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AXIOMATIZATIONS OF THE VALUE OF MATRIX GAMES

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Abstract

The function that assigns to each matrix game (i.e., the mixed extension of a finite zero-sum two-player game) its value is axiomatized by a number of intuitive properties.

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1 Introduction

The minimax theorem of Von Neumann (1928) states that every finite zero-sum two-player game (or matrix game) has a well-defined value. Formally, an $m \times n$ matrix game is represented by a matrix $A \in \mathbb{R}^{m \times n}$, with $m, n \in \mathbb{N}$. The entry $a_{ij} \in \mathbb{R}$ indicates the payoff to player 1 if he chooses row $i \in \{1, \ldots, m\}$ and player 2 chooses column $j \in \{1, \ldots, n\}$. The game is zero-sum, so the payoff to player 2 then equals $-a_{ij}$. Payoffs are extended to mixed strategies in the usual way. Let $G = \cup_{m,n \in \mathbb{N}} \mathbb{R}^{m \times n}$ be the set of all matrix games. For $k \in \mathbb{N}$, let $\Delta_k = \{x \in \mathbb{R}^k_+ \mid \sum_{i=1}^k x_i = 1\}$ denote the unit simplex in $\mathbb{R}^k$. The *value* of a game $A \in \mathbb{R}^{m \times n}$ is defined by

$$v(A) = \max_{x \in \Delta_m} \min_{y \in \Delta_n} x^t A y = \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^t A y,$$

where the superscript t denotes transposition.

The purpose of this paper is to axiomatize the function $v : G \to \mathbb{R}$ that assigns to each matrix game $A \in G$ its value $v(A) \in \mathbb{R}$. Recalling that the value can be interpreted as the expected payoff that player 1 (the row player) can guarantee himself, regardless of the strategy choice of his opponent, the following properties are intuitive:

Monotonicity: If all payoffs in the matrix are weakly increased, this should have a nonnegative effect on the value.

Symmetry: If the role of the row player and the column player is exchanged, the value of the new game equals minus the value of the original game.

Objectivity: In a trivial game where each of the two players has only one strategy, the value of the game is the payoff corresponding with this strategy combination.

Subgame property: Removing one of the rows implies a decrease in the strategic possibilities of the row player. This should have a nonpositive effect on the value.

Independence of irrelevant alternatives: Adding a row that is payoff equivalent with a mixed strategy of the row player does not change his strategic possibilities. This should not affect the value.

Lower bound property: The value of the game is at least as large as the smallest payoff in the matrix.

Strictly dominated action property: If one of the rows is strictly dominated (i.e., if the row player has a mixed strategy that is strictly better than that row, regardless of the choice of the column player), then removing that row leaves the value of the game unaffected.

These properties are formalized in Section 2 and are used to give several axiomatizations of the value function. In Section 3, it is shown that the axiomatizations employ logically independent axioms. Potential relaxations of some axioms and directions for further research are discussed in Section 4.

2 Axiomatizations

In this section, the properties from the introduction are formally defined and four axiomatizations of the value function are provided. For a function $f : G \to \mathbb{R}$ we introduce the following properties:

Monotonicity: For each $m, n \in \mathbb{N}$, and $A, B \in \mathbb{R}^{m\times n}$, if $A \geqq B$ (i.e., $a_{ij} \geqq b_{ij}$ for all entries), then $f(A) \geqq f(B)$.

Symmetry: For each $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$, $-f(A) = f(-A^t)$.

Objectivity: If $A = [a] \in \mathbb{R}^{1\times 1}$, then $f(A) = a$.

Subgame property: For each $m, n \in \mathbb{N}$ with $m \geqq 2$ and $A \in \mathbb{R}^{m\times n}$, if $B \in \mathbb{R}^{(m-1)\times n}$ is obtained from A by deleting one of its rows, then $f(A) \geqq f(B)$.

Independence of irrelevant alternatives: For each $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$, if $B \in \mathbb{R}^{(m+1)\times n}$ is obtained from A by inserting a row that is a convex combination of the rows of A, then $f(A) = f(B)$.

Lower bound property: For each $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$, $f(A) \geqq \min_{i,j} a_{ij}$.

Strictly dominated action property: For each $m, n \in \mathbb{N}$ with $m \geqq 2$ and $A \in \mathbb{R}^{m\times n}$, if $B \in \mathbb{R}^{(m-1)\times n}$ is obtained from A by deleting a strictly dominated row[1], then $f(A) = f(B)$.

[1]For $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$, let $e_i \in \mathbb{R}^k$ denote the i-th standard basis vector (with one as the i-th coordinate and zero everywhere else). Row i is strictly dominated if there exists a strategy $x \in \Delta_m$ for which $e_i^t Ay < x^t Ay$ for all $y \in \Delta_n$. Similarly, column j is strictly dominated if there exists a strategy $y \in \Delta_n$ for which $x^t Ae_j > x^t Ay$ for all $x \in \Delta_m$.

Remark 2.1 A simple argumentation shows that symmetry and independence of irrelevant alternatives of f imply: for each $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m \times n}$, if $B \in \mathbb{R}^{m \times (n+1)}$ is obtained from A by inserting a column that is a convex combination of the columns of A, then $f(A) = f(B)$. This property is used in the proof of Theorem 2.2.

Similarly, if f satisfies symmetry and the strictly dominated action property, then deleting a strictly dominated strategy of the column player does not affect the function value. In particular, this implies that in a game $A \in \mathbb{R}^{1 \times n}$ where the row player has only one strategy, $f(A)$ depends exclusively on the smallest entry in the matrix. This property is used in the proof of Theorem 2.4. ◁

Theorem 2.2 *The value function v is the unique function on G satisfying monotonicity, symmetry, objectivity, the subgame property, and independence of irrelevant alternatives.*

Proof. The value function v clearly satisfies the five properties. Let $f : G \to \mathbb{R}$ also satisfy them. To show: $f(A) = v(A)$ for all $A \in G$. Let $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m \times n}$. Let $z \in \Delta_m$ be a maximin strategy of the row player: $\min_{y \in \Delta_n} z^t Ay = v(A)$. This implies

$$\forall j \in \{1, \ldots, n\} : z^t Ae_j \geqq \min_{y \in \Delta_n} z^t Ay = v(A). \tag{1}$$

Let $B \in \mathbb{R}^{(m+1) \times n}$ be the matrix obtained from A by inserting $z^t A$ as a final row:

$$B = \begin{bmatrix} A \\ z^t A \end{bmatrix}.$$

Independence of irrelevant alternatives implies

$$f(A) = f(B). \tag{2}$$

Deleting all rows of B, except the final one, and repeatedly applying the subgame property yields

$$f(B) \geqq f([z^t A]). \tag{3}$$

Property (1) implies that $[z^t A] \geqq [v(A) \cdots v(A)] \in \mathbb{R}^{1 \times n}$. By monotonicity:

$$f([z^t A]) \geqq f([v(A) \cdots v(A)]). \tag{4}$$

Repeated application of symmetry and independence of irrelevant alternatives (see Remark 2.1) and objectivity yield

$$f([v(A) \cdots v(A)]) = f([v(A)]) = v(A). \tag{5}$$

Combining (2) to (5) yields $f(A) \geqq v(A)$. Since A is an arbitrary game, this implies that $f \geqq v$. Conversely, for every $A \in G$: $f(A) = -f(-A^t) \leqq -v(-A^t) = v(A)$ by symmetry. So $f \leqq v$. Hence $f(A) = v(A)$ for all $A \in G$. □

Theorem 2.3 *The value function v is the unique function on G satisfying symmetry, the subgame property, independence of irrelevant alternatives, and the lower bound property.*

Proof. The value function v clearly satisfies the four properties. Let $f : G \to \mathbb{R}$ also satisfy them. To show: $f(A) = v(A)$ for all $A \in G$. Let $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m \times n}$. Let $z \in \Delta_m$ be a maximin strategy of the row player: $\min_{y \in \Delta_n} z^t Ay = v(A)$. Since $\min_j z^t Ae_j = \min_{y \in \Delta_n} z^t Ay = v(A)$, the smallest entry of the matrix $[z^t A] \in \mathbb{R}^{1 \times n}$ equals $v(A)$. By the lower bound property:

$$f([z^t A]) \geqq v(A). \tag{6}$$

Consecutively using independence of irrelevant alternatives, the subgame property (m times), and (6) yields

$$f(A) = f\left(\begin{bmatrix} A \\ z^t A \end{bmatrix}\right) \geqq f([z^t A]) \geqq v(A).$$

Since A is an arbitrary game, this implies that $f \geqq v$. Conversely, for every $A \in G$: $f(A) = -f(-A^t) \leqq -v(-A^t) = v(A)$ by symmetry. So $f \leqq v$. Hence $f(A) = v(A)$ for all $A \in G$. □

Theorem 2.4 *The value function v is the unique function on G satisfying monotonicity, symmetry, objectivity, independence of irrelevant alternatives, and the strictly dominated action property.*

Proof. The value function v clearly satisfies the five properties. Let $f : G \to \mathbb{R}$ also satisfy them. To show: $f(A) = v(A)$ for all $A \in G$. Let $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m \times n}$. Let $z \in \Delta_m$ be a maximin strategy of the row player: $\min_{y \in \Delta_n} z^t Ay = v(A)$. Let $C \in \mathbb{R}^{m \times n}$ be a matrix with all entries strictly smaller than $\min_{i,j} a_{ij}$ and consequently also strictly smaller than $v(A)$. Then

$$f(A) = f\left(\begin{bmatrix} A \\ z^t A \end{bmatrix}\right) \geqq f\left(\begin{bmatrix} C \\ z^t A \end{bmatrix}\right) = f([z^t A]) = f([v(A)]) = v(A),$$

where the first equality follows from independence of irrelevant alternatives, the inequality from monotonicity, the second equality from repeated application of the strictly dominated action property. To establish the third equality, remember that $\min_{y \in \Delta_n} z^t Ay = \min_j z^t Ae_j = v(A)$. Symmetry and the strictly dominated action property (see Remark 2.1) allow us to delete from $[z^t A] \in \mathbb{R}^{1 \times n}$ all columns with a number larger than $v(A)$, while independence of irrelevant alternatives allows us to delete multiple occurrences of the coordinate $v(A)$ if necessary. This proves the third equality. The final equality follows from objectivity. Hence $f(A) \geqq v(A)$. Since A is an arbitrary game, this implies that $f \geqq v$. Conversely, for every $A \in G$: $f(A) = -f(-A^t) \leqq -v(-A^t) = v(A)$ by symmetry. So $f \leqq v$. Hence $f(A) = v(A)$ for all $A \in G$. □

Vilkas (1963) was the first to provide an axiomatization of the value function $v : G \to \mathbb{R}$. In addition to monotonicity, symmetry, and objectivity, he used the following axiom:

Dominance: For each $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m \times n}$, if $B \in \mathbb{R}^{(m+1) \times n}$ is obtained from A by inserting a row $[b_1 \cdots b_n]$ such that $[b_1 \cdots b_n] \leqq z^t A$ for some $z \in \Delta_m$, then $f(A) = f(B)$.

It is easy to see that the value function v satisfies dominance and that dominance implies both independence of irrelevant alternatives and the strictly dominated action property. Thus, Vilkas' characterization is a direct consequence of Theorem 2.4:

Corollary 2.5 [Vilkas, 1963] *The value function v is the unique function on G satisfying monotonicity, symmetry, objectivity, and dominance.*

Notice that dominance is stronger than the conjunction of independence of irrelevant alternatives and the strictly dominated action property: it also makes statements about weakly dominated actions. Yet, according to Theorem 2.4, the latter properties, together with symmetry, objectivity, and monotonicity, suffice to axiomatize the value function. This makes Theorem 2.4 much more appealing than Corollary 2.5.

3 Logical independence of the axioms

In this section, it is shown that the four axiomatizations use logically independent properties.

Proposition 3.1 *The axioms from Theorem 2.2 are logically independent.*

The result is proven by five examples of real-valued functions on G, each of which violates exactly one of the axioms. In all but one of the examples, it is straightforward to check that certain axioms are satisfied. This part is left to the reader.

The construction of a function that satisfies all axioms in Theorem 2.2 except monotonicity is the most troublesome. This example is discussed in detail. For each $n \in \mathbb{N}$ and each $A = [a_1 \cdots a_n] \in \mathbb{R}^{1\times n}$, define

$$g(A) = 2 \min_i a_i - \max_i a_i.$$

For each $m \in \mathbb{N}$ and each $A = [a_1 \cdots a_m]^t \in \mathbb{R}^{m\times 1}$, define

$$h(A) = 2 \max_i a_i - \min_i a_i.$$

Moreover, for each $m, n \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$, define

$$L(A) = \max_{x\in\Delta_m} g(x^t A) \text{ and } U(A) = \min_{y\in\Delta_n} h(Ay).$$

Since the functions g and h are continuous and the unit simplices are compact, these maxima and minima exist. Notice that for an $m \times 1$ matrix $x = [x_1 \cdots x_m]^t$:

$$\begin{aligned} g(-x^t) &= g([-x_1 \cdots - x_m]) = 2 \min_i(-x_i) - \max_i(-x_i) \\ &= -2 \max_i x_i + \min_i x_i = -h\left(\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}\right) = -h(x) \end{aligned}$$

and consequently that for an arbitrary $m \times n$ matrix A:

$$L(-A^t) = \max_{x\in\Delta_n} g(x^t(-A^t)) = \max_{x\in\Delta_n} g(-(Ax)^t) = \max_{x\in\Delta_n} -h(Ax) = -\min_{x\in\Delta_n} h(Ax) = -U(A). \quad (7)$$

Finally, define for each $A \in G$:

$$f_1(A) = \frac{1}{2}L(A) + \frac{1}{2}U(A).$$

By (7), the function f_1 satisfies symmetry: for every $A \in G$, $f_1(-A^t) = \frac{1}{2}L(-A^t) + \frac{1}{2}U(-A^t) = \frac{1}{2}(-U(A)) + \frac{1}{2}(-L(A)) = -f_1(A)$.

The function f_1 satisfies objectivity: if $A = [a] \in \mathbb{R}^{1\times 1}$, then $f_1(A) = \frac{1}{2}L([a]) + \frac{1}{2}U([a]) = \frac{1}{2}a + \frac{1}{2}a = a$.

The function f_1 satisfies the subgame property: let $m, n \in \mathbb{N}$ with $m \geqq 2$, let $A \in \mathbb{R}^{m\times n}$, and let $B \in \mathbb{R}^{(m-1)\times n}$ be obtained from A by deleting one of its rows, say the i-th one. Then

$$L(A) = \max_{x \in \Delta_m} g(x^t A) \geqq \max_{x \in \Delta_m : x_i = 0} g(x^t A) = \max_{x \in \Delta_{m-1}} g(x^t B) = L(B).$$

Moreover, for every $y \in \Delta_n$, the vector By is obtained from Ay by deleting its i-th coordinate. Hence $\max_j (Ay)_j \geqq \max_j (By)_j$ and $\min_j (Ay)_j \leqq \min_j (By)_j$, which implies that $h(Ay) \geqq h(By)$. Consequently

$$U(A) = \max_{y \in \Delta_n} h(Ay) \geqq \max_{y \in \Delta_n} h(By) = U(B).$$

Since both $L(A) \geqq L(B)$ and $U(A) \geqq U(B)$, it follows that $f_1(A) \geqq f_1(B)$.

The function f_1 satisfies independence of irrelevant alternatives: let $m, n \in \mathbb{N}$, $A \in \mathbb{R}^{m\times n}$, and let $B \in \mathbb{R}^{(m+1)\times n}$ be obtained from A by inserting a row (for notational convenience, we will take this to be the final row) that is a convex combination of the rows of A, i.e., there is a $z \in \Delta_m$ such that

$$B = \begin{bmatrix} A \\ z^t A \end{bmatrix}.$$

The convex hull of the rows of A is the same as the convex hull of the rows of B, so $L(A) = L(B)$. Moreover, for every $y \in \Delta_n$,

$$h(By) = h\left(\begin{bmatrix} Ay \\ z^t Ay \end{bmatrix}\right) = h(Ay),$$

since $z^t Ay$ is a convex combination of the numbers in Ay and hence $\min_i (Ay)_i \leqq z^t Ay \leqq \max_i (Ay)_i$. Hence also $U(A) = U(B)$. Conclude that $f_1(A) = f_1(B)$.

The function f_1 does not satisfy monotonicity:

$$L([0\ 0]) = U([0\ 0]) = 0, \text{ so } f_1([0\ 0]) = 0, \text{ but } L([0\ 1]) = -1, U([0\ 1]) = 0, \text{ so } f_1([0\ 1]) = -\frac{1}{2}.$$

The function $f_2 : G \to \mathbb{R}$ defined by $f_2(A) = \max_{i,j} a_{ij}$ for all $A \in G$ satisfies all axioms in Theorem 2.2 except symmetry:

$$-f_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -1 \neq 0 = f_2([-1\ 0]).$$

The function $f_3 : G \to \mathbb{R}$ defined by $f_3(A) = 0$ for all $A \in G$ satisfies all axioms in Theorem 2.2 except objectivity. The function $f_4 : G \to \mathbb{R}$ defined by $f_4(A) = \frac{1}{2}(\max_{i,j} a_{ij} + \min_{i,j} a_{ij})$ for all $A \in G$ satisfies all axioms in Theorem 2.2 except the subgame property:

$$f_4\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{2} < 1 = f_4([1]).$$

The function $f_5 : G \to \mathbb{R}$ defined by $f_5(A) = \frac{1}{2}(\max_i \min_j a_{ij} + \min_j \max_i a_{ij})$ for all $A \in G$ satisfies all axioms in Theorem 2.2 except independence of irrelevant alternatives:

$$f_5\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2} \neq \frac{3}{4} = f_5\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right).$$

Proposition 3.2 *The axioms in Theorem 2.3 are logically independent.*

This is easily checked using the functions defined earlier:

The function	satisfies all properties in Thm 2.3 except
f_2	symmetry
f_4	subgame property
f_5	independence of irrelevant alternatives
f_3	lower bound property

Proposition 3.3 *The axioms in Theorem 2.4 are logically independent.*

For each $A \in G$, let $d(A) \in G$ be the game obtained from A by the iterated elimination of strictly dominated rows and columns. Recall, for instance from Osborne and Rubinstein (1994, Section 4.2.2), that $d(A)$ is well-defined. The function $f_6 : G \to \mathbb{R}$ defined by $f_6(A) = \frac{1}{2}(\max_{i,j} d(A)_{ij} + \min_{i,j} d(A)_{ij})$ satisfies all axioms in Theorem 2.4, except monotonicity:

$$f_6\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \frac{1}{2} < 1 = f_6\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right).$$

This function, together with those defined earlier, prove the proposition:

The function	satisfies all properties in Thm 2.4 except
f_6	monotonicity
f_2	symmetry
f_3	objectivity
f_5	independence of irrelevant alternatives
f_4	strictly dominated action property

Proposition 3.4 *The axioms in Corollary 2.5 are logically independent.*

Vilkas (1963) does not prove this result: his example violating monotonicity also violates objectivity. The proposition follows from earlier examples:

The function	satisfies all properties in Cor. 2.5 except
f_6	monotonicity
f_2	symmetry
f_3	objectivity
f_5	dominance

4 Concluding remarks

The full strength of the different axioms is not always required in the proofs. We briefly discuss a number of relaxations of the axioms. In the proof of Theorem 2.2, a much weaker form of monotonicity suffices:

Restricted monotonicity: $f : G \to \mathbb{R}$ is monotonic on $\mathbb{R}^{1\times 2}$.

To see this, consider $[z^t A] \in \mathbb{R}^{1\times n}$ as in (3). Independence of irrelevant alternatives allows us to eliminate $n-2$ elements that are not equal to the maximal and minimal elements of this matrix. This yields an $[y_1\ y_2] \in \mathbb{R}^{1\times 2}$ whose coordinates are $\max_j z^t A e_j$ and $\min_j z^t A e_j = v(A)$ with $f([z^t A]) = f([y_1\ y_2])$. Since $[y_1\ y_2] \geqq [v(A)\ v(A)]$, restricted monotonicity and objectivity give

$$f([z^t A]) = f([y_1\ y_2]) \geqq f([v(A)\ v(A)]) = v(A),$$

just like in (4) and (5).

For the same reason, the lower bound property in Theorem 2.3 needs to hold only for 1×2 games. Although it suffices to require monotonicity and the lower bound property only on a small set of matrix games, the original axioms are more elegant: they avoid the consideration of a somewhat artificial subclass of games.

In conjunction with symmetry, the objectivity axiom can be relaxed as follows:

Restricted objectivity: If $A = [a] \in \mathbb{R}^{1\times 1}$, then $f(A) \geqq a$.

Indeed, let $A = [a] \in \mathbb{R}^{1\times 1}$. Then $f(A) \geqq a$ by restricted objectivity and $f(A) = -f(-A^t) \leqq -(-a) = a$ by symmetry and restricted objectivity. Thus $f(A) = a$, as required by objectivity.

Let us conclude by briefly mentioning two potential topics for further research. Tijs (1975, 1981a) extends the characterization of Vilkas (1963) to different classes of two-person zero-sum games in which the players may have an infinite set of pure strategies. Following his line of proof, we believe that our axiomatizations of the value function can be extended to this context of infinite games.

A more interesting question is the following: it is well-known that von Neumann's minimax theorem and the duality theorem from linear programming are closely related. Is it possible to use properties similar to the ones presented in our paper to characterize the value function on the set of feasible linear programs? The results from Tijs (1981b) suggest a positive answer; this topic is taken up in further research by the authors.

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