounting effect*. An increase of uncertainty raises the discount rate via the risk premium component. This reduces the net present value of the investment and thus raises the investment threshold. The second effect is the *volatility effect*, which affects the value of the option to wait positively: higher uncertainty increases the upside potential payoff from the option, leaving the downside payoff unchanged at zero (since the option will not be exercised at low payoff values). This increased option value implies that the firm has more incentive to wait, which also increases the investment threshold. The third effect of an increase of uncertainty on the investment threshold is the *convenience yield effect*. The increase of asset riskiness raises the discount rate and thus also the convenience yield of the investment opportunity. This decreases the value of waiting, so that it is more attractive to invest earlier resulting in a lower investment threshold.

¹Some more recent contributions include studies of implications of learning (Decamps and Mariotti (2004), Thijssen, Huisman and Kort (2006)), agency (Grenadier and Wang (2005)), business cycle (Guo, Miao and Morellec (2005)), policy change (Pawlina and Kort (2005)), and implications to capital structure choices (Miao (2005)), mergers and acquisitions dynamics (Morellec and Zhdanov (2005)), or exit strategies (Murto (2004)).

²Notably, Majd and Pindyck (1987) discuss some implications of finite project life on real options modeling. While they provide some arguments for the finite project life assumption to be omitted, these considerations turn out to play an important role in our analysis.

The discounting and volatility effects thus raise the investment threshold, while the convenience yield effect works in the opposite direction. It can be shown, and it is also intuitively clear, that the size of the discounting effect is small in case of a short project life, while effect two, the volatility effect, is small when uncertainty is limited. Consequently, it is possible for the negative convenience yield effect to dominate the two other effects when the project life is finite and uncertainty is limited. In that case it thus holds that the investment threshold decreases with uncertainty.

We examine the robustness of the non-monotonic effect of uncertainty on investment in the case of a finite project life by considering several variations of the problem. First, we show that this result survives in case the opportunity to invest in the project is available only for a limited amount of time. Next, we prove that this also holds for other relaxations of the infinite project life assumption, like uncertain project duration, capital depreciation or catastrophe risk. Furthermore, we find that general functional forms of the convenience yield preserve the observed relationships.

The impact of uncertainty on investments has been of interest to economists for a long time. One strand of literature relies on convex costs of capital adjustment and convexity of marginal profits in prices. As shown by Hartman (1972) and Abel (1983), in such a setting uncertainty hastens investment. The other important strand of literature, based on the real options theory, acknowledges (partial) irreversibility of investments and predicts that uncertainty delays investment. This paper verifies the latter prediction and shows that the investment trigger is not necessarily increasing in uncertainty. Most closely related papers are Caballero (1991) and Bar-Ilan and Strange (1996). Caballero (1991) considers a perfect competition setting with convex adjustment costs, and he obtains that irreversibility does not lead to the usual negative investment-uncertainty relationship. Bar-Ilan and Strange (1996) assume that there are lags between investment decisions and realizations. Firms have abilities to abandon uncompleted projects in bad times, which creates a convexity in the output and value functions. Bar-Ilan and Strange (1996) find that uncertainty may accelerate as well as decelerate investment depending on specific parameter values. Both papers have in common that they depart from the conventional result of the real options literature, because the models create convexities in line of Hartman (1972) and Abel (1983). Thus it comes with little surprise that in these papers uncertainty may either accelerate or decelerate investment. The result of our paper is unique in the sense that uncertainty may hasten irreversible investment without building on the convexity of the marginal product of capital. Our model remains in the pure real options framework and the reversal of the conventional result builds solely on the contingent claims valuation of investment opportunities and the finite capital lifetime. Moreover, since we only depart from the standard real option framework by imposing a finite lifetime, our model is more general and is thus applicable to more investment situations than Caballero (1991) and Bar-Ilan and Strange (1996).

A different approach to study the relationship between uncertainty and irreversible investments is taken by Sarkar (2000). Sarkar analyzes the probability of investment taking place within a certain time period and points at the fact that an increasing trigger does not automatically mean that investment will be delayed. The difference with our result is that we show

that increased uncertainty may not even lead to an increased trigger.

Beyond this introduction the paper is organized as follows. In the next section we consider the model of the finitely-lived project and derive the optimal investment trigger. Section 3 studies how uncertainty influences the investment decision. In Section 4 we discuss robustness, while Section 5 concludes. All proofs are contained in the Appendix.

2 The model and the optimal investment decision

The basic real options problem, extensively treated in Dixit and Pindyck (1994), considers the problem of a single firm considering investment in a perpetual project. The assumption of a project having an infinite life is useful mostly due to its simplicity. However, in corporate practice the investment projects are usually considered to have a finite life. Certainly, in the fast-changing technology environment that many businesses face, very few investment projects are accurately approximated by the assumption of infinite life.

We thus consider an irreversible investment project with finite life that can be undertaken at any time. After the investment has taken place, the project generates a stochastic revenue of Q_t per unit time. Q_t evolves exogenously according to a geometric Brownian motion

$$dQ_t = \mu Q_t dt + \sigma Q_t dZ_t, \quad (1)$$

where dZ is the increment of a standard Wiener process, μ is the drift parameter and $\sigma (> 0)$ is the volatility parameter that introduces the uncertainty in our model. When the project is undertaken, a one-time investment cost I is paid. For simplicity, the marginal costs are put equal to zero.

The standard methods in real options theory to value an investment opportunity are dynamic programming and contingent claims valuation (Dixit and Pindyck (1994)). In this paper we employ the latter approach. Compared to dynamic programming, the contingent claims approach offers a better treatment of the discount rate, because it is endogenously determined as an implication of the overall equilibrium in capital markets. On the other hand, the contingent claims approach requires that markets are sufficiently complete so that the project's risk can be spanned by traded assets. Making this assumption allows us to analyze the equilibrium impact of the systematic risk on the discount rate and, further, on the value of the investment option and the investment policy by using the intertemporal Capital Asset Pricing Model (CAPM) of Merton (1973).³ The CAPM formula relates the expected return of the project π , the risk-free interest rate r , the correlation of the project return with the return of the market portfolio ρ , and the market price of risk λ as follows:

$$\pi = r + \lambda \rho \sigma. \quad (2)$$

The difference between π , the expected return of the project, and μ , the expected rate

³The assumption that the CAPM holds is in accordance with the related literature. The CAPM brings a linear relationship between the discount rate and σ . In Section 4.3 we show that the results we present also hold for more general discount rates functions. So other equilibrium asset pricing models would yield similar results as long as the risk premium earned on an asset is increasing in the riskiness of the asset in a non-increasing rate.

of change of Q , is referred to as the convenience yield (or return shortfall) of the investment opportunity. The later is denoted by δ and satisfies

$$\delta \equiv \pi - \mu = r + \lambda\rho\sigma - \mu. \quad (3)$$

We assume that $\delta > 0$, which ensures that the investment is ever undertaken; otherwise it is never optimal to exercise the option.

The paper's aim is to study the impact of the level of uncertainty σ on optimal investment behavior. To do so, a special care must be taken in assigning endogenous variables. From (3) we obtain that a change in σ results in a change of π , which must lead to an adjustment of either μ or δ or both. In general, this relation depends on what is assumed to be an endogenous parameter affected by changes in volatility. A certain guideline in this respect could be Pindyck (2004), which relates commodity inventories, spot and future prices and the level of volatility. The equilibrium in cash (spot transactions) and storage markets implies that both the spot price and the convenience yield rise with the volatility, but the convenience yield is affected more directly. The model is estimated for several commodities and the results show that in particular the dynamics of the convenience yields are relatively well explained by changes in volatility. Dynamic simulations of the model show that a volatility shock has a significant effect on the convenience yield and only a small effect on the price. These findings suggest that in our model it is more plausible to assume that δ (the convenience yield) rather than μ (the drift of the revenue or of the commodity price) changes with σ . Moreover, it also seems to be more common in the related literature on the investment-uncertainty relationship to assume that μ is fixed and δ changes with σ (e.g. Sarkar (2000) and Sarkar (2003)). To sum up, in the context of contingent claims valuation of real investment opportunities, the assumption of uncertainty affecting the discount rate and convenience yield appears to be the most plausible one.

The value of the project $V(Q)$ evolves over time and depends on the current realization of Q . Upon installation the project value is equal to the expected present value of the revenue stream discounted by the risk-adjusted discount rate. If the project has a finite life of T years, then the project value at the time of the investment is

$$V(Q) = E \left[\int_0^T e^{-\pi t} Q_t dt \right] = \int_0^T e^{-(\pi-\mu)t} Q dt = Q \frac{1 - e^{-(r+\lambda\rho\sigma-\mu)T}}{r + \lambda\rho\sigma - \mu}. \quad (4)$$

Before the project is installed, the firm holds an option to invest. The option is held until the stochastic revenue flow reaches a sufficiently high level at which it is optimal to exercise the option and invest. The option value $F(Q)$ can be found by the replicating portfolio argument. Employing the standard methods (cf. Dixit and Pindyck (1994)) yields the differential equation

$$\frac{1}{2}\sigma^2 Q^2 F''(Q) + (\mu - \lambda\rho\sigma)QF'(Q) - rF(Q) = 0. \quad (5)$$

The general solution to (5) is given by

$$F(Q) = A_1 Q^{\beta_1} + A_2 Q^{\beta_2}, \quad (6)$$

where β_1 and β_2 are the roots of the quadratic equation

$$L_0 \equiv \frac{1}{2}\sigma^2\beta(\beta - 1) + (\mu - \lambda\rho\sigma)\beta - r = 0, \quad (7)$$

with $\beta_1 > 1$ and $\beta_2 < 0$. The unknown constants A_1 and A_2 together with the optimal investment threshold Q^* are determined by employing the following boundary conditions:

$$F(0) = 0, \quad (8)$$

$$F(Q^*) = V(Q^*) - I, \quad (9)$$

$$F'(Q^*) = V'(Q^*). \quad (10)$$

Condition (8) ensures that the option value will be zero if the revenues are zero (note that from (1) it follows that zero is an absorbing barrier). As β_2 in (6) is negative, for the option value to go to zero as Q goes to zero, we need to impose $A_2 = 0$. Equation (9) is the value matching condition, which equates the value of the option at the exercise moment (at $Q = Q^*$) to the net payoff the firm receives. Condition (10) is the smooth pasting condition. Solving this system yields the investment trigger

$$Q^* = \frac{\beta_1}{\beta_1 - 1} \frac{r + \lambda\rho\sigma - \mu}{1 - e^{-(r+\lambda\rho\sigma-\mu)T}} I, \quad (11)$$

while

$$\beta_1 = \frac{1}{2} - \frac{\mu - \lambda\rho\sigma}{\sigma^2} + \sqrt{\left(\frac{\mu - \lambda\rho\sigma}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}. \quad (12)$$

From (11) the important result of the real options theory follows: Q^* is higher than the level of the revenue flow that would induce investment under the net present value (NPV) rule. In the latter case the investment is undertaken as soon as the risk-adjusted project value exceeds the investment cost, that is at the revenue level equal to $\frac{r+\lambda\rho\sigma-\mu}{1-e^{-(r+\lambda\rho\sigma-\mu)T}} I$. This value is always lower than Q^* in (11), as $\beta_1 > 1$. So there are states where the expected payoff of investment is positive and the firm chooses to wait and not to invest. The option to invest captures this positive value of waiting.

3 The effects of uncertainty on the investment trigger

This section studies the effect of uncertainty on the value of waiting. First, we show that, as usual, the value of waiting, reflected in the level of investment trigger, always increases with uncertainty when the project life is infinite or when discount rates are unaffected by uncertainty. Second, if the equilibrium discount rate contains a positive risk premium, we derive that the value of waiting decreases with uncertainty in case of finite project lives and limited uncertainty. Finally, we provide an economic analysis of these results.

3.1 Monotonicity results

We start out with the basic real options result for the investment project with infinite life.

Proposition 1 *If the project life is infinite, the investment trigger increases with uncertainty.*

Hence, in case of an infinite project life the effect of uncertainty on the investment trigger is unambiguously positive. This is the standard real options result, which says that the value of waiting increases with uncertainty. This is reflected by higher trigger values, because then the revenue must reach a higher level before investment is optimally undertaken.

Now, let us move on to the finite life project case. We first consider the scenario where the impact of systematic risk is absent or not priced by the market. This implies that the discount rate is constant, and requires that either the market price of risk is zero, $\lambda = 0$, or that the correlation of the project return with the return of the market portfolio is zero, $\rho = 0$.

Proposition 2 *If the discount rate is constant, the relationship between the investment trigger and uncertainty is always positive.*

Proposition 2 states that, in the absence of the risk premium effect the investment trigger always increases with uncertainty irrespective of the project lifetime, which is again the usual real options result. It is important to point out, however, that the conditions necessary for constant discount rates ($\lambda = 0$ or $\rho = 0$) are in general difficult to accept in the context of investment models; see discussions in e.g. Zeira (1990) and Sarkar (2003).

We can also show that in case of a negative risk premium (possible if either the correlation of the project return with the return of the market portfolio or the market price of risk is negative), the usual positive relationship arises.

Proposition 3 *If $\lambda\rho < 0$, then the relationship between the investment trigger and uncertainty is always positive.*

3.2 Non-monotonicity result

We proved in the previous subsection that both in the model with a project of infinite life and in the model without or with negative risk premium, the impact of uncertainty on the investment trigger is always positive. These are interesting special and limit cases; however, the assumptions of Propositions 1 and 2 are serious abstractions from reality, and the negative risk premium condition of Proposition 3 is a relatively rare phenomenon in the markets. Next, we turn to the most common situation where the project life is finite and the discount rate is set in the capital market equilibrium with a positive risk premium. We now show that, the effect on the trigger is no longer monotonic in uncertainty.

Proposition 4 *If the project life is finite and $\lambda\rho > 0$, the uncertainty effect on the investment trigger is non-monotonic: it decreases in σ for low levels of σ and then increases. The length of the σ -interval where the negative effect occurs, is negatively related to the project lifetime.*

Figure 1 presents some numerical examples, where the parameter values correspond to earlier work on the investment-uncertainty relationship, in particular to Sarkar (2000). We see that indeed there is a negative relation between σ and Q^* for lower values of σ . The effect is more pronounced for short-term projects, but even in the case of a 30-year project Q^* decreases

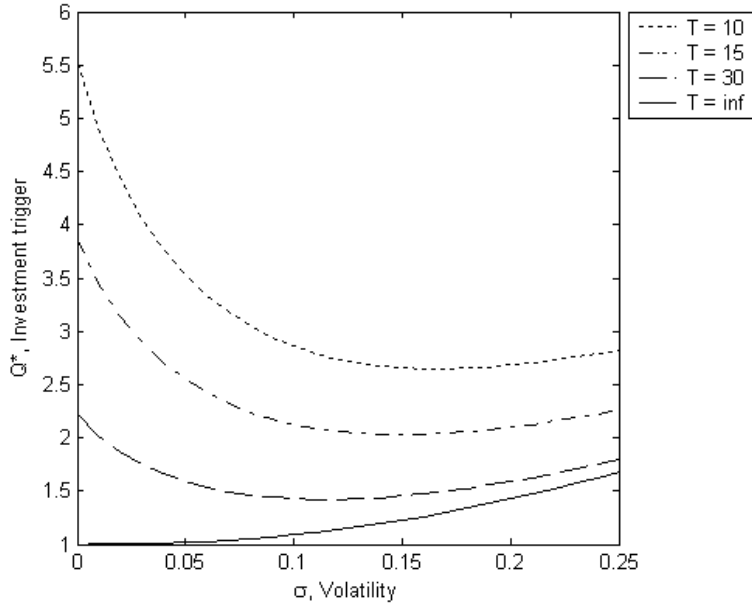


Figure 1: Investment trigger as a function of volatility for various project lengths T and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$.

until σ is around 0.12. The example shows that the positive effect of uncertainty on investment (negative on the trigger) arises for economically relevant parameter values. The figure, of course, also confirms that for an infinitely long project the relation is monotonic and increasing in line of the results in Proposition 1.

3.3 Economic analysis of the non-monotonicity result

From (3) and (11) it follows that the trigger can also be expressed as

$$Q^* = \frac{\beta_1}{\beta_1 - 1} \frac{\delta}{1 - e^{-\delta T}} I. \quad (13)$$

At this point it is convenient to trace all the variables that are affected by uncertainty and consider the trigger as a function of three parameters: $Q^*(\sigma, \delta(\sigma), \beta_1(\sigma, \delta(\sigma)))$. Then the derivative of the investment trigger with respect to σ can be decomposed into three effects in the following way:

$$\frac{d}{d\sigma} Q^*(\sigma, \delta(\sigma), \beta_1(\sigma, \delta(\sigma))) = \underbrace{\frac{\partial Q^*}{\partial \delta} \frac{\partial \delta}{\partial \sigma}}_{\text{Discounting effect}} + \underbrace{\frac{\partial Q^*}{\partial \beta_1} \frac{\partial \beta_1}{\partial \sigma}}_{\text{Volatility effect}} + \underbrace{\frac{\partial Q^*}{\partial \beta_1} \frac{\partial \beta_1}{\partial \delta} \frac{\partial \delta}{\partial \sigma}}_{\text{Convenience yield effect}}. \quad (14)$$

The first term on the right-hand side measures the impact of revenue uncertainty on the rate used to discount the project value. Rising uncertainty increases the discount rate, which reduces the net present value of the investment project. This implies that it is less profitable to invest in this project, which leads to an increase of the trigger value. Consequently, as is straightforward to derive, the discounting effect is always positive.

Since they both affect the trigger value via β_1 , the second and the third term of (14) reflect the influence of uncertainty on the value of the option to wait. Below we refer to these two effects combined as the (combined) option effect. The volatility effect, which is represented by the derivative $\frac{\partial Q^*}{\partial \beta_1} \frac{\partial \beta_1}{\partial \sigma}$, captures the direct impact of uncertainty on the value of the option to wait. Higher uncertainty increases the upside potential payoff from the option, leaving the downside payoff unchanged at zero (since the option will not be exercised at low payoff values). This is the well-known positive impact of uncertainty on the option value. An increased option value implies that the firm has more incentive to wait. This raises the opportunity cost of investing so that the investment trigger will increase. Hence, the effect is unequivocally positive.

The product $\frac{\partial Q^*}{\partial \beta_1} \frac{\partial \beta_1}{\partial \delta} \frac{\partial \delta}{\partial \sigma}$ in (14) represents the impact of uncertainty on the option value through the convenience yield. Increased uncertainty raises the risk premium of the expected rate of return and thus also the convenience yield, which in turn elevates the opportunity cost of holding the option and consequently decreases its value. For this reason it is attractive to invest earlier, which reduces the trigger.

Summarizing, we conclude that the discounting and volatility effects are positive, while the convenience yield effect is negative. The following proposition shows how the uncertainty level and the project length influence the relative size of the three effects.

Proposition 5 (i) Define $\hat{\sigma} = \{ \sigma : (\beta_1 - 1) \sigma - \lambda \rho = 0 \}$. The combined option effect is negative at $\sigma < \hat{\sigma}$ and positive at $\sigma > \hat{\sigma}$.

(ii) The shorter is the project life T , the smaller is the discounting effect and the larger in absolute terms are the two option effects.

The proposition states that the sign of the effect of uncertainty on the option value is ambiguous but separable into two regions: a negative influence at relatively low uncertainty and positive at relatively high uncertainty.⁴ The positive influence is governed by the volatility effect, the size of which depends on the differences between the magnitudes of upside and downside potential payoffs. This implies that the volatility effect is absent at $\sigma = 0$ and increasing in σ .

On the other hand, the negative effect of uncertainty depends on its marginal impact on the convenience yield. It exists for $\sigma = 0$, but it is vanishing at high σ . To understand the last point, note that the negative channel works through the discounted opportunity cost of waiting. Since the discount factor is convex and decreasing in the discount rate and thus in σ , the discount factor is almost constant and almost equal to zero at high σ . Thus at high σ the discount factor is insensitive to σ and the effect of the convenience yield diminishes. For these reasons it is not surprising that the proposition says that the negative influence of the convenience yield effect dominates the positive influence of the volatility effect at relatively low σ and the opposite occurs at high σ .

The proposition also states that the project- and option-related effects react differently to changes in the project life. The discounting effect becomes smaller with shorter project lives. Clearly, short-lived projects are relatively insensitive to marginal changes of the discount rate.

⁴From Proposition 5 (i) it is clear that in a setup where only the option effects are present, the non-monotonic investment-uncertainty relationship would arise irrespective of the project lifetime. This could be the case for example, if the project value V behaves according to geometric Brownian process. However such a setup is a rather serious abstraction from reality (see Dixit and Pindyck (1994; 175) for arguments).

σ	$T = 10$					$T = 30$				
	Q^*	(1)	(2)	(3)	(4)	Q^*	(1)	(2)	(3)	(4)
0.00	5.52	7.47	0.00	-77.23	-69.77	2.22	8.38	0.00	-31.03	-22.65
0.02	4.44	5.95	1.19	-48.51	-41.37	1.87	6.86	0.50	-20.44	-13.08
0.04	3.77	5.01	2.15	-33.70	-26.54	1.66	5.92	0.95	-14.86	-8.00
0.06	3.34	4.39	2.99	-25.03	-17.65	1.54	5.29	1.38	-11.53	-4.86
0.08	3.05	3.97	3.75	-19.47	-11.75	1.46	4.88	1.80	-9.35	-2.67
0.10	2.86	3.68	4.40	-15.66	-7.57	1.43	4.60	2.20	-7.82	-1.02
0.12	2.74	3.49	4.95	-12.92	-4.48	1.42	4.43	2.57	-6.70	0.29
0.14	2.67	3.38	5.36	-10.90	-2.17	1.44	4.34	2.88	-5.86	1.36
0.16	2.65	3.31	5.65	-9.38	-0.42	1.47	4.30	3.15	-5.22	2.22
0.18	2.65	3.28	5.85	-8.24	0.90	1.53	4.30	3.37	-4.74	2.93
0.20	2.68	3.28	5.99	-7.37	1.91	1.59	4.34	3.55	-4.37	3.52
0.22	2.73	3.31	6.09	-6.70	2.69	1.67	4.39	3.72	-4.09	4.02
0.24	2.79	3.34	6.17	-6.19	3.32	1.75	4.46	3.87	-3.89	4.45

Table 1: The three effects of uncertainty affecting the position of the investment trigger for the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$, $Q = 1$. The columns present: the discounting effect (1), the volatility effect (2), the convenience yield effect (3), and the total effect (4).

On the other hand, the option-related effects increase with shorter project lives. A shorter project life implies that the current revenue flow needs to be larger for the investment to be optimal. This implies that also the option effects are enlarged.

Now we are ready to establish when and why increasing uncertainty may lower the investment threshold. First we note that if the combined option effect is positive, that is $\sigma > \hat{\sigma}$, the overall effect is always positive. Furthermore, the combined option effect is negative for $\sigma = 0$ and remains so for relatively small levels of uncertainty. Then the convenience yield effect determines the sign of the combined option effect. For the negative combined option effect to dominate the positive impact of the discounting effect, the project must be relatively short-lived, as was argued in the previous paragraph. We conclude that the overall effect of uncertainty on the investment trigger is positive when uncertainty is small and project life is short.

These mechanisms are illustrated in a numerical example presented in Table 1. It allows for a closer inspection of the magnitude of the effects of uncertainty affecting the position of the investment trigger. The volatility and convenience yield effects increase with shortening the project life. The discounting effect decreases with smaller T . The combined option effect is negative but increasing in σ (it becomes positive for $\sigma > \hat{\sigma} = 0.241$). The longer the project life, the faster is the negative convenience yield effect offset by the positive impact of the discounting and volatility effects. If $T = 10$, the total effect is negative for σ between 0 and 0.16, while for $T = 30$ the total effect remains negative for σ between 0 and 0.10.

It should be noted that the project value is unaffected by the revenue uncertainty if the discount rate, and thus the convenience yield, is constant (i.e. if $\lambda\rho = 0$). In this case both the discounting effect of uncertainty and also the convenience yield effect are absent. From Proposition 5 it can be seen that $\lambda\rho = 0$ implies that $\hat{\sigma} = 0$, so that we have a positive option effect for all levels of uncertainty. Therefore, increasing uncertainty directly raises the option

value and, consequently, the investment trigger through the volatility effect. This explains the difference between the results of Propositions 2 and 4.

4 Robustness

The model of the previous sections was geared to show our results in the simplest setting. The aim of this section is to demonstrate that our main result, i.e. the value of waiting decreases with uncertainty in case of a short project life and a limited amount of uncertainty, can be generalized. First we consider a scenario where the investment opportunity is available only for a limited amount of time. After that we analyze the case where the project has an uncertain duration. The latter case is interesting also because it allows for alternative interpretations, such as depreciation or catastrophe risk. Finally, we consider more general, thus not necessary linear, convenience yield functions in uncertainty.

4.1 Finite-life option

We now assume that the project and the option to invest both have finite durability.⁵ The project life is T years and its value $V(Q)$ is given by equation (4). Denote the life length of the option as T_F . Since the option expires at T_F , its value $F(Q, t)$ depends on calendar time t . To find the differential equation defining the option value we follow the same steps as in Section 2. The resulting partial differential equation includes the time derivative and is given by

$$\frac{1}{2}\sigma^2Q^2F_{QQ} + (\mu - \lambda\rho\sigma)QF_Q + F_t - rF = 0. \quad (15)$$

The option value must satisfy the terminal condition at the expiry date T_F :

$$F(Q, T_F) = \max(V(Q) - I, 0),$$

which states that at $t = T_F$ the option is exercised (the investment is undertaken) if the project's expected present value exceeds the investment cost. The option satisfies also the boundary conditions at $Q = 0$ and $Q = Q^*$ similar to the ones used in Section 2:

$$\begin{aligned} F(0, t) &= 0, \\ F(Q^*, t) &= V(Q^*) - I, \\ F_Q(Q^*, t) &= V'(Q^*). \end{aligned}$$

Unlike in the previous problem, in which Q^* was a single point, here the optimal investment trigger $Q^*(t)$ is a function of time.

The problem we have to solve is analogous to the valuation of American-style options with a finite expiry date, to which no closed-form solutions exist. We solve equation (15) together with the boundary conditions using standard finite-difference numerical methods.

⁵Note that McDonald and Siegel (1986) also allow for a finite life of the investment opportunity, but their project is implicitly perpetual.

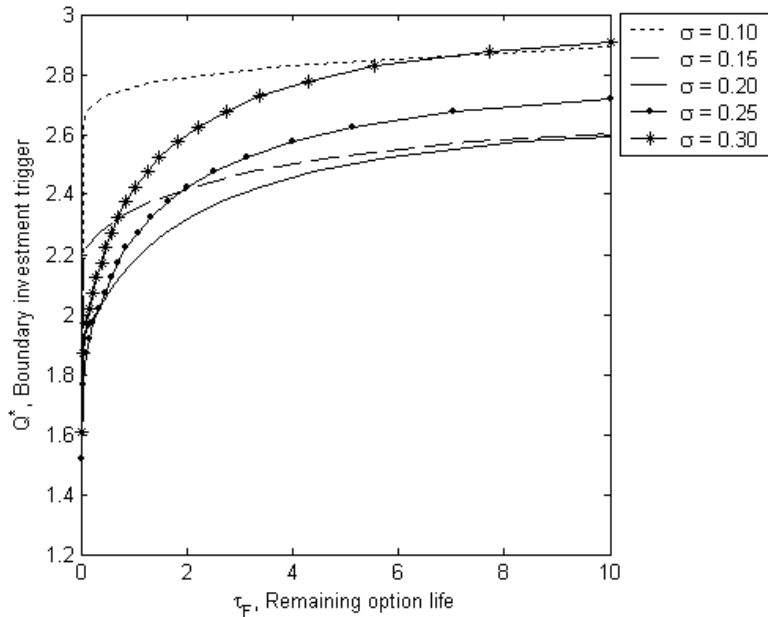


Figure 2: Project and option with finite life: Boundary investment trigger, $Q^*(\tau_F)$, for various levels of volatility and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$, $T = 10$, $T_F = 10$.

Figures 2 and 3 present our results for the optimal investment trigger boundary $Q^*(t)$. We assumed the option life T_F to be 10 years and the project life T to be either 10 years (Figure 2), or perpetual (Figure 3). All other parameters are as in the numerical example of Figure 1. The triggers $Q^*(t)$ are drawn for various levels of σ ranging from 0.10 to 0.30. The horizontal axis depicts the remaining option life τ_F , defined as $\tau_F \equiv T_F - t$ (at $\tau_F = 0$ the option to invest expires).

As expected, the right-hand-side of both figures at $\tau_F = T_F = 10$ is well approximated by the model with a perpetual real option, so that the trigger boundary values are very close to those in Figure 1 ($T = 10$ and $T = \infty$ curves). At $\tau_F = 0$, when the investment decision becomes a now-or-never decision, all curves are at the values implied by the NPV investment rule.

Figure 2 clearly confirms our result that a finite project life may cause the real option investment rule to be non-monotonic in uncertainty. An increase of σ from 0.10 to 0.15 moves the curve downwards.⁶ But an increase of σ from 0.20 to 0.25 and 0.30 shifts the optimal triggers upwards. The important finding of this numerical analysis is that after comparing Figure 2 and Figure 1, we can conclude that the levels of σ at which the trigger decreases and increases with uncertainty, remain roughly the same. In both cases the revenue uncertainty level at which the change of sign occurs lies between $\sigma = 0.15$ and $\sigma = 0.20$. Thus the finite-life option assumption neither mitigates nor augments the positive relationship between investment and uncertainty due to the decreasing trigger.

Figure 2 shows also that the effect of uncertainty may differ depending on the remaining

⁶Except at the expiry date $\tau_F = 0$, at which $Q^*(t)$ increases in σ for all σ .

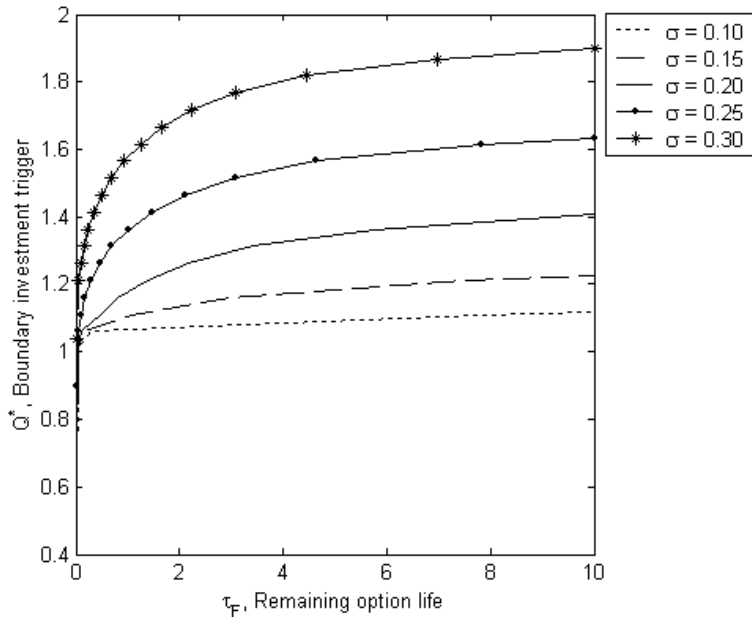


Figure 3: A perpetual project and a finitely-lived option to invest: Boundary investment trigger, $Q^*(\tau_F)$, for various levels of volatility and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$, $T = \infty$, $T_F = 10$.

option life. The dashed curve of $\sigma = 0.15$ is below the dot-marked curve of $\sigma = 0.25$ at high τ_F and above at low τ_F . The reason is the nearly flat horizontal shape of the optimal investment trigger curve at relatively low σ ($\sigma = 0.10$ or $\sigma = 0.15$) for most of the option life and a sudden drop close to $\tau_F = 0$. This shape may be caused by the convenience yield being low at lower σ , implying that there is only a small gain of undertaking the investment early (recall that a call option is never prematurely exercised if the convenience (dividend) yield is zero).

The behavior of the investment boundary in Figure 2 can be contrasted with the case of the perpetual project. Figure 3 shows that when the project life is infinite then $Q^*(t)$ moves upwards with increasing uncertainty. This is the usual monotonic relation consistent with the model with perpetual opportunity to invest.

4.2 Stochastic project life

An alternative for assuming a deterministic finite project life is to impose that a Poisson arrival brings the project to an end. We study this here and assume that the project lifetime (after installation) follows a Poisson process with rate γ .

Using equation (4) and the probability density of the stochastic lifetime, we obtain the project value

$$V(Q) = \int_0^\infty Q \frac{1 - e^{-(r+\lambda\rho\sigma-\mu)t}}{r + \lambda\rho\sigma - \mu} \gamma e^{-\gamma t} dt = \frac{Q}{r + \lambda\rho\sigma - \mu + \gamma}.$$

The resulting formula prompts other economic interpretations of a stochastic project life. It is equivalent to the assumption of a perpetual project that is exponentially depreciated with rate γ (see Dixit and Pindyck (1994), p.200). In yet another interpretation, γ may represent a

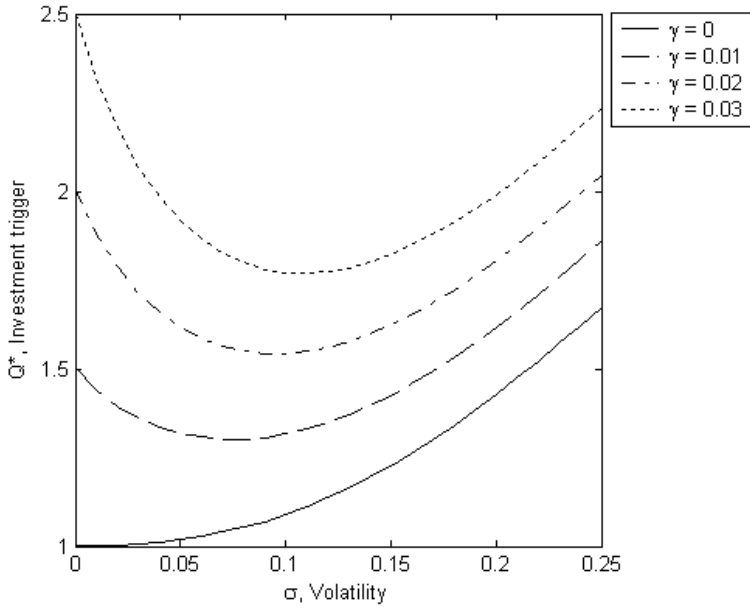


Figure 4: Investment trigger as a function of volatility for various Poisson arrival rates γ and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$.

catastrophe risk that may hit the whole economy or sector or an individual firm.

Analogous to the previous analyses, the optimal investment trigger can be derived:

$$Q^* = \frac{\beta_1}{\beta_1 - 1} (r + \lambda\rho\sigma - \mu + \gamma) I. \quad (16)$$

We can now show that the non-monotonic uncertainty effect carries over to the case of a stochastic project life.

Proposition 6 *If $\gamma > 0$ and $\lambda\rho > 0$, then the uncertainty effect on the investment trigger is non-monotonic: it decreases in σ for low levels of σ and then increases. The length of the σ -interval where the negative effect occurs, increases γ .*

This result points out how strongly the monotonic relationship between the investment trigger and uncertainty hinges on the assumption of the project being perpetual. If there exists just a small probability that the project will be finished in finite time, the investment trigger will be decreasing with increasing uncertainty for a small enough σ . To illustrate this result, a numerical example is presented in Figure 4. Here we indeed see that even a very small γ causes the trigger to decrease in uncertainty at small but realistic levels of uncertainty.

4.3 General convenience yield

The previous results stated in Propositions 1-6 are obtained for the framework of Section 2 (and Section 4.2 in the stochastic life case). In that model, the equilibrium discount rate, and also the convenience yield, are determined by the standard CAPM and thus linear on σ . Here we check whether this linearity is crucial for the results that we obtained. This issue is relevant as,

apart from the standard CAPM, there exist theory and some evidence in favour of nonlinearity. For example, it is well-known that the presence of finite heterogeneous investment horizons leads to a nonlinear CAPM with a nonlinear relationship between returns and risk (see, e.g., Lee, Wu and Wei (1990)). Moreover, there is a growing literature on factor pricing models with nonlinearities (see Bansal and Viswanathan (1993)).

Let the convenience yield be a non-decreasing, continuous, twice differentiable function of uncertainty $\delta(\sigma)$ for $\sigma \geq 0$. In the previous sections we obtained results for the linear case, i.e. $\delta''(\sigma) = 0$. We now show propositions that generalize those results. Corresponding to Proposition 1 we have that

Proposition 7 *If the project life is infinite and $\delta'(\sigma) \geq 0$, then the investment trigger increases with uncertainty.*

Proposition 2 is, of course, as general as one can get in terms of $\delta(\sigma)$. Proposition 4 can be generalized as follows.

Proposition 8 *If the project life is finite, $\delta'(\sigma) > 0$, and $\delta''(\sigma) \leq 0$, then the uncertainty effect on the investment trigger is non-monotonic: it decreases in σ for low levels of σ and then increases. The length of the σ -interval where the negative effect occurs, decreases with project lifetime.*

So in the case of a finite project life, the previously observed properties for linear $\delta(\sigma)$ carry over to a concave $\delta(\sigma)$. In case of a convex $\delta(\sigma)$, we can have either a U-shaped relationship and a monotonic one.⁷

5 Conclusions

Our paper shows that a finite life of an investment project in combination with a risk premium in expected rates of return may reverse of the usual effect of uncertainty on irreversible investments. In particular, we determined a scenario under which increased uncertainty reduces the value of waiting with investment. We now briefly discuss some implications of this result.

In corporate practice investment projects are usually considered to have a finite life, which supports the importance of our result. It thus seems that assuming the project life to be infinite, which is done in the overwhelming majority of real options contributions, is useful for simplicity reasons but dangerous since adverse uncertainty effects are lost.

From a policy point of view our results demonstrate that there exists a positive level of uncertainty at which the investment trigger admits its lowest value. If the policy aim is to increase investment, then the implication is that it is not necessarily optimal in all cases to decrease the level of uncertainty of policy instruments. However, any specific recommendation may be a bit far-reaching in the current single-firm model with a general source of uncertainty. To derive policy implications out of our non-monotonic investment-uncertainty relationship deserves a separate study. Similarly, in order to focus on the main features of the described

⁷To check it, take, for instance, $\delta(\sigma) = r + \lambda\rho\sigma^{3/2} - \mu$ with the parameter values as in Table 1 and the uncertainty effect is U-shaped. However, if $\delta(\sigma) = r + \lambda\rho\sigma^3 - \mu$, the effect of uncertainty is always positive.

mechanism, we have not attempted to construct a richer model of industry equilibrium. This can be done by considering a competitive industry (as in Caballero and Pindyck (1996) and others) or imperfect competition (as in Smets (1991), Grenadier (1996) and others). However, we are quite confident that, qualitatively spoken, our result carries over to these frameworks.

Our non-monotonicity result accords with empirical findings of Bo and Lensink (2005). In a panel of Dutch firms, the investment-uncertainty relationship is positive at low levels of uncertainty and negative at high levels. Until now, a clear theoretical explanation for such empirical results is missing. The factors hastening investment with greater uncertainty indicated in this paper lend themselves to empirical tests.

A Appendix: Proofs

A.1 Deterministic project life

The derivative of the investment trigger (given in (11)) with respect to σ is

$$\frac{dQ^*}{d\sigma} = \frac{I\beta_1}{(\beta_1 - 1)^2} \frac{1}{\sigma^2 (\beta_1 - \frac{1}{2}) + \mu - \lambda\rho\sigma} \frac{1}{1 - e^{-(r+\lambda\rho\sigma-\mu)T}} (M - N\Delta), \quad (17)$$

where

$$\begin{aligned} M &= (\beta_1 - 1) \left(\beta_1 + \frac{1}{2} \right) \lambda\rho\sigma^2 + (\beta_1 - 1) (r - \mu) \sigma + \beta_1 (\mu - \lambda\rho\sigma) \lambda\rho - r\lambda\rho, \\ N &= (\beta_1 - 1) \left(\beta_1 - \frac{1}{2} \right) \lambda\rho\sigma^2 + (\beta_1 - 1) (\mu - \lambda\rho\sigma) \lambda\rho, \\ \Delta &= (r + \lambda\rho\sigma - \mu) T \left[e^{(r+\lambda\rho\sigma-\mu)T} - 1 \right]^{-1}. \end{aligned}$$

Denote the term $M - N\Delta$ by L_1 . The first three fractions of (17) are always positive (recall that $\sigma^2 (\beta_1 - \frac{1}{2}) + \mu - \lambda\rho\sigma = \partial L_0 / \partial \beta|_{\beta=\beta_1} > 0$, as the derivative is evaluated at the higher root of the convex quadratic L_0). The sign of L_1 thus determines for the sign of the derivative. From (7) we observe that

$$(\mu - \lambda\rho\sigma) \beta_1 = -\frac{1}{2}\beta_1^2\sigma^2 + \frac{1}{2}\beta_1\sigma^2 + r,$$

which can be substituted twice into M and N to obtain

$$M = \frac{1}{2} (\beta_1 - 1)^2 \lambda\rho\sigma^2 + (\beta_1 - 1) (r + \lambda\rho\sigma - \mu) \sigma \quad (18)$$

and

$$N = \frac{1}{2} (\beta_1 - 1)^2 \lambda\rho\sigma^2 + (r + \lambda\rho\sigma - \mu) \lambda\rho. \quad (19)$$

Proof of Proposition 1. First, suppose that $\lambda\rho > 0$. Combining $T \rightarrow \infty$ with (17) and (18),

we obtain that

$$\begin{aligned}\frac{dQ^*}{d\sigma} &= \frac{I\beta_1}{(\beta_1 - 1)^2} \frac{1}{\sigma^2 (\beta_1 - \frac{1}{2}) + \mu - \lambda\rho\sigma} \left[(\beta_1 - 1) \sigma \left(r + \frac{1}{2} (\beta_1 + 1) \lambda\rho\sigma - \mu \right) \right] \\ &> \frac{I\beta_1}{(\beta_1 - 1)^2} \frac{1}{\sigma^2 (\beta_1 - \frac{1}{2}) + \mu - \lambda\rho\sigma} [(\beta_1 - 1) \sigma (r + \lambda\rho\sigma - \mu)] \\ &\geq 0,\end{aligned}$$

where the first inequality stems from the observation that $\frac{1}{2}(\beta_1 + 1) > 1$ and the second from the assumption that $r + \lambda\rho\sigma - \mu = \delta > 0$.

The two other possibilities $\lambda\rho = 0$ and $\lambda\rho < 0$ are covered by the proofs of Propositions 2 and 3, respectively. ■

Proof of Proposition 2. Within our model we can impose absence of the impact of systematic risk by setting $\rho = 0$. The derivative of the investment trigger (given in equation (11)) with respect to σ is

$$\frac{dQ^*}{d\sigma} = \frac{I\beta_1}{(\beta_1 - 1)^2} \frac{1}{\sigma^2 (\beta_1 - \frac{1}{2}) + \mu} \frac{1}{1 - e^{-(r-\mu)T}} (\beta_1 - 1) \sigma (r - \mu)$$

The resulting expression is always positive if $r > \mu$, which holds by the assumption that $\delta > 0$. ■

Proof of Proposition 3. Suppose that $\lambda\rho < 0$. Then the assumption that $\delta > 0$ holds if and only if $\sigma \in [0, \bar{\sigma})$, where $\bar{\sigma} = \frac{\mu - r}{\lambda\rho}$. We have that, denoting $\delta(\cdot)$ and $\beta_1(\cdot)$ as functions of σ , $\delta(\bar{\sigma}) = 0$ and $\beta_1(\bar{\sigma}) = 1$. So $[0, \bar{\sigma})$ is the relevant domain for σ in this case. Next, we claim that

$$\frac{1}{2} (\beta_1 + 1) \sigma < \bar{\sigma}, \text{ for all } \sigma \in [0, \bar{\sigma}). \quad (20)$$

To verify, note that $\frac{d}{d\sigma} \frac{1}{2} (\beta_1 + 1) \sigma = [\sigma^2 (\beta_1 - \frac{1}{2}) + \mu - \lambda\rho\sigma]^{-1} [\frac{1}{2} (3\beta_1 - 1) \sigma^2 + (\beta_1 - 1) \mu - \lambda\rho\sigma] > 0$ and $\frac{1}{2} (\beta_1(\bar{\sigma}) + 1) \bar{\sigma} = \bar{\sigma}$. So, for positive σ less than $\bar{\sigma}$, the inequality (20) is true.

Now, $\lambda\rho < 0$ implies that $N < 0$. Combining (20) and (18) we have that

$$\begin{aligned}M &= (\beta_1 - 1) \sigma \left[r + \frac{1}{2} (\beta_1 + 1) \lambda\rho\sigma - \mu \right] \\ &> (\beta_1 - 1) \sigma (r + \lambda\rho\bar{\sigma} - \mu) = (\beta_1 - 1) \sigma \delta(\bar{\sigma}) = 0.\end{aligned}$$

Since $M > 0$, $N < 0$, and $1 \geq \Delta > 0$, the derivative (17) is also positive and the proposition is proved. ■

Proof of Proposition 4. Suppose that T is finite and $\lambda\rho > 0$. We want to show that L_1 is negative for low $\sigma \geq 0$ and becomes positive when σ increases. First, it is useful to observe the simple fact that $1 \geq \Delta > 0$ and

$$\frac{d\Delta}{d\sigma} > 0. \quad (21)$$

It can also be verified that

$$L_1 \leq 0 \Rightarrow (\beta_1 - 1) \sigma - \lambda \rho < 0 \iff \frac{d\beta_1}{d\sigma} > 0. \quad (22)$$

Then note that at $\sigma = 0$, $L_1 = -(r - \mu) \lambda \rho \Delta < 0$. So $\frac{dQ^*}{d\sigma}$ is also negative at $\sigma = 0$. As σ increases, Δ converges to zero and L_1 becomes positive. We show now that L_1 changes its sign from negative to positive only once with increasing σ . If $L_1 = 0$, then $\Delta = \frac{M}{N}$ and

$$\begin{aligned} \frac{dL_1}{d\sigma} &= \frac{dM}{d\sigma} - \frac{dN}{d\sigma} \Delta - N \frac{d\Delta}{d\sigma} > \frac{dM}{d\sigma} - \frac{dN}{d\sigma} \Delta = \frac{1}{N} \left(\frac{dM}{d\sigma} N - \frac{dN}{d\sigma} M \right) \\ &= \frac{\delta \lambda \rho}{N} \left[\frac{d\beta_1}{d\sigma} \sigma + \beta_1 - 1 \right] \{ (\beta_1 - 1) [\lambda \rho - (\beta_1 - 1) \sigma] \sigma + \delta \} \\ &> 0, \end{aligned} \quad (23)$$

The inequalities follow from (21) and (22). So L_1 always increases in σ if $L_1 = 0$. Now, continuity of L_1 implies that it changes its sign only once from negative to positive at some $\sigma^* > 0$. Hence the first part of the proposition is proved.

To verify that the σ -interval where the negative effect occurs is larger the shorter is the project life, we consider

$$\frac{d\sigma^*}{dT} = - \frac{\frac{\partial L_1}{\partial T}}{\frac{\partial L_1}{\partial \sigma}} \Big|_{\sigma=\sigma^*} = \frac{N \frac{d\Delta}{dT}}{\frac{\partial L_1}{\partial \sigma}} \Big|_{\sigma=\sigma^*} < 0.$$

The inequality follows from the fact that $\frac{d\Delta}{dT} < 0$ and (23). ■

Proof of Proposition 5. The sum of the two option effects is

$$\frac{\partial Q^*}{\partial \beta_1} \frac{\partial \beta_1}{\partial \sigma} + \frac{\partial Q^*}{\partial \beta_1} \frac{\partial \beta_1}{\partial \delta} \frac{\partial \delta}{\partial \sigma} = \frac{I \beta_1}{(\beta_1 - 1)^2} \frac{\delta(\sigma)}{1 - e^{-\delta(\sigma)T}} \frac{(\beta_1 - 1) \sigma - \lambda \rho}{\sigma^2 (\beta_1 - \frac{1}{2}) + \mu - \lambda \rho \sigma}. \quad (24)$$

As $\beta_1 > 1$ and $\sigma^2 (\beta_1 - \frac{1}{2}) + r - \delta(\sigma) > 0$, the sign of expression (24) depends on the sign of $L_2 \equiv (\beta_1 - 1) \sigma - \lambda \rho$ in the way stated in the proposition.

It remains to be shown that there exists a unique non-negative $\hat{\sigma}$. Note that, if $\lambda \rho > 0$, at $\sigma = 0$ we have that $L_2 = -\lambda \rho < 0$ and the combined option effect is negative. To verify that the option effect changes its sign only once from negative to positive with increasing σ , we show that L_2 (being continuous in $\sigma > 0$) always increases with σ if $L_2 \leq 0$. That is,

$$\frac{dL_2}{d\sigma} = \frac{\lambda \rho \sigma - (\beta_1 - 1) \sigma^2}{\sigma^2 (\beta_1 - 1) + \mu - \lambda \rho \sigma} + \beta_1 - 1 \geq \beta_1 - 1 > 0,$$

if $L_2 \leq 0$.

The discounting effect is given by

$$\frac{\partial Q^*}{\partial \delta} \frac{\partial \delta}{\partial \sigma} = \frac{I \beta_1}{\beta_1 - 1} \frac{1 - e^{-\delta(\sigma)T} - \delta(\sigma)T e^{-\delta(\sigma)T}}{(1 - e^{-\delta(\sigma)T})^2} \lambda \rho,$$

which is always positive and increasing in T . It is straightforward from derivations leading to (24) that $\frac{\partial Q^*}{\partial \beta_1} \frac{\partial \beta_1}{\partial \sigma}$ and $\frac{\partial Q^*}{\partial \beta_1} \frac{\partial \beta_1}{\partial \delta} \frac{\partial \delta}{\partial \sigma}$ decrease in absolute terms in T . ■

A.2 Stochastic project life

Let $\delta(\sigma)$ be a continuous twice differentiable convenience yield function. The derivative of Q^* given in (16) with respect to σ eventually becomes:

$$\frac{dQ^*}{d\sigma} = \frac{I\beta_1}{(\beta_1 - 1)^2 \sigma^2 (\beta_1 - \frac{1}{2}) + r - \delta(\sigma)} L_3, \quad (25)$$

where

$$L_3 = \frac{1}{2} (\beta_1 - 1)^2 \delta'(\sigma) \sigma^2 + (\beta_1 - 1) \delta(\sigma) \sigma + [(\beta_1 - 1) \sigma - \delta'(\sigma)] \gamma. \quad (26)$$

The first two fractions of the right-hand side of (25) are always positive, so the sign of the derivative is determined by the sign of L_3 .

Proof of Proposition 6. The proof follows from the proof of Proposition 8 below with linear $\delta(\sigma)$. ■

We prove Propositions 7 and 8 only for stochastic project lifetime; similar proofs can be obtained for the deterministic case.

Proof of Proposition 7. Note that if $\gamma = 0$ and $\delta'(\sigma) > 0$ then $L_3 = \frac{1}{2} (\beta_1 - 1)^2 \delta'(\sigma) \sigma^2 + (\beta_1 - 1) \delta(\sigma) \sigma > 0$. ■

Proof of Proposition 8. We want to show that for $\gamma > 0$ and $\delta'(\sigma)$, L_3 is negative for low $\sigma \geq 0$ and turns to positive with increasing σ . First we note that at $\sigma = 0$, $L_3 = -\delta'(0)\gamma < 0$. Then observe that a straightforward consequence of (26) is that

$$L_3 \leq 0 \Rightarrow (\beta_1 - 1) \sigma - \delta'(\sigma) < 0 \iff \frac{d\beta_1}{d\sigma} > 0. \quad (27)$$

Using this, if $L_3 \leq 0$, we have that

$$\begin{aligned} \frac{dL_3}{d\sigma} &= \frac{d\beta_1}{d\sigma} [(\beta_1 - 1) \delta'(\sigma) \sigma^2 + \sigma (\delta(\sigma) + \gamma)] + (\beta_1 - 1)^2 \delta'(\sigma) \sigma \\ &\quad + (\beta_1 - 1) (\delta(\sigma) + \gamma + \delta'(\sigma) \sigma) + \left[\frac{1}{2} (\beta_1 - 1)^2 \sigma^2 - \gamma \right] \delta''(\sigma) \\ &> \left[-\frac{1}{\delta'(\sigma)} (\beta_1 - 1) \sigma (\delta(\sigma) + \gamma) \right] \delta''(\sigma) > 0. \end{aligned}$$

So L_3 always increases in σ if $L_3 \leq 0$. From the continuity of L_3 now follows that L_3 changes its sign only once from negative to positive at some $\sigma^* > 0$. This proves the first part of proposition.

To verify that the σ -interval where the negative effect occurs is larger, the shorter is the project life we consider

$$\frac{d\sigma^*}{d\gamma} = - \frac{\frac{\partial L_3}{\partial \gamma}}{\frac{\partial L_3}{\partial \sigma}} \bigg|_{\sigma=\sigma^*} = \frac{\delta'(\sigma) - (\beta_1 - 1) \sigma}{\frac{\partial L_3}{\partial \sigma}} \bigg|_{\sigma=\sigma^*} > 0,$$

where for the inequality we employ (27) and the first part of the proof of this proposition ■

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