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CORE STABILITY IN CHAIN-COMPONENT ADDITIVE GAMES

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Core stability in chain-component additive games

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Abstract

Chain-component additive games are graph-restricted superadditive games, where an exogenously given line-graph determines the cooperative possibilities of the players. These games can model various multi-agent decision situations, such as strictly hierarchical organisations or sequencing / scheduling related problems, where an order of the agents is fixed by some external factor, and with respect to this order only consecutive coalitions can generate added value.

In this paper we characterise core stability of chain-component additive games in terms of polynomial many linear inequalities and equalities that arise from the combinatorial structure of the game. Furthermore we show that core stability is equivalent to essential extendibility. We also obtain that largeness of the core as well as extendibility and exactness of the game are equivalent properties which are all sufficient for core stability. Moreover, we also characterise these properties in terms of linear inequalities.

Keywords: Core stability, graph-restricted games, large core, exact game.
JEL classification: C71

1 Introduction

Chain-component additive games were first introduced in [4]. At these superadditive games the cooperative possibilities of the players are restricted by an exogenously given line-graph. In particular, only consecutive coalitions can generate added value. The class of chain-component additive games contains the well-known classes of sequencing games (cf. [3; 6; 8]) and neighbour games (cf. [7; 10]). Due to the combinatorial structure of chain-component additive games, and because it covers many interesting classes of games, chain-component additive games are extensively studied in game theory. Non-emptiness of the core is shown in [4]. Furthermore, it is proven in [14] that the core coincides with the bargaining set and that the kernel only consists of the nucleolus. In [17] a primal and in [11] a dual type algorithm is presented for the efficient computation of the nucleolus.

The main focus of this paper is core stability. Core stability combines the well-known concept "core" with the classical solution concept "stable set" proposed in [20]. The core is stable if all non-core members of the imputation set are dominated by a core element. In general, the existence (cf. [13]) and uniqueness of stable sets is not guaranteed. However, if the core is stable, then it is the unique stable set. A class of games that satisfies core stability is that of convex games (cf. [15]). Apart from the result on convex games, only few results are known with respect to core stability. These results include a characterisation of core stability for symmetric games (cf. [2]) and a characterisation for assignment games in terms of the underlying matrix (cf. [18]). We characterise core stability for chain-component additive games by introducing covering families.

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These covering families give rise to (in the number of players) polynomially many linear equalities and inequalities. We show the necessity of these linear equalities and inequalities using a dual approach. First we appoint a certain subset of the imputation set and we show, using a variant of Farkas' Lemma, that this subset contains an undominated imputation outside the core if and only if all vectors from a related polyhedron satisfy a well-chosen linear inequality. We then decompose each member of this polyhedron into a sum of three types of basis vectors. Finally we use these basis vectors to show that the well-chosen linear inequality is indeed satisfied for each member of the polyhedron.

In this paper we also investigate largeness of the core, as well as extendibility and exactness of the game. Largeness is the property that the lower boundary of the upper core coincides with the core. A game is called extendible if each core element of each subgame can be extended to a core element of the game. Finally, if a game is exact, then for every coalition there exists a core element that allocates its value to its members. In [16] it is proven that largeness of the core is a sufficient condition for core stability. In [9] extendibility of the game is shown to be necessary for largeness and sufficient for core stability. Extendibility had been conjectured to be equivalent to core stability, but in [19] a counter-example was given. We will show for chain-component additive games that largeness, extendibility and exactness are equivalent. Moreover, we characterise these concepts in terms of linear inequalities arising from the combinatorial structure of the game.

Finally, we refine the concept of extendibility in the following way. We call a game essential extendibility if each core element of each subgame corresponding to an essential coalition can be extended to a core element of the game. We show that essential extendibility is equivalent to core stability for chain-component additive games.

The remainder of this paper is organised as follows. In Section 2 we recall some concepts from cooperative game theory and in Section 3 we state and prove our main results. In the Appendix we state and prove some technical lemmas.

2 Preliminaries

In this section we recall some concepts from cooperative game theory and we introduce chain-component additive games.

A *cooperative TU-game* (N, v) consists of a finite player set $N = \{1, \ldots, n\}$ and a map $v : 2^N \to \mathbb{R}$ that expresses the worth of each coalition. By convention, $v(\emptyset) = 0$. A game (N, v) is called *superadditive* if for each $S, T \subseteq N$ with $S \cap T = \emptyset$ it holds that $v(S) + v(T) \leq v(S \cup T)$. Coalition $S \subseteq N$ is called *essential* if for each partition P of S it holds that $\sum_{T \in P} v(T) < v(S)$. A coalition that is not essential is said to be *inessential.* For each $T \subseteq N$, the *subgame* (T, v_T) is the game with player set T and $v_T(S) = v(S)$ for each $S \subseteq T$. The imputation set $I(v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N\}$ is the set of efficient payoff vectors respecting the worth each player can obtain on its own. The *upper core* $U(v) = \{x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$ is the set of payoff vectors at which each coalition is satisfied. The *core* $C(v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$ is the set of efficient payoff vectors in the upper core. If a core element is proposed as a payoff vector, then no coalition has an incentive to leave the grand coalition. Note that the core of a game can be empty. Games with a non-empty core are called *balanced.* If for a balanced game each subgame is also balanced, then the game is said to be *totally balanced.* An *order* on the player set is a bijection $\sigma : \{1, \ldots, n\} \to N$. Let σ be an order on the player set. The marginal vector $m^\sigma(v)$ is defined by $m^\sigma_{\sigma(j)}(v) = v(\{\sigma(i) : 1 \leq i \leq j\}) - v(\{\sigma(i) : 1 \leq i < j\})$. A collection $B \subseteq 2^N \backslash \{\emptyset\}$ is called *balanced* if there exists a map $\lambda : B \to [0, 1]$ such that $\lambda(S) > 0$ for each $S \in B$ and $\sum_{S \in B} \lambda(S) e(S) = e(N)$, where $e_j(S) = 1$ if $j \in S$ and $e_j(S) = 0$ otherwise.

Let $x, y \in I(v)$. Now x is said to dominate y via coalition $S \subseteq N$ if $\sum_{i \in S} x_i \leq v(S)$ and $x_i > y_i$ for all $i \in S$. The core is called *stable* if for each imputation y outside the core there is a core element x and a coalition $S \subseteq N$ such that x dominates y via S. A game (N, v) is said to be *exact* if for each $S \subseteq N$ there is an $x \in C(v)$ with $\sum_{i \in S} x_i = v(S)$. The core is said to be *large* if for each $x \in U(v)$ there is a $y \in C(v)$ with $y_i \leq x_i$ for each $i \in N$. Finally, a game is said to be *extendible* if each core element of each subgame can be extended to a core element of (N, v). In other words, the game is extendible if for each $x \in C(v_T)$, $T \subseteq N$, there exists a $y \in C(v)$ with $y_i = x_i$ for each $i \in T$.

In [16] it is shown that largeness of the core is a sufficient condition for core stability. It was proven in [9] that extendibility is necessary for largeness of the core and sufficient for core stability. For totally balanced games it is shown in [16] that exactness is implied by largeness. Moreover, in [1] it is shown for totally balanced games that exactness is necessary for extendibility. In general, exactness and core stability do not imply one another (cf. [1; 19]).

Finally we introduce chain-component additive games. Let $\sigma_0 : \{1, \ldots, n\} \to N$ be a bijective map. Coalition $S \subseteq N$ is said to be *connected* with respect to σ_0 if for each $1 \leq i < j \leq n$ with $\sigma_0(i), \sigma_0(j) \in S$ it holds that $\sigma_0(k) \in S$ for all $i \leq k \leq j$. For coalition $T \subseteq N$, let $T \backslash \sigma_0$ denote the partition of T into maximally connected components. A game (N, v) is a *chain-component additive game*, with respect to σ_0, if it is superadditive and if $v(T) = \sum_{S \in T \backslash \sigma_0} v(S)$ for each $T \subseteq N$. That is, the worth of a coalition is equal to the sum of the worths of its connected parts. Note that disconnected coalitions are inessential. It is shown in [4] that chain-component additive games are balanced. Obviously this implies that chain-component additive games are totally balanced as well, since subgames inherit the chain-component additive structure.

In the remainder of this paper we assume without loss of generality that $\sigma_0(i) = i$ for each $i \in N$. Let $\mathcal{S}$ be the set of connected coalitions with respect to σ_0. It will be convenient to write (with a little abuse of notation) a connected coalition $S \in \mathcal{S}$ as an (ordered) set of players $S = \{s_1, \ldots, s_2\}$ with the convention that $s_1 = \min S$ and $s_2 = \max S$. Note that for chain-component additive games both the upper core and the core are completely determined by the connected coalitions, i.e., $U(v) = \{x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \mathcal{S}\}$ and $C(v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \mathcal{S}\}$.

3 Core stability of chain-component additive games

In this section we state and prove our main results. First we show that for chain-component additive games a large class of marginal vectors are core elements. Then we show that largeness, extendibility and exactness are all equivalent. Furthermore we will characterise these concepts in terms of inequalities. Subsequently we introduce a refinement of extendibility, called essential extendibility, and show that is equivalent to core stability for chain-component additive games. As a final result we will characterise core stability and essential extendibility in terms of (in the number of players) polynomially many linear equalities and inequalities.

The following theorem shows that a large class of marginal vectors are core elements. In particular, if an order $\sigma : \{1, \ldots, n\} \to N$ is such that $N \backslash \{\sigma(i) : 1 \leq i \leq l\}$ is connected with respect to σ_0 for all $1 \leq l \leq n$, then the corresponding marginal vector $m^\sigma(v)$ is a core element.

Theorem 3.1 Let (N, v) be a chain-component additive game. Let $\sigma : \{1, \ldots, n\} \to N$ be such that $N \backslash \{\sigma(i) : 1 \leq i \leq l\} \in \mathcal{S}$ for each $1 \leq l \leq n$. Then $m^\sigma(v) \in C(v)$.

Proof: Let $\sigma : \{1, \ldots, n\} \to N$ be such that $N \backslash \{\sigma(i) : 1 \leq i \leq l\}$ is connected with respect to σ_0 for each $1 \leq l \leq n$. Because of the definition of σ and the chain-component additivity of (N, v) it holds that $m^\sigma_{\sigma(n)}(v) = v(N) - v(\{1, \ldots, \sigma(n) - 1\}) - v(\{\sigma(n) + 1, \ldots, n\})$, $m^\sigma_i(v) = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\})$ for each $i \in N \backslash \{\sigma(n)\}$ with $i < \sigma(n)$, and $m^\sigma_i(v) = v(\{i, \ldots, n\}) - v(\{i+1, \ldots, n\})$ for all $i \in N \backslash \{\sigma(n)\}$ with $i > \sigma(n)$.

Since $m^\sigma(v)$ is efficient by definition, we only need to show that for all $S \in \mathcal{S}$ it holds that $\sum_{j\in S} m_j^\sigma(v) \geq v(S)$. Let $S \in \mathcal{S}$ and write $S = \{s_1, \ldots, s_2\}$. First assume that $\sigma(n) \notin S$. Then either $s_2 < \sigma(n)$ or $s_1 > \sigma(n)$. Without loss of generality assume that $s_2 < \sigma(n)$. It holds that

$$\sum_{i\in S} m_i^\sigma(v) = v(\{1, \ldots, s_2\}) - v(\{1, \ldots, s_1 - 1\}) \geq v(\{s_1, \ldots, s_2\}) = v(S),$$

where the inequality follows from superadditivity. Now assume that $\sigma(n) \in S$. Observe that

$$\sum_{i\in S} m_i^\sigma(v) = v(N) - v(\{1, \ldots, s_1 - 1\}) - v(\{s_2 + 1, \ldots, n\}) \geq v(\{s_1, \ldots, s_2\}) = v(S),$$

where again the inequality follows because of superadditivity. □

Note that from Theorem 3.1 it follows that $m^\sigma(v), m^\tau(v) \in C(v)$ with σ and τ such that $\sigma(i) = i$, $\tau(i) = n + 1 - i$ for all $i \in \{1, \ldots, n\}$. This is also indirectly proven in [4].

In the upcoming part of this section we will characterise largeness, extendibility and exactness in terms of inequalities. In order to do so we introduce covering families. We call an ordered set $\{T_1, \ldots, T_m\} \subseteq \mathcal{S}$ an *m-covering family of N* if

(P1) for all $i \in N$ there is a $1 \leq j \leq m$ with $i \in T_j$;

(P2) $T_i \cap T_{i+1} \neq \emptyset$ for all $1 \leq i \leq m - 1$;

(P3) for each $1 \leq j \leq m$ there is an $i \in N$ with $i \in T_j \setminus (\cup_{k\neq j} T_k)$.

Requirement (P1) states that every player is contained in least one element of a covering family, and (P2) states that two subsequent elements of a covering family should not be disjoint. Notice that (P3) is equivalent to stating that each proper subset of a covering family is not a covering family itself. The following example illustrates the concept of covering families.

Example 3.1 Let $N = \{1, 2, 3, 4, 5\}$. Then $\{\{1, 2\}, \{2, 3, 4\}, \{4, 5\}\}$ forms a 3-covering family, since it satisfies (P1),(P2) and (P3). Also note that $\{\{1\}, \{1, 2, 3\}, \{3, 4, 5\}\}$ and $\{\{1, 2\}, \{2, 3, 4\}, \{3, 4, 5\}\}$ do not form 3-covering families, since for both sets condition (P3) is violated. ◇

Observe that an m-covering family could equivalently be described by the alternating sequence of the $2m - 1$ nonempty blocks of consecutive players who are covered by exactly one or exactly two family-member coalitions. It follows that in an n-player chain-component additive game the number of different m-covering families is $\binom{n-1}{2m-2}$, provided, of course, that $2m - 1 \leq n$.

With each m-covering family $\mathcal{T} = \{T_1, \ldots, T_m\}$ we will associate a corresponding covering family inequality:

$$\sum_{i=1}^{m} v(T_i) \leq v(N) + \sum_{1\leq i\leq m-1} v(T_i \cap T_{i+1}).$$

The following theorem characterises largeness, extendibility and exactness in terms of covering family inequalities.

Theorem 3.2 Let (N, v) be a chain-component additive game. The following statements are equivalent:

1. For each covering family the corresponding inequality is satisfied;

2. The game has a large core;

3. The game is extendible;

4. The game is exact.

Proof: We will show 1 ⇒ 2 and 4 ⇒ 1. The proof of 2 ⇒ 3 follows from [9] and 3 ⇒ 4 is shown in [1], since chain-component additive games are totally balanced.

First we show that if for each covering family the corresponding inequality is satisfied, then the core is large. Assume that (N, v) is such that the inequalities corresponding to the covering families hold. Let $x \in U(v)$. If $x \in C(v)$, i.e. if $\sum_{i\in N} x_i = v(N)$, then we are done, so assume that $\sum_{i\in N} x_i > v(N)$. We need to show the existence of a $y \in C(v)$ with $y_i \leq x_i$ for each $i \in N$. Instead, we will show the existence of an $x^1 \in U(v)$ with $x^1_j < x_j$ for some $j \in N$ and $x^1_i = x_i$ for all $i \in N\backslash\{j\}$. Observe that $\sum_{j\in N} x^1_j < \sum_{j\in N} x_j$. We will then argue that in a finite number of steps we can recursively find an $x^p \in U(v)$ for some $p \geq 1$ with $x^p \leq x$ and $\sum_{j\in N} x^p_j = v(N)$, i.e. $x^p \in C(v)$.

Define $S(x) = \{S \in \mathcal{S} : \sum_{i\in S} x_i = v(S)\}$ to be the set of connected coalitions which are tight at x. We first show, by contradiction, that there is a $j \in N$ such that $j \notin T$ for all $T \in S(x)$. Suppose that for all $j \in N$ there is a $T \in S(x)$ with $j \in T$. According to Lemma A.2 $S(x)$ contains at least one covering family. Let $\{T_1, \dots, T_m\}$ be such a covering family. Since $T_i \in S(x)$ for each $1 \leq i \leq m$, it follows that $\sum_{j\in T_i} x_j = v(T_i)$ for all $1 \leq i \leq m$. This yields

$$\sum_{1\leq i\leq m}\sum_{j\in T_i} x_j = \sum_{1\leq i\leq m} v(T_i) \leq v(N) + \sum_{1\leq i\leq m-1} v(T_i \cap T_{i+1})$$

$$< \sum_{j\in N} x_j + \sum_{1\leq i\leq m-1} v(T_i \cap T_{i+1}) \leq \sum_{j\in N} x_j + \sum_{1\leq i\leq m-1}\sum_{j\in T_i\cap T_{i+1}} x_j.$$

The first inequality holds because $\{T_1, \dots, T_m\}$ is a covering family, and by our assumption, the corresponding inequality is satisfied. The strict inequality holds because we have assumed that $\sum_{j\in N} x_j > v(N)$. The last inequality holds because $x \in U(v)$, and therefore $\sum_{j\in T_i\cap T_{i+1}} x_j \geq v(T_i \cap T_{i+1})$ for all $1 \leq i \leq m-1$. So we have obtained that

$$\sum_{1\leq i\leq m}\sum_{j\in T_i} x_j < \sum_{j\in N} x_j + \sum_{1\leq i\leq m-1}\sum_{j\in T_i\cap T_{i+1}} x_j. \tag{1}$$

However, since $\{T_1, \dots, T_m\}$ is a covering family, $\sum_{1\leq i\leq m}\sum_{j\in T_i} x_j = \sum_{j\in N} x_j + \sum_{1\leq i\leq m-1}\sum_{j\in T_i\cap T_{i+1}} x_j$, which contradicts (1). We conclude that it cannot hold that for every $j \in N$ there is a $T \in S(x)$ with $j \in T$.

Now let $j \in N$ be such that $j \notin T$ for every $T \in S(x)$. Define $\epsilon = \min\{\sum_{i\in S} x_i - v(S) : j \in S, S \in \mathcal{S}\}$. It clearly holds that $\epsilon > 0$ since $\sum_{j\in S} x_j > v(S)$ for all $S \in \mathcal{S}$ with $j \in S$. Define x^1 by $x^1_i = x_i$ for all $i \in N\backslash\{j\}$ and $x^1_j = x_j - \epsilon$. Trivially it holds for all $T \in \mathcal{S}$ with $j \notin T$ that $\sum_{i\in T} x^1_i = \sum_{i\in T} x_i \geq v(T)$. For each $T \in \mathcal{S}$ with $j \in T$ we have that $\sum_{i\in T} x^1_i = \sum_{i\in T} x_i - \epsilon \geq v(T)$, where the inequality holds since $\epsilon = \min\{\sum_{i\in S} x_i - v(S) : j \in S, S \in \mathcal{S}\} \leq \sum_{i\in T} x_i - v(T)$. We conclude that $x^1 \in U(v)$, $x^1 \leq x$ and $\sum_{j\in N} x^1_j < \sum_{j\in N} x_j$. Also note that $S(x) \subsetneq S(x^1)$ since all coalitions that are tight at x are also tight at x^1, while at x^1 at least one more coalition is tight.

If $\sum_{j\in N} x^1_j = v(N)$, then we are done. If $\sum_{j\in N} x^1_j > v(N)$, we can apply the same procedure to find an element $x^2 \in U(v)$ with $x^2 \leq x^1 \leq x$ and $\sum_{j\in N} x^2_j < \sum_{j\in N} x^1_j$. Recursively, we obtain a sequence $(x^m)_{1\leq m\leq p}$ with $x^m \in U(v)$ for all $1 \leq m \leq p$, $x^p \leq \ldots \leq x^1 \leq x$ and $\sum_{j\in N} x^p_j < \ldots < \sum_{j\in N} x^1_j < \sum_{j\in N} x_j$, for some $p \geq 1$. Observe that, by definition of ϵ, $S(x) \subsetneq S(x^1) \subsetneq \ldots \subsetneq S(x^p)$. Since the set of players not covered by the families $S(x) \subsetneq S(x^1) \subsetneq \ldots \subsetneq S(x^p)$ is strictly shrinking, it follows that we construct an $x^p \in C(v)$ with $x^p \leq x$ in $p \leq n$ steps. Hence, the core of the game is large.

The second and final statement we need to show is that if the game is exact, then the inequalities corresponding to covering families hold. Assume that the game is exact and let $\{T_1, \ldots, T_m\}$ be a covering family. Since the game is exact, there is an $x \in C(v)$ with $\sum_{1 \le i \le m-1} \sum_{j \in T_i \cap T_{i+1}} x_j = v(\cup_{1 \le i \le m-1}(T_i \cap T_{i+1})) = \sum_{1 \le i \le m-1} v(T_i \cap T_{i+1})$, where the last equality holds because of (P3) and because of the chain-component additive nature of (N, v). Since $x \in C(v)$ it holds that $\sum_{j \in T_i} x_j \ge v(T_i)$ for all $1 \le i \le m$. Therefore

$$\sum_{1 \le i \le m} v(T_i) \le \sum_{1 \le i \le m} \sum_{j \in T_i} x_j = \sum_{j \in N} x_j + \sum_{1 \le i \le m-1} \sum_{j \in T_i \cap T_{i+1}} x_j = v(N) + \sum_{1 \le i \le m-1} v(T_i \cap T_{i+1}).$$

We conclude that the inequality corresponding to covering family $\{T_1, \ldots, T_m\}$ is satisfied. □

Observe that for an n-player chain component additive game our characterisation requires the checking of $\sum_{m=2}^{\frac{n+1}{2}} \binom{n-1}{2m-2} \approx 2^{n-2}$ linear inequalities.

In the final part of this section we will investigate core stability of chain-component additive games. Of course, if the condition of Theorem 3.2 is satisfied, then it easily follows that the core is stable. However, this condition is too strong. We will obtain polynomially many linear equalities and inequalities arising from 2− and 3-covering families that characterise core stability. Furthermore we prove that core stability for chain-component additive games is equivalent to essential extendibility. A TU-game (N, v) is called *essential extendible* if for each essential $T \subseteq N$ and each $x \in C(v_T)$ there exists a $y \in C(v)$ with $y_i = x_i$ for each $i \in T$. Note that if a game is extendible, then it is essential extendible. The following proposition shows that for each TU-game essential extendibility is a sufficient condition for core stability.

Proposition 3.1 Let (N, v) be essential extendible. Then its core is stable.

Proof: For core stability we need to show that each imputation outside the core is dominated by a core element. Let $x \in I(v) \backslash C(v)$. Let $S \subseteq N$ be a smallest coalition that is not satisfied at imputation x. In other words, S is such that $\sum_{i \in S} x_i < v(S)$ and for all $T \subseteq S$, $T \ne S$, it holds that $\sum_{i \in T} x_i \ge v(T)$. Now observe that S is essential. Indeed, if S is not essential, then there is a partition P of S with $\sum_{T \in P} v(T) \ge v(S)$, implying that one of the coalitions in P is not satisfied at x as well.

Now define $z_i = x_i + \frac{v(S) - \sum_{i \in S} x_i}{|S|}$ for all $i \in S$. Clearly, $\sum_{i \in S} z_i = v(S)$. From $\sum_{i \in T} z_i > \sum_{i \in T} x_i \ge v(T)$ for each $T \subseteq S$, $T \ne S$, we conclude that $z \in C(v_S)$. Since S is essential, it follows from the essential extendibility of (N, v) that there is a $y \in C(v)$ with $y_i = z_i$ for each $i \in S$. Now observe that y dominates x via S. □

We now approach the main theorem of this paper. In the proof of this theorem we apply a variant of Farkas' Lemma, proven in [5]. Note that we abuse notation by omitting all transpose signs.

Lemma 3.1 ([5]) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n : Ax \le b\} \ne \emptyset$. Let $c \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$. It holds for all $x \in P$ that $cx \le \delta$ if and only if there exists a $y \in \mathbb{R}^m_+$ with $yA = c$ and $yb \le \delta$.

We are now ready to state and prove our main result.

Theorem 3.3 Let (N, v) be a chain-component additive game. The following statements are equivalent:

1. Each 2-covering family inequality is satisfied. For each 3-covering family $\{T_1, T_2, T_3\}$, if the corresponding inequality is not satisfied, then T_2 is inessential;

2. (N, v) is essential extendible;

3. The core of (N, v) is stable.

Proof: We first show $1 \Rightarrow 2$. So assume that all 2-covering family inequalities are satisfied. Furthermore assume for each 3-covering family $\{T_1, T_2, T_3\}$ with essential T_2 that the corresponding inequality is satisfied. We need to show that (N, v) is essential extendible, so let $S \subseteq N$ be essential. Now note that if $\{T_1, S, T_3\}$ is a 3-covering family, then the corresponding inequality is satisfied by assumption.

Let $x \in C(v_S)$. We will construct a $y \in C(v)$ with $y_i = x_i$ for each $i \in S$. First we make some agreements on notation. Since $S \in \mathcal{S}$ and $S \neq N$ it holds that $N \backslash S$ consists of at least one and at most two components. For simplicity, let $S_1, \ldots, S_m$ denote these components (so $m = 1$ or $m = 2$). For each $1 \leq i \leq m$ there is an $s_i \in S_i$ and a $t_i \in S$ such that $\{s_i, t_i\}$ is connected with respect to σ_0. For each $1 \leq i \leq m$, let $\sigma_i : \{1, \ldots, |S_i|\} \to S_i$ be an order on S_i such that $N \backslash \{\sigma_i(k) : 1 \leq k \leq l\}$ is connected with respect to σ_0 for each $1 \leq l \leq |S_i|$. Note that this implies that $\sigma_i(|S_i|) = s_i$ for each $1 \leq i \leq m$. With each order σ_i on S_i associate a partial marginal vector as follows. Let $m^{\sigma_i}_{\sigma_i(l)}(v) = v(\{\sigma_i(j) : 1 \leq j \leq l\}) - v(\{\sigma_i(j) : 1 \leq j < l\})$ for each $l \in \{1, \ldots, |S_i| - 1\}$, $1 \leq i \leq m$. Now we are ready to construct y. Let

$$y_l = \begin{cases} x_l, & \text{if } l \in S; \\ m^{\sigma_i}_l(v), & \text{if } l \in S_i \backslash \{s_i\}, 1 \leq i \leq m; \\ \max\{v(T) - \sum_{k \in T \backslash \{l\}} y_k : T \subseteq S \cup (\bigcup_{1 \leq q \leq i} S_q), l \in T, T \in \mathcal{S}\}, & \text{if } l = s_i,\ 1 \leq i \leq m. \end{cases}$$

We first show that $y \in U(v)$. Secondly we show that y is efficient and thus that $y \in C(v)$. Observe that $\sum_{i \in T} y_i = \sum_{i \in T} x_i \geq v(T)$ for each $T \subseteq S$. By definition of y_{s_m} it holds that $y_{s_m} + \sum_{k \in T \backslash \{s_m\}} y_k \geq v(T)$ for all $T \in \mathcal{S}$ with $s_m \in T$. Furthermore, in case of $m = 2$ by definition of y_{s_1} we also have that $y_{s_1} + \sum_{k \in T \backslash \{s_1\}} y_k \geq v(T)$ for all $T \in \mathcal{S}$ with $s_2 \notin T$ and $s_1 \in T$. For $T \subseteq S$, $T \in \mathcal{S}$ we note that $\sum_{i \in T} y_i > \sum_{i \in T} x_i \geq v(T)$. Finally, let $T \subseteq S_i \backslash \{s_i\}$, $1 \leq i \leq m$ and $T \in \mathcal{S}$. It follows from Theorem 3.1 applied to the subgame $(S_i \backslash \{s_i\}, v_{S_i \backslash \{s_i\}})$ and our choice of σ_i that $\sum_{k \in T} y_k \geq v(T)$. We conclude that $y \in U(v)$.

It remains to show that $\sum_{i \in N} y_i \leq v(N)$, since by definition of y_{s_m} it holds that $\sum_{i \in N} y_i \geq v(N)$. For each $1 \leq i \leq m$ let $R_i = \text{argmax}\{v(T) - \sum_{k \in T \backslash \{s_i\}} y_k : T \subseteq S \cup (\cup_{1 \leq q \leq i} S_q)\}$. So $\sum_{j \in R_i} y_j = v(R_i)$ for each $1 \leq i \leq m$. We assume that R_i, $1 \leq i \leq m$, is maximal with respect to inclusion. To be more precise, if $\bar{R}_i = \text{argmax}\{v(T) - \sum_{k \in T \backslash \{s_i\}} y_k : T \subseteq S \cup (\cup_{1 \leq q \leq i} S_q)\}$, $\bar{R}_i \neq R_i$, then $R_i \not\subseteq \bar{R}_i$. Under this assumption the following three properties hold for R_i, $1 \leq i \leq m$:

(Q1) If $m = 2$, then $R_1 \cap R_2 \neq \emptyset$ or $R_1 \cup R_2 \notin \mathcal{S}$;

(Q2) $S_i \subseteq R_i$ for each $1 \leq i \leq m$;

(Q3) $t_i \in R_i$ for each $1 \leq i \leq m$.

We will only show (Q1), the other two properties are immediate from the assumed maximality of R_i for each $1 \leq i \leq m$. If $m = 1$ there is nothing to prove, so assume that $m = 2$. Suppose $R_1 \cap R_2 = \varnothing$ and $R_1 \cup R_2 \in \mathcal{S}$. By definition of R_1 it holds that $\sum_{j \in R_1} y_j = v(R_1)$. This yields

$$v(R_2) - \sum_{j \in R_2 \backslash \{s_2\}} y_j = v(R_2) - \sum_{j \in R_2 \backslash \{s_2\}} y_j + v(R_1) - \sum_{j \in R_1} y_j \leq v(R_1 \cup R_2) - \sum_{j \in R_1 \cup R_2 \backslash \{s_2\}} y_j.$$

The inequality comes from superadditivity. We have now obtained that $R_1 \cup R_2 = \text{argmax}\{v(T) - \sum_{k\in T\setminus\{l\}} y_k : T \subseteq S \cup_{1\leq q\leq 2} S_q\}$, and this clearly contradicts our assumption that R_2 is maximal with respect to inclusion. We conclude that (Q1) holds.

We now distinguish between two possibilities.

`Case 1:` $\cup_{i=1}^m R_i = N$.

If $m = 1$, then we have that $R_1 = N$. By definition of y_{s_m} it now follows that $\sum_{j\in N} y_j = v(N)$. So assume that $m = 2$ and that $R_2 \neq N$. Since $R_1 \cup R_2 = N$, it follows that $\{R_1, R_2\}$ is a 2-covering family. This yields

$$\begin{aligned}\sum_{j\in N} y_j &= \sum_{j\in R_1} y_j + \sum_{j\in R_2} y_j - \sum_{j\in R_1\cap R_2} y_j = v(R_1) + v(R_2) - \sum_{j\in R_1\cap R_2} y_j \\ &\leq v(R_1) + v(R_2) - v(R_1 \cap R_2) \leq v(N).\end{aligned}$$

The second equality holds by definition of R_1 and R_2. The first inequality is satisfied because $\sum_{j\in T} y_j \geq v(T)$ for each $T \in \mathcal{S}$. The last inequality is satisfied because $\{R_1, R_2\}$ is a 2-covering family and we assumed that 2-covering family inequalities are satisfied.

`Case 2:` $\cup_{i=1}^m R_i \neq N$.

If $m = 1$, then R_1 and S form a 2-covering family. Consequently,

$$\begin{aligned}\sum_{j\in N} y_j &= \sum_{j\in R_1} y_j + \sum_{j\in S} y_j - \sum_{j\in R_1\cap S} y_j = v(R_1) + v(S) - \sum_{j\in R_1\cap S} y_j \\ &\leq v(R_1) + v(S) - v(R_1 \cap S) \leq v(N).\end{aligned}$$

The second equality holds by definition of y and R_1. The first inequality because $\sum_{j\in T} y_j \geq v(T)$ for all $T \in \mathcal{S}$. The second inequality is satisfied because R_1 and S form a 2-covering family and we assumed that 2-covering family inequalities are satisfied.

If $m = 2$, then it follows from $R_1 \cup R_2 \neq N$ that $R_1 \cup R_2 \notin \mathcal{S}$. Because of (Q3) we have that $R_1 \cap S \neq \emptyset$ and $S \cap R_2 \neq \emptyset$. Therefore $\{R_1, S, R_2\}$ forms a 3-covering family. We conclude that

$$\begin{aligned}\sum_{j\in N} y_j &= \sum_{j\in R_1} y_j + \sum_{j\in S} y_j + \sum_{j\in R_2} y_j - \sum_{j\in R_1\cap S} y_j - \sum_{j\in S\cap R_2} y_j \\ &= v(R_1) + v(S) + v(R_2) - \sum_{j\in R_1\cap S} y_j - \sum_{j\in S\cap R_2} y_j \\ &\leq v(R_1) + v(S) + v(R_2) - v(R_1 \cap S) - v(S \cap R_2) \leq v(N).\end{aligned}$$

Again, the second equality holds by definition of R_1, R_2 and y. The first inequality because $\sum_{j\in T} y_j \geq v(T)$ for each $T \in \mathcal{S}$ and the second because the inequality corresponding to the 3-covering family $\{R_1, S, R_2\}$ is satisfied. This concludes the proof of $1 \Rightarrow 2$.

It remains to show $3 \Rightarrow 1$, since $2 \Rightarrow 3$ follows from Proposition 3.1. We first show that the inequalities corresponding to 2-covering families are necessary. We then proceed with the necessity of the condition involving 3-covering families.

Suppose that that the inequality corresponding to the 2-covering family $\{T_1, T_2\}$ is not satisfied. In other words, suppose that $v(T_1) + v(T_2) > v(N) + v(T_1 \cap T_2)$. We will show that the core is not stable by constructing a non-core imputation that cannot be dominated by any core element.

Let t^* be such that $T_1 = \{1, \ldots, t^*\}$, and consider the order $\sigma : \{1, \ldots, n\} \to N$ with $\sigma(i) = t^* + 1 - i$ for each $i \in \{1, \ldots, t^*\}$ and $\sigma(i) = n + t^* + 1 - i$ for each $i \in \{t^* + 1, \ldots, n\}$. It holds that $\sum_{i\in T_2} m_i^\sigma(v) = v(N) - v(T_1) + v(T_1 \cap T_2) < v(T_2)$, where the inequality follows by assumption. Thus, $m^\sigma(v) \notin C(v)$. Furthermore, from Theorem 3.1 and the choice of σ it follows straightforwardly that $\sum_{i\in S} m_i^\sigma(v) \geq v(S)$ for all $S \subseteq N\setminus\{t^* + 1\}$. Therefore $m^\sigma(v)$ can only be dominated via coalitions that contain player $t^* + 1$. However, at $m^\sigma(v)$ player $t^* + 1$ receives a payoff of $m^\sigma_{t^*+1}(v) = v(N) - v(\{1, \ldots, t^*\}) - v(\{t^*+2, \ldots, n\}) = v(N) - v(N \backslash \{t^*+1\})$. But for any $x \in C(v)$ it holds that $x_{t^*+1} \leq v(N) - v(N\backslash\{t^* + 1\})$. We therefore conclude that $m^\sigma(v)$ cannot be dominated by a core element via a coalition containing player $t^* + 1$. This implies that $m^\sigma(v)$ cannot be dominated by any core element. So the core is not stable. Consequently, the inequalities arising from 2-covering families are necessary for core stability.

We now show that the conditions for 3-covering families are necessary. Suppose that the inequalities corresponding to 2-covering families are satisfied. Furthermore suppose that for the 3-covering family $\{T_1, T_2, T_3\}$ the corresponding condition is not satisfied, i.e. suppose that $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ while T_2 is essential. Again, we show that the core is not stable by showing the existence of a non-core imputation that can not be dominated by any core element. Before we actually start the proof, we first introduce some notation.

Define $T^* = N\backslash(T_1 \cup T_3)$ and $\mathcal{T} = \{T \in \mathcal{S} : T^* \not\subseteq T\}$. So $\mathcal{T}$ is the set of connected coalitions not containing T^*. Define P by

$$P = \{x \in \mathbb{R}^N : \sum_{i \in T} x_i \geq v(T) \text{ for all } T \in \mathcal{T}, \sum_{i \in N} x_i \leq v(N), \sum_{i \in T^*} x_i \geq v(N) - v(T_1) - v(T_3)\}.$$

Firstly we show that $P \neq \emptyset$ and that $P \subseteq I(v)$. Secondly we show, by applying Lemma 3.1, the existence of an $x \in P\backslash C(v)$ that cannot be dominated by any core element. This implies the necessity of the condition involving 3-covering families.

Let $T_1 = \{1, \ldots, t_1\}$ and consider $\sigma : \{1, \ldots, n\} \to N$ with $\sigma(i) = i$ for all $i \in \{1, \ldots, t_1\}$ and $\sigma(i) = n + t_1 + 1 - i$ for all $i \in \{t_1 + 1, \ldots, n\}$. From Theorem 3.1 it follows that $m^\sigma(v) \in C(v)$. Consequently, we have $\sum_{i \in T} m^\sigma_i(v) \geq v(T)$ for all $T \in \mathcal{T}$. Furthermore, observe that $\sum_{i \in T^*} m^\sigma_i(v) = v(N) - v(T_1) - v(T_3)$. We conclude that $m^\sigma(v) \in P$, and thus that $P \neq \emptyset$.

Next we show that $P \subseteq I(v)$. Let $x \in P$. We need to show that $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$. Since $T_1, T_3 \in \mathcal{T}$ it holds that $\sum_{i \in T_1} x_i \geq v(T_1)$ and $\sum_{i \in T_3} x_i \geq v(T_3)$. Because $\sum_{i \in T^*} x_i \geq v(N) - v(T_1) - v(T_3)$ it follows that $\sum_{i \in N} x_i \geq v(N)$. By definition of P it holds that $\sum_{i \in N} x_i \leq v(N)$. So we conclude that $\sum_{i \in N} x_i = v(N)$. Because $\{i\} \in \mathcal{T}$ for each $i \notin T^*$ it holds that $x_i \geq v(\{i\})$ for all $i \notin T^*$. If $|T^*| > 1$, then it also holds that $\{i\} \in \mathcal{T}$ for all $i \in T^*$. So in this case we have that $x_i \geq v(\{i\})$ for all $i \in N$. If $|T^*| = 1$, then $T^* = \{i\}$ for some $i \in N$, and consequently we have that $\{i\} \notin \mathcal{T}$. However, observe that $x_i = \sum_{j \in T^*} x_j \geq v(N) - v(T_1) - v(T_3) \geq v(T^*) = v(\{i\})$, where the first inequality follows since $x \in P$ and the second by superadditivity. So if $|T^*| = 1$, it also holds that $x_i \geq v(\{i\})$ for all $i \in N$. We conclude that $P \subseteq I(v)$.

It remains to show the existence of an $x \in P\backslash C(v)$ that cannot be dominated by any core element. In order to do so, define a matrix A and a vector b such that $x \in P$ if and only if $Ax \leq b$. So for each $T \in \mathcal{T} \cup \{T^*\}$ there is a corresponding row $-e(T)$ in A and for coalition N there is a row $e(N)$ in A. Similarly, $b_i = -v(T)$ if the i-th row in A is $-e(T)$ for some $T \in \mathcal{T}$. Furthermore, $b_i = -v(N) + v(T_1) + v(T_3)$ if the i-th row in A is $-e(T^*)$, and $b_i = v(N)$ if the i-th row in A is $e(N)$. Since P is nonempty it holds that $Ax \leq b$ has a solution. So we can apply Lemma 3.1, with $c = -e(T_2)$ and $\delta = -v(T_2)$, to conclude that for all $x \in P$ it holds that $-e(T_2)x = -\sum_{i \in T_2} x_i \leq -v(T_2)$ if and only if there is a $y \geq 0$ with $yA = -e(T_2)$ and $yb \leq -v(T_2)$. However, we will show that for all $y \geq 0$ with $yA = -e(T_2)$ it holds that $yb > -v(T_2)$. This means there is an $x \in P$ with $-\sum_{i \in T_2} x_i > -v(T_2)$. Hence, $\sum_{i \in T_2} x_i < v(T_2)$ and therefore $x \notin C(v)$. By definition of P, x can only be dominated by coalitions containing T^*. But for every $y \in C(v)$ we have that $\sum_{j \in T_1} y_j \geq v(T_1)$, $\sum_{j \in T_3} y_j \geq v(T_3)$ and $\sum_{j \in N} y_j = v(N)$. Consequently, $\sum_{j \in T^*} y_j \leq v(N) - v(T_1) - v(T_3)$ for every $y \in C(v)$. That is, at x coalition T^* receives a payoff that is at least as much as its highest payoff at any core allocation. So x can not be dominated by a core element via a coalition that contains T^*. Consequently, the core is not stable. This implies that for core stability the conditions corresponding to 3-covering families are necessary.

It remains to show that for all $y \geq 0$ with $yA = -e(T_2)$ it holds that $yb > -v(T_2)$. For each $|\mathcal{T} \cup \{T^*, N\}|$ -dimensional vector $u \geq 0$ we write, with abuse of notation, u_S instead of u_i if the i-th row of A is the row corresponding to coalition S. Define $\mathcal{Y}(u) = \{S \in \mathcal{T} \cup \{T^*, N\} : u_S > 0\}$ as the set of coalitions that u assigns a positive weight to. Let $y \geq 0$ be such that $yA = -e(T_2)$. Instead of calculating yb directly, we first decompose y by using Lemmas A.4, A.5 and A.6. These lemmas are stated and proven in the Appendix. Then we derive the product of these decomposition vectors with b. This enables us to obtain a bound for yb.

According to Lemma A.4 we can decompose y into $\sum_{k=1}^{a_1} \lambda_k u^k + r^1$, with $r^1 \geq 0$, $r^1 A = -e(T_2)$, $\mathcal{Y}(r^1)\backslash\{N\}$ contains no partition of N and for all $1 \leq k \leq a_1$ it holds that $\lambda_k > 0$ and u^k satisfies (A1). Observe that r^1 satisfies the conditions of Lemma A.5. It follows that $r^1 = \sum_{k=1}^{a_2} \mu_k w^k + r^2$ with $r^2 \geq 0$, $r^2_N = 0$, $\sum_{k=1}^{a_2} \mu_k \leq 1$ and for all $1 \leq k \leq a_2$ it holds that $\mu_k > 0$ and w^k satisfies (A2). This implies, because $r^1 A = -e(T_2)$ and $w^k A = -e(T_2)$ for all $1 \leq k \leq a_2$, that $r^2 A = (1 - \sum_{k=1}^{a_2} \mu_k)(-e(T_2))$. Note that, since $0 \leq (1 - \sum_{k=1}^{a_2} \mu_k) \leq 1$, it follows that r^2 satisfies the condition of Lemma A.6. Therefore we can write $r^2 = \sum_{k=1}^{a_3} \nu_k z^k$ with $\sum_{k=1}^{a_3} \nu_k = 1 - \sum_{k=1}^{a_2} \mu_k$ and for all $1 \leq k \leq a_3$ it holds that $\nu_k > 0$ and z^k satisfies (A3). Concluding, we have $y = \sum_{k=1}^{a_1} \lambda_k u^k + \sum_{k=1}^{a_2} \mu_k w^k + \sum_{k=1}^{a_3} \nu_k z^k$ with $\sum_{k=1}^{a_2} \mu_k + \sum_{k=1}^{a_3} \nu_k = 1$.

Before we show that $yb > -v(T_2)$ we first find bounds for $u^k b$, $w^k b$ and $z^k b$. Let $1 \leq k \leq a_1$. First suppose that $T^* \notin \mathcal{Y}(u^k)$. It holds that

$$u^k b = \sum_{S \in \mathcal{Y}(u^k)\backslash\{N\}} [-v(S)] + v(N) \geq 0.$$

Here the inequality is satisfied because of superadditivity and because $\mathcal{Y}(u^k)\backslash\{N\}$ is a partition of N. Now suppose that $T^* \in \mathcal{Y}(u^k)$. Since $\mathcal{Y}(u^k)\backslash\{N\}$ is a partition of N it follows that $\mathcal{Y}(u^k)\backslash\{T^*, N\}$ consists of a partition A of T_1 and a partition B of T_3. It follows that

$$\begin{aligned} u^k b &= \sum_{S \in \mathcal{Y}(u^k)\backslash\{T^*, N\}} [-v(S)] + [v(T_1) + v(T_3) - v(N)] + v(N) \\ &= \sum_{S \in A} [-v(S)] + \sum_{S \in B} [-v(S)] + v(T_1) + v(T_3) \geq 0. \end{aligned}$$

The inequality holds because of superadditivity and because A is a partition of T_1 and B a partition of T_3. Concluding, for all $1 \leq k \leq a_1$ it holds that

$$u^k b \geq 0. \tag{2}$$

Now let $1 \leq k \leq a_2$. First suppose that $T^* \notin \mathcal{Y}(w^k)$. It holds that $\mathcal{Y}(w^k) = U_k \cup V_k \cup \{N\}$, where U_k is a partition of $T_1 \cup T_2$ and V_k a partition of $T_2 \cup T_3$ with $U_k \cap V_k = \emptyset$. Let $\bar{V}_k$ consist of those elements of V_k that are not a subset of T_2, i.e. $\bar{V}_k = \{T \in V_k : T \not\subseteq T_2\}$. It holds that

$$\begin{aligned}
w^k b &= \sum_{S\in\mathcal{Y}(w^k)\backslash\{N\}} [-v(S)] + v(N) \\
&= \sum_{S\in U_k} [-v(S)] + \sum_{S\in V_k} [-v(S)] + v(N) \\
&\geq \sum_{S\in U_k} [-v(S)] + \sum_{S\in V_k} [-v(S)] + v(T_1 \cup T_2) + v(\cup_{T\in\bar{V}_k} T) - v(T_2 \cap (\cup_{T\in\bar{V}_k} T)) \\
&\geq \sum_{S\in V_k} [-v(S)] + v(\cup_{T\in\bar{V}_k} T) - v(T_2 \cap (\cup_{T\in\bar{V}_k} T)) \\
&\geq \sum_{S\in V_k} [-v(S)] + \sum_{T\in\bar{V}_k} v(T) - v(T_2 \cap (\cup_{T\in\bar{V}_k} T)) \\
&= \sum_{S\in V_k\backslash\bar{V}_k} [-v(S)] - v(T_2 \cap (\cup_{T\in\bar{V}_k} T)) \\
&\geq -v(\cup_{S\in V_k\backslash\bar{V}_k} S) - v(T_2 \cap (\cup_{T\in\bar{V}_k} T)) \\
&> -v(T_2).
\end{aligned}$$

We first explain the first inequality. According to Lemma A.9 there is a $T \in \bar{V}_k$ with $T \cap T_2 \neq \emptyset$. This implies that $\{T_1 \cup T_2, \cup_{T\in\bar{V}_k} T\}$ forms a 2-covering family. Observe that because of Lemma A.9 it holds that $(T_1 \cup T_2) \cap (\cup_{T\in\bar{V}_k} T) = T_2 \cap (\cup_{T\in\bar{V}_k} T)$. Since we have assumed that the inequalities corresponding to 2-covering families hold, the first inequality is satisfied. The second inequality holds because of superadditivity and because U_k is a partition of $T_1 \cup T_2$. The third and fourth inequalities are satisfied due to superadditivity. Finally we explain the last inequality. According to Lemma A.9 it holds that $\bar{V}_k$ and $V_k\backslash\bar{V}_k$ are both nonempty, and that there is a $T \in \bar{V}_k$ with $T \cap T_2 \neq \emptyset$. This means that $\{\cup_{S\in V_k\backslash\bar{V}_k} S, T_2 \cap (\cup_{T\in\bar{V}_k} T)\}$ forms a partition of T_2. Because of our assumption that T_2 is essential the last inequality is satisfied.

Now suppose that $T^* \in \mathcal{Y}(w^k)$. Since $U_k \cap V_k = \emptyset$ it either holds that $T^* \in U_k$ or $T^* \in V_k$. Without loss of generality assume that $T^* \in U_k$. Now observe, since U_k is a partition of $T_1 \cup T_2$, that $U_k\backslash\{T^*\}$ consists of a partition C of T_1 and a partition D of $T_2 \cap T_3$. Therefore

$$\begin{aligned}
w^k b &= \sum_{S\in\mathcal{Y}(w^k)\backslash\{T^*,N\}} [-v(S)] + [v(T_1) + v(T_3) - v(N)] + v(N) \\
&> \sum_{S\in\mathcal{Y}(w^k)\backslash\{T^*,N\}} [-v(S)] + [v(T_1 \cap T_2) + v(T_2 \cap T_3) - v(T_2)] + v(N) \\
&= \sum_{S\in C} [-v(S)] + \sum_{S\in D} [-v(S)] + \sum_{S\in V_k} [-v(S)] + v(T_1 \cap T_2) + v(T_2 \cap T_3) - v(T_2) + v(N) \\
&\geq -v(T_1) - v(T_2 \cap T_3) - v(T_2 \cup T_3) + v(T_1 \cap T_2) + v(T_2 \cap T_3) - v(T_2) + v(N) \\
&\geq -v(T_2).
\end{aligned}$$

The first inequality holds since we have assumed that the 3-covering family inequality corresponding to $\{T_1, T_2, T_3\}$ is not satisfied. The second inequality because of superadditivity and because C is a partition of T_1, D is a partition of $T_2 \cap T_3$ and V_k is a partition of $T_2 \cup T_3$. The last inequality is satisfied because $\{T_1, T_2 \cup T_3\}$ forms a 2-covering family with $T_1 \cap (T_2 \cup T_3) = T_1 \cap T_2$, and because of our assumption that all 2-covering family inequalities are satisfied. Concluding, we have for all $1 \leq k \leq a_2$ that

$$w^k b > -v(T_2). \tag{3}$$

Finally let $1 \leq k \leq a_3$. According to Lemma A.6 it holds that $\mathcal{Y}(z^k)$ is a partition of T_2. Now first suppose that $T^* \notin \mathcal{Y}(z^k)$. Then

$$z^k b = \sum_{S \in \mathcal{Y}(w^k)} -v(S) > -v(T_2).$$

Here the inequality holds because $\mathcal{Y}(z^k)$ is a partition of T_2, our assumption that T_2 is essential.

Now suppose that $T^* \in \mathcal{Y}(z^k)$. Since $\mathcal{Y}(z^k)$ is a partition of T_2, it follows that $\mathcal{Y}(z^k)\backslash\{T^*\}$ can be split into a partition A of $T_1 \cap T_2$ and a partition B of $T_2 \cap T_3$. It holds that

$$\begin{aligned}
z^k b &= \sum_{S \in \mathcal{Y}(z^k)\backslash\{T^*\}} [-v(S)] - v(N) + v(T_1) + v(T_3) \\
&= \sum_{S \in A} [-v(S)] + \sum_{S \in B} [-v(S)] - v(N) + v(T_1) + v(T_3) \\
&\geq -v(T_1 \cap T_2) - v(T_2 \cap T_3) - v(N) + v(T_1) + v(T_3) \\
&> -v(T_2).
\end{aligned}$$

The first inequality follows by superadditivity and the second by our assumption that $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$.

Concluding, we have for all $1 \leq k \leq a_3$ that

$$z^k b > -v(T_2). \tag{4}$$

Summarising we find for yb that

$$\begin{aligned}
yb &= \sum_{k=1}^{a_1} \lambda_k u^k b + \sum_{k=1}^{a_2} \mu_k w^k b + \sum_{k=1}^{a_3} \nu_k z^k b \\
&\geq \sum_{k=1}^{a_2} \mu_k w^k b + \sum_{k=1}^{a_3} \nu_k z^k b \\
&> \sum_{k=1}^{a_2} \mu_k(-v(T_2)) + \sum_{k=1}^{a_3} \nu_k(-v(T_2)) \\
&= -v(T_2).
\end{aligned}$$

The first inequality holds because of (2). The second inequality holds because of (3) and (4). The last equality is satisfied since $\sum_{k=1}^{a_2} \mu_k + \sum_{k=1}^{a_3} \nu_k = 1$. □

The next example illustrates the decomposition lemmas that are used in the proof 3 ⇒ 1 of Theorem 3.3. In particular we decompose a specific $y \geq 0$ with $yA = -e(T_2)$ and show that $yb > -v(T_2)$.

Example 3.2 Let $N = \{1, \ldots, 7\}$ and consider the 3-covering family $\{T_1, T_2, T_3\}$ with $T_1 = \{1, 2\}$, $T_2 = \{2, 3, 4, 5, 6\}$ and $T_3 = \{6, 7\}$. Let (N, v) be a chain-component additive game for which all 2-covering family inequalities are satisfied, but for which $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ with T_2 essential.

It now holds that $T^* = N\backslash(T_1 \cup T_3) = \{3, 4, 5\}$ and $\mathcal{T} = \{S \in \mathcal{S} : \{3, 4, 5\} \not\subseteq S\}$. Define the matrix A and the vector b as described in the proof of Theorem 3.3. Now let y be given by

$$y_S = \begin{cases} \frac{1}{2}, & \text{if } S = \{1,2,3\}, \{2\}, \{2,3,4\}, \{3,4\}, \{4,5,6\}, \{5,6\}; \\ 1, & \text{if } S = \{1,2,3,4\}; \\ 1\frac{1}{2}, & \text{if } S = \{5,6,7\}, N; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $yA = \sum_{S\in\mathcal{T}\cup\{T^*\}} y_S(-e(S)) + y_N e(N) = -e(T_2)$. So we need to show that $yb > -v(T_2)$. We will do this by decomposing y. Note that $\mathcal{Y}(y)\backslash\{N\}$ contains a partition of N, for instance $U = \{\{1,2,3,4\},\{5,6,7\}\}$. Therefore we write $y = u^1 + r^1$, with

$$u^1_S = \begin{cases} 1, & \text{if } S = \{1,2,3,4\},\{5,6,7\},N; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$r^1_S = \begin{cases} \frac{1}{2}, & \text{if } S = \{1,2,3\},\{2\},\{2,3,4\},\{3,4\},\{4,5,6\},\{5,6\},\{5,6,7\},N; \\ 0, & \text{otherwise.} \end{cases}$$

Now $\mathcal{Y}(r^1)\backslash\{N\}$ does not contain a partition of N. However, it contains a subset that covers each player of $N\backslash T_2$ exactly once, and each player of T_2 exactly twice. For instance $\{\{1,2,3\},\{4,5,6\},\{2\},\{3,4\},\{5,6,7\}\}$ is such a subset. Therefore we decompose r^1 into $\frac{1}{2}w^1 + r^2$, with

$$w^1_S = \begin{cases} 1, & \text{if } S = \{1,2,3\},\{4,5,6\},\{2\},\{3,4\},\{5,6,7\},N; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$r^2_S = \begin{cases} \frac{1}{2}, & \text{if } S = \{2,3,4\},\{5,6\}; \\ 0, & \text{otherwise.} \end{cases}$$

Finally we note that $\mathcal{Y}(r^2) = \{\{2,3,4\},\{5,6\}\}$ is a partition of T_2. Hence, we write $r^2 = \frac{1}{2}z^1$ with

$$z^1_S = \begin{cases} 1, & \text{if } S = \{2,3,4\},\{5,6\}; \\ 0, & \text{otherwise.} \end{cases}$$

So we have decomposed y into $u^1 + \frac{1}{2}w^1 + \frac{1}{2}z^1$. We will now show that $yb = (u^1 + \frac{1}{2}w^1 + \frac{1}{2}z^1)b > 0$. First note that superadditivity of (N,v) implies

$$u^1 b = -v(\{1,2,3,4\}) - v(\{5,6,7\}) + v(N) \geq 0.$$

Furthermore it holds that

$$\begin{aligned} w^1 b &= -v(\{1,2,3\}) - v(\{4,5,6\}) - v(\{2\}) - v(\{3,4\}) - v(\{5,6,7\}) + v(N) \\ &\geq -v(\{1,2,3,4,5,6\}) - v(\{2,3,4\}) - v(\{5,6,7\}) + v(N) \\ &\geq -v(\{2,3,4\}) - v(\{5,6\}) \\ &> -v(T_2). \end{aligned}$$

Here the first inequality is satisfied due to superadditivity. The second holds because the 2-covering family inequality corresponding to $\{\{1,2,3,4,5,6\},\{5,6,7\}\}$ is satisfied by assumption. The strict inequality is satisfied by our assumption that T_2 is essential. Finally observe that this assumption also proves that

$$z^1 b = -v(\{2,3,4\}) - v(\{5,6\}) > -v(T_2).$$

We conclude that $yb = (u^1 + \frac{1}{2}w^1 + \frac{1}{2}z^1)b > 0 - \frac{1}{2}v(T_2) - \frac{1}{2}v(T_2) = -v(T_2)$. $\diamond$

Observe that for an n-player chain component additive game our characterisation of core stability requires the checking of polynomially many linear inequalities and equations. Indeed, there are $\binom{n-1}{2}$ 2-covering and $\binom{n-1}{4}$ 3-covering inequalities, and for each 3-covering inequality that is not satisfied there are at most $n-2$ linear equations to consider.

The last theorem we present also deals with core stability and essential extendibility. The condition involving 3-covering families of Theorem 3.3 is slightly strengthened. In particular, for considering core stability and essential extendibility one may take into account a more restricted set of equalities.

Theorem 3.4 Let (N, v) be a chain-component additive game. The following statements are equivalent:

1. For each 2-covering family the corresponding inequality is satisfied. For each 3-covering family $\{T_1, T_2, T_3\}$, if the corresponding inequality is not satisfied, then $v(T_2) = v(A)+v(B)$ for some partition $\{A, B\}$ of T_2 with $T_1 \cap T_2 \subsetneq A$, $T_2 \cap T_3 \subsetneq B$, and $A, B \in \mathcal{S}$;

2. (N, v) is essential extendible;

3. The core of (N, v) is stable.

Proof: From Theorem 3.3 it follows that $1 \Rightarrow 2$ and Proposition 3.1 shows $2 \Rightarrow 3$. Therefore we only show $3 \Rightarrow 1$.

Assume that the core is stable. From the proof of Theorem 3.3 it follows that each 2-covering family inequality is satisfied. Assume that there is some 3-covering family $\{T_1, T_2, T_3\}$ with $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ and $v(A) + v(B) < v(T_2)$ for every partition $\{A, B\}$ of T_2 with $T_1 \cap T_2 \subsetneq A$, $T_2 \cap T_3 \subsetneq B$ and $A, B \in \mathcal{S}$. We show that this leads to a contradiction.

Assume that $\{T_1, T_2, T_3\}$ is a smallest 3-covering family with $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ and $v(A) + v(B) < v(T_2)$ for every partition $\{A, B\}$ of T_2 with $T_1 \cap T_2 \subsetneq A$, $T_2 \cap T_3 \subsetneq B$ and $A, B \in \mathcal{S}$ in the following sense: for each 3-covering family $\{S_1, S_2, S_3\}$ with $S_2 \subset T_2$ either $v(S_1) + v(S_2) + v(S_3) \leq v(N) + v(S_1 \cap S_2) + v(S_2 \cap S_3)$ or $v(A) + v(B) = v(S_2)$ for some partition $\{A, B\}$ of S_2 with $S_1 \cap S_2 \subsetneq A$, $S_2 \cap S_3 \subsetneq B$ and $A, B \in \mathcal{S}$.

Since the core is stable, it follows from Theorem 3.3 that T_2 is inessential. Hence, there is a partition $P = \{P_1, \ldots, P_k\}$, $k \geq 2$, of T_2 with $\sum_{T \in P} v(T) \geq v(T_2)$. From superadditivity of (N, v) we conclude that $v(\cup_{i=1}^{k-1} P_i) + v(P_k) \geq \sum_{T \in P} v(T) \geq v(T_2)$. Again using superadditivity we obtain $v(\cup_{i=1}^{k-1} P_i) + v(P_k) = v(T_2)$. Since (N, v) is chain-component additive, we may assume that $\cup_{i=1}^{k-1} P_i$ and P_k are connected. We conclude there are $A, B \in \mathcal{S}$ with $v(A) + v(B) = v(T_2)$ and $\{A, B\}$ a partition of T_2. By assumption, either $A \subseteq T_1 \cap T_2$ or $B \subseteq T_2 \cap T_3$. Without loss of generality assume that $A \subseteq T_1 \cap T_2$.

First suppose that $A = T_1 \cap T_2$. Then obviously $B = T_2 \backslash T_1$. Consequently

$$\begin{aligned} v(T_1) + v(T_2) + v(T_3) &= v(T_1) + v(T_1 \cap T_2) + v(T_2 \backslash T_1) + v(T_3) \\ &\leq v(T_1 \cup T_2) + v(T_1 \cap T_2) + v(T_3) \\ &\leq v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3). \end{aligned}$$

The first inequality holds because of superadditivity and the second because $\{T_1 \cup T_2, T_3\}$ is a 2-covering family with $(T_1 \cup T_2) \cap T_3 = T_2 \cap T_3$. Since $v(T_1) + v(T_2) + v(T_3) \leq v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ is a contradiction to our assumption, we conclude that $A \neq T_1 \cap T_2$.

Now suppose that $A \subseteq T_1 \cap T_2$, but $A \neq T_1 \cap T_2$. Observe that

$$\begin{aligned} v(T_1) + v(B) + v(T_3) &= v(T_1) + v(A) + v(B) + v(T_3) - v(A) \\ &= v(T_1) + v(T_2) + v(T_3) - v(A) \\ &> v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3) - v(A) \\ &\geq v(N) + v((T_1 \cap T_2) \backslash A) + v(T_2 \cap T_3) \\ &= v(N) + v(T_1 \cap B) + v(B \cap T_3). \end{aligned}$$

The first inequality holds by assumption and the second one because of superadditivity. The last equality comes from $(T_1 \cap T_2) \backslash A = T_1 \cap B$ and $T_2 \cap T_3 = B \cap T_3$. Obviously, $\{T_1, B, T_2\}$ is a 3-covering family with $B \subset T_2$ and $v(T_1) + v(B) + v(T_3) > v(N) + v(T_1 \cap B) + v(B \cap T_3)$. By assumption, there is some partition $\{C, D\}$ of B with $T_1 \cap B \subsetneq C$, $B \cap T_3 \subsetneq D$, and $C, D \in \mathcal{S}$. From $v(A) + v(B) = v(T_2)$ and $v(C) + v(D) = v(B)$ it follows that $v(A) + v(C) + v(D) = v(T_2)$. From superadditivity we conclude that $v(A \cup C) + v(D) = v(T_2)$. Note that $B \cap T_3 \subsetneq D$ and therefore $T_2 \cap T_3 \subsetneq D$. Furthermore, since $T_1 \cap B \subsetneq C$, we have that $T_1 \cap T_2 \subsetneq (A \cup C)$. This contradicts our initial assumption. □

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Appendix

In this section we show some technical lemmas needed to prove Theorem 3.2 and Theorem 3.3.

Lemma A.1 Let (N, v) be a chain-component additive game. Let $x \in U(v)$. If $A, B \in S(x)$ with $A \cap B = \emptyset$ and $A \cup B \in \mathcal{S}$, then $A \cup B \in S(x)$.

Proof: Since $x \in U(v)$ it holds that $\sum_{j \in A \cup B} x_j \geq v(A \cup B)$. Also

$$\sum_{j \in A \cup B} x_j = \sum_{j \in A} x_j + \sum_{j \in B} x_j = v(A) + v(B) \leq v(A \cup B),$$

where the inequality holds because of superadditivity. Since $\sum_{j \in A \cup B} x_j = v(A \cup B)$ it holds that $A \cup B \in S(x)$. □

Lemma A.2 Let (N, v) be a chain-component additive game and let $x \in U(v)$. If for all $j \in N$ there is a $T \in S(x)$ with $j \in T$, then $S(x)$ contains a covering family.

Proof: Suppose that for all $j \in N$ there is a $T \in S(x)$ with $j \in T$. We will construct a covering family consisting of elements of $S(x)$. Consider the following algorithm.

Algorithm: Finding a covering family within $S(x)$.

Set $\mathcal{T} = \emptyset$ and $m = 1$.

While $\cup_{T \in \mathcal{T}} T \neq N$.
Let $a_m = min\{j \in N : j \notin \cup_{T \in \mathcal{T}} T\}$. Let T_m be such that $\cup_{T \in \mathcal{T}} T \cup S \subseteq \cup_{T \in \mathcal{T}} T \cup T_m$ for all $S \in S(x)$ with $a_m \in S$. Set $\mathcal{T} = \mathcal{T} \cup \{T_m\}$ and $m = m + 1$.

Let $\mathcal{T} = \{T_1, \ldots, T_m\}$ be the set produced by the algorithm. Since for all $j \in N$ there is a $T \in S(x)$ with $j \in T$, it is obvious that, for each $j \in N$, the algorithm selects at least one subset containing j. That is, for all $j \in N$, there is a $T_i \in \mathcal{T}$ with $j \in T_i$.

We now show that $T_i \cap T_{i+1} \neq \emptyset$ for all $1 \leq i \leq m-1$ by contradiction. Suppose that $T_i \cap T_{i+1} = \emptyset$ for some $1 \leq i \leq m-1$. By definition of T_{i+1} it holds that $T_i \cup T_{i+1} \in \mathcal{S}$. From Lemma A.1 we obtain that $T_i \cup T_{i+1} \in S(x)$. This contradicts the choice of T_i, since now $\cup_{T \in \mathcal{T}} T \cup T_i \subsetneq \cup_{T \in \mathcal{T}} T \cup (T_i \cup T_{i+1})$.

Finally, we show that $T_i \cup T_{i+2} \notin \mathcal{S}$ for all $1 \leq i \leq m-2$ by contradiction. Suppose that for some $1 \leq i \leq m-2$ it holds that $T_i \cap T_{i+2} \neq \emptyset$. Then $a_{i+1} \in T_{i+2}$. But this contradicts the choice of T_{i+1} since now $\cup_{T \in \mathcal{T}} T \cup T_{i+1} \subsetneq \cup_{T \in \mathcal{T}} T \cup T_{i+2}$. So $T_i \cup T_{i+2} \notin \mathcal{S}$.

We conclude that $\{T_1, \ldots, T_m\}$ is a covering family. □.

The following lemma is Proposition 3.1 of [12]. It states that each balanced collection that is a subset of $\mathcal{S}$ necessarily contains a partition.

Lemma A.3 ([12]) Let $B \subseteq \mathcal{S}$ be a balanced collection. Then B contains a partition of N as a subset.

We will now prove the decomposition lemmas needed for the proof of Theorem 3.3.

Lemma A.4 Let $y \geq 0$ be such that $yA = -e(T_2)$. We can decompose y into $\sum_{k=1}^{a_1} \lambda_k u^k + r^1$, with $r^1 \geq 0$, $r^1 A = -e(T_2)$, $\mathcal{Y}(r^1)\backslash\{N\}$ does not contain a partition of N, and for all $1 \leq k \leq a_1$ it holds that $\lambda_k > 0$ and u^k satisfies

(A1) $u_S^k \in \{0,1\}$ for all $S \in \mathcal{T} \cup \{T^*, N\}$, $u^k A = 0$ and $\mathcal{Y}(u^k) = U_k \cup \{N\}$ for some partition U_k of N.

Proof: Let $y \geq 0$ be such that $yA = -e(T_2)$. We show the decomposition by recursion. Suppose that for some $a^* \geq 0$ we have written $y = \sum_{k=1}^{a^*} \lambda_k u^k + r^1$, with $r^1 \geq 0$, $r^1 A = -e(T_2)$ and for all $1 \leq k \leq a^*$ it holds that $\lambda_k > 0$ and that u^k satisfies (A1). Note that this certainly holds for $a^* = 0$ and $r^1 = y$.

Now if $\mathcal{Y}(r^1)\backslash\{N\}$ does not contain a partition of N, then we are done, so suppose that $\mathcal{Y}(r^1)\backslash\{N\}$ contains a partition, say U, of N. Define

- $u_S^{a^*+1} = 1$ if $S \in U \cup \{N\}$;
- $u_S^{a^*+1} = 0$ if $S \notin U$;
- $\lambda_{a^*+1} = \min\{r_S^1 : S \in U \cup \{N\}\}$.

Note that $\lambda_{a^*+1} > 0$ and that $\mathcal{Y}(u^{a^*+1}) = U \cup \{N\}$. Observe, since U is a partition of N, that $u^{a^*+1}A = 0$. Thus, u^{a^*+1} satisfies (A1). Furthermore, by definition of λ_{a^*+1} and u^{a^*+1} it holds that $\bar{r}^1 = r^1 - \lambda_{a^*+1} u^{a^*+1} \geq 0$. Finally, note, because $u^k A = 0$ for all $1 \leq k \leq a^* + 1$ and because $yA = -e(T_2)$, that $\bar{r}^1 A = yA - \sum_{k=1}^{a^*+1} \lambda_k u^k A = -e(T_2)$.

So it now holds that $y = \sum_{k=1}^{a^*+1} \lambda_k u^k + \bar{r}^1$, with $\bar{r}^1 \geq 0$, $\bar{r}^1 A = -e(T_2)$, and for all $1 \leq k \leq a^* + 1$ it holds that $\lambda_k > 0$ and that u^k satisfies (A1).

Observe that because of our choice of λ_{a^*+1} it holds that $\mathcal{Y}(\bar{r}^1) \subsetneq \mathcal{Y}(r^1)$. This implies that in a finite number of steps we can decompose y into $\sum_{k=1}^{a_1} \lambda_k u^k + r^1$, with $r^1 \geq 0$, $r^1 A = -e(T_2)$, $\mathcal{Y}(r^1)\backslash\{N\}$ does not contain a partition of N, and for all $1 \leq k \leq a_1$ it holds that $\lambda_k > 0$ and that u^k satisfies (A1). □

Lemma A.5 Let $y \geq 0$ be such that $yA = -e(T_2)$ and $\mathcal{Y}(y)\backslash\{N\}$ does not contain a partition of N. Then we can decompose y into $\sum_{k=1}^{a_2} \mu_k w^k + r^2$, with $r^2 \geq 0$, $r_N^2 = 0$, $\sum_{k=1}^{a_2} \mu_k \leq 1$ and for all $1 \leq k \leq a_2$ it holds that $\mu_k > 0$ and w^k satisfies

(A2) $w_S^k \in \{0,1\}$ for all $S \in \mathcal{T} \cup \{T^*, N\}$, $w^k A = -e(T_2)$ and $\mathcal{Y}(w^k) = U_k \cup V_k \cup \{N\}$ for some partition U_k of $T_1 \cup T_2$ and some partition V_k of $T_2 \cup T_3$ with $U_k \cap V_k = \emptyset$.

Proof: Let $y \geq 0$ be such that $yA = -e(T_2)$, and such that $\mathcal{Y}(y)\backslash\{N\}$ does not contain a partition of N. We show the decomposition recursively. Suppose that for some $a^* \geq 0$ we have written $y = \sum_{k=1}^{a^*} \mu_k w^k + r^2$, with $r^2 \geq 0$, $\sum_{k=1}^{a^*} \mu_k \leq 1$ and for all $1 \leq k \leq a^*$ it holds that $\mu_k > 0$ and that w^k satisfies (A2). Note that this certainly holds for $a^* = 0$ and $r^2 = y$. If $r_N^2 = 0$ then we are done, so suppose that $r_N^2 > 0$. Observe that since $\mathcal{Y}(y)\backslash\{N\}$ does not contain a partition of N, and since $\mathcal{Y}(r^2) \subseteq \mathcal{Y}(y)$, it follows that $\mathcal{Y}(r^2)\backslash\{N\}$ does not contain a partition of N.

We will first show that $\sum_{k=1}^{a^*} \mu_k < 1$ by contradiction. Suppose that $\sum_{k=1}^{a^*} \mu_k = 1$. Then it follows, using $yA = -e(T_2)$ and $w^k A = -e(T_2)$ for all $1 \leq k \leq a^*$, that $r^2 A = yA - \sum_{k=1}^{a^*} \mu_k w^k A = (1 - \sum_{k=1}^{a^*} \mu_k)(-e(T_2)) = 0$. Since $r_N^2 > 0$, it follows that $\mathcal{Y}(r^2)\backslash\{N\}$ is a balanced collection of N. From Lemma A.3 we conclude that $\mathcal{Y}(r^2)\backslash\{N\}$ contains a partition of N. Since $\mathcal{Y}(r^2) \subseteq \mathcal{Y}(y)$, it follows that $\mathcal{Y}(y)\backslash\{N\}$ contains a partition of N. However, we assumed that this was not the case. Therefore we conclude that $\sum_{k=1}^{a^*} \mu_k < 1$.

Since $r^2 A = (1 - \sum_{k=1}^{a^*} \mu_k)(-e(T_2))$ with $\sum_{k=1}^{a^*} \mu_k < 1$, $r_2^N > 0$ and $\mathcal{Y}(r^2)\backslash\{N\}$ does not contain a partition of N, it follows that r^2 satisfies the conditions of Lemma A.7. According to Lemma A.7 $\mathcal{Y}(r^2)$ contains a partition U_{a^*+1} of $T_1 \cup T_2$ and a partition V_{a^*+1} of $T_2 \cup T_3$ with $U_{a^*+1} \cap V_{a^*+1} = \emptyset$.

Define

- $w_S^{a^*+1} = 1$ if $S \in U_{a^*+1} \cup V_{a^*+1} \cup \{N\}$;
- $w_S^{a^*+1} = 0$ if $S \notin U_{a^*+1} \cup V_{a^*+1} \cup \{N\}$;
- $\mu_{a^*+1} = \min\{r_S^2 : S \in U_{a^*+1} \cup V_{a^*+1} \cup \{N\}\}$

Note that $\mu_{a^*+1} > 0$. We will now show that $w^{a^*+1}A = -e(T_2)$. For each $i \in T_2$ there are unique $S \in U_{a^*+1}$ and $T \in V_{a^*+1}$ with $i \in S$ and $i \in T$. Note that since $U_{a^*+1} \cap V_{a^*+1} = \emptyset$, it holds that $S \neq T$. Furthermore, for each $i \in T_1 \backslash T_2$ there is a unique $S \in U_{a^*+1}$ with $i \in S$ and for each $i \in T_3 \backslash T_2$ there is a unique $T \in V_{a^*+1}$ with $i \in T$. We conclude that $w^{a^*+1}A = -e(T_2)$.

Since $w^{a^*+1}A = -e(T_2)$ we now observe that w^{a^*+1} satisfies (A2). Also note that $\bar{r}^2 = r^2 - \mu_{a^*+1}w^{a^*+1} \geq 0$. We will now show by contradiction that it holds that $\sum_{k=1}^{a^*+1} \mu_k \leq 1$. Suppose that $\sum_{k=1}^{a^*+1} \mu_k > 1$. Because $\sum_{k=1}^{a^*} \mu_k \leq 1$, it follows that there is a $0 \leq d < \mu_{a^*+1}$ with $\sum_{k=1}^{a^*} \mu_k + d = 1$. It trivially holds that $y = \sum_{k=1}^{a^*} \mu_k w^k + dw^{a^*+1} + (r^2 - dw^{a^*+1})$. By definition of d it holds that $(r^2 - dw^{a^*+1}) \gneq (r^2 - \mu_{a^*+1}w^{a^*+1}) \geq 0$. Since $yA = -e(T_2)$, $w^kA = -e(T_2)$ for all $1 \leq k \leq a^* + 1$ and $\sum_{k=1}^{a^*} \mu_k + d = 1$, it follows that $(r^2 - dw^{a^*+1})A = 0$. Because $d < \mu_{a^*+1}$, it holds that $\mathcal{Y}(r^2 - dw^{a^*+1}) \neq \emptyset$. Therefore it follows that $\mathcal{Y}(r^2 - dw^{a^*+1}) \backslash \{N\}$ is a balanced collection. By Lemma A.3 it now follows that $\mathcal{Y}(r^2 - dw^{a^*+1}) \backslash \{N\}$ contains a partition of N. Since $\mathcal{Y}(r^2 - dw^{a^*+1}) \subseteq \mathcal{Y}(r^2)$ it holds that $\mathcal{Y}(r^2) \backslash \{N\}$ contains a partition of N. This is clearly a contradiction to our initial assumption, so we conclude that $\sum_{k=1}^{a^*+1} \mu_k \leq 1$.

It follows that $y = \sum_{k=1}^{a^*+1} \mu_k w^k + \bar{r}^2$, with $\bar{r}^2 \geq 0$, $\sum_{k=1}^{a^*+1} \mu_k \leq 1$ and for all $1 \leq k \leq a^* + 1$ it holds that $\mu_k > 0$ and w^k satisfies (A2).

Observe that because of our choice of μ_{a^*+1} it holds that $\mathcal{Y}(\bar{r}^2) \subsetneq \mathcal{Y}(r^2)$. This implies that in a finite number of steps we can decompose y into $\sum_{k=1}^{a_2} \mu_k w^k + r^2$, with $r^2 \geq 0$, $r_N^2 = 0$, $\sum_{k=1}^{a_2} \mu_k \leq 1$ and for all $1 \leq k \leq a_2$ it holds that $\mu_k > 0$ and w^k satisfies (A2). □

Lemma A.6 Let $y \geq 0$ be such that $y_N = 0$ and $yA = d(-e(T_2))$, for some $0 \leq d \leq 1$. Then we can decompose y into $\sum_{k=1}^{a_3} \nu_k z^k$ with $\sum_{k=1}^{a_3} \nu_k = d$, and for all $1 \leq k \leq a_3$ it holds that $\nu_k > 0$ and z^k satisfies

(A3) $z_S^k \in \{0, 1\}$ for all $S \in \mathcal{T} \cup \{T^*, N\}$, $z^kA = -e(T_2)$ and $\mathcal{Y}(z^k) = U_k$ for some partition U_k of T_2.

Proof: Let $y \geq 0$ be such that $y_N = 0$, and $yA = d(-e(T_2))$ for some $0 \leq d \leq 1$. We recursively show the decomposition. Suppose that for some $a^* \geq 0$ we have written $y = \sum_{k=1}^{a^*} \nu_k z^k + r^3$, with $\sum_{k=1}^{a^*} \nu_k \leq d$, $r^3 \geq 0$ and for all $1 \leq k \leq a^*$ it holds that $\nu_k > 0$ and that z^k satisfies (A3). Note that this certainly holds for $a^* = 0$ and $r^3 = y$.

Now if $\sum_{k=1}^{a^*} \nu_k = d$, then it follows, because $yA = d(-e(T_2))$ and $z^kA = -e(T_2)$ for all $1 \leq k \leq a^*$, that $r^3A = yA - \sum_{k=1}^{a^*} \nu_k z^k A = 0$. Because $r_N^3 = 0$, $r^3 \geq 0$ and because A has only non-positive entries in each row that does not correspond to N, we conclude that $r^3 = 0$. So $y = \sum_{k=1}^{a^*} \nu_k z^k$ and we are done. Therefore suppose that $\sum_{k=1}^{a^*} \nu_k < d$.

Now $r^3A = yA - \sum_{k=1}^{a^*} \nu_k z^k A = (d - \sum_{k=1}^{a^*} \nu_k)(-e(T_2))$, with $d - \sum_{k=1}^{a^*} \nu_k > 0$. Since $r_N^3 = 0$, and because in A the only row with positive entries is the row corresponding to N, this means that $r_S^3 = 0$ for all $S \in \mathcal{T} \cup \{T^*, N\}$ with $S \not\subseteq T_2$. This implies that $\mathcal{Y}(r^3)$ is a balanced collection on T_2. From Lemma A.3 it follows that $\mathcal{Y}(r^3)$ contains a partition of T_2. Now let U be such a partition. Define

- $z_S^{a^*+1} = 1$ if $S \in U$;
- $z_S^{a^*+1} = 0$ if $S \notin U$;
- $\nu_{a^*+1} = \min\{r_S^3 : S \in U\}$.

Note that $\nu_{a^*+1} > 0$. Since $z^{a^*+1}A = -e(T_2)$, it follows that z^{a^*+1} satisfies (A3). Also observe that by definition of ν_{a^*+1} and z^{a^*+1} it holds that $\bar{r}^3 \geq 0$. It remains to show that $\sum_{k=1}^{a^*+1} \nu_k \leq d$.

Suppose that $\sum_{k=1}^{a^*+1} \nu_k > d$. Then it follows that $\bar{r}^3 A = (d - \sum_{k=1}^{a^*+1} \nu_k)(-e(T_2))$, where $d - \sum_{k=1}^{a^*+1} \nu_k < 0$. Hence, $\bar{r}^3 A = fe(T_2)$ for some $f > 0$. However, this is impossible, since $\bar{r}^3 \geq 0$, $\bar{r}^3_N = 0$ and because A contains only non-positive entries in the rows not corresponding to N. Therefore we obtain that $\sum_{k=1}^{a^*+1} \nu_k \leq d$.

Hence, we have that $y = \sum_{k=1}^{a^*+1} \nu_k z^k + \bar{r}^3$, with $\sum_{k=1}^{a^*+1} \nu_k \leq d$, $\bar{r}^3 \geq 0$ and for all $1 \leq k \leq a^*+1$ it holds that $\nu_k > 0$ and that z^k satisfies (A3).

Observe that by definition of ν_{a^*+1} and z^{a^*+1} it holds that $\mathcal{Y}(\bar{r}^3) \subsetneq \mathcal{Y}(r^3)$. Hence, in a finite number of steps we obtain that $y = \sum_{k=1}^{a_3} \nu_k z^k$, where $\nu_k > 0$ and z^k satisfies (A3) for all $1 \leq k \leq a_3$. Since $yA = d(-e(T_2))$ and that $z^k A = -e(T_2)$ for all $1 \leq k \leq a_3$ it follows that $\sum_{k=1}^{a_3} \nu_k = d$. □

Lemma A.7 Let $r^2 \geq 0$ be such that $r^2 A = f(-e(T_2))$ for some $0 < f \leq 1$, $r_2^N > 0$ and $\mathcal{Y}(r^2)\backslash\{N\}$ does not contain a partition of N. Then $\mathcal{Y}(r^2)$ contains a partition U of $T_1 \cup T_2$ and a partition V of $T_2 \cup T_3$ with $U \cap V = \emptyset$.

Proof: We will show how to obtain a partition of $T_1 \cup T_2$. Analogously one can find a partition of $T_2 \cup T_3$. First we will show that we can find disjoint elements $S_k \in \mathcal{Y}(r^2)$, $1 \leq k \leq q$, such that $(T_1 \cup T_2) \subseteq \cup_{k=1}^q S_k$. We will do this by giving a recursive argument.

Because $r^2 A = f(-e(T_2))$ for some $0 < f \leq 1$ and $1 \notin T_2$, it holds that $\sum_{S \in \mathcal{T} \cup \{T^*\}: 1 \in S} r_S^2 = r_N^2$. By assumption $r_N^2 > 0$ and we conclude that $\sum_{S \in \mathcal{T} \cup \{T^*\}: 1 \in S} r_S^2 > 0$. Hence, there exists an $S_1 \in \mathcal{Y}(r^2)$, with $1 \in S_1$.

Now suppose that we have selected disjoint $S_k \in \mathcal{Y}(r^2)$, $1 \leq k \leq t$, such that $N\backslash(\cup_{k=1}^t S_k) = \{b, \ldots, n\}$ for some $b \in N$. Or in other words, suppose that $\cup_{k=1}^t S_k$ is a head of σ_0. Note that $t = 1$ and S_1 satisfy this property.

If $b \notin T_1 \cup T_2$, then we are done, so suppose that $b \in T_1 \cup T_2$. According to Lemma A.8, with $a = b$, it holds that $\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: b \in S} r_S^2 \geq \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: b-1 \in S} r_S^2$. Since for S_t it holds that $b-1 \in S_t$, $b \notin S_t$ and $r_{S_t}^2 > 0$, it follows that there is an $S_{t+1} \in \mathcal{Y}(r^2)$ with $b-1 \notin S_{t+1}$, $b \in S_{t+1}$ and $r_{S_{t+1}}^2 > 0$. We conclude that $N \setminus \cup_{k=1}^{t+1} S_k = \{c, \ldots, n\}$ with $c > b$. By recursion we obtain disjoint $S_k \in \mathcal{Y}(r^2)$, $1 \leq k \leq q$, with $(T_1 \cup T_2) \subseteq \cup_{k=1}^q S_k$.

We will now show that $(T_1 \cup T_2) = \cup_{k=1}^q S_k$ by contradiction. Suppose that $(T_1 \cup T_2) \subsetneq \cup_{k=1}^q S_k$. Then $N\backslash(\cup_{k=1}^q S_k) = \{b, \ldots, n\}$ for some $b \in T_3\backslash T_2$ with $b-1 \in T_3\backslash T_2$. According to Lemma A.8, with $a = b$ it follows that $\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: b \in S} r_S^2 = \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: b-1 \in S} r_S^2$. Since for S_q it holds that $b-1 \in S_q$, $b \notin S_q$ and $r_{S_q}^2 > 0$, it follows that there is an $S_{q+1} \in \mathcal{Y}(r^2)\backslash\{N\}$ with $b-1 \notin S_{q+1}$, $b \in S_q$ and $r_{S_{q+1}}^2 > 0$. Note that $N\backslash \cup_{k=1}^{q+1} S_k = \{c, \ldots, n\}$ with $c > b$. By recursion we therefore obtain a partition of N. However, initially we assumed that $\mathcal{Y}(r^2)\backslash\{N\}$ does not contain a partition of N. From this contradiction we conclude that $(T_1 \cup T_2) = \cup_{k=1}^q S_k$.

Now let $U = \{S_1, \ldots, S_q\} \subseteq \mathcal{Y}(r^2)$ be a partition of $T_1 \cup T_2$ and $V = \{R_1, \ldots, R_m\} \subseteq \mathcal{Y}(r^2)$ a partition of $T_2 \cup T_3$. If $U \cap V \neq \emptyset$, then $S_i = R_j$ for some $1 \leq i \leq q$, $1 \leq j \leq m$. Hence, $\{S_1, \ldots, S_i, R_{j+1}, \ldots, R_m\}$ is a partition of N. This contradicts the assumption that $\mathcal{Y}(r^2) \setminus \{N\}$ does not contain a partition of N. We conclude that $U \cap V = \emptyset$. □

Lemma A.8 Let $r^2 \geq 0$ be such that $r^2 A = f(-e(T_2))$ for some $0 < f \leq 1$. Then it holds for all $a \in T_1 \cup T_2$ with $a > 1$, that $\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a \in S} r_S^2 \geq \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a-1 \in S} r_S^2$. Furthermore it holds for all $a, a+1 \in T_3\backslash T_2$ that $\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a+1 \in S} r_S^2 = \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a \in S} r_S^2$.

Proof: Let $r^2 \geq 0$ be such that $r^2 A = f(-e(T_2))$ for some $0 < f \leq 1$. Let $a \in T_1 \cup T_2$ with $a > 1$.

If $a \in T_1\backslash T_2$, then it follows that $a-1 \in T_1\backslash T_2$. It follows from $r^2 A = f(-e(T_2))$ that

$$\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a \in S} r_S^2 = r_N^2 = \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a-1 \in S} r_S^2.$$

If $a \in T_2$ and $a-1 \in T_1 \backslash T_2$, then it follows from $r^2 A = f(-e(T_2))$ that

$$\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a \in S} r_S^2 = f + r_N^2 > r_N^2 = \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a-1 \in S} r_S^2.$$

Finally, if $a-1 \in T_2$, then it follows from $r^2 A = f(-e(T_2))$ that

$$\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a \in S} r_S^2 = f + r_N^2 = \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a-1 \in S} r_S^2.$$

So we conclude that

$$\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a \in S} r_S^2 \geq \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a-1 \in S} r_S^2.$$

Similarly it can be shown that for all $a, a+1 \in T_3 \backslash T_2$ that

$$\sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a+1 \in S} r_S^2 = \sum_{S \in \mathcal{Y}(r^2)\backslash\{N\}: a \in S} r_S^2.$$ □

Lemma A.9 Let $r^2 \geq 0$ be such that $r^2 A = f(-e(T_2))$ for some $0 < f \leq 1$, $r_N^2 > 0$ and $\mathcal{Y}(r^2)\backslash\{N\}$ does not contain a partition of N. Consider a partition $V_k \subseteq \mathcal{Y}(r^2)\backslash\{N\}$ of $T_2 \cup T_3$ and let $\bar{V}_k = \{T \in V_k : T \nsubseteq T_2\}$. Then $\bar{V}_k \neq \emptyset$ and $V_k \backslash \bar{V}_k \neq \emptyset$. Furthermore it holds that $T \cap T_1 = \emptyset$ for all $T \in \bar{V}_k$ and that there is a $T \in \bar{V}_k$ with $T \cap T_2 \neq \emptyset$.

Proof: Observe that from Lemma A.7 it follows that a partition V_k of $T_2 \cup T_3$ exists. Note that $\bar{V}_k \neq \emptyset$, since V_k is a partition of $T_2 \cup T_3$ and $T_3 \neq \emptyset$.

We will now show that for all $T \in \bar{V}_k$ it holds that $T_1 \cap T = \emptyset$ by contradiction. Suppose that there is a $T \in \bar{V}_k$ with $T_1 \cap T \neq \emptyset$. Since $T \in \bar{V}_k$ it follows that $T \nsubseteq T_2$. That is, there is a $j \in T$ with $j \in T_3 \backslash T_2$. Since T is connected it follows that $T^* \subsetneq T$. This is a contradiction since the coalitions containing T^* are not in $\mathcal{T}$ and therefore also not in $\mathcal{Y}(r^2)\backslash\{N\}$. Hence it holds for all $T \in \bar{V}_k$ that $T \cap T_1 = \emptyset$. Because $T_1 \cap T_2 \neq \emptyset$, there is an $S \in V_k$ with $S \cap T_1 \neq \emptyset$. This implies that $S \notin \bar{V}_k$ and hence that $V_k \neq \bar{V}_k$. Finally, we prove that there is a $T \in \bar{V}_k$ with $T \cap T_2 \neq \emptyset$. Suppose that for all $T \in \bar{V}_k$ it holds that $T \cap T_2 = \emptyset$. According to Lemma A.7 there is a partition U_k of $T_1 \cup T_2$. This implies that $U_k \cup \bar{V}_k$ forms a partition of N, which is a contradiction to our initial assumption. □