**FIXED TREE GAMES WITH REPEATED
PLAYERS**

By S. Miquel, S. van Velzen, H.J.M. Hamers, H.W. Norde

September 2003

TILBURG ◆ UNIVERSITY

Fixed tree games with repeated players

S. Miquel[a]   B. van Velzen[b, c]   H. Hamers[b]   H. Norde[b]

Abstract

This paper introduces fixed tree games with repeated players (FRP games) which are a generalization of standard fixed tree games. This generalization consists in allowing players to be located in more than one vertex. As a consequence, these players can choose among several ways of connection with the root.

In this paper we show that FRP games are balanced. Moreover, we prove that the core of an FRP game coincides with the core of a related concave fixed tree game. We show how to find the nucleolus and we characterize the orders which provide marginal vectors in the core of an FRP game.

Classification: C71
Keywords: Cooperative game, fixed tree game, core.

1   Introduction

In this paper we consider a generalization of the fixed tree problem, introduced by Megiddo (1978). In a fixed tree problem a rooted tree Γ and a set of agents N is given, each agent being located at precisely one vertex of Γ and each vertex containing precisely one agent. Megiddo (1978) associates with such a problem a cooperative cost game (N, c), a *fixed tree game*, where, for every coalition $S \subseteq N$, $c(S)$ denotes the minimal cost needed to connect all members of S to the root via a subtree of Γ.

Fixed tree games and variants of fixed tree games have also been studied in Galil (1980), Maschler et al. (1995), Granot et al. (1996), Koster (1999), Van Gellekom (2000) and Koster et al. (2001). The special case where the tree is a chain corresponds to airport games, which have been considered in Littlechild (1974), Littlechild and Owen (1977) and Littlechild and Thompson

[a]Departament de Matemàtiques, Universitat de Lleida, Spain. Financial support by Universitat de Lleida.

[b]CentER and Department of Econometrics and OR, Tilburg University, P.O.Box 90153, 5000 LE Tilburg, The Netherlands.

[c]Corresponding author. Email: s.vanvelzen@uvt.nl

(1977). Variants of fixed tree games, where it is allowed that one vertex is occupied by more players or by no player, are considered in e.g. Koster (1999) and Van Gellekom (2000). However, in these variants it is still required that every player occurs in precisely one vertex.

In this paper we generalize the model of Megiddo (1978) in the sense that players may occur repeatedly, i.e. in more than one vertex of the given tree. As a motivation for this generalized model one can consider for example the following irrigation problem. Consider a set of parcels in a desert environment which need to be irrigated from a well. For that reason, a network has been designed which allows the transportation of water from the well to at least one of the corners of each parcel. Consider, for example, the situation with eight parcels in Figure 1.
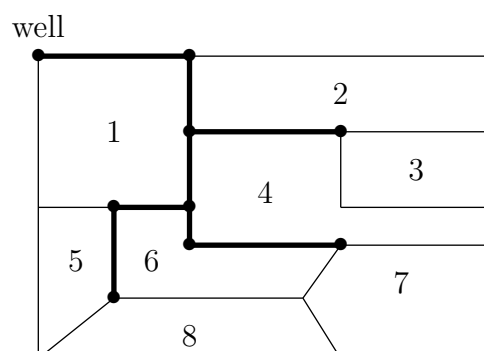

Figure 1: An irrigation problem with eight parcels.

Bold lines indicate the network that has already been constructed. The players are facing the problem of dividing the maintenance costs of this network, so they are facing a fixed tree problem where the tree is as depicted in Figure 2.

With this fixed tree we can associate in a natural way a cooperative cost game (N, c). For every coalition $S \subseteq N$ the number $c(S)$ denotes the minimal cost needed to connect all members of S at least once to the root via a subtree. This leads to the class of *fixed tree games with repeated players.*

Standard fixed tree games and the variants of these games are known to be concave. We will show that this needs not be true for fixed tree games with repeated players. However, we will show that these games are balanced, by showing that the core of a fixed tree game with repeated players coincides with the core of some related standard fixed tree game. Moreover, we analyze which marginal vectors provide core elements.

This paper is organized as follows. In Section 2 we recall some notions from graph theory and cooperative game theory. In Section 3 we formally introduce fixed tree games with repeated players and in Section 4 we focus on the structure of the core of these games.
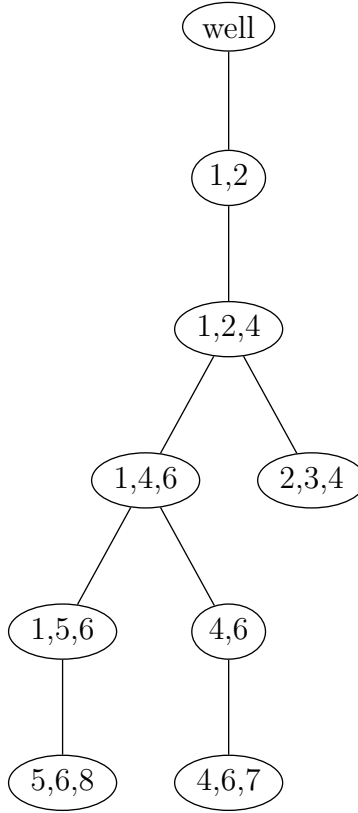
Figure 2: Fixed tree of the problem in Figure 1.

2 Preliminaries

In this section we recall some notions of graph theory and cooperative game theory.

A *tree* $\Gamma = (V, E)$ is an undirected connected graph without cycles, with set of vertices V and set of edges E. The set V contains a vertex which has a special meaning. We denote this vertex by 0 and refer to it as the root. For each vertex $v \in V$ there is a unique path from the root to vertex v, which we denote by $P(v)$.

The precedence relation $(V, \preceq)$ on V is defined by $v \preceq v'$ if and only if $v \in P(v')$. Analogously we define the precedence relation $(E, \preceq)$ on the set of edges. A *trunk* of (V, E) is a set of vertices $T \subseteq V$ containing the root which is closed under the precedence relation $(V, \preceq)$, i.e. if $v \in T$ and $v' \preceq v$, then $v' \in T$. Hence, the subtree corresponding to this set of vertices T is connected. The set of *followers* of a vertex v is the set $F(v) = \{v' \in V | v \preceq v'\}$. A vertex v is called a *leaf* if $F(v) = \{v\}$. Analogously we define the set of edges $F(e)$ following an edge e. Note that it holds that $e \in F(e)$.

A *cooperative (cost) game* is a tuple (N, c) where $N = \{1, 2, ..., n\}$ is the

set of players, and $c : 2^N \to \mathbb{R}$ is its characteristic (cost) function, in which, $c(S)$ is interpreted as the total cost of coalition S. By convention, $c(\emptyset) = 0$.

The *core* of a cost game (N, c) is the set

$$C(c) := \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

If $x \in C(c)$, then no coalition $S \subseteq N$ has an incentive to split off from the grand coalition N if x is the proposed vector of cost shares. A game (N, c) is called *balanced* if it has a nonempty core and *totally balanced* if the core of every subgame is nonempty, where the subgame corresponding to some coalition $T \subseteq N$, $T \neq \emptyset$, is the game (T, c_T) with $c_T(S) = c(S)$ for all $S \subseteq T$.

An *order* on the players is a bijection $\theta : N \to \{1, \ldots, n\}$, where $\theta(i) = j$ means that player i is at position j. For every order $\theta : N \to \{1, \ldots, n\}$ we define the *marginal vector* $m^\theta(c)$ by

$$m_k^\theta(c) = c(\{l \in N | \theta(l) \leq \theta(k)\}) - c(\{l \in N | \theta(l) < \theta(k)\})$$

for every $k \in N$.

A game (N, c) is called *monotone* if for every $S, T \in 2^N$ with $S \subseteq T$, we have $c(S) \leq c(T)$. Moreover, for a balanced monotonic game (N, c) it holds that $x_i \geq 0$ for every $x \in C(c)$.

A game (N, c) is *concave* if for every $i \in N$ and every $S \subseteq T \subseteq N \backslash \{i\}$, we have $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$. Hence, for concave games the marginal contribution of a player to any coalition is larger than his marginal contribution to a larger coalition. It is a well known result that a cost game is concave if and only if all marginal vectors are core elements (cf. Shapley (1971) and Ichiishi (1981)). Hence, concave cost games are balanced.

3 Model and game

In this section we introduce fixed tree problems with repeated players and its associated cooperative games.

Definition 1. *A fixed tree problem with repeated players*, FRP problem for short, is a 5-tuple

$$\Gamma = (N, (V, E), 0, (S(v))_{v \in V}, (a(e))_{e \in E})$$

where

    1. $N = \{1, 2, ..., n\}$ is a finite set of players;

    2. (V, E) is a tree with vertex set V and edge set E;

    3. 0 is a special element of V, called the root of the tree;

4. $S(v) \subseteq N$ is, for every $v \in V$, a (possibly empty) subset of players occupying vertex v;

5. $a(e) > 0$ is, for every edge $e \in E$, the cost associated with edge e;

which satisfies the following assumptions

(A1) for every $i \in N$ there is a $v \in V$ with $i \in S(v)$;

(A2) for each leaf $t \in V$, there is an $i \in N$ such that $i \in S(t)$ and $i \notin S(v)$ for every $v \in V \backslash \{t\}$.

Assumption (A1) states that every player should occupy at least one vertex in the tree. Assumption (A2) states that the tree (V, E) is "optimal" for the grand coalition N, in the sense that no proper subtree of (V, E) provides at least one connection with the root for every $i \in N$.

Players who occupy at least two vertices are called *repeated players*. Players who occupy precisely one vertex are called *essential players*. If an FRP problem does not have repeated players, then it is called a *standard FRP*.
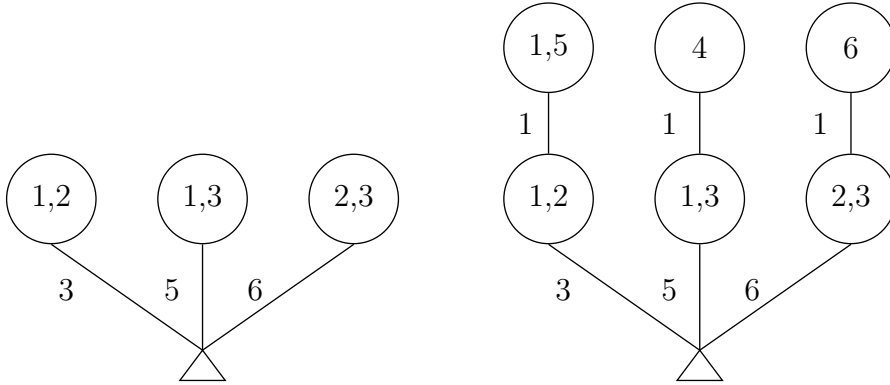

Figure 3: Two trees which depict cost sharing problems.

Example 1. In Figure 3 two cost sharing problems are depicted.

We can easily see that the tree on the left does not correspond to an FRP problem because after removing one of the edges, all three players remain connected to the root. Hence, Assumption (A2) is violated.

The tree on the right, on the contrary, corresponds to an FRP problem. It has three leaves and each of them is occupied by one essential player. Nevertheless, this is not a standard tree problem as defined in Megiddo (1978) since each of the players 1, 2 and 3 is located in more than one vertex.

For an FRP problem we can define a corresponding cost game.

Definition 2. Let $\Gamma = (N, (V, E), 0, (S(v))_{v \in V}, (a(e))_{e \in E})$ be an FRP problem. The corresponding *fixed tree game with repeated players*, FRP game for short, is the cost game (N, c) defined by

$$c(S) = \min_{T_S \in A_S} \left\{ \sum_{e \in T_S} a(e) \right\} \text{ for every } S \subseteq N, \ S \neq \emptyset$$

$$c(\emptyset) = 0$$

where A_S is the collection of admissible subtrees for coalition S. A subtree is admissible for coalition $S \subseteq N$ if it provides at least one connection with the root for every $i \in S$.

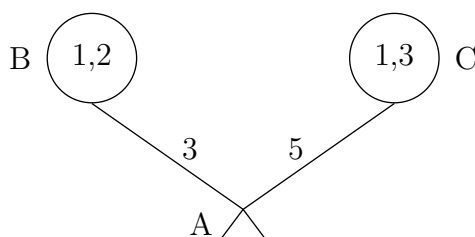In the following example we illustrate the concepts of FRP games and admissible trees.

Figure 4: An FRP problem.

Example 2. In Figure 4 an FRP problem is depicted. The set of admissible trees for player 1 is $A_{\{1\}} = \{\{(A,B)\}, \{(A,C)\}, \{(A,B),(A,C)\}\}$. It now holds that $c(\{1\}) = \min\{3, 5, 3+5\} = 3$.

From the definition of FRP games it easily follows that FRP games are monotonic games since for every $S \subseteq T \subseteq N$, we have $A_S \supseteq A_T$.

Note that in the tree on the right depicted in Figure 3, the position of player 1 in the vertex with player 5 seems irrelevant since the path from the root to this vertex contains another vertex occupied by player 1. We formalize this idea below.

Given an FRP problem

$$\Gamma = (N, (V, E), 0, (S(v))_{v \in V}, (a(e))_{e \in E})$$

we define the *reduced problem*

$$\Gamma^{red} = (N, (V, E), 0, (S^{red}(v))_{v \in V}, (a(e))_{e \in E})$$

where for every $v \in V$

$$S^{red}(v) = \{i \in S(v) : \text{ there is no } v' \in P(v), v' \neq v \text{ with } i \in S(v')\}.$$

That is, from the set of players occupying vertex v, those which also occupy a preceding vertex are dropped. Observe that the essential players remain in their initial vertex. The cost game associated with the reduced problem will be denoted by (N, c^{red}). This reduction is illustrated in Example 3.
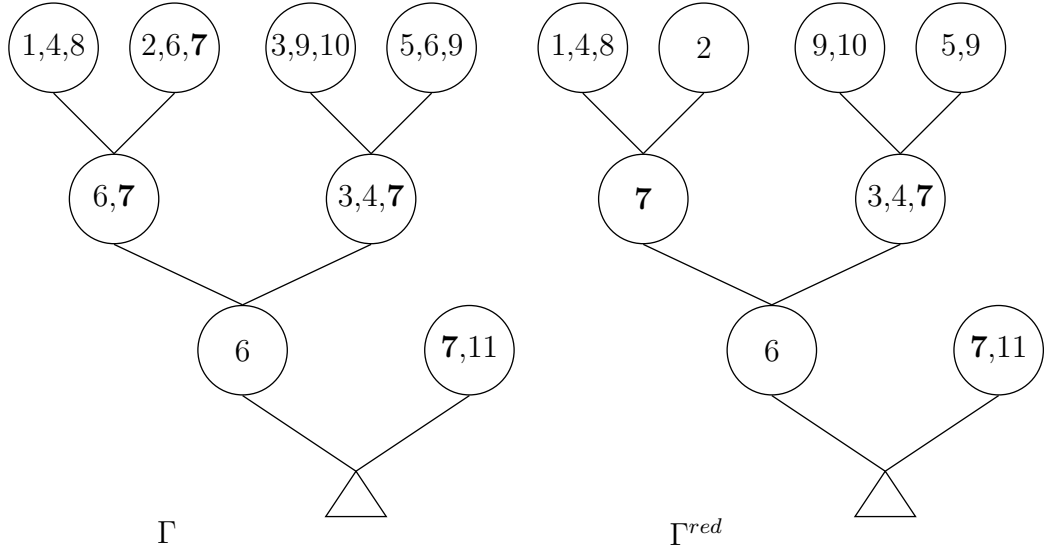
Figure 5: An FRP problem Γ and the corresponding reduced problem Γ^{red}.

Example 3. In Figure 5 an FRP problem Γ is depicted and its corresponding reduced problem Γ^{red}. In the first tree, player 7 is located in four different vertices. So player 7 can choose among four different paths to be connected to the root. However, the path that ends in the vertex occupied by players 6 and 7 is part of the path that ends in the vertex occupied by players 2, 6 and 7. Therefore player 7 will never choose this second path to connect himself to the root, since this path yields a higher cost. Therefore we can delete player 7 from the vertex occupied by players 2, 6 and 7 without changing the game.

Proceeding in this way we obtain the reduced problem Γ^{red} which is depicted in the second tree of Figure 5.

The proof of the following proposition is straightforward and therefore omitted.

Proposition 1. Let (N, c) be an FRP game corresponding to an FRP problem and let (N, c^{red}) be the game corresponding to its associated reduced FRP problem. Then (N, c) and (N, c^{red}) coincide.

Henceforth we assume in the sequel of this paper, without loss of generality, that an FRP problem is reduced.

4   The core

In this section we show that FRP games are balanced and we focus on the structure of the core.

Standard fixed tree games are known to be concave. Hence, it follows that these games are balanced. The following example illustrates that this

argument can not be used for FRP games, since these games need not be totally balanced, and therefore not concave.

Example 4. Let (N, c) be the FRP game obtained from the second FRP problem depicted in Figure 3. The characteristic function of the subgame restricted to coalition $\{1, 2, 3\}$ is the following

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$c_{\{1,2,3\}}(S)$	3	3	5	3	5	6	8

Now suppose that $C(c_{\{1,2,3\}}) \neq \emptyset$. Then there is an $x \in C(c_{\{1,2,3\}})$. It follows that $x_1 + x_2 \leq c_{\{1,2,3\}}(\{1, 2\}) = 3$. Similarly we have that $x_1 + x_3 \leq c_{\{1,2,3\}}(\{1, 3\}) = 5$ and $x_2 + x_3 \leq c_{\{1,2,3\}}(\{2, 3\}) = 6$. Adding these three expressions yields $2(x_1 + x_2 + x_3) \leq 14$, or equivalently $x_1 + x_2 + x_3 \leq 7$. However, since $x \in C(c_{\{1,2,3\}})$ it holds that $x_1 + x_2 + x_3 = 8$. Because of this contradiction we conclude that $C(c_{\{1,2,3\}}) = \emptyset$, and therefore that (N, c) is not totally balanced.

Next we will show that FRP games have a nonempty core. In order to do so we will show that the core of an FRP game coincides with the core of a related standard FRP game. Since this standard FRP game is concave and thus balanced, we then conclude that FRP games are balanced as well.

Consider the FRP problem $\Gamma = (N, (V, E), 0, (S(v))_{v \in V}, (a(e))_{e \in E})$. We obtain the related standard FRP problem Γ^{st} by relocating the repeated players. In particular, each repeated player gets relocated to precisely one vertex. This new situation is defined by the 5-tuple

$$\Gamma^{st} = (N, (V, E), 0, (\bar{S}(v))_{v \in V}, (a(e))_{e \in E}),$$

where $\bar{S}(v)$ is obtained from $S(v)$ as follows.

If player $i \in N$ is a repeated player in Γ, then there is more than one path connecting him with the root. Since (V, E) is a tree, the common part of all these paths is again a path. Let $v^* \in V$ be the furthest vertex from the root on this common path, then in the new problem, player i is located only in vertex v^*.

If player $i \in N$ is essential in Γ then in Γ^{st} this player remains in the same vertex. That is, if i is an essential player in Γ and $i \in S(v)$, then $i \in \bar{S}(v)$.

In the following example the construction of Γ^{st} is illustrated.

Example 5. Consider the FRP problem Γ corresponding to the tree depicted in Figure 6.

From Γ we obtain Γ^{st} by changing only the position of the repeated players. For example player 4 can choose between two paths in order to be connected to the root. The intersection of these two paths is the path which contains the root and the vertex occupied by player 6. In Γ^{st}, player 4 occupies the furthest vertex from the root on this intersection path. Note that repeated
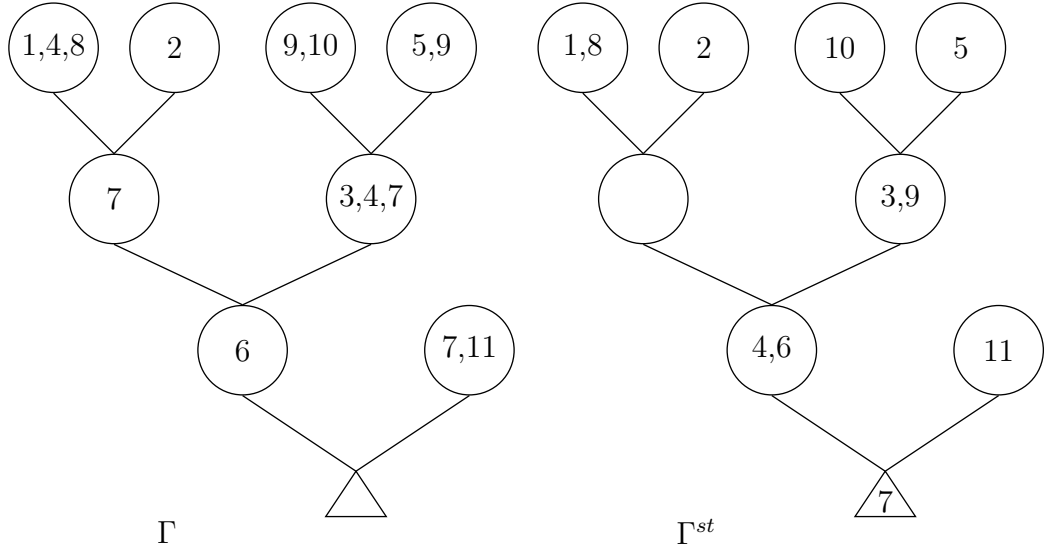
Figure 6: An FRP problem and the corresponding standard fixed tree problem.

players 4, 7 and 9, also move to vertices which are closer to the root. Observe that player 7 is reallocated to the root. Meanwhile all essential players remain in the same vertex.

Granot et al. (1996) show that standard fixed tree games are concave. The following proposition weakly generalizes this result and can be proved in a similar way. Therefore the proof is omitted.

Proposition 2. Cost games corresponding to standard FRP's are concave.

In the following theorem we show that the cores of the cost games corresponding to an FRP problem and its associated standard FRP problem coincide.

Theorem 1. Let $\Gamma = (N, (V, E), 0, (S(v))_{v \in V}, (a(e))_{e \in E})$ be an FRP problem and let $\Gamma^{st} = (N, (V, E), 0, (\bar{S}(v))_{v \in V}, (a(e))_{e \in E})$ be the corresponding standard FRP problem. Let (N, c) and (N, c^{st}) be the corresponding cost games. Then it holds that $C(c) = C(c^{st})$.

Proof. First note that $c^{st}(S) \leq c(S)$ for every $S \subseteq N$ and $c^{st}(N) = c(N)$. As a consequence we get $C(c^{st}) \subseteq C(c)$.

Now we show that $C(c) \subseteq C(c^{st})$. Let $x \in C(c)$. From the monotonicity of (N, c) it follows that $x \geq 0$. We need to show that for every $S \subseteq N$ it holds that $x(S) \leq c^{st}(S)$, or equivalently that $x(N \backslash S) \geq c(N) - c^{st}(S)$.

Let $S \subset N$, and let V_S be the set of vertices in which the members of S are located in the standard FRP problem, i.e. $V_S = \{v \in V | \bar{S}(v) \cap S \neq \emptyset\}$. Let T_S be the smallest trunk containing V_S, and let E_S be the subset of

9

edges corresponding to this trunk. By definition of T_S it holds that $c^{st}(S) = \sum\limits_{e \in E_S} a(e)$.

Now let O_S denote the set of outgoing edges of T_S, i.e. $O_S = \{(i, j) \in E | i \in T_S \text{ and } j \notin T_S\}$. Furthermore let, for all $e \in O_S$, V_e be the set of vertices corresponding to edges of $F(e) \backslash \{e\}$. Finally, let $I_e^c = \bigcup\limits_{v \notin V_e} S(v)$ and let $I_e = N \backslash I_e^c$. In other words, I_e are those players which appear only in vertices of V_e, and I_e^c is its complement. Because of assumption (A2) it follows that $I_e \neq \emptyset$ for each $e \in O_S$.

Let $e \in O_S$. Since each member of I_e^c appears at least once in a vertex of $V \backslash V_e$, the edges in $F(e)$ are not needed to connect the members of I_e^c to the root.

Hence, $c(I_e^c) \leq \sum\limits_{\substack{f \in E \\ f \notin F(e)}} a(f)$ and therefore $x(I_e^c) \leq \sum\limits_{\substack{f \in E \\ f \notin F(e)}} a(f)$. From the efficiency of x we deduce that

$$x(I_e) \geq c(N) - \sum_{\substack{f \in E \\ f \notin F(e)}} a(f) = \sum_{f \in F(e)} a(f). \tag{1}$$

Hence, the players which appear only in one branch of the tree pay the entire cost of that branch.

If $j \in S$, then it follows that $j \notin I_e$ for every $e \in O_S$. Therefore, $\cup_{e \in O_S} I_e \subset N \backslash S$. By definition of I_e it follows that $I_e \cap I_{\bar{e}} = \emptyset$ for all $e, \bar{e} \in O_S$, $e \neq \bar{e}$. That is, the I_e's are pairwise disjoint. Hence,

$$\sum_{e \in O_S} \sum_{f \in F(e)} a(f) \leq \sum_{e \in O_S} x(I_e) = x(\cup_{e \in O_S} I_e) \leq x(N \backslash S),$$

where the first inequality follows from (1), and the second from $x \geq 0$. It also holds that $\sum\limits_{e \in O_S} \sum\limits_{f \in F(e)} a(f) = \sum\limits_{f \in E} a(f) - \sum\limits_{f \in E_S} a(f) = c(N) - c^{st}(S)$.

We conclude that $x(N \backslash S) \geq c(N) - c^{st}(S)$. $\qquad\qquad\square$

The following corollary is an immediate consequence of Theorem 1 and the fact that concave games have a nonempty core.

Corollary 1. The core of an FRP game is nonempty.

Theorem 1 also enables us to obtain the nucleolus of an FRP game. The nucleolus is a well known one-point solution concept introduced by Schmeidler (1969). The nucleolus has the property that it is a core element whenever the core is nonempty. Potters and Tijs (1994) showed that if two games have the same core, with one of the games being concave, then both games have the same nucleolus as well. Hence, we conclude using Theorem 1 that the nucleolus of an FRP game coincides with the nucleolus of its associated standard FRP game. In Megiddo (1978) it is proved that the nucleolus of a fixed

tree game can be computed within $O(n^3)$ operations. Later on, Galil (1980) reduced the complexity of determining the nucleolus to $O(n \log n)$. Another algorithm, presented by Granot et al. (1996), has a complexity of $O(n^2)$, and in some trees only $O(n)$. In Maschler et al. (1995) other algorithms are given, and also a painting story that shows monotonicity properties of the nucleolus in a fixed tree game.

The last part of this section is dedicated to marginal vectors. We will characterize the marginal vectors that are core elements. First we need some definitions.

Let $\Gamma = (N, (V, E), 0, (S(v))_{v \in V}, (a(e))_{e \in E})$ be an FRP problem and $\Gamma^{st} = (N, (V, E), 0, (\bar{S}(v))_{v \in V}, (a(e))_{e \in E})$ be the corresponding standard FRP problem. For every $i \in N$ we define

$$N_i = \{j \in N | \text{ there exists } v, v' \in V \text{ with } v' \in F(v), i \in S(v), j \in \bar{S}(v')\}.$$

Note that for every essential player we have $i \in N_i$. A coalition $S \subseteq N$ is called *proper* if for every $i \in S$ there exists a player $j \in S \cap N_i$. These definitions are illustrated in Example 6.

Example 6. Consider the FRP problem Γ depicted in Figure 6. Player 4 is a repeated player who appears in two different vertices, say v_1 and v_2. Consider now these two vertices v_1 and v_2 in Γ^{st} and the corresponding sets of vertices $F(v_1)$ and $F(v_2)$. The set of players located in these sets of vertices in Γ^{st} is N_4. More precisely, $N_4 = \{1, 8\} \cup \{3, 9, 10, 5\} = \{1, 3, 5, 8, 9, 10\}$.

Consider now coalition $S = \{4, 6\}$. We can easily see that $N_4 \cap S = \emptyset$. Therefore coalition S is not proper. On the contrary, coalition $T = \{1, 4, 6\}$ is proper. Since, 1 and 6 are both essential it holds that $1 \in N_1 \cap T$ and $6 \in N_6 \cap T$. It is also obvious that $1 \in N_4 \cap T$.

The following lemma states that proper coalitions have the same cost in the FRP game and in its associated standard FRP game, while non proper coalitions have a strictly larger cost in the FRP game than in its associated standard FRP game.

Lemma 1. Let $\Gamma = (N, (V, E), 0, (S(v))_{v \in V}, (a(e))_{e \in E})$ be an FRP problem and let $\Gamma^{st} = (N, (V, E), 0, (\bar{S}(v))_{v \in V}, (a(e))_{e \in E})$ be the corresponding standard FRP problem. Let (N, c) and (N, c^{st}) be the corresponding cost games. Then the following holds:
  (i) $c(S) = c^{st}(S)$ for all proper $S \subseteq N$,
  (ii) $c(S) > c^{st}(S)$ for all non proper $S \subseteq N$.

Proof. Let $S \subseteq N$. Let T^* be the optimal tree for S in Γ^{st} and let $V(T^*)$ be the vertex set corresponding to this tree. Since for every $T \in A_S(\Gamma)$ we have $T^* \subseteq T$, it holds that $c(S) \geq c^{st}(S)$. Moreover, $c(S) = c^{st}(S)$ if and only

if $T^* \in A_S(\Gamma)$. Hence we only need to prove that S is proper if and only if $T^* \in A_S(\Gamma)$.

First assume that S is proper. Let $i \in S$ and let $j \in N_i \cap S$. There are $v, v' \in V$ with $v' \in F(v)$, $i \in S(v)$ and $j \in \bar{S}(v')$. Since T^* is optimal for S in Γ^{st} and $j \in \bar{S}(v')$, it must hold that $v' \in V(T^*)$. Since $v' \in F(v)$ this implies that $v \in V(T^*)$. We conclude that T^* connects all $i \in S$ to the root in Γ. Therefore we have that $T^* \in A_S(\Gamma)$.

To show the reverse, assume that $T^* \in A_S(\Gamma)$. We want to show that S is proper. Let $i \in S$. Since T^* is admissible for S in Γ there is a $v \in V(T^*)$ with $i \in S(v)$. Because T^* is optimal for S in Γ^{st}, there is a $j \in S$ and a $v' \in V(T^*)$ with $v' \in F(v)$ and $j \in \bar{S}(v')$. Thus $j \in S \cap N_i$. Hence S is proper. $\qquad \square$

The next theorem characterizes those orders that provide marginal vectors that are core elements.

Theorem 2. Let $(N, (V, E), 0, (S(v))_{v \in V}, (a(e))_{e \in E})$ be an FRP problem, (N, c) the corresponding FRP game and $\theta : N \to \{1, \ldots, n\}$ an ordering of the set of players. Then $m^{\theta}(c) \in C(c)$ if and only if for every $i \in N$ there exists a $j \in N_i$ such that $\theta(j) \leq \theta(i)$.

Proof. Let $(N, (V, E), 0, (\bar{S}(v))_{v \in V}, (a(e))_{e \in E})$ be the associated standard FRP problem, and (N, c^{st}) its corresponding game. First we show the "if" part. Assume that $\theta : N \to \{1, \ldots, n\}$ is such that for every player $i \in N$ there exists a player $j \in N_i$ such that $\theta(j) \leq \theta(i)$. Now let $k \in N$. By assumption it holds that for all $i \in \{l \in N | \theta(l) \leq \theta(k)\}$ there is a $j \in N_i$ with $\theta(j) \leq \theta(i)$. Hence, $j \in \{l \in N | \theta(l) \leq \theta(k)\}$ and we conclude that $\{l \in N | \theta(l) \leq \theta(k)\}$ is proper. This implies that for all $S \subseteq N$ it holds that

$$
\begin{aligned}
\sum_{k \in S} m_k^{\theta}(c) &= \sum_{k \in S} [c(\{l \in N | \theta(l) \leq \theta(k)\}) - c(\{l \in N | \theta(l) < \theta(k)\})] \\
&= \sum_{k \in S} [c^{st}(\{l \in N | \theta(l) \leq \theta(k)\}) - c^{st}(\{l \in N | \theta(l) < \theta(k)\})] \\
&= \sum_{k \in S} m_k^{\theta}(c^{st}) \\
&\leq c^{st}(S) \leq c(S),
\end{aligned}
$$

where the second equality follows by Lemma 1 and the fact that $\{l \in N | \theta(l) \leq \theta(k)\}$ and $\{l \in N | \theta(l) < \theta(k)\}$ are proper, the first inequality by the concavity of (N, c^{st}) and the last inequality again by Lemma 1. As a result we have that $m^{\theta}(c) \in C(c)$.

Second we show the "only if" part. Assume $\theta : N \to \{1, \ldots, n\}$ is such that there exists a player $i \in N$ such that for any $j \in N_i$ it holds that $\theta(j) > \theta(i)$. Let i^* be the first player in the order θ with this property. Consider now coalition $S = \{i \in N | \theta(i) \leq \theta(i^*)\}$. Then, by Lemma 1, $c(S) > c^{st}(S)$.
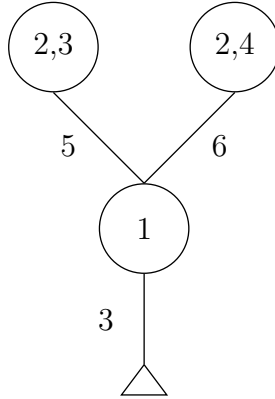
Figure 7: A fixed tree problem.

Since $\sum_{i \in S} m_i^{\theta}(c) = c(S)$, it follows that $\sum_{i \in S} m_i^{\theta}(c) > c^{st}(S)$, and therefore $m^{\theta}(c) \notin C(c^{st})$. Hence, by Theorem 1 we have $m^{\theta}(c) \notin C(c)$. $\square$

Every extreme point of $C(c^{st})$ coincides with a marginal vector of (N, c^{st}). The following example shows that this need not hold for $C(c)$, i.e. there are extreme points in $C(c)$ that do not coincide with marginal vectors of (N, c).

Example 7. Consider the FRP problem depicted in Figure 7.

Then $C(c) = C(c^{st}) = Co\{(3, 0, 5, 6), (0, 0, 8, 6), (0, 0, 5, 9), (0, 3, 5, 6)\}$. It is straightforward that there is no $\theta : N \to \{1, \ldots, n\}$ such that $m^{\theta}(c) = (0, 3, 5, 6)$. Moreover, not all marginal vectors of (N, c) are core elements. For instance, $m^{\theta}(c) = (0, 8, 0, 6) \notin C(c)$ where $\theta = (3, 4, 1, 2)$.

5 References

Galil, Z. (1980) *Applications of efficient mergeable heaps for optimization problems on trees.* Acta Informatica 13, 53-58.

Granot, D., Maschler, M., Owen, G. and Zhu, WR. (1996) *The Kernel/nucleolus of a standard tree game.* International Journal of Game Theory 25, 219-244.

Ichiishi, T. (1981) *"Super-modularity: Applications to concave games and the greedy algorithm for LP"* Journal of Economic Theory 25, 283-286.

Koster, M. (1999) *Cost sharing in production situations and network exploitation.* Ph. D. thesis, Tilburg University, Tilburg, The Netherlands.

Koster, M., Molina, E., Sprumont, Y. and Tijs, S. (2001) *Sharing the cost of a network: core and core allocations.* International Journal of Game Theory 30, 567-600.

Littlechild, S. (1974) *A simple expression for the nucleolus in a especial case.* International Journal of Game Theory 3, 21-29.

Littlechild, S. and Owen, G. (1977) *A further note on the nucleolus of the airport game.* International Journal of Game Theory 5, 91-95.

Littlechild, S. and Thompson, G. (1977) *Aircraft landing fees: a game theory approach.* The Bell Journal of Economics 8, 186-204.

Maschler, M., Potters, J. and Reijnierse, H. (1995) *Monotonicity properties of the nucleolus of standard tree games.* Report 9556, Department of Mathematics, University of Nijmegen, Nijmegen, The Netherlands.

Megiddo, N. (1978) *Computational complexity of the game theory approach to cost allocation for a tree.* Mathematics of Operations Research 3, 189-196.

Potters, J. and Tijs, S. (1994) *Essays in Game Theory. Essay 14: On the locus of the nucleolus.* 193-203. Springer-Verlag

Schmeidler, D. (1969) *The nucleolus of a characteristic function game.* Siam Journal on Applied Mathematics 17, 1163-1170.

Shapley, L.S. (1971) *Cores of convex games.* International Journal of Game Theory 1, 11-26.

Van Gellekom, A. (2000) *Cost and profit sharing in a cooperative environment.* Ph. D. thesis, University of Nijmegen, Nijmegen, The Netherlands.