CentER

Discussion Paper

No. 2003–89

ON THE AXIOMATIC CHARACTERIZATION OF "WHO IS A J?"

By S.C. Sung, D. Dimitrov

October 2003

ISSN 0924-7815

TILBURG UNIVERSITY

On the Axiomatic Characterization of "Who is a J?"*

Shao Chin Sung
School of Information Science
Japan Advanced Institute of Science and Technology
1-1 Asahidai, Tatsunokuchi, Ishikawa 923-1292, Japan
Email: son@jaist.ac.jp

Dinko Dimitrov
CentER and Department of Econometrics and Operations Research
Tilburg University
P.O. Box 90153, 5000 LE Tilburg, The Netherlands
Email: d.a.dimitrov@uvt.nl

August 2003

Abstract

Recent work by Kasher and Rubinstein (1997) considers the problem of group identification from a social choice perspective. These authors provide an axiomatic characterization of a "liberal" aggregator whereby the group consist of those and only those individuals each of which views oneself a member of the group. In the present paper we show that the five axioms used in Kasher and Rubinstein's characterization of the "liberal" aggregator are not independent and prove that only three of their original axioms are necessary and sufficient for

*Dinko Dimitrov gratefully acknowledges financial support from a Marie Curie Research Fellowship of the European Community programme "Improving the Human Research Potential and the Socio-Economic Knowledge Base" under contract number HPMF-CT-2002-02121.

the required characterization. *Journal of Economic Literature* Classification Numbers: D63, D71.
Key Words: group identification, liberalism, social identity

1 Introduction

The problem of group identification can be formulated as follows: Given a group of individuals, how to define the extent of a subgroup of it? In very recent papers (Billot (2003), Kasher and Rubinstein (1997), Samet and Schmeidler (2003)) this problem has been related to formal models from social choice and voting theory.

Kasher's (1993) paper on collective identity can be considered as a first, non-formal attempt to look at the group identification problem as an aggregation task. In that paper the author views that each individual of a society has an opinion about every individual, including oneself, whether the latter is a member of a group to be formed. The collective identity of the group to be formed is then determined by aggregating opinions of all the individuals in the society. The formal link between Kasher's approach and the theory of aggregators mainly developed in economic theory (Rubinstein and Fishburn (1986)) was made by Kasher and Rubinstein (1997). In the latter paper the authors provide axiomatic characterization of three aggregators: the "dictatorship" aggregator whereby a pre-designated member of the society determines who deserves to became a group member; the "oligarchical" aggregator whereby the decision is taken by consensus among the members of a pre-designated subgroup of the society; and the "liberal" aggregator whereby the group consist of those and only those individuals each of which views oneself a member of the group. The first two characterizations are based on previous results by Fishburn and Rubinstein (1986) and

Rubinstein and Fishburn (1986) whereas the characterization of the "liberal" aggregator is new.

Two of the five axioms used for the characterization of this new "liberal" aggregator are called *consensus* (C) and *monotonicity* (MON). These axioms are very familiar in the social choice literature and, in fact, sound plausible when imposed as requirements on a collective identity aggregator. Consensus says that if an individual is defined as a group member by every one in the society, then this individual should be considered as a socially accepted group member; and, correspondingly, if no one defines this individual as a group member, then he or she should not deserve the social acceptance as a group member. On the other hand, monotonicity describes what happens if someone changes his opinion in favor of a given individual provided that this individual already enjoys the social acceptance as a group member. The exact definitions of (C) and (MON) are given in the next section.

These two axioms, in combination with other three different axioms are used by Kasher and Rubinstein to reach logically the "liberal" aggregator. In order to show the independence of the axioms these authors construct examples (one for each axiom) that satisfy all axioms but the considered one. However, a careful check of the examples for (C) and (MON) convince us that both do not satisfy some of the other proposed axioms either and, in addition, that these examples can not be repaired.

This fact shadows the characterization result of Kasher and Rubinstein and constitutes the main motivation for this paper. Section 2 presents the basic notation and axioms used for the characterization of the "liberal" aggregator, as well as the examples for (C) and (MON). Section 3 collects our results and it first shows that (C) is implied by three of the other axioms used in Kasher and Rubinstein's original characterization. Moreover, we prove that the same three axioms (being independent) are necessary and sufficient to reach the axiomatic characterization of the "liberal" aggregator. The latter fact indicates that these axioms imply (MON) as well. Hence, a simplification of the corresponding axiomatic system is reached.

2 Basic notation and axioms

Let $N = \{1, \ldots, n\}$ denote the set of all individuals in the society. Each individual $i \in N$ forms a set $V_i \subseteq N$ consisting of all society members that in the view of i deserve to be accepted as group members. A *profile of views* is an n-tuple $V = (V_1, \ldots, V_n)$ where $V_i \subseteq N$ for every $i \in N$. Let $\mathcal{V}$ be the set of all profiles of views, i.e., $\mathcal{V} = (P(N))^n$ where $P(N)$ is the power set of N. A *collective identity function* (CIF) $J : \mathcal{V} \to P(N)$ assigns to each profile $V \in \mathcal{V}$ a set $J(V) \subseteq N$ of socially accepted group members.

Definition 1 *The strong liberal CIF J^* is defined as follows.*

$$J^*(V) = \{i \in N \mid i \in V_i\} \quad \textit{for every } V \in \mathcal{V}.$$

The five axioms used for characterization of J^* in Kasher and Rubinstein (1997) are consensus (C), symmetry (SYM), monotonicity (MON), independence (I), and liberal principle (L). Each of these axioms is defined as follows.

- A CIF J satisfies *consensus* (C) if for every $V \in \mathcal{V}$,
  - $i \in V_k$ for every $k \in N$ implies $i \in J(V)$, and
  - $i \notin V_k$ for every $k \in N$ implies $i \notin J(V)$.
- A CIF J satisfies *symmetry* (SYM) if, for every $V \in \mathcal{V}$ and for every $i, j \in N$,

- $V_i - \{i,j\} = V_j - \{i,j\}$,
- $i \in V_k \Leftrightarrow j \in V_k$, for every $k \in N - \{i,j\}$,
- $i \in V_i \Leftrightarrow j \in V_j$,
- $i \in V_j \Leftrightarrow j \in V_i$,

imply $i \in J(V) \Leftrightarrow j \in J(V)$.

- A CIF J satisfies *monotonicity* (MON) if, for every $V \in \mathcal{V}$ and for every $i, j \in N$,

  - $i \in J(V)$
  - $V'_j = V_j \cup \{i\}$,
  - $V'_k = V_k$ for every $k \in N - \{j\}$,

  imply $i \in J(V')$.

- A CIF J satisfies *independence* (I) if, for every $V, V' \in \mathcal{V}$ and for every $i \in N$,

  - $k \in J(V) \Leftrightarrow k \in J(V')$ for every $k \in N - \{i\}$,
  - $i \in V_k \Leftrightarrow i \in V'_k$ for every $k \in N$,

  imply $i \in J(V) \Leftrightarrow i \in J(V')$.

- A CIF J satisfies *liberal principle* (L) if, for every $V \in \mathcal{V}$,

  - $k \in V_k$ for some $k \in N$ implies $J(V) \neq \emptyset$, and
  - $k \notin V_k$ for some $k \in N$ implies $J(V) \neq N$.

Kasher and Rubinstein prove that the above five axioms characterize J^* (Theorem 1(a) in Kasher and Rubinstein (1997, p. 389). That is, the following proposition.

Proposition 1 *The strong liberal CIF J^* is the only CIF that satisfies axioms (C), (SYM), (MON), (L), and (I).*

Moreover, in order to show the independence of these axioms (Theorem 1(b) in Kasher and Rubinstein (1997, p. 390-391) the authors construct examples (one for each axiom) that satisfy all axioms but the considered one. They use the following examples for (C) and (MON).

Example (C): Let n be an odd number. Consider the CIF J defined as follows. For every profile $V \in \mathcal{V}$, $J(V) = \{i \in N \mid i \in V_i\}$ if the cardinality of $\{i \in N \mid i \in V_i\}$ is odd and $J(V) = \{i \in N \mid i \notin V_i\}$ otherwise.

Example (MON): Consider the CIF J defined by $J(V) = \{i \in N \mid V_i = \{i\}\}$ for every $V \in \mathcal{V}$, i.e. a J is anyone who considers only oneself to be a J.

However, the example for (C) does not satisfy (L) either. To see this, let $n = 3$ (odd number) and let $V \in \mathcal{V}$ be a profile such that $V_1 = V_2 = V_3 = \emptyset$. Then the proposed CIF produces $J(V) = \{1, 2, 3\}$ since $\#\{i \in N \mid i \in V_i\} = \#\emptyset = 0$ (even number). Hence, there is a profile of views such that $i \notin V_i$ for some $i \in N$ and $J(V) = N$. It contradicts (L).

On the other hand, the example for (MON) does not satisfy (C), (L), and (I) either. Take $n = 3$ and let $V \in \mathcal{V}$ be a profile such that $V_1 = V_2 = V_3 = \{1, 2\}$. According to the proposed aggregator we have $J(\{1, 2\}, \{1, 2\}, \{1, 2\}) =$

$\emptyset$. Hence, although 1 and 2 are defined as group members by every one in the society, they are not socially accepted, and it contradicts (C); and although 1 and 2 define themselves as group members, the final group is empty, and it contradicts (L). To see that this aggregator does not satisfy (I) as well take again $n = 3$ and let $V, V' \in \mathcal{V}$ with $V = (\{1\}, \{2,3\}, \{1\})$, $V' = (\{1\}, \{2\}, \{1\})$. According to the proposed aggregator we have $J(V) = \{1\}$ and $J(V') = \{1,2\}$. Notice that $k \in J(V) \Leftrightarrow k \in J(V')$ for $k = 1,3$ and $2 \in V_k \Leftrightarrow 2 \in V'_k$ for $k = 1,2,3$. Nevertheless, $2 \notin J(V)$ and $2 \in J(V')$, i.e. (I) is violated.

In what follows we show that these examples can not be "repaired".

3 Axiomatization of the strong liberal aggregator

In this section, we provide an axiomatic characterization of the strong liberal CIF as defined by Kasher and Rubinstein (1997) by using only three of their original axioms: (SYM), (I), and (L). For that purpose we first show that (C) is implied by (SYM), (I), and (L). Given an arbitrary 4-partition of N and two special profiles depending on it, we then point out a very useful connection between any CIF satisfying the above three axioms and the strong liberal CIF. This connection is used to prove our characterization result. Finally, we show the independence of the axioms as well.

3.1 The consensus axiom

Let us first have a look at some properties of CIFs satisfying (I), (L), and (SYM). For each $S \subseteq N$, let $V^S \in \mathcal{V}$ be a profile such that

$$V^S = (V_1^S, V_2^S, \ldots, V_n^S) \quad \text{with} \quad V_k^S = S \quad \text{for every } k \in N.$$

Obviously, for every CIF J satisfying (C), we have $J(V^S) = S$ for every $S \subseteq N$. The following lemma says that the same holds for every CIF J satisfying (SYM), (I), and (L).

Lemma 1 *If a CIF J satisfies (SYM), (I), and (L), then $J(V^S) = S$ for every $S \subseteq N$.*

Proof. In order to proof the lemma, we show that, for every CIF J satisfying (SYM), (I), and (L),

$$J(V^S) = S \text{ and } J(V^{N-S}) = N - S \text{ for every } S \subseteq N$$

by induction of the cardinality $\#S$ of S. Obviously, $J(V^S) = S$ and $J(V^{N-S}) = N - S$ for every $S \subseteq N$ is equivalent to $J(V^S) = S$ for every $S \subseteq N$.

Let J be a CIF satisfying (SYM), (I), and (L). From (SYM), each of $J(V^S)$ and $J(V^{N-S})$ must be one of $\emptyset$, S, $N - S$ and N, i.e., $J(V^S), J(V^{N-S}) \in \{\emptyset, S, N - S, N\}$ for every $S \subseteq N$.

Basis Step: Suppose $\#S = 0$ (i.e., $S = \emptyset$). Then, we have $J(V^S), J(V^{N-S}) \in \{\emptyset, N\}$. From (L), we have $J(V^S) \neq N$ and $J(V^{N-S}) \neq \emptyset$. Therefore, $J(V^S) = \emptyset = S$ and $J(V^{N-S}) = N = N - S$.

Induction Step: For each $m \geq 0$, assume $J(V^S) = S$ and $J(V^{N-S}) = N - S$ for every $S \subseteq N$ with $\#S = m$, and show $J(V^S) = S$ and $J(V^{N-S}) = N - S$ for every $S \subseteq N$ with $\#S = m + 1$.

Recall that $J(V^{\emptyset}) = \emptyset$ and $J(V^N) = N$. Thus, $J(V^S) = S$ and $J(V^{N-S}) = N - S$ when $S = N$. Suppose $S \neq N$. Obviously, from $\#S > 0$, S is nonempty. Since $S \neq N$ and $S \neq \emptyset$, from (L) each of $J(V^S)$ and $J(V^{N-S})$ is neither $\emptyset$ nor N. Thus, we have $J(V^S), J(V^{N-S}) \in \{S, N - S\}$.

Let $i \in S$, and let $\hat{S} = S - \{i\}$. Obviously, $\#\hat{S} = \#S - 1 = m$. By the induction hypothesis, we have $J(V^{\hat{S}}) = \hat{S}$ and $J(V^{N-\hat{S}}) = N - \hat{S}$. From (I), we have $J(V^S) = S$; otherwise, i.e., $J(V^S) = N - S$, (I) is violated, because $k \in J(V^S)$ if and only if $k \in J(V^{N-\hat{S}})$ for every $k \in N - \{i\}$, and $i \in V_k^S$ and $i \in V_k^{N-\hat{S}}$ for every $k \in N$, but $i \notin J(V^S)$ and $i \in J(V^{N-\hat{S}})$. Again from (I), we have $J(V^{N-S}) = N - S$; otherwise, i.e., $J(V^{N-S}) = S$, (I) is violated because $k \in J(V^{N-S})$ if and only if $k \in J(V^{\hat{S}})$ for every $k \in N - \{i\}$, and $i \in V_k^{N-S}$ and $i \in V_k^{\hat{S}}$ for every $k \in N$, but $i \in J(V^{N-S})$ and $i \notin J(V^{\hat{S}})$.

Therefore, $J(V^S) = S$ and $J(V^{N-S}) = N - S$ for every $S \subseteq N$. ■

With the help of Lemma 1 we reach our first refinement of Kasher and Rubinstein's axiomatic system.

Theorem 1 *If a CIF satisfies (SYM), (I), and (L), then it also satisfies (C).*

Proof. The theorem is proven by contradiction. Suppose there exists a CIF J that satisfies (SYM), (I), and (L), but (C). Then, there exists a profile $V \in \mathcal{V}$ such that, for some $i \in N$,

- $i \in V_k$ for every $k \in N$ but $i \notin J(V)$, or
- $i \notin V_k$ for every $k \in N$ but $i \in J(V)$.

Suppose there exists $i \in N$ such that $i \in V_k$ for every $k \in N$ but $i \notin J(V)$. Let $S = J(V) \cup \{i\}$. From Lemma 1, we have $J(V^S) = S = J(V) \cup \{i\}$.

However, (I) is violated, because, $k \in J(V)$ if and only if $k \in J(V^S)$ for every $k \in N - \{i\}$, and $i \in V_k$ and $i \in V_k^S$ for every $k \in N$, but $i \notin J(V)$ and $i \in J(V^S)$.

Suppose there exists $i \in N$ such that $i \notin V_k$ for every $k \in N$ but $i \in J(V)$. Let $S' = J(V) - \{i\}$. Them from Lemma 1, we have $J(V^{S'}) = S' = J(V) - \{i\}$. However, (I) is violated, because, $k \in J(V)$ if and only if $k \in J(V^{S'})$ for every $k \in N - \{i\}$, and $i \notin V_k$ and $i \notin V_k^{S'}$ for every $k \in N$, but $i \in J(V)$ and $i \notin J(V^{S'})$.

Now we can conclude that every CIF that satisfies (SYM), (I), and (L) also satisfies (C), and the proof is completed. ■

Remark 1 *Notice that (L) and (SYM) do not appear in the proof of Theorem 1, but both of them are applied in Lemma 1.*

3.2 A partition lemma

Before we show that the strong liberal CIF J^* is the only CIF that satisfies (SYM), (I), and (L), let us slightly extend Lemma 1. Let $P = (P_1, P_2, P_3, P_4)$ be an arbitrary 4-partition of N, and let $V^{(P,0)}, V^{(P,1)} \in \mathcal{V}$ be profiles defined as follows. For each $k \in N$,

$$V_k^{(P,0)} = \begin{cases} P_1 \cup P_2 & \text{if } k \in P_1 \cup P_3, \\ P_1 \cup P_2 \cup P_3 & \text{if } k \in P_2 \cup P_4, \end{cases}$$

$$V_k^{(P,1)} = \begin{cases} P_1 & \text{if } k \in P_1 \cup P_3, \\ P_1 \cup P_2 & \text{if } k \in P_2 \cup P_4. \end{cases}$$

By definition of the strong liberal CIF J^*, we have $J^*(V^{(P,0)}) = J^*(V^{(P,1)}) = P_1 \cup P_2$.

Lemma 2 *If a CIF J satisfies (SYM), (I), and (L), then* $J(V^{(P,0)}) =$

$J(V^{(P,1)}) = P_1 \cup P_2$ *for every 4-partition* P *of* N.

Proof. Let J be a CIF satisfying (SYM), (I), and (L). From Theorem 1, J satisfies (C), and thus,

- $P_1 \cup P_2 \subseteq J(V^{(P,0)})$ and $P_4 \cap J(V^{(P,0)}) = \emptyset$, and
- $P_1 \subseteq J(V^{(P,1)})$ and $(P_3 \cup P_4) \cap J(V^{(P,1)}) = \emptyset$.

Moreover, from (SYM), $P_3 \subseteq J(V^{(P,0)})$ or $P_3 \cap J(V^{(P,0)}) = \emptyset$, and $P_2 \subseteq J(V^{(P,1)})$ or $P_2 \cap J(V^{(P,1)}) = \emptyset$. Therefore, $J(V^{(P,0)})$ is $P_1 \cup P_2$ or $P_1 \cup P_2 \cup P_3$, and $J(V^{(P,1)})$ is P_1 or $P_1 \cup P_2$. In the following, we show by contradiction that $J(V^{(P,0)}) \neq P_1 \cup P_2 \cup P_3$ with $P_3 \neq \emptyset$, and $J(V^{(P,1)}) \neq P_1$ with $P_2 \neq \emptyset$.

Suppose $J(V^{(P,0)}) = P_1 \cup P_2 \cup P_3$ with $P_3 \neq \emptyset$. Let $i \in P_3$. Notice that $i \notin V_i^{(P,0)}$ and $\{k \in N \mid i \in V_k^{(P,0)}\} = P_2 \cup P_4$. Consider the profile $V' \in \mathcal{V}$ defined as follows. For every $k \in N$,

$$V'_k = \begin{cases} N & \text{if } k \in P_2, \\ (P_1 \cup P_2 \cup P_3) - \{i\} & \text{if } k \in (P_1 \cup P_3) - \{i\}, \\ N - \{k\} & \text{if } k \in P_4 \cup \{i\}. \end{cases}$$

Then, we have $(P_1 \cup P_2 \cup P_3) - \{i\} \subseteq J(V')$ from Theorem 1, and either $P_4 \cup \{i\} \subseteq J(V')$ or $(P_4 \cup \{i\}) \cap J(V') = \emptyset$ from (SYM). Thus, $J(V')$ is either $(P_1 \cup P_2 \cup P_3) - \{i\}$ or N. From (L) and $i \notin V'_i$, we have $J(V') \neq N$. Therefore, $J(V') = (P_1 \cup P_2 \cup P_3) - \{i\}$. However, (I) is violated, because $k \in J(V^{(P,0)})$ if and only if $k \in J(V')$ for every $k \in N - \{i\}$, and $\{k \in N \mid i \in V_k^{(P,0)}\} = \{k \in N \mid i \in V'_k\} = P_2 \cup P_4$ (i.e., $i \in V_k^{(P,0)}$ if and only if $i \in V'_k$ for every $k \in N$), but $i \in J(V^{(P,0)})$ and $i \notin J(V')$.

Suppose $J(V^{(P,1)}) = P_1$ with $P_2 \neq \emptyset$. Let $i \in P_2$. Notice that $i \in V_j^{(P,1)}$ and $\{k \in N \mid i \in V_k^{(P,1)}\} = P_2 \cup P_4$. Consider the profile $V'' \in \mathcal{V}$ defined as follows. For every $k \in N$,

$$V''_k = \begin{cases} \{k\} & \text{if } k \in P_1 \cup \{i\}\,, \\ P_1 \cup \{i\} & \text{if } k \in (P_2 \cup P_4) - \{i\}\,, \\ \emptyset & \text{if } k \in P_3. \end{cases}$$

Then we have $((P_2 \cup P_3 \cup P_4) - \{i\}) \cap J(V'') = \emptyset$ from Theorem 1, and either $P_1 \cup \{i\} \subseteq J(V'')$ or $(P_1 \cup \{i\}) \cap J(V'') = \emptyset$ from (SYM). Thus, $J(V'')$ is either $\emptyset$ or $P_1 \cup \{i\}$. From (L) and $i \in V''_i$, we have $J(V'') \neq \emptyset$. Therefore $J(V'') = P_1 \cup \{i\}$. However, (I) is violated, because, $k \in J(V^{(P,1)})$ if and only if $k \in J(V'')$ for every $k \in N - \{i\}$, and $\{k \in N \mid i \in V_k^{(P,1)}\} = \{k \in N \mid i \in V''_k\} = P_2 \cup P_4$ (i.e., $i \in V_k^{(P,1)}$ if and only if $i \in V''_k$ for every $k \in N$), but $i \notin J(V^{(P,1)})$ and $i \in J(V'')$. Now we can conclude that $J(V^{(P,0)}) = J(V^{(P,1)}) = P_1 \cup P_2$ for every 4-partition P of N. ■

Remark 2 *Lemma 2 can be considered as an extension of Lemma 1, since, for every $S \subseteq N$, $V^S = V^{(P,0)} = V^{(P,1)}$ with $P = (S, \emptyset, \emptyset, N - S)$.*

Remark 3 *From Lemma 2 and $J^*(V^{(P,0)}) = J^*(V^{(P,1)}) = P_1 \cup P_2$ for every 4-partition P of N, it follows that $J(V^{(P,0)}) = J^*(V^{(P,0)})$ and $J(V^{(P,1)}) = J^*(V^{(P,1)})$ for every 4-partition P of N.*

3.3 The characterization

Now we are ready for our characterization result.

Theorem 2 *The strong liberal CIF J^* is the only CIF that satisfies (SYM), (I), and (L).*

Proof. Obviously, the strong liberal CIF J^* satisfies (SYM), (I), and (L). Suppose there exists a CIF J that satisfies (SYM), (I), and (L), and $J \neq J^*$. It follows that there exists a profile V for which $J(V) \neq J^*(V)$. That is, there exists $i \in N$ such that

- $i \in J(V)$ and $i \notin J^*(V)$, or
- $i \notin J(V)$ and $i \in J^*(V)$.

Then, let

$$\begin{aligned} M_0 &= \{k \in J(V) - \{i\} \mid i \notin V_k\}, \\ M_1 &= \{k \in J(V) - \{i\} \mid i \in V_k\}, \\ N_0 &= \{k \in (N - J(V)) - \{i\} \mid i \notin V_k\}, \\ N_1 &= \{k \in (N - J(V)) - \{i\} \mid i \in V_k\}. \end{aligned}$$

Notice that $M_0 \cup M_1 = J(V) - \{i\}$, $N_0 \cup N_1 = (N - J(V)) - \{i\}$, $M_0 \cup N_0 = \{k \in N - \{i\} \mid i \notin V_k\}$ and $M_1 \cup N_1 = \{k \in N - \{i\} \mid i \in V_k\}$.

Suppose $i \in J(V)$ and $i \notin J^*(V)$. Then we have $i \notin V_i$. Let $V' \in \mathcal{V}$ be the profile defined as follows. For each $k \in N$,

$$V'_k = \begin{cases} M_0 \cup M_1 & \text{if } k \in M_0, \\ M_0 \cup M_1 \cup N_0 \cup \{i\} & \text{if } k \in M_1, \\ M_0 \cup M_1 & \text{if } k \in N_0 \cup \{i\}, \\ M_0 \cup M_1 \cup N_0 \cup \{i\} & \text{if } k \in N_1. \end{cases}$$

From Lemma 2 and $V' = V^{(P,0)}$ with $P = (M_0, M_1, N_0 \cup \{i\}, N_1)$, we have $J(V') = J^*(V') = M_0 \cup M_1 = J(V) - \{i\}$ with $i \in J(V)$. However, (I) is violated, because $k \in J(V)$ if and only if $k \in J(V')$ for every $k \in N - \{i\}$, and $\{k \in N \mid i \in V_k\} = \{k \in N \mid i \in V'_k\} = M_1 \cup N_1$ (i.e., $i \in V_k$ if and only if $i \in V'_k$ for every $k \in N$), but $i \in J(V)$ and $i \notin J(V'')$.

Suppose $i \notin J(V)$ and $i \in J^*(V)$. Then, we have $i \in V_i$. Let $V'' \in \mathcal{V}$ be the profile defined as follows. For each $k \in N$,

$$V''_k = \begin{cases} M_0 & \text{if } k \in M_0, \\ M_0 \cup M_1 \cup \{i\} & \text{if } k \in M_1 \cup \{i\}, \\ M_0 & \text{if } k \in N_0, \\ M_0 \cup M_1 \cup \{i\} & \text{if } k \in N_1. \end{cases}$$

From Lemma 2 and $V'' = V^{(P,1)}$ with $P = (M_0, M_1 \cup \{i\}, N_0, N_1)$, we have $J(V'') = J^*(V'') = M_0 \cup M_1 \cup \{i\} = J(V) \cup \{i\}$ with $i \notin J(V)$. However, (I) is violated, because, $k \in J(V)$ if and only if $k \in J(V'')$ for every $k \in N - \{i\}$, and $\{k \in N \mid i \in V_k\} = \{k \in N \mid i \in V''_k\} = M_1 \cup N_1 \cup \{i\}$ (i.e., $i \in V_k$ if and only if $i \in V''_k$ for every $k \in N$), but $i \notin J(V)$ and $i \in J(V')$.

Now we can conclude that if a CIF J satisfies (SYM), (I), and (L), then $J(V) = J^*(V)$ for every profile $V \in \mathcal{V}$, i.e., $J = J^*$. Therefore, the strong liberal CIF J^* is the only CIF that satisfies (SYM), (I), and (L). ■

From the monotonicity of the strong liberal CIF J^*, the following corollary can be obtained immediately from Theorem 2.

Corollary 1 *If a CIF satisfies (SYM), (I), and (L), then it also satisfies (MON).*

Furthermore, in the following, we show that the three axioms (SYM), (I), and (L) are independent.

Theorem 3 *The strong liberal CIF J^* is not the only CIF that satisfies some but not all of (SYM), (I), and (L).*

Proof. The proof consists of three examples, each of which satisfies exactly two of the three axioms.

- Consider a CIF J defined as follows. For every $V \in \mathcal{V}$,

$$J(V) = \begin{cases} J^*(V) & \text{if } n = 1 \text{ (i.e., } N = \{1\}\text{)}, \\ \{1\} & \text{otherwise.} \end{cases}$$

  Obviously, J satisfies (I) and (L) when $n = 1$. Suppose $n > 1$. Since $k \in J(V)$ if and only $k \in J(V')$ for every $k \in N$ and for every $V, V' \in \mathcal{V}$, J satisfies (I). Moreover $J(V)$ is neither $\emptyset$ nor N, and thus, J satisfies (L).

  Now we show that J does not satisfy (SYM). It is obvious from $J \neq J^*$ when $n > 1$. But, in order to show it directly, let $n > 1$ and consider a profile V such that $V_k = \{1, 2\}$ for every $k \in N$. Notice that every CIF satisfying (SYM) must either include both 1 and 2 or exclude both 1 and 2. However, $J(V) = \{1\}$, and thus J does not satisfy (SYM).

- Consider a CIF J defined as follows. For every $V \in \mathcal{V}$,

$$J(V) = \begin{cases} J^*(V) & \text{if } J^*(V) \in \{\emptyset, N\}, \\ N - J^*(V) & \text{otherwise.} \end{cases}$$

  Since J^* satisfies (L) and $J(V) \in \{\emptyset, N\}$ if and only if $J^*(V) \in \{\emptyset, N\}$, J satisfies (L). Since J^* satisfies (SYM), $\overline{J^*}$, defined by $\overline{J^*}(V) = N - J^*(V)$ for every $V \in \mathcal{V}$, also satisfies (SYM). Moreover, since $J(V) \in \{J^*(V), \overline{J^*}(V)\}$ for every $V \in \mathcal{V}$, it follows that $J(V)$ satisfies (SYM).

  Now we show that J does not satisfy (I). Let $n > 1$ and consider profiles V and V' such that $V_k = N - \{1\}$ and $V'_k = \emptyset$ for every $k \in N$. Notice that $J(V) = \{1\}$ and $J(V') = \emptyset$, and $1 \notin V_k$ and $1 \notin V'_k$ for every $k \in N$. Thus, J does not satisfy (I).

- Consider a CIF J defined as follows. For every $V \in \mathcal{V}$,

$$J(V) = \emptyset.$$

Since $k \in J(V)$ if and only if $k \in J(V')$ for every $k \in N$ and for every $V, V' \in \mathcal{V}$, J satisfies (SYM) and (I). Consider a profile V such that $V_i = \{i\}$ for every $i \in N$. We have $J(V) = \emptyset$, but every CIF satisfying (L) must not be empty for such a profile V. Thus, J does not satisfy (L). ■

References

[1] A. Billot, 2003. How Liberalism Kills Democracy or Sen's Theorem Revised, Public Choice 116 247-270.

[2] P.C. Fishburn and A. Rubinstein, 1986. Aggregation of Equivalence Relations, Journal of Classification 3 61-65.

[3] A. Kasher, 1993. Jewish Collective Identity, in: D.T. Goldberg and M. Krausz (eds.), Jewish Identity, Temple University Press, Philadelphia, 56-78.

[4] A. Kasher and A. Rubinstein, 1997. On the Question "Who is a J?": A Social Choice Approach, Logique & Analyse 160 385-395.

[5] A. Rubinstein and P.C. Fishburn, 1986. Algebraic Aggregation Theory, Journal of Economic Theory 38 63-77.

[6] D. Samet and D. Schmeidler, Between Liberalism and Democracy, Journal of Economic Theory 110 213-233.