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Sequencing games with repeated players.

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Abstract:

Two classes of one machine sequencing situations are considered in which each job corresponds to exactly one player but a player may have more than one job to be processed, so called RP(repeated player) sequencing situations. In max-RP sequencing situations it is assumed that each player's cost function is linear with respect to the maximum completion time of his jobs, whereas in min-RP sequencing situations the cost functions are linear with respect to the minimum completion times. For both classes, following explicit procedures to go from the initial processing order to an optimal order for the coalition of all players, equal gain splitting rules are defined. It is shown that these rules lead to core elements of the associated RP sequencing games. Moreover, it is seen that min-RP sequencing games are convex.

Keywords: cooperative game theory, sequencing, equal gain splitting, core, convexity.
JEL classification: C71

1 Introduction

Scheduling problems were first studied from an interactive cooperative point of view by Curiel, Pederzoli and Tijs (1989) in the framework of one-machine sequencing with an initial processing order on the jobs. Identifying jobs with players and introducing cost functions for the players dependent on the completion time of their jobs, an associated cooperative game is defined in which the value of the coalition reflects the maximum cost savings this coalition can achieve by reordering their jobs from their initial position to a feasible, optimal one. Different types of sequencing games have been studied in the literature depending on the structure of the underlying cost functions, the number of machines, further restrictions such as ready times and due dates etc. For a survey we refer to Curiel, Hamers and Klijn (2002). The common feature in this stream of literature is the assumption that each job is of interest to exactly one player and that each player has exactly one job to be processed.

An exception is the recent contribution of Calleja, Estévez-Fernández, Borm and Hamers (2004). Here the latter assumption is dropped, so a job may correspond to several players and a player may have interest in more than one job. It is shown that the corresponding sequencing games are balanced if the underlying cost functions satisfy a specific type of additivity relative to the initial order. In the current paper we focus on the specific subclass of RP (repeated players) sequencing situations in which (just as in the classical approach) each job corresponds to exactly one player, but a player may have more than one job to be processed. Moreover we restrict attention to two types of RP sequencing situations and their related games. In max-RP sequencing

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situations it is assumed that each player's cost function is linear with respect to the maximum completion time of his jobs, whereas in min-RP sequencing situations the cost function of a player is linear with respect to the minimum completion time of his jobs. It was already pointed out in Calleja et al. (2004) that both types of cost functions satisfy the additivity condition, so the corresponding games are balanced. Next, we offer two motivating examples for these types of cost functions.

In a garage, a car may need more than one reparation (change of tires, of oil, etc). Here, it seems reasonable to assume that each repair job not only has a certain fixed cost but the owner also incurs variable costs that are proportional to the total time that the car has to spend in the garage, i.e. to the completion time of the last reparation carried out in the car. Note that the reparations in the car are complementary since the car can not leave the garage until it is completely repaired.

The classrooms of a faculty are equipped with an overhead projector and a beamer. If one of the devices breaks down, lecturers have to report to the maintenance service for the device(s) to be repaired, incurring a fixed cost for each reparation. A lecturer needs at least one of the devices to start the lesson. Hence, when both devices are out of order, there is an extra variable cost which is proportional to the time that she has to wait until she can start her lecture, i.e. until one of the devices is fixed. Note that the reparations in this setting are substitutes since the lecturer can start her lecture as soon as one of the reparations is carried out.

The contributions of the current paper are the following. For both max-RP and min-RP sequencing situations explicit procedures are devised to go from the initial order on all jobs to an optimal one. Following the steps of this procedure an EGS (equal gain splitting) mechanism is adopted to construct an allocation rule for the maximal cost savings of the grand coalition. It is shown that this EGS-allocation is in the core of the associated game, and that it in fact is PMAS-extendable (cf. Sprumont (1990)). In particular this implies that for calculating a core allocation one does not need the data on all coalitional values. In addition it is shown that min-RP sequencing games are convex.

The structure of the paper is as follows. Section 2 recalls some basic game theoretic notions and provides the formal definition of RP sequencing situations and related games. Section 3 considers max-RP sequencing whereas Section 4 analyzes min-RP sequencing. An appendix contains the lemmas used in the proofs of the main results.

2 Preliminaries

A cooperative TU-game in characteristic function form is an ordered pair (N, v) where N is a finite set (the set of players) and $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. The *core* of a cooperative TU-game (N, v) is defined by

$$Core(v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N\},$$

i.e. the core is the set of efficient allocations of v(N) such that there is no coalition with an incentive to split off. A game is said to be *balanced* (see Bondareva (1963) and Shapley (1967)) if the core is nonempty.

An important subclass of balanced games is the class of convex games (Shapley (1971)). A game (N, v) is said to be *convex* if

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$$

for every $i \in N$ and for every $S \subset T \subset N \setminus \{i\}$.

A *sequencing situation* is a 4-tuple (N, σ_0, p, c), where $N = \{1, \dots, n\}$ is the set of players (or jobs), $\sigma_0 : N \to \{1, \dots, n\}$ is a bijection that represents the initial order on the jobs (job i is in position $\sigma_0(i)$), $p \in \mathbb{R}^N$ is the vector of processing times of the jobs and $c = (c_i)_{i \in N} \in \mathbb{R}^N$ is the vector of cost functions of the players depending on the completion time of their jobs, so $c_i : [0, +\infty) \to \mathbb{R}$. Costs are assumed to be linear, i.e. $c_i(t) = \alpha_i t$ with $\alpha_i > 0$. Alternatively, a sequencing situation (N, σ_0, p, c) is denoted by (N, σ_0, p, α) with $\alpha \in \mathbb{R}^N_{++}$. Let $\Pi(N)$ denote the set of all possible orders of the jobs. Given an order $\sigma \in \Pi(N)$ the jobs will be processed in a semi-active way, i.e. there will not exist a job that could be processed earlier without altering the processing order. Therefore, the completion time of player i is given by $C_i^\sigma = \displaystyle\sum_{j \in N : \sigma(j) \leq \sigma(i)} p_j$. For simplicity we denote $c_i(C_i^\sigma)$ by $c_i(\sigma)$.

Given a sequencing situation the associated *sequencing game*, (N, v_C), is defined by

$$v_C(S) = \max_{\sigma \in \mathcal{F}(S)} (c_S(\sigma_0) - c_S(\sigma))$$

for every $S \subset N$, where, for all $\sigma \in \Pi(N)$, $c_S(\sigma) = \displaystyle\sum_{i \in S} c_i(\sigma)$ and $\mathcal{F}(S)$ is the set of feasible orders for coalition S. An order $\sigma \in \Pi(N)$ is said to be *feasible* for S if $P_i(\sigma) = P_j(\sigma_0)$ for all $i, j \in N$, where $P_i(\sigma) = \{k \in N \mid \sigma(k) < \sigma(i)\}$ is the set of predecessors of i with respect to σ. Note that feasible orders will only allow reordering within connected components of S with respect to σ_0. The set of connected components of S with respect to σ_0 is denoted S/σ_0. Assuming $\sigma_0 = (12 \dots n)$ the associated coalitional values can be expressed as

$$v_C(S) = \sum_{T \in S/\sigma_0} \sum_{i,j \in T:\, i<j} g_{ij}$$

for every $S \subset N$, where $g_{ij} = \max\{0, \alpha_j p_i - \alpha_i p_j\}$ is the cost savings that players i and j can achieve by means of a neighbour switch when i is in front of j (cf. Curiel et al. (1989)).

Since an optimal order for the grand coalition can be derived from the initial order by non-negative neighbour switches, a natural allocation rule in sequencing situations is provided by the equal gain splitting rule or EGS rule introduced in Curiel et al. (1989), where the cost savings attained by neighbour switches are divided equally among the players involved. Formally,

$$\mathrm{EGS}_i(N, \sigma_0, p, \alpha) = \frac{1}{2} \sum_{j=1}^{i-1} g_{ji} + \frac{1}{2} \sum_{j=i+1}^{n} g_{ij}$$

for every $i \in N$.

Sequencing games are convex and the EGS rule provides a core allocation.

An *RP (repeated players) sequencing situation* is a 6-tuple $(N, M, J, \sigma_0, p, c)$, where $N = \{1, \ldots, n\}$ is the set of players, M is the finite set of jobs, $J : N \twoheadrightarrow M$ is a correspondence where $J(i)$ denotes the non-empty set of jobs in which player i is involved with the extra condition that $\bigcup_{i \in N} J(i) = M$ and $J(i) \cap J(j) = \emptyset$ for all $i, j \in N$, $i \neq j$, $\sigma_0 : M \to \{1, \ldots, |M|\}$ is a bijection representing the initial order on the jobs, $p \in \mathbb{R}^M$ is the vector of processing times of the jobs and $c = (c_i)_{i \in N} \in \mathbb{R}^N$ is the vector of cost functions associated to the players. Let $\Pi(M)$ denote the set of all bijections $\sigma : M \to \{1, \ldots, |M|\}$. Given an order $\sigma \in \Pi(M)$ it is assumed that the jobs will be processed in a semi-active way.
Given an RP sequencing situation $(N, M, J, \sigma_0, p, c)$ the associated *RP sequencing game* (N, v) is defined by

$$v(S) = \max_{\sigma \in \mathcal{A}(S)} (c_S(\sigma_0) - c_S(\sigma))$$

for every $S \subset N$, where for all $\sigma \in \Pi(M)$, $c_S(\sigma) = \sum_{i \in S} c_i(\sigma)$ and $\mathcal{A}(S)$ is the set of admissible orders for coalition S. An order $\sigma \in \Pi(M)$ is said to be *admissible* for S if $P_d(\sigma) = P_d(\sigma_0)$ for all $d \notin \bigcup_{i \in S} J(i)$, where $P_d(\sigma) = \{e \in M \mid \sigma(e) < \sigma(d)\}$ is the set of predecessors of job d with respect to σ. Note that if an order is admissible for S, the completion time of each job belonging to a player in $N \setminus S$ does not change. Moreover, only within connected components of $\bigcup_{i \in S} J(i)$ w.r.t. σ_0, jobs can be reordered.

It has been shown in Calleja et al. (2004) that RP sequencing games are balanced if the cost functions of the players are "additive relative to the initial order".

3 Max-RP sequencing

In this section we will consider max-RP sequencing situations and associated games.

A *max-RP sequencing situation* is an RP sequencing situation where $c_i(\sigma) = \alpha_i \max_{d \in J(i)} \{C_d^\sigma\}$ for some $\alpha_i > 0$, for every $i \in N$ and all $\sigma \in \Pi(M)$. Usually, a max-RP sequencing situation like this will be described by $(N, M, J, \sigma_0, p, \alpha)$ with $\alpha \in \mathbb{R}^N_{++}$. It has been pointed out in Calleja et al. (2004) that the above type of cost functions satisfies the additivity condition needed for balancedness of the associated sequencing game.

Given an order $\sigma \in \Pi(M)$, we denote by $l_i^\sigma \in M$ the job of $i \in N$ that is processed in last position according to σ, i.e. $\sigma(l_i^\sigma) \geq \sigma(d)$ for all $d \in J(i)$. Note that an order $\sigma \in \Pi(M)$ induces an order $\bar{\sigma} \in \Pi(N)$ on the players in the following way: $\bar{\sigma}(i) < \bar{\sigma}(j)$ if and only if $\sigma(l_i^\sigma) < \sigma(l_j^\sigma)$. Throughout this section we will assume w.l.o.g. that $\sigma_0 \in \Pi(M)$ is such that $\bar{\sigma}_0 = (12 \ldots n)$ and we will write $l_i = l_i^{\sigma_0}$. Hence, $i < j$ if and only if $\sigma_0(l_i) < \sigma_0(l_j)$.

We say that the jobs of player i are *clustered* according to an order $\sigma \in \Pi(M)$ if they are processed consecutively, i.e. if $d_1, d_2 \in J(i)$ and $\sigma(d_1) < \sigma(e) < \sigma(d_2)$ imply that $e \in J(i)$. It is easy to see that all jobs of a player will be clustered in an optimal order for max-RP sequencing situations. To derive an optimal order on all jobs we next turn to classical sequencing: the optimal order of the clusters will be in non-decreasing order of urgencies (cf. Smith (1956)). Here, the urgency of a cluster obviously will be the quotient of cost coefficient α_i of the corresponding player while the processing time of the cluster is given by $\sum_{d \in J(i)} p_d$.

An explicit procedure to derive an optimal order for M from the initial order σ_0 by non-negative switches is described in the following way.

First, we put all the jobs of player n at the back of the queue[5]. After this, all jobs of player $n-1$ are clustered in front of jobs of player n, and so on. Note that the cost savings induced on i by clustering the jobs of j $(i < j)$ during this step are given by: $b_{ij}^N = \alpha_i \sum_{e \in J(j):\, \sigma_0(e) < \sigma_0(l_i)} p_e$.

Second, consider the (remaining) classical sequencing situation (on the constructed clusters) given by $(N, \bar{\sigma_0}, q, \alpha)$ with $q \in \mathbb{R}^N$ such that $q_i = \sum_{d \in J(i)} p_d$. The cost savings in this step can be obtained by non-negative (cluster) neighbour switches and equal $\sum_{i,j \in N:\, i<j} g_{ij}^N$ with $g_{ij}^N = \max\{0, \alpha_j q_i - \alpha_i q_j\}$.

Summarizing, the total maximal cost savings $v(N)$ are given by $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r(i,j,N)$, where

$$r(i,j,N) := b_{ij}^N + g_{ij}^N$$

for all $i, j \in N$ with $i < j$.

Adopting the equal gain splitting mechanism in the procedure above we can define the *max-EGS* rule in the following way:

$$\text{max-EGS}_i(N, M, J, \sigma_0, p, \alpha) = \frac{1}{2} \sum_{j=i+1}^{n} r(i,j,N) + \frac{1}{2} \sum_{j=1}^{i-1} r(j,i,N).$$

We will show that the max-EGS rule leads to core elements of the associated max-RP sequencing game. With this purpose, we first will give an explicit construction and expression for the coalitional values in a max-RP sequencing game similar to the derivation of $v(N)$ provided above.

For this, we need some notation. Given $S \subset N$ let $[J(S)/\sigma_0] = \{U_1, \ldots, U_u\}$ be the set of maximal connected components of the set of jobs $J(S)$ such that $U_r \cap \{l_i\}_{i \in S} \neq \emptyset$ for all r. The collection $\{U_1, \ldots, U_u\}$ is called the *induced job partition of S by σ_0*. Associated to each U_r we define S_r by $S_r := \{i \in S \mid l_i \in U_r\}$. Observe that $\{S_1, \ldots, S_u\}$ is a partition of S. We call this partition the *induced partition of S by σ_0*.

Now, consider $i \in S_r$ and $j \in S$ with $i < j$. We denote by $b_{ij}^{S_r}$ the cost savings induced on i when moving the jobs of player j that are in U_r to be clustered to the back, i.e. $b_{ij}^{S_r} = \alpha_i \sum_{\substack{e \in J(j) \cap U_r \\ \sigma_0(e) < \sigma_0(l_i)}} p_e$.

[5]Remember that we assume that $\sigma_0(l_n) > \sigma_0(l_i)$ for all $i \in N$, $i \neq n$.

Moreover, with $i, j \in S_r$ and $i < j$ we denote by $g_{ij}^{S_r}$ the cost savings obtainable by means of a neighbour switch between the clusters corresponding to i and j within U_r, i.e. $g_{ij}^{S_r} = \max\{0, \alpha_j \sum_{d \in J(i) \cap U_r} p_d - \alpha_i \sum_{e \in J(j) \cap U_r} p_e\}$.

For $i, j \in S$ with $i < j$ we define

$$r(i,j,S) = \begin{cases} b_{ij}^{S_r} + g_{ij}^{S_r}, & \text{if } i, j \in S_r; \\ b_{ij}^{S_r}, & \text{if } i \in S_r, j \notin S_r. \end{cases}$$

It is readily established that the coalitional value $v(S)$ in the corresponding max-RP sequencing game is given by

$$v(S) = \sum_{i \in S} \sum_{j \in S:\, i<j} r(i,j,S)$$

Theorem 3.1. For any max-RP sequencing game, the max-EGS rule provides a core element.

Proof: Let $(N, M, J, \sigma_0, p, \alpha)$ be a max-RP sequencing situation and let (N, v) be the associated max-RP sequencing game.

Efficiency holds by definition. Next, we will show that the rule is stable. Let $S \subset N$, then

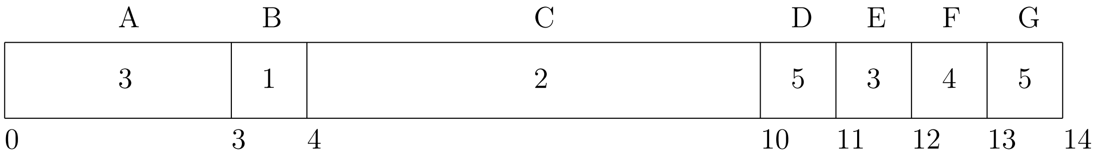
$$\begin{aligned}
\sum_{i \in S} \text{max-EGS}_i(N, M, J, \sigma_0, p, \alpha) &= \sum_{i \in S} \Big[\frac{1}{2} \sum_{j=i+1}^{n} r(i,j,N) + \frac{1}{2} \sum_{j=1}^{i-1} r(j,i,N)\Big] \\
&= \sum_{i \in S} \sum_{j \in S:\, i<j} r(i,j,N) + \frac{1}{2} \sum_{i \in S} \sum_{j \in N \setminus S:\, i<j} r(i,j,N) \\
&\quad + \frac{1}{2} \sum_{i \in S} \sum_{j \in N \setminus S:\, j<i} r(j,i,N) \\
&\geq \sum_{i \in S} \sum_{j \in S:\, i<j} r(i,j,N) \\
&\geq \sum_{i \in S} \sum_{j \in S:\, i<j} r(i,j,S) = v(S),
\end{aligned}$$

where the last inequality holds by Lemma A.1. □

We want to note that in fact the max-EGS core allocation is PMAS extendable (cf. Sprumont (1990)) by considering the max-EGS allocations for all subgames, and the use of Lemma A.1.

The following example provides a non-convex max-RP sequencing game.

Example 3.2. Let $(N, M, J, \sigma_0, p, c)$ be a max-RP sequencing situation with $N = \{1, 2, 3, 4, 5\}$, $M = \{A, B, C, D, E, F, G\}$ and $J(1) = \{B\}$, $J(2) = \{C\}$, $J(3) = \{A, E\}$, $J(4) = \{F\}$, $J(5) = \{D, G\}$. Let $\sigma_0 = (A, B, C, D, E, F, G)$, $p = (3, 1, 6, 1, 1, 1, 1)$, and $\alpha = (10, 6, 4, 1, 1)$. Note that $\bar{\sigma}_0 = (12345)$ This situation is depicted below:



Let $S = \{2, 3\}$, $T = \{1, 2, 3\}$, and $i = 5$. It is readily checked that the optimal order for coalition S is σ_0 and consequently $v(S) = 0$, the optimal order for coalition $S \cup \{i\}$ is A-B-E-C-D-F-G and $v(S \cup \{i\}) = 22$, the optimal order for coalition T is B-C-A-D-E-F-G and $v(T) = 48$, while finally, the optimal order for coalition $T \cup \{i\}$ is B-C-A-E-D-F-G and $v(T \cup \{i\}) = 52$. Therefore, $v(T \cup \{i\}) - v(T) = 4 < 22 = v(S \cup \{i\}) - v(S)$ and the game is not convex. □

4 Min-RP sequencing

In this section we will analyze min-RP sequencing situations and related games.

A *min-RP sequencing situation* is an RP sequencing situation where $c_i(\sigma) = \alpha_i \min_{d \in J(i)} \{C_d^\sigma\}$ for some $\alpha_i > 0$, for every $i \in N$ and all $\sigma \in \Pi(M)$. Usually, a min-RP sequencing situation like this will be described by $(N, M, J, \sigma_0, p, \alpha)$ with $\alpha \in \mathbb{R}^N_{++}$. It has been pointed out in Calleja et al. (2004) that also this type of cost functions satisfy the additivity condition needed for balancedness of the associated sequencing game.

Given an order $\sigma \in \Pi(M)$, we denote by $f_i^\sigma \in M$ the job of $i \in N$ that is processed in first position according to σ, i.e. $\sigma(f_i^\sigma) \leq \sigma(d)$ for all $d \in J(i)$. Note that an order σ on the jobs induces an order $\tilde{\sigma} \in \Pi(N)$ on the players in the following way: $\tilde{\sigma}(i) < \tilde{\sigma}(j)$ if and only if $\sigma(f_i^\sigma) < \sigma(f_j^\sigma)$. Throughout this section we will assume w.l.o.g. that σ_0 is such that $\tilde{\sigma}_0 = (12 \ldots n)$ and we will write $f_i = f_i^{\sigma_0}$. Hence, $i < j$ if and only if $\sigma_0(f_i) < \sigma_0(f_j)$.

It is easy to see that for min-RP sequencing situations in every optimal order on M the first n jobs will belong to different players. Moreover, these jobs will have minimum processing time among the jobs of the corresponding player and they will be processed in decreasing order with respect to their urgencies.

An optimal order can be constructed from σ_0 by non-negative switches in the following way. First, we put all the jobs of player 1 that are not f_1 and are in front of f_n at the back of the queue[6]. After this, we do the same with all the jobs of player 2 that are not f_2 and are in front of f_n, and so on. Once we finish this step, the first n jobs of the queue belong to different players. Observe that the cost savings induced on j by moving jobs of players i ($i < j$) to the back are given by: $\gamma_{ij}^N = \alpha_j \sum_{d \in J(i) \setminus \{f_i\} : \, \sigma_0(d) < \sigma_0(f_j)} p_d.$

Next, we switch f_i[7] with the job of player i that has shortest processing time, if necessary, in the order $1, 2, \ldots, n$. The cost savings induced on j by the internal job switch of i (with $i \leq j$)

[6]Remember that we have assumed that $\sigma_0(f_1) < \sigma_0(f_j)$ for all $j \in N$, $j \neq 1$.

[7]Note that for the order $\sigma \in \Pi(M)$ derived of the first step we have that $f_i = f_i^\sigma$ for all $i \in N$.

in this step are: $\delta_{ij}^N = \alpha_j (p_{f_i} - \min_{d \in J(i)} \{p_d\})$.

Now, consider the (remaining) classical sequencing situation (on these first n jobs) given by $(N, \tilde{\sigma_0}, q, \alpha)$ with $q \in \mathbb{R}^N$ such that $q_i = \min_{d \in J(i)} p_d$. Clearly, the maximal cost savings in this third step equal $\sum_{i,j \in N:\, i<j} g_{ij}^N$ with $g_{ij}^N = \max\{0, \alpha_j q_i - \alpha_i q_j\}$.

Summarizing, the total maximal cost savings $v(N)$ are given by $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s(i,j,N) + \sum_{i=1}^{n} s(i,N)$, where

$$s(i,N) = \delta_{ii}^N$$

and

$$s(i,j,N) = \gamma_{ij}^N + \delta_{ij}^N + g_{ij}^N$$

for all $i, j \in N$ with $i < j$.

Adopting the equal gain splitting mechanism in the procedure above we can define the *min-EGS* rule in the following way

$$\text{min-EGS}_i(N, M, J, \sigma_0, p, \alpha) = \frac{1}{2} \sum_{j=1}^{i-1} s(j,i,N) + \frac{1}{2} \sum_{j=i+1}^{n} s(i,j,N) + s(i,N).$$

It will be shown that the min-EGS rule leads to core allocations of the associated min-RP sequencing games. With this purpose, we first will give an explicit construction and expression for the coalitional values in a min-RP sequencing game similar to max-EGS games.

For this, we introduce first some notation. Given $S \subset N$ let $[J(S)/\sigma_0] = \{U_1, \ldots, U_u\}$ be the set of maximal connected components of $J(S)$ such that $U_r \cap \{f_i\}_{i \in S} \neq \emptyset$ for all r. The collection $\{U_1, \ldots, U_u\}$ is called the *induced job partition of S by σ_0*. Associated to each U_r we define S_r by $S_r := \{i \in S \mid f_i \in U_r\}$. Observe that $\{S_1, \ldots, S_u\}$ is a partition of S. We call this partition the *induced partition of S by σ_0*. Note that $\{U_1, \ldots, U_u\}$ and $\{S_1, \ldots, S_u\}$ have a different interpretation than in max-RP sequencing situations, although we use the same notation.

Now, consider $i \in S_r$, $j \in S$ with $i < j$. We will denote by $\gamma_{ij}^{S_r}$ the cost savings that player j obtains when the jobs of player i within U_r (unequal to f_i) that are in front of f_j are moved to the back of U_r, i.e. $\gamma_{ij}^{S_r} = \alpha_j \sum_{\substack{d \in (J(i) \setminus \{f_i\}) \cap U_r \\ \sigma_0(d) < \sigma_0(f_j)}} p_d$.

Next, consider $i, j \in S_r$ with $i \leq j$. With $\delta_{ij}^{S_r}$ we symbolize the cost savings that player j obtains when player i ($i \leq j$) switches its first job with the one with minimum processing time in U_r, i.e. $\delta_{ij}^{S_r} = \alpha_j (p_{f_i} - \min_{d \in J(i) \cap U_r} \{p_d\})$.

Finally, take $i, j \in S_r$ with $i < j$. By $g_{ij}^{S_r}$ we denote the cost savings obtainable by means of a neighbour switch between the two jobs of players i and j within U_r with shortest processing times, i.e. $g_{ij}^{S_r} = \max\{0, \alpha_j \min_{d \in J(i) \cap U_r} \{p_d\} - \alpha_i \min_{e \in J(j) \cap U_r} \{p_e\}\}$.

Now consider $S \subset N$. For $i, j \in S$ and $i < j$ define

$$s(i,j,S) = \begin{cases} \gamma_{ij}^{S_r} + \delta_{ij}^{S_r} + g_{ij}^{S_r}, & \text{if } i, j \in S_r; \\ \gamma_{ij}^{S_r}, & \text{if } i \in S_r, j \notin S_r; \end{cases}$$

and for $i \in S$,

$$s(i,S) = \delta_{ii}^{S_r}$$

if $i \in S_r$.

Then, it readily follows that the coalitional value $v(S)$ in the corresponding min-RP sequencing game is given by

$$v(S) = \sum_{i\in S}\sum_{j\in S: i<j} s(i,j,S) + \sum_{i\in S} s(i,S).$$

Theorem 4.1. For any min-RP sequencing game, the min-EGS rule provides a core element.

Proof: Let $(N, M, J, \sigma_0, p, \alpha)$ be a min-RP sequencing situation and let (N, v) be the associated min-RP sequencing game.

Efficiency holds by definition. Next, it will be shown that the rule is stable. Let $S \subset N$, then

$$\begin{aligned}
\sum_{i\in S} \text{min-EGS}_i(N, M, J, \sigma_0, p, \alpha) &= \sum_{i\in S} \big[\frac{1}{2}\sum_{j=1}^{i-1} s(j,i,N) + \frac{1}{2}\sum_{j=i+1}^{n} s(i,j,N) + s(i,N)\big] \\
&= \sum_{i\in S}\sum_{j\in S\colon j<i} s(j,i,N) + \frac{1}{2}\sum_{i\in S}\sum_{j\in N\setminus S\colon j<i} s(j,i,N) \\
&\quad + \frac{1}{2}\sum_{i\in S}\sum_{j\in N\setminus S\colon i<j} s(i,j,N) + \sum_{i\in S} s(i,N) \\
&\geq \sum_{i\in S}\sum_{j\in S\colon j<i} s(j,i,N) + \sum_{i\in S} s(i,N) \\
&\geq \sum_{i\in S}\sum_{j\in S\colon j<i} s(j,i,S) + \sum_{i\in S} s(i,S) = v(S),
\end{aligned}$$

where the second inequality holds by Lemma A.2 and Lemma A.3. □

Also here, note that in fact the min-EGS core allocation is PMAS extendable by considering the min-EGS allocations for all subgames and the use of Lemma A.2 and Lemma A.3.

Moreover, we have that min-RP sequencing games are convex.

Theorem 4.2. Min-RP sequencing games are convex.

Proof: Let $(N, M, J, \sigma_0, p, \alpha)$ be a min-RP sequencing situation and let (N, v) be the associated min-RP sequencing game. Note that the characteristic function of the game can be written as the sum of two characteristic functions: $v(S) = w(S) + u(S)$ with $w(S) = \sum_{i\in S}\sum_{j\in S: i<j} s(i,j,S)$ and $u(S) = \sum_{i\in S} s(i,S)$. We will show that both (N,u) and (N,w) are convex and therefore that (N,v) is convex.

First, we will prove that (N,u) is convex. We have to show that

$$u(S \cup \{j\}) - u(S) \leq u(T \cup \{j\}) - u(T)$$

for every $j \in N$ and every $S \subset T \subset N \setminus \{j\}$.

Let $S \subset T$ and take $j \in N \setminus T$. Since,

$$u(S \cup \{j\}) - u(S) = \sum_{k \in S} \big(s(k, S \cup \{j\}) - s(k, S)\big) + s(j, S \cup \{j\})$$

and

$$u(T \cup \{j\}) - u(T) = \sum_{k \in T} \big(s(k, T \cup \{j\}) - s(k, T)\big) + s(j, T \cup \{j\})$$

it is sufficient to show that the following three inequalities hold:

$$s(k, S \cup \{j\}) - s(k, S) \leq s(k, T \cup \{j\}) - s(k, T) \text{ for every } k \in S, \tag{1}$$

$$0 \leq s(k, T \cup \{j\}) - s(k, T) \text{ for every } k \in T \setminus S, \tag{2}$$

$$s(j, S \cup \{j\}) \leq s(j, T \cup \{j\}) \tag{3}$$

Statement (1) is true by convexity of $s(*, \cdot)$ (Lemma A.4) and statements (2) and (3) are true by monotonicity of $s(*, \cdot)$ (Lemma A.2).

Secondly, we will show that (N,w) is convex. We have to show that

$$w(S \cup \{j\}) - w(S) \leq w(T \cup \{j\}) - w(T)$$

for every $j \in N$ and every $S \subset T \subset N \setminus \{j\}$.

Let $S \subset T$ and take $j \in N \setminus T$. Since

$$\begin{aligned} w(S \cup \{j\}) - w(S) = &\sum_{k \in S} \sum_{i \in S : k < i} \big(s(k, i, S \cup \{j\}) - s(k, i, S)\big) \\ &+ \sum_{k \in S : k < j} s(k, j, S \cup \{j\}) + \sum_{i \in S : j < i} s(j, i, S \cup \{j\}) \end{aligned}$$

and

$$\begin{aligned} w(T \cup \{j\}) - w(T) = &\sum_{k \in T} \sum_{i \in T : k < i} \big(s(k, i, T \cup \{j\}) - s(k, i, T)\big) \\ &+ \sum_{k \in T : k < j} s(k, j, T \cup \{j\}) + \sum_{i \in T : j < i} s(j, i, T \cup \{j\}) \end{aligned}$$

it is sufficient to show that

$$s(k, i, S \cup \{j\}) - s(k, i, S) \leq s(k, i, T \cup \{j\}) - s(k, i, T) \text{ for every } k, i \in S, \text{ with } k < i, \tag{4}$$

$$0 \leq s(k, i, T \cup \{j\}) - s(k, i, T) \text{ for every } k \in T \setminus S,\ i \in T \text{ or } k \in S,\ i \in T \setminus S, \text{ with } k < i, \tag{5}$$

$$s(k, j, S \cup \{j\}) \leq s(k, j, T \cup \{j\}) \text{ for every } k \in S,\ k < j, \tag{6}$$

$$s(j, i, S \cup \{j\}) \leq s(j, i, T \cup \{j\}) \text{ for every } i \in S,\ j < i, \tag{7}$$

$$0 \leq s(k, j, T \cup \{j\}) \text{ for every } k \in T \setminus S,\ k < j, \tag{8}$$

$$0 \leq s(j, i, T \cup \{j\}) \text{ for every } i \in T \setminus S,\ j < i, \tag{9}$$

Statement (4) is true by convexity of $s(*, *, \cdot)$ (Lemma A.5). Statements (5), (6), and (7) are true by monotonicity of $s(*, *, \cdot)$ (Lemma A.3). Finally, statements (8) and (9) follow from the definition of $s(*, *, \cdot)$. $\square$

Appendix

Lemma A.1. Let $S \subset T \subset N$. For every $i, j \in S$ with $i < j$, $r(i,j,S) \leq r(i,j,T)$.

Proof: Let $(N, M, J, \sigma_0, p, \alpha)$ be a max-RP sequencing situation and take $i, j \in S$ with $i < j$. Let $i \in S_r \subset T_\rho$, with S_r and T_ρ components within the induced partition of S and T, respectively. Moreover, let U_r, V_ρ be the correspondent components within the job partition associated to S_r and T_ρ, respectively. Note that $U_r \subset V_\rho$. We will distinguish two cases.
Case 1: $j \notin S_r$. Here,

$$r(i,j,S) = b_{ij}^{S_r} = \alpha_i \sum_{\substack{e \in J(j) \cap U_r \\ \sigma_0(e) < \sigma_0(l_i)}} p_e \leq \alpha_i \sum_{\substack{e \in J(j) \cap V_\rho \\ \sigma_0(e) < \sigma_0(l_i)}} p_e = b_{ij}^{T_\rho} \leq r(i,j,T),$$

where the first inequality holds because $U_r \subset V_\rho$ and the second inequality holds by definition of $r(i,j,T)$.
Case 2: $j \in S_r$. Here, two subcases will be distinguished.

Subcase 2.1: $g_{ij}^{S_r} = 0$. Then

$$r(i,j,S) = b_{ij}^{S_r} + g_{ij}^{S_r} = \alpha_i \sum_{\substack{e \in J(j) \cap U_r \\ \sigma_0(e) < \sigma_0(l_i)}} p_e \leq \alpha_i \sum_{\substack{e \in J(j) \cap V_\rho \\ \sigma_0(e) < \sigma_0(l_i)}} p_e = b_{ij}^{T_\rho} \leq r(i,j,T).$$

Subcase 2.2: $g_{ij}^{S_r} > 0$. Then $l_i, l_j \in U_r$ and $g_{ij}^{S_r} = \alpha_j \sum_{d \in J(i) \cap U_r} p_d - \alpha_i \sum_{e \in J(j) \cap U_r} p_e$. We will distinguish two new subcases.

Subcase 2.2.1: $g_{ij}^{T_\rho} = 0$. Hence, by definition of $g_{ij}^{T_\rho}$ it holds

$$\alpha_j \sum_{d \in J(i) \cap V_\rho} p_d - \alpha_i \sum_{e \in J(j) \cap V_\rho} p_e \leq 0. \tag{10}$$

Therefore,

$$\begin{aligned}
r(i,j,T) &= b_{ij}^{T_\rho} + g_{ij}^{T_\rho} = b_{ij}^{T_\rho} - g_{ij}^{S_r} + g_{ij}^{S_r} \\
&= \alpha_i \sum_{\substack{e \in J(j) \cap V_\rho \\ \sigma_0(e) < \sigma_0(l_i)}} p_e - (\alpha_j \sum_{d \in J(i) \cap U_r} p_d - \alpha_i \sum_{e \in J(j) \cap U_r} p_e) + g_{ij}^{S_r} \\
&= \alpha_i \sum_{\substack{e \in J(j) \cap U_r \\ \sigma_0(e) < \sigma_0(l_i)}} p_e + \alpha_i \sum_{e \in J(j) \cap (V_\rho \setminus U_r)} p_e - \alpha_j \sum_{d \in J(i) \cap U_r} p_d + \alpha_i \sum_{e \in J(j) \cap U_r} p_e + g_{ij}^{S_r} \\
&= b_{ij}^{S_r} + \alpha_i \sum_{e \in J(j) \cap V_\rho} p_e - \alpha_j \sum_{d \in J(i) \cap U_r} p_d + g_{ij}^{S_r} \\
&\geq b_{ij}^{S_r} + \alpha_i \sum_{e \in J(j) \cap V_\rho} p_e - \alpha_j \sum_{d \in J(i) \cap V_\rho} p_d + g_{ij}^{S_r} \\
&\geq b_{ij}^{S_r} + g_{ij}^{S_r} = r(i,j,S),
\end{aligned}$$

where the fourth equality holds because $l_i, l_j \in U_r$, $\sigma_0(l_i) < \sigma_0(l_j)$ and then $\{e \in J(j) \cap (V_\rho \setminus U_r) \mid \sigma_0(e) < \sigma_0(l_i)\} = J(j) \cap (V_\rho \setminus U_r)$. Last inequality follows from (10).

Subcase 2.2.2: $g_{ij}^{T_\rho} > 0$. Hence, $g_{ij}^{T_\rho} = \alpha_j \sum_{d \in J(i) \cap V_\rho} p_d - \alpha_i \sum_{e \in J(j) \cap V_\rho} p_e$ by definition. Then

$$\begin{aligned}
r(i,j,T) = b_{ij}^{T_\rho} + g_{ij}^{T_\rho} &= \alpha_i \sum_{\substack{e \in J(j) \cap V_\rho \\ \sigma_0(e) < \sigma_0(l_i)}} p_e + \alpha_j \sum_{d \in J(i) \cap V_\rho} p_d - \alpha_i \sum_{e \in J(j) \cap V_\rho} p_e \\
&= \alpha_i \sum_{\substack{e \in J(j) \cap U_r \\ \sigma_0(e) < \sigma_0(l_i)}} p_e + \alpha_i \sum_{e \in J(j) \cap (V_\rho \setminus U_r)} p_e + \alpha_j \sum_{d \in J(i) \cap V_\rho} p_d \\
&\quad - \alpha_i \sum_{e \in J(j) \cap U_r} p_e - \alpha_i \sum_{e \in J(j) \cap (V_\rho \setminus U_r)} p_e \\
&= b_{ij}^{S_r} + \alpha_j \sum_{d \in J(i) \cap V_\rho} p_d - \alpha_i \sum_{e \in J(j) \cap U_r} p_e \\
&\geq b_{ij}^{S_r} + \alpha_j \sum_{d \in J(i) \cap U_r} p_d - \alpha_i \sum_{e \in J(j) \cap U_r} p_e \\
&= b_{ij}^{S_r} + g_{ij}^{S_r} = r(i,j,S).
\end{aligned}$$

where the third equality holds because $l_i, l_j \in U_r$, $\sigma_0(l_i) < \sigma_0(l_j)$ and then $\{e \in J(j) \cap (V_\rho \setminus U_r) \mid \sigma_0(e) < \sigma_0(l_i)\} = J(j) \cap (V_\rho \setminus U_r)$. □

Lemma A.2. Let $S \subset T \subset N$. For every $i \in S$, $s(i,S) \leq s(i,T)$.

Proof: Let $(N, M, J, \sigma_0, p, \alpha)$ be a min-RP sequencing situation and let $i \in S$. Let $i \in S_r \subset T_\rho$, with S_r and T_ρ components within the induced partition of S and T, respectively. Moreover, let U_r, V_ρ be the components within the job partition associated to S_r and T_ρ, respectively. Note that $U_r \subset V_\rho$. Hence,

$$s(i,S) = \delta_{ii}^{S_r} = \alpha_i (p_{f_i} - \min_{e \in J(i) \cap U_r} \{p_e\}) \leq \alpha_i (p_{f_i} - \min_{e \in J(i) \cap V_\rho} \{p_e\}) = \delta_{ii}^{T_\rho} = s(i,T).$$

□

Lemma A.3. Let $S \subset T \subset N$. For every $i, j \in S$ with $i < j$, $s(i,j,S) \leq s(i,j,T)$.

Proof: Let $(N, M, J, \sigma_0, p, \alpha)$ be a min-RP sequencing situation and let $i, j \in S$. Let $i \in S_r \subset T_\rho$, with S_r and T_ρ components within the induced partition of S and T, respectively. Moreover, let U_r, V_ρ be the components within the job partition associated to S_r and T_ρ, respectively. Note that $U_r \subset V_\rho$.

Case 1: $j \notin S_r$. In this case $s(i,j,S) = \gamma_{ij}^{S_r}$.

$$s(i,j,S) = \gamma_{ij}^{S_r} = \alpha_j \sum_{\substack{e \in J(i) \cap (U_r \setminus \{f_i\}) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e \leq \alpha_j \sum_{\substack{e \in J(i) \cap (V_\rho \setminus \{f_i\}) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e = \gamma_{ij}^{T_\rho} \leq s(i,j,T).$$

Case 2: $j \in S_r \subset T_\rho$. Here, $s(i,j,S) = \gamma_{ij}^{S_r} + \delta_{ij}^{S_r} + g_{ij}^{S_r}$ and $s(i,j,T) = \gamma_{ij}^{T_\rho} + \delta_{ij}^{T_\rho} + g_{ij}^{T_\rho}$. We will distinguish two subcases.

Subcase 2.1: $g_{ij}^{S_r} = 0$. Hence,

$$\begin{aligned}
s(i,j,S) &= \gamma_{ij}^{S_r} + \delta_{ij}^{S_r} + g_{ij}^{S_r} = \gamma_{ij}^{S_r} + \delta_{ij}^{S_r} \\
&= \alpha_j \sum_{\substack{e \in J(i) \cap (U_r \setminus \{f_i\}) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e + \alpha_j (p_{f_i} - \min_{e \in J(i) \cap U_r} \{p_e\}) \\
&\leq \alpha_j \sum_{\substack{e \in J(i) \cap (V_\rho \setminus \{f_i\}) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e + \alpha_j (p_{f_i} - \min_{e \in J(i) \cap V_\rho} \{p_e\}) \\
&= \gamma_{ij}^{T_\rho} + \delta_{ij}^{T_\rho} \leq \gamma_{ij}^{T_\rho} + \delta_{ij}^{T_\rho} + g_{ij}^{T_\rho} = s(i,j,T).
\end{aligned}$$

Subcase 2.2: $g_{ij}^{S_r} > 0$. In this case, $g_{ij}^{S_r} = \alpha_j \min_{e \in J(i) \cap U_r} \{p_e\} - \alpha_i \min_{d \in J(j) \cap U_r} \{p_d\}$ by definition. Here, two new cases are needed.

Subcase 2.2.1: $g_{ij}^{T_\rho} = 0$. Hence, by definition of $g_{ij}^{T_\rho}$ it holds

$$\alpha_j \min_{e \in J(i) \cap V_\rho} \{p_e\} - \alpha_i \min_{d \in J(j) \cap V_\rho} \{p_d\} \leq 0. \tag{11}$$

It will be shown that $s(i,j,T) - s(i,j,S) \geq 0$.

$$\begin{aligned}
s(i,j,T) - s(i,j,S) &= (\gamma_{ij}^{T_\rho} + \delta_{ij}^{T_\rho} + g_{ij}^{T_\rho}) - (\gamma_{ij}^{S_r} + \delta_{ij}^{S_r} + g_{ij}^{S_r}) \\
&= \alpha_j \sum_{\substack{e \in J(i) \cap (V_\rho \setminus U_r) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e - \alpha_j \min_{e \in J(i) \cap V_\rho} \{p_e\} + \alpha_i \min_{d \in J(j) \cap U_r} \{p_d\} \\
&\geq \alpha_j \sum_{\substack{e \in J(i) \cap (V_\rho \setminus U_r) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e - \alpha_j \min_{e \in J(i) \cap V_\rho} \{p_e\} + \alpha_i \min_{d \in J(j) \cap V_\rho} \{p_d\} \\
&\geq 0,
\end{aligned}$$

where the equality holds by definition and the second inequality holds by equation (11) and because $\alpha_j \sum_{\substack{e \in J(i) \cap (V_\rho \setminus U_r) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e \geq 0$.

Subcase 2.2.2: $g_{ij}^{T_\rho} > 0$. Hence, $g_{ij}^{T_\rho} = \alpha_j \min_{e \in J(i) \cap V_\rho} \{p_e\} - \alpha_i \min_{d \in J(j) \cap V_\rho} \{p_d\}$ by definition. Therefore,

$$\begin{aligned}
s(i,j,S) &= \gamma_{ij}^{S_r} + \delta_{ij}^{S_r} + g_{ij}^{S_r} \\
&= \alpha_j \sum_{\substack{e \in J(i) \cap (U_r \setminus \{f_i\}) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e + \alpha_j p_{f_i} - \alpha_i \min_{d \in J(j) \cap U_r} \{p_d\} \\
&\leq \alpha_j \sum_{\substack{e \in J(i) \cap (V_\rho \setminus \{f_i\}) \\ \sigma_0(e) < \sigma_0(f_j)}} p_e + \alpha_j p_{f_i} - \alpha_i \min_{d \in J(j) \cap V_\rho} \{p_d\} \\
&= \gamma_{ij}^{T_\rho} + \delta_{ij}^{T_\rho} + g_{ij}^{T_\rho} = s(i,j,T).
\end{aligned}$$

where the first and second equalities hold by definition. □

Lemma A.4. Let $S \subset T \subset N$. For every $j \in N \setminus T$ and every $i \in S$,

$$s(i, S \cup \{j\}) - s(i, S) \leq s(i, T \cup \{j\}) - s(i, T).$$

Proof: Let $(N, M, J, \sigma_0, p, \alpha)$ be a min-RP sequencing situation. Let $i \in S$ and $j \in N \setminus T$. Let $i \in S_r \subset T_\rho$, with S_r and T_ρ components within the induced partition of S and T, respectively. Moreover, let U_r, V_ρ be the components within the job partition associated to S_r and T_ρ, respectively. Let $i \in S_t^{(j)} \subset T_\tau^{(j)}$, with $S_t^{(j)}$ and $T_\tau^{(j)}$ components within the induced partition of $S \cup \{j\}$ and $T \cup \{j\}$, respectively. Besides, let $U_t^{(j)}$, $V_\tau^{(j)}$ be the components within the job partition associated to $S_t^{(j)}$ and $T_\tau^{(j)}$, respectively. Note that $S_r \subset S_t^{(j)} \subset T_\tau^{(j)}$, $S_r \subset T_\rho \subset T_\tau^{(j)}$, $U_r \subset U_t^{(j)} \subset V_\tau^{(j)}$ and $U_r \subset V_\rho \subset V_\tau^{(j)}$.

Define $\mathrm{fol}(U_r)$ to be the job that is processed in position $\max\limits_{e \in U_r}\{\sigma_0(e)\} + 1$. Note that $\mathrm{fol}(U_r)$ may not exist. We will distinguish three cases.

Case 1: $\mathrm{fol}(U_r)$ either belongs to a player in $N \setminus (T \cup \{j\})$ or $\mathrm{fol}(U_r)$ does not exist. Here, we face the following situation:

	f_i					$\mathrm{fol}(U_r)$	
	i	m	i	m	m		

where $m \in S$ and the grid job either belongs to a player in $N \setminus (T \cup \{j\})$ or represents the end of the queue. In this case, $J(i) \cap U_r = J(i) \cap V_\rho = J(i) \cap U_t^{(j)} = J(i) \cap V_\tau^j$ and the inequality holds.

Case 2: $\mathrm{fol}(U_r)$ belongs to j. In this case, we face the following situation:

	f_i					$\mathrm{fol}(U_r)$	
	i	m	i	m	m	j	

where $m \in S$. As a result, $J(i) \cap U_r = J(i) \cap V_\rho$ and the inequality boils down to $s(i, S \cup \{j\}) \leq s(i, T \cup \{j\})$, which holds by Lemma A.2.

Case 3: $\mathrm{fol}(U_r)$ belongs to a player in $T \setminus S$. In this case we face the following situation:

	f_i					$\mathrm{fol}(U_r)$	
	i	m	i	m	m	l	

where $m \in S$ and $l \in T \setminus S$. Here, $J(i) \cap U_r = J(i) \cap U_t^{(j)}$ and the inequality comes down to $0 \leq s(i, T \cup \{j\}) - s(i, T)$, which is true by Lemma A.2. □

Lemma A.5. Let $S \subset T \subset N$. For every $k \in N \setminus T$ and every $i, j \in S$ with $i < j$,

$$s(i, j, S \cup \{k\}) - s(i, j, S) \leq s(i, j, T \cup \{k\}) - s(i, j, T).$$

PROOF: Let $(N, M, J, \sigma_0, p, \alpha)$ be a min-RP sequencing situation. Let $k \in N \setminus T$ and $i, j \in S$ with $i < j$. Let $i \in S_r \subset T_\rho$, with S_r and T_ρ components within the induced partition of S and T, respectively. Moreover, let U_r, V_ρ be the components within the job partition associated to S_r and T_ρ, respectively. Let $i \in S_t^{(k)} \subset T_\tau^{(k)}$, with $S_t^{(k)}$ and $T_\tau^{(k)}$ components within the induced partition of $S \cup \{k\}$ and $T \cup \{k\}$, respectively. Besides, let $U_t^{(k)}$, $V_\tau^{(k)}$ be the components within the job partition associated to $S_t^{(k)}$ and $T_\tau^{(k)}$, respectively. Note that $S_r \subset S_t^{(k)} \subset T_\tau^{(k)}$, $S_r \subset T_\rho \subset T_\tau^{(k)}$, $U_r \subset U_t^{(k)} \subset V_\tau^{(k)}$ and $U_r \subset V_\rho \subset V_\tau^{(k)}$.

In order to show the statement, we will study two cases.

Case 1: $j \in S_r$. Define $\text{fol}(U_r)$ to be the job that is processed in position $\max_{e \in U_r}\{\sigma_0(e)\} + 1$. Note that $\text{fol}(U_r)$ may not exist. We will distinguish three subcases.

Subcase 1.1: $\text{fol}(U_r)$ either belongs to a player in $N \setminus (T \cup \{k\})$ or $\text{fol}(U_r)$ does not exist. In this case we face the following situation:

	f_i			f_j				$\text{fol}(U_r)$	
	i	m	i	j	m	j	m		

where $m \in S$ and the grid job either belongs to a player in $N \setminus (T \cup \{k\})$ or represents the end of the queue. Here, $J(i) \cap U_r = J(i) \cap V_\rho = J(i) \cap U_t^{(k)} = J(i) \cap V_\tau^{(k)}$ and the inequality holds.

Subcase 1.2: $\text{fol}(U_r)$ belongs to k. In this case we face the following situation:

	f_i			f_j				$\text{fol}(U_r)$	
	i	m	i	j	m	j	m	k	

where $m \in S$. As a result, $J(i) \cap U_r = J(i) \cap V_\rho$ and the inequality boils down to $s(i, j, S \cup \{k\}) \leq s(i, j, T \cup \{k\})$, which is true by Lemma A.3.

Subcase 1.3: $\text{fol}(U_r)$ belongs to a player in $T \setminus S$. In this case we face the following situation:

	f_i			f_j				$\text{fol}(U_r)$	
	i	m	i	j	m	j	m	l	

where $m \in S$ and $l \in T \setminus S$. In this case, $J(i) \cap U_r = J(i) \cap U_t^{(k)}$ and the inequality comes down to $0 \leq s(i, j, T \cup \{k\}) - s(i, j, T)$, which holds by Lemma A.3.

Case 2: $j \notin S_r$, $j \in S_{r^*} \subset T_{\rho^*}$ ($j \in S_{t^*}^{(k)} \subset T_{\tau^*}^{(k)}$).
Define $\text{pred}(U_{r^*})$ to be the job that is processed in position $\min_{e \in U_{r^*}}\{\sigma_0(e)\} - 1$. Note that $\text{pred}(U_{r^*})$ may not exist. We will distinguish three subcases.

Subcase 2.1: $\text{pred}(U_{r^*})$ either belongs to a player in $N \setminus (T \cup \{k\})$ or $\text{pred}(U_{r^*})$ does not exist. In this case we face the following situation:

	$\text{pred}(U_{r^*})$		f_j				
		m	j	m	j	m	

where $m \in S$ and the grid job either belongs to a player in $N \setminus (T \cup \{k\})$ or represents the beginning of the queue. Here, $\{e \in J(i) \cap U_{r^*} : e < f_j\} = \{e \in J(i) \cap V_{\rho^*} : e < f_j\} = \{e \in J(i) \cap U^{(k)}_{t^*} : e < f_j\} = \{e \in J(i) \cap V^{(k)}_{\tau^*} : e < f_j\}$ and the inequality holds.

Subcase 2.2: pred(U_{r^*}) belongs to k. In this case we face the following situation:

	pred(U_{r^*})		f_j				
	k	m	j	m	j	m	

where $m \in S$. As a result, $\{e \in J(i) \cap U_{r^*} : e < f_j\} = \{e \in J(i) \cap V_{\rho^*} : e < f_j\}$ and the inequality boils down to $s(i, j, S \cup \{k\}) \leq s(i, j, T \cup \{k\})$, which is true by Lemma A.3.

Subcase 2.3: pred(U_{r^*}) belongs to a player in $T \setminus S$. In this case we face the following situation:

	pred(U_{r^*})		f_j				
	l	m	j	m	j	m	

where $m \in S$ and $l \in T \setminus S$. In this case, $\{e \in J(i) \cap U_{r^*} : e < f_j\} = \{e \in J(i) \cap U^{(k)}_{t^*} : e < f_j\}$ and the inequality comes down to $0 \leq s(i, j, T \cup \{k\}) - s(i, j, T)$, which holds by Lemma A.3. □

References

Bondareva, O. N. (1963), 'Some applications of linear programming methods to the theory of cooperative games', *Problemy Kibernitiki* **10**, 119–139 (in Russian).

Calleja, P., Estévez-Fernández, A., Borm, P. and Hamers, H. (2004), 'Job scheduling, cooperation and control', *CenTER Discussion Paper* . Tilburg University, Tilburg, The Netherlands.

Curiel, I., Hamers, H. and Klijn, F. (2002), Sequencing games: a survey, *in* P. Borm and H. Peters, eds, 'Chapters in Game Theory: in Honor of Stef Tijs', Kluwer Academic Publishers, Boston, pp. 27–50.

Curiel, I., Pederzoli, G. and Tijs, S. (1989), 'Sequencing games', *European Journal of Operational Research* **40**, 344–351.

Shapley, L. S. (1967), 'On balanced sets and cores', *Naval Research Logistics Quarterly* **14**, 453–460.

Shapley, L. S. (1971), 'Cores of convex games', *International Journal of Game Theory* **1**, 11–26.

Smith, W. (1956), 'Various optimizers for single-stage production', *Naval Research Logistics Quarterly* **3**, 59–66.

Sprumont, Y. (1990), 'Population monotonic allocation schemes for cooperative games with transferable utility', *Games and Economic Behavior* **2**, 378–394.