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THE NUCLEOLUS AS A CONSISTENT POWER INDEX IN NONCOOPERATIVE MAJORITY GAMES

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Discussion paper

The Nucleolus as a Consistent Power Index in Noncooperative Majority Games*

(Running title: Nucleolus as Power Index)

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Abstract

This paper studies noncooperative bargaining with random proposers in proper simple games. A power index is called *consistent* if it can be obtained as an equilibrium of the game with random proposers using the index itself as probability vector. Unlike the Shapley-Shubik and Banzhaf indices, the nucleolus has this property. The proof uses the balancedness result in Kohlberg (1971) reinterpreting the balancing weights as mixed strategies.

Keywords: noncooperative bargaining, random proposers, nucleolus, consistency, balancedness.

JEL Classification Numbers: C71, C72, C78.

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1 Introduction

Consider the classical problem of dividing a dollar by majority rule. Suppose there are n players, player i has w_i votes and q votes ($q > \frac{\sum_{i \in N} w_i}{2}$) are needed to achieve a majority. What are the expected shares for the players in this game? Power indices provide possible answers to this question. Each power index can be interpreted as corresponding to some model of coalition formation and payoff division in the divide-the-dollar game. In the case of the Deegan-Packel (1978) index, this model is very simple: the dollar will be divided equally between the members of a minimal winning coalition and all minimal coalitions are equally likely. Models for the Shapley-Shubik (1954) and Banzhaf (1965) indices are slightly more complicated: all winning coalitions are possible, and not all of them are equally likely.

One may wish to choose between the different power indices on the basis of the plausibility of their properties[1]. An alternative approach, consistent with the Nash (1953) program, is to set up a noncooperative bargaining game that we consider intuitively plausible and check whether expected equilibrium shares coincide with any of the well-known power indices. If the answer is positive, we will have improved our understanding of the corresponding power index. If the answer is negative, the equilibrium of the game can be interpreted as a new power index with noncooperative foundations.

This paper takes the latter approach, using the natural extension of the classical Baron and Ferejohn (1989) bargaining game[2]. Baron and Ferejohn consider the case of symmetric players, each of them being recognized to be the proposer with the same probability. The natural extension of this game to general weighted majority games, already hinted by Baron and Ferejohn in one of their examples, would be to select each player with a probability proportional to his number of votes. This extension has a straightforward interpretation if players are parties, different number of votes correspond to different number of representatives, and each representative is selected to be

[1] See Laruelle (1998).

[2] Baron and Ferejohn (1989) build on Rubinstein (1982) and Binmore (1987).

the proposer with equal probability[3].

The first part of the paper deals with constant-sum homogeneous games without dummies. We show that the equilibria of the generalized Baron and Ferejohn game do not generally correspond to any of the well-known power indices. Instead, the normalized weights arise as expected equilibrium shares, vindicating the nucleolus as a power index with noncooperative foundations. This is perhaps surprising, since it is common wisdom that weights and power seldom coincide, even for homogeneous games.

One may argue that the result is not that surprising: the recognition probabilities clearly affect the power distribution in the noncooperative game, and we are using the weights themselves as recognition probabilities. What would be the result for other recognition probabilities? For example, suppose we make the recognition probabilities coincide with the Shapley value. Will the expected equilibrium shares also coincide with the Shapley value? In general, we will call a power index *consistent* if it is an equilibrium of the generalized Baron-Ferejohn game in which the power index itself is used as probability vector. We provide examples that show that neither the Shapley-Shubik index nor the Banzhaf index are consistent for constant-sum homogeneous majority games; furthermore, the nucleolus is always consistent, even if it is not a representation of the game. The proof uses Kohlberg's (1971) result on balanced collections of coalitions, reinterpreting the balancing weights as weights corresponding to a mixed-strategy equilibrium.

The rest of the paper is organized as follows. Section 2 serves as an in-

[3] Many other papers have extended the Baron and Ferejohn game to different preferences of the players, different voting rules and/or different sets of alternatives. We consider general voting rules, but keep unchanged the set of alternatives (different divisions of a dollar) and the preferences of the players (identical and risk-neutral in money). Closely related papers are Harrington (1990), Winter (1996), Banks and Duggan (2000) and Eraslan (2001). Harrington (1990) and Eraslan (2001) study symmetric voting games with rules other than simple majority; Winter (1996) allows for the presence of veto players; Banks and Duggan (2000) show the existence of stationary equilibria for arbitrary voting rules in general environments.

troduction to weighted majority games and the nucleolus. Section 3 presents the noncooperative game. Section 4 contains the results for constant-sum homogeneous games; section 5 is devoted to the consistency property of the nucleolus. Section 6 concludes.

2 Preliminaries

2.1 Weighted majority games

Let $N = \{1, ..., n\}$ be the set of players. $S \subseteq N$ ($S \neq \varnothing$) represents a generic coalition of players, and $v : 2^n \to \mathbb{R}$ with $v(\varnothing) = 0$ denotes the characteristic function. The (cooperative) game (N, v) is a *simple game* iff $v(S) \in \{0, 1\}$ for all $S \subseteq N$, $v(\varnothing) = 0$, and $v(N) = 1$. A coalition S is called *winning* iff $v(S) = 1$ and *losing* iff $v(S) = 0$. It is called *minimal winning* iff $v(S) = 1$ and $v(T) = 0$ for all T such that $T \subset S$. We will denote the set of winning coalitions by $\mathbf{W}$ and the set of minimal winning coalitions by $\mathbf{W}^m$. The set of minimal winning coalitions containing player i is denoted by $\mathbf{W}_i^m$. A player with $\mathbf{W}_i^m = \varnothing$ is called a *dummy player.*

A simple game is *proper* iff $v(S) = 1$ implies $v(N \backslash S) = 0$ for all $S \subseteq N$. It is *constant-sum* iff $v(S) + v(N \backslash S) = 1$ for all $S \subseteq N$. It is a *weighted majority game* iff there exist n nonnegative numbers (weights) $w_1, ..., w_n$ and a nonnegative number q such that $v(S) = 1$ if and only if $\sum_{i \in S} w_i := w(S) \geq q$. We will denote a weighted majority game by $(q; w_1, ..., w_n)$. The pair (q, w) is called a *representation* of the game v. A weighted majority game has many possible representations, but not all of them are equally convenient. A representation w is called *normalized* iff $\sum_{i \in N} w_i = 1$; it is *homogeneous* iff $\sum_{i \in S} w_i = q$ for all $S \in \mathbf{W}^m$. Not all weighted majority games admit a homogeneous representation. A weighted majority game admitting a homogeneous representation is called a *homogeneous game.*

2.2 The nucleolus

Let (N, v) be a zero-normalized game (that is, $v(i) = 0$ for all i) and $x = (x_1, ..., x_n)$ with $x_i \geq 0$ and $x(N) = v(N)$ be a *payoff vector*. For any coalition S the value $e(S, x) = v(S) - x(S)$ is called the *excess* of S at x.

For any payoff vector x let $S_1, ..., S_{2^{|N|}-1}$ be an ordering of the coalitions for which $e(S_l, x) \geq e(S_{l+1}, x)$ for all $l = 1, ..., 2^{|N|} - 2$ and let $E(x)$ be the vector of excesses defined by $E_l(x) = e(S_l, x)$ for all $l = 1, ..., 2^{|N|} - 1$. We say that $E(x)$ is *lexicographically less* than $E(y)$ if $E_l(x) < E_l(y)$ for the smallest l for which $E_l(x) \neq E_l(y)$. The *nucleolus* is the set of payoff vectors x for which the vector $E(x)$ is lexicographically minimal. Schmeidler (1969) shows that the nucleolus consists of a unique payoff vector. It is contained in the classical bargaining set (Davis and Maschler, 1967) and in the kernel (Davis and Maschler, 1965).

We now recall a very useful result of Kohlberg (1971). For this we need some definitions.

Let b_0, b_1,..., b_p be a sequence of sets whose elements are coalitions of N. The sequence is a *coalition array* whenever

1. Every coalition of N is contained in exactly one of the sets b_1, ..., b_p.
2. b_0 contains only singletons.

For every game v and every payoff vector x, let $b_1(x, v)$ be the set of those $S \subset N$ for which $\max\{v(S) - x(S) : S \subset N\}$ is attained. Similarly, $b_2(x, v)$ is the set of those $S \subset N$ where $\max\{v(S) - x(S) : S \subset N,\ S \notin b_1(x, v)\}$, and so on. Finally, let $b_0(x) = \{\{i\} : x_i = 0\}$. This is the coalition array that *belongs* to x.

Let $\mathcal{C}$ be a collection of nonempty subsets of N. We say that the collection is (weakly) *balanced* iff there exist (nonnegative) positive numbers $(\lambda_S)_{S\in\mathcal{C}}$ such that, for each $i \in N$, $\sum_{S \ni i} \lambda_S = 1$.

Theorem 1 *(Kohlberg, 1971) A payoff vector x is in the nucleolus of v if and only if the coalition array that belongs to x has the following property:*

$$\forall\ k = 1, ..., p\ \exists\ b_0^k \subseteq b_0 \text{ such that } b_0^k \cup b_1 \cup ... \cup b_k \text{ is balanced.}$$

3 The noncooperative game

Let (N, v) be a proper simple game. We interpret this game as a transferable payoff game where n risk-neutral players decide by majority rule on the division of a (perfectly divisible) budget.

Given the underlying cooperative game, bargaining proceeds as follows: At every round $t = 1, 2, ...$ Nature selects a player randomly to be the proposer. This player proposes a coalition $S \subseteq N$ to which he belongs and a feasible division of $v(S)$, denoted by $x^S = (x_i^S)_{i \in S}$. Given a proposal, the rest of players in S (called responders) accept or reject sequentially (the order does not affect the results). If all players in S accept, the proposal is implemented and the game ends. If at least one player rejects, the game proceeds to the next period in which nature selects a new proposer (always with the same probability distribution). Players are risk-neutral and share a discount factor $\delta < 1$. Thus, if a proposal x^S is accepted by all players in S at time t, each player in S receives a payoff $\delta^{t-1} x_i^S$. A player not in S remains a singleton and receives zero.

We will call the probability distribution Nature uses to select proposers a *protocol*. We will denote the protocol by $\theta = (\theta_i)_{i \in N}$. Two natural protocols are the *egalitarian protocol*, $\theta_i = \frac{1}{n}$ for all i in N, and (if weights are assigned to the players) the *proportional protocol*, $\theta_i = \frac{w_i}{w(N)}$ for all i in N.

A pure strategy for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$, where σ_i^t, the tth round strategy of player i, prescribes

1. A *proposal* (S, x^S).

2. A *response function* assigning "yes" or "no" to all possible proposals of the other players.

The solution concept is *stationary subgame perfect equilibrium* (SPE). Stationarity requires that players follow the same strategy at every round t.

We denote the noncooperative game described above by $G(v, \theta)$.

Given a SPE σ^* we will denote the associated *expected payoff* for player i (computed at the beginning of the game, before Nature chooses the proposer) by $y_i(\sigma^*)$ -we will drop σ^* to simplify notation-. The expected payoff given that a proposal is rejected, δy_i, is called the *continuation value.*

We will denote the probability that player i proposes coalition S (conditional on i being the proposer) by λ_S^i and the probability that i receives a proposal by r_i. Thus, $r_i = \sum_{j \in N \backslash \{i\}} \theta_j \lambda_S^j I_i(S)$ ($I_i(S) = 1$ if $i \in S$ and 0 otherwise). Like y_i, λ_S^i and r_i depend on σ^*.

3.1 Properties of subgame perfect equilibria

The first thing to notice is that no delay occurs in a SPE. This is shown by Okada (1996) for $\theta = (\frac{1}{n})_{i \in N}$ and superadditive games; a direct extension for a general θ can be found in Montero (1999a). Okada's proof can also be adapted to proper simple games, as the following lemma shows

Lemma 2 *Let (N, v) be a proper simple game. If σ^* is a SPE of $G(v, \theta)$ and $(y_i)_{i \in N}$ the associated equilibrium expected payoffs, then σ^* is such that every player i only proposes coalitions that solve the following maximization problem*

$$\max_{S: S \ni i} v(S) - \sum_{j \in S \backslash \{i\}} \delta y_i \qquad (1)$$

and all proposals are accepted.

Proof. Consider the situation of player i as a proposer. Let m_i be the maximum value of (1). Subgame perfection implies that a proposal that gives $\delta y_j + \epsilon$ to each responder j must be accepted, thus player i can achieve a payoff arbitrarily close to m_i.

If instead player i makes a proposal (S, x^S) with $x_i > m_i$, the proposal will be rejected (otherwise at least one responder j is getting less than δy_j and could do better by rejecting the proposal) and i will get δy_i. We will show that $m_i > \delta y_i$.

Since the game is simple we have $1 = v(N) \geq \sum_{j \in N} y_j$. Clearly, in a SPE $\sum_{j \in N} y_j > 0$. Thus, $v(N) > \delta \sum_{j \in N} y_j$. This means that player i can get more than δy_i by making an acceptable proposal, and therefore all proposals must be accepted in equilibrium.

Let S be a coalition with $\lambda_S^i > 0$. It must be the case that player i offers every responder j exactly δy_j (otherwise player i could reduce the payoff offered to j and we would not have an equilibrium). Moreover, S must be a solution to (1); otherwise player i would prefer to propose other coalition. ■

4 Constant-sum homogeneous games

In this section we show that, given a constant-sum homogeneous game without dummies, the proportional payoffs can be obtained as an equilibrium of the game with a proportional protocol. We will state the results in terms of the normalized representation[4]. It should be clear that the result does not depend on the actual weights being normalized (though it does require the actual weights to be homogeneous).

The proof of the proposition will require the following lemma:

Lemma 3 *If (N, v) is a constant-sum homogeneous game without dummies, then the set of minimal winning coalitions is balanced.*

Proof. We know from Peleg (1968) that the nucleolus is a homogeneous representation. Since we consider games without dummy players and nondummy players always have a positive weight, the set b_0 in Kohlberg's theorem is empty. Furthermore, the set b_1 coincides with the set of minimal winning coalitions $\mathbf{W}^m$ (the excess of a minimal winning coalition S at the nucleolus is $1 - q$; all other coalitions have smaller excesses). Thus, $\mathbf{W}^m$ is balanced. ■

Proposition 4 *Let (q, w) be the normalized representation of a constant-sum homogeneous game without dummies. Let $\theta_i = w_i$. There is an equilibrium of the game $G(v, \theta)$ such that $y_i = w_i$ for all i in N. In this equilibrium*

[4] Peleg (1968) shows that a constant-sum homogeneous game without dummies has a unique normalized homogeneous representation, coinciding with the nucleolus.

only minimal winning coalitions are proposed, and the proposer offers δw_j to each responder j.

Proof. We will construct an equilibrium of the game $G(v, \theta)$ such that θ itself is the equilibrium expected payoff vector. Let σ^* be the equilibrium we are going to construct. There are two conditions σ^* must satisfy: first, σ^* must prescribe an optimal behavior for the players (both as proposers and as responders) given the vector w; second, w should be the expected payoff vector corresponding to σ^*. We attend to these aspects in turn.

1. The behavior of responders in equilibrium is going to be very simple: they just accept any offer that gives them their continuation value. Formally, player i accepts any offer x^S such that $x_i^S \geq \delta y_i$. As for the proposers, they will always propose a solution to the maximization problem in (1) with $y = w$. Since the game is homogeneous and each responder j must receive a payoff equal to δw_j, the proposer is indifferent between proposing any of the minimal winning coalitions including him. Larger or smaller coalitions are not optimal. Formally, if $\lambda_S^i > 0$, then $S \in \mathbf{W}_i^m$. Furthermore, the payoff player i gets conditional on being a proposer is $1 - \delta \sum_{j \in \overline{S} \setminus \{i\}} y_j$, where $\overline{S}$ is an arbitrary coalition such that $\overline{S} \in \mathbf{W}_i^m$. Clearly, if expected payoffs coincide with y, no proposer has an incentive to make unacceptable proposals.

2. We have fully specified the responders' behavior and part of the proposers' behavior (all minimal winning coalitions to which he belongs are optimal for a proposer; we have not yet chosen an exact probability distribution). Now all we need to do is to choose the mixed strategies in such a way that a player's expected payoff is indeed w.

A player's expected payoff is given by the following equation

$$y_i = \theta_i[1 - \delta \sum_{j \in \overline{S} \setminus \{i\}} y_j] + r_i \delta y_i.$$

By assumption, $\theta_i = y_i = w_i$. Thus, we need to find mixed strategies such that $\sum_{j\in\overline{S}\setminus\{i\}} y_j = r_i$. By assumption, $\sum_{j\in\overline{S}\setminus\{i\}} y_j = \sum_{j\in\overline{S}\setminus\{i\}} w_j$; since the game is homogeneous $\sum_{j\in\overline{S}\setminus\{i\}} w_j = q - w_i$. Thus, we need $q - w_i = r_i$.

We know from lemma 3 that the set of minimal winning coalitions is balanced. Let $(\lambda_S)_{S\in\mathbf{W}^m}$ be a collection of balancing weights. Let each player in S propose coalition S with probability λ_S; balancedness ensures that $\sum_{S\ni i}\lambda_S = 1$ for all i in N. All we have to show now is that, given these mixed strategies, players' payoffs are indeed w.

Consider player i. The probability of player i being a responder, r_i, equals $\sum_{S:i\in S\in\mathbf{W}^m}\sum_{j\in S\setminus\{i\}}\theta_j\lambda_S = \sum_{S:i\in S\in\mathbf{W}^m}(q-w_i)\lambda_S$ (because of homogeneity and our assumption about the protocol). Since $\sum_{S\ni i}\lambda_S = 1$, we indeed get $q - w_i = r_i$. ■

It is easy to see that neither the Shapley value nor the normalized Banzhaf index can in general be obtained as an equilibrium of the game with the proportional protocol. Consider the apex game with four players $(3; 2111)$. Montero (1999a) shows that the unique equilibrium payoffs of the game with the proportional protocol are $(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$, the nucleolus of the game. Instead, the Shapley-Shubik index and the normalized Banzhaf index are both $(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6})$ the Deegan-Packel index is $(\frac{3}{8},\frac{5}{24},\frac{5}{24},\frac{5}{24})$ and the Johnston (1978) index is $(\frac{9}{14},\frac{5}{42},\frac{5}{42},\frac{5}{42})$.

It is well-known that the equilibrium outcomes of the Baron-Ferejohn divide-the-dollar game may be very asymmetric (see Harrington (1990)). Indeed, it is easy to see that, since expected payoffs are proportional to the number of votes of the players and the proposer only needs to "buy" less than half of the votes, the proposer, however insignificant he may be, always gets strictly more than $\frac{1}{2}$. Despite this ex post deviation from proportionality, expected payoffs conditional on a certain minimal winning coalition being formed are also proportional in the equilibrium we have constructed. The reason is that all players propose each coalition with the same probability.

Corollary 5 *If the equilibrium σ^* is played, the expected payoff for player i conditional on coalition $S\in\mathbf{W}^m$, $S\ni i$ being formed is $\frac{w_i}{q}$.*

Proof. The probability that coalition $S \in \mathbf{W}^m$ is formed in equilibrium is $\sum_{j\in S}\theta_j\lambda_S^j = \sum_{j\in S} w_j\lambda_S = q\lambda_S$. The probability that it is formed with player $i \in S$ being the proposer is $\theta_i\lambda_S^i = w_i\lambda_S$; in this case, player i gets $1-\delta\sum_{j\in S\setminus\{i\}} w_j = 1-\delta(q-w_i)$. The probability that it is formed with player i being a responder is $\sum_{j\in S\setminus\{i\}}\theta_j\lambda_S^j$, that is, $\sum_{j\in S\setminus\{i\}} w_j\lambda_S = (q-w_i)\lambda_S$; in this case, player i gets δw_i. Thus, the expected equilibrium payoff for player i conditional on S being formed, $y_i(\sigma,S)$, equals

$$y_i(\sigma,S) = \frac{w_i\lambda_S\left(1-\delta(q-w_i)\right)+(q-w_i)\lambda_S\delta w_i}{q\lambda_S} = \frac{w_i}{q}.$$

■

Remark 6 *Corollary 5 implies that the equilibrium payoffs induced by* σ^* *correspond (in expected terms) not only to nucleolus but also to solution concepts like the von Neumann-Morgenstern (1944) main simple solution, the set of balanced aspirations (Cross, 1967) and the stable demand set (Morelli and Montero, 2001). These solution concepts predict that any minimal winning coalition may form and the payoff division will be proportional to the weights.*

5 The nucleolus as a consistent power index

It is well-known in random proposer games that the protocol θ affects the expected payoffs of the players[5]. We have argued that a natural choice for θ in majority games is to select each player with a probability proportional to his weight. However, if the underlying game is not a weighted majority game there is no natural choice for the protocol. Yan (2000) interprets the protocol as reflecting different bargaining power of the players. One may then argue that the Baron-Ferejohn game should not use the actual weights as probabilities but rather a measure of the bargaining power of the players. In short, θ should correspond to a power index.

[5] Many papers make reference to this question, including Eraslan and Merlo (1999), Montero (1999a, 1999b), Yan (2000), Gomes (2001) and Eraslan (2001).

We will call a power index *BF-consistent* (that is, consistent in the Baron-Ferejohn game) iff it is an equilibrium of the game that uses the index itself as protocol. This is a very natural property of a power index. If we use the Shapley-Shubik index in the protocol but obtain the nucleolus as expected payoff vector, then it seems that the Shapley-Shubik index was not a good measure of the players' bargaining power in the noncooperative game. Instead, a consistent power index is *self-confirming*: if we assume that the power distribution in the game corresponds to θ, there is an equilibrium of the game that confirms this idea.

Definition 7 *Let (N, v) be a proper simple game, and $\theta \in \mathbb{R}^n_+$ be a protocol. We say that θ is **BF-consistent** for v iff the game $G(v, \theta)$ has an equilibrium σ^* such that $y(\sigma^*) = \theta$.*

In this paper we will refer to BF-consistency simply as consistency.

We have chosen to define consistency for any vector and any game. Analogously, one can define consistency of a solution concept for a given family of games. Let $\psi : \mathbb{R}^{2^n-1} \to \mathbb{R}^n$ be a solution concept and let $\mathbb{S}$ be the set of proper simple games. We say that ψ is consistent for proper simple games iff

$$\forall v \in \mathbb{S}, \forall \theta \in \psi(v) : \theta(v) \text{ is consistent for } v.$$

We now show that the nucleolus is consistent for proper simple games.

Proposition 8 *Let (N, v) be a proper simple game. If $\mu \in \mathbb{R}^n$ is the nucleolus of v, then the game $G(v, \mu)$ has an equilibrium σ^* with $y(\sigma^*) = \mu$.*

Proof. Clearly, the result holds if the game has a dictator. If the game has no dictators (that is, $v(i) = 0$ for all i) we will construct an equilibrium σ^* with the desired property. The strategy combination σ^* will prescribe the following:

1. Each player accepts any proposal that gives him at least $\delta\mu_i$. In particular, if $\mu_i = 0$ player i accepts any nonnegative payoff.

2. If a player is selected to be the proposer, he proposes only coalitions of maximum excess (that is, coalitions in $b_1(\mu, v)$). If expected equilibrium payoffs coincide with the nucleolus, coalitions with maximum excess correspond to coalitions that solve the maximization problem in (1). Kohlberg's theorem implies that each player i belongs to at least one coalition in $b_1(\mu, v)$. Furthermore, since the set $b_1(\mu, v) \cup b_0(\mu)$ is weakly balanced, there exists a collection of balancing weights $(\lambda_S)_{S \in b_0(\mu) \cup b_1(\mu,v)}$. Let i be a player such that $\mu_i > 0$. We set $\lambda^i_S = \lambda_S$ for $S \in b_1(\mu, v)$, $S \ni i$ and $\lambda^i_S = 0$ otherwise. If $\mu_i = 0$, player i cannot be selected to be the proposer, so we do not need to specify what would he do if he were selected. It is also clear that players prefer to propose a coalition in $b_1(\mu, v)$ rather than make unacceptable proposals.

3. Given these strategies, all we have to check is that equilibrium expected payoffs given that players follow strategy combination σ^* actually coincide with μ.

   Recall that expected payoffs are given by the following equation

$$y_i = \theta_i[1 - \delta \sum_{j \in \overline{S} \setminus \{i\}} y_j] + r_i \delta y_i.$$

   where $\overline{S}$ is an arbitrary coalition with $\overline{S} \in b_1(\mu, v)$, $i \in \overline{S}$.

   Substituting $y = \theta = \mu$, we obtain

$$\mu_i = \mu_i[1 - \delta \sum_{j \in \overline{S} \setminus \{i\}, \mu_j > 0} \mu_j] + r_i \delta \mu_i.$$

   In order for this equation to hold we need $\sum_{j \in \overline{S} \setminus \{i\}, \mu_j > 0} \mu_j := \overline{\mu}_i = r_i$.

   We know $r_i = \sum_{S \in b_1(\mu,v), S \ni i} \sum_{j \in S \setminus \{i\}, \mu_j > 0} \mu_j \lambda^j_S = \sum_{S \in b_1(\mu,v), S \ni i} \overline{\mu}_i \lambda_S = \overline{\mu}_i$.

■

It is easy to see that neither the Shapley value nor the normalized Banzhaf index have this property. Consider the case of apex games with four players. Both the Shapley value of the game and the normalized Banzhaf index are $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. If there is an equilibrium with expected payoffs are $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, all the minor players must propose the minor player coalition. Suppose $\theta = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. Then the expected equilibrium payoff for the apex player equals $\frac{1}{2}[1 - \frac{\delta}{6}] \neq \frac{1}{2}$. Something analogous happens with the Johnston index, $(\frac{9}{14}, \frac{5}{42}, \frac{5}{42}, \frac{5}{42})$. The Deegan-Packel index is $(\frac{3}{8}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24})$. If there is an equilibrium with these payoffs, the minor players prefer to form a coalition with the apex player and the equilibrium expected payoff for a minor player would be $\frac{5}{24}[1 - \delta\frac{9}{24}] \neq \frac{5}{24}$.

A property of power indices that makes them unlikely to coincide with expected equilibrium payoffs in the Baron-Ferejohn model is that they usually induce strict preferences for the players over the minimal winning coalitions they can propose. The same can be said about the kernel (Davis and Maschler, 1965)[6]. Instead, the nucleolus makes the players indifferent between several coalitions and provides us with more degrees of freedom when constructing an equilibrium (indeed, we managed to construct an equilibrium with the nucleolus as expected payoff vector even after imposing the restriction that $\lambda_S^i = \lambda_S^j$ for all $i, j \in S$). Another solution concept that makes the players indifferent over several coalitions is the modified nucleolus (Sudhölter, 1996). The modified nucleolus is a representation of all homogeneous games[7], and therefore would make each player i indifferent between *all* coalitions in $\mathbf{W}_i^m$ in a homogeneous game (whereas the nucleolus only makes him indifferent over the set $\mathbf{W}_i^m \cap b_1(\mu, v)$). Rosenmüller and Sudhölter (1994) show that there are homogeneous games for which the set of minimal winning coalitions cannot be even weakly balanced. Thus, we cannot construct mixed

[6] Consider the game $(6; 422111)$. The payoff vector $(\frac{14}{34}, \frac{7}{34}, \frac{7}{34}, \frac{2}{34}, \frac{2}{34}, \frac{2}{34})$ is in the kernel but is not consistent.

[7] The nucleolus is not always a representation of the game. For a study of the conditions under which the nucleolus is a representation of homogeneous games, see Peleg and Rosenmüller (1992).

strategies for the modified nucleolus as we did for the nucleolus. It does not follow immediately that the modified nucleolus is not consistent (only that it cannot be supported by strategies such that $\lambda_S^i = \lambda_S^j$ for all $i, j \in S$). We now show by an example that indeed the modified nucleolus is not consistent.

Example 9 *Consider the game* $(5; 3, 2, 2, 1)$. *If* $\theta = (\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8})$, *expected equilibrium payoffs cannot coincide with* θ.

Suppose we have an equilibrium with expected payoffs $(\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8})$. Player 1 is indifferent between proposing $\{1, 2\}$ and $\{1, 3\}$; players 2 and 3 are indifferent between proposing to player 1 and proposing $\{2, 3, 4\}$; player 4 prefers proposing coalition $\{2, 3, 4\}$. The equilibrium strategies would have to satisfy (among others) the following equations

$$\begin{aligned} y_1 &= \frac{3}{8}[1 - \delta\frac{2}{8}] + [\frac{2}{8}\lambda^2_{\{1,2\}} + \frac{2}{8}\lambda^3_{\{1,3\}}]\delta\frac{3}{8} \\ y_4 &= \frac{1}{8}[1 - \delta\frac{4}{8}] + [\frac{2}{8}\lambda^2_{\{2,3,4\}} + \frac{2}{8}\lambda^3_{\{2,3,4\}}]\delta\frac{1}{8} \\ \lambda^i_{\{1,i\}} + \lambda^i_{\{2,3,4\}} &= 1;\ i = 2, 3. \end{aligned}$$

In order for y_4 to be $\frac{1}{8}$, we need $\lambda^2_{\{2,3,4\}} = \lambda^3_{\{2,3,4\}} = 1$. But then $y_1 = \frac{3}{8}[1 - \delta\frac{2}{8}] \approx \frac{9}{32}$.

The nucleolus is not the only consistent solution concept. The set of balanced aspirations (Cross 1967, Bennett 1983) normalized so as to obtain an imputation, and the least core (Maschler et al., 1979) also have this property. Both sets coincide with the nucleolus for constant-sum homogeneous weighted majority games.

6 Concluding remarks

We have provided a noncooperative interpretation of the nucleolus of proper simple games. The balancing weights in Kohlberg (1971) are interpreted as mixed strategies of the players. This is quite a reasonable interpretation of the balancing weights in a balanced collection, indeed more reasonable than the interpretation of players distributing their time over several coalitions, which often requires the players to be "in two places at once[8]" besides requiring constant returns.

The paper also makes a case for the nucleolus as a power index in divide-the-dollar games, especially in constant-sum homogeneous games. Each power index has its own advantages and indeed the Shapley-Shubik index can also be obtained as an equilibrium of a game with random proposers (see Hart and Mas-Colell 1996). However, Hart and Mas-Colell (1996) assume bargaining under *unanimity rule*, though subcoalitions matter through the possibility of partial breakdown.

Yan's (2000) result on core implementation can be restated in the language of proposition 8: each element in the core is a consistent power index for the corresponding game, and the corresponding strategies have $\lambda_N = 1$.

Other papers have provided noncooperative interpretations of the nucleolus. Serrano (1997) provides a noncooperative interpretation of the kernel (and therefore of the nucleolus) based on the reduced game property (Davis and Maschler, 1965). Potters and Tijs (1992) show that the nucleolus can be interpreted as the optimal strategy of one of the players in a matrix game. This paper can be viewed as complementary to the existing literature. It is less general, but, especially for constant-sum homogeneous games, it is based on a very natural noncooperative game.

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