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Discussion paper

The constrained equal award rule for bankruptcy problems with a priori unions

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Abstract

In this paper, we provide four extensions of the constrained equal award rule for bankruptcy situations to the class of bankruptcy situations with a priori unions. We present some characterisations and relations with corresponding games. The four new extensions are illustrated by a specific application.

JEL code: C71
Key words: bankruptcy, a priori unions, constrained equal award

1 Introduction

In many situations in which agents interact, they do so in groups. Cooperative game theory studies such situations by taking into account what each particular coalition of players can achieve on its own. These values of the coalitions are subsequently taken into account in determining a fair division of the value of the grand coalition between all players.

Often, however, some coalitions play a special role, in that they arise in a natural way from the underlying situation. If these naturally arising groups form a partition of the grand coalition, they are usually referred to as a priori unions.

One interesting class of problems in which the role of a priori unions has been studied is the class of bankruptcy problems. In a bankruptcy problem, there is

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an estate to be divided among a number of claimants, whose total claim exceeds the estate available. In many situations, these claimants can be divided in a priori unions, based on the nature or cause of their claims. Eg, when a firm goes bankrupt, the creditors can usually be grouped in a natural way by distinguishing between claims on the basis of outstanding bonds, equity or commercial transactions.

The main focus of the bankruptcy literature is on finding rules assigning to each bankruptcy situation an allocation of the estate, which satisfies some appealing properties. This branch of cooperative game theory was initiated by O'Neill (1982) and has gained in popularity over the years.

One natural way to analyse the class of bankruptcy situations with a priori unions is to extend well-known standard bankruptcy rules to this class. Eg, Casas-Méndez et al. (2000) extend the adjusted proportional rule by considering a two-stage procedure in which the estate is first divided among the unions, and subsequently the amount that each union receives is divided among its members.

In this paper, we present some extensions of the constrained equal award (CEA) rule. The first extension involves a similar two-stage procedure as in Casas-Méndez et al. (2000). We relate this extension to the CEA solution of a corresponding TU game with a priori unions, which is inspired by Owen (1977). We provide two characterisations of this two-stage extension, based on previous results by Dagan (1996) and Herrero and Villar (2001).

The second extension uses the concept of multi-issue allocation situations as introduced in Calleja et al. (2001). Two further extensions of the CEA rule are based on the recursive completion rule introduced in O'Neill (1982), one of which is characterised by a consistency property.

We illustrate and compare our four extensions of the CEA rule by applying them to the bankruptcy case of the Pacific Gas and Electric Company.

The outline of the paper is as follows. In section 2, we formally define the class of bankruptcy situations with a priori unions and some related concepts that are used throughout the paper. In section 3, the problem of extending standard bankruptcy rules is addressed and the first two extensions are presented. In section 4, we provide the two characterisations of the two-stage extension of the CEA rule. Section 5 contains the last two extensions and deals with the concept of consistency. Finally, in section 6 we present the application.

2 Bankruptcy with a priori unions

A bankruptcy problem arises when there is an estate to be divided and this estate is not enough to satisfy all the claims on it. In this kind of problems the question is how to divide the available estate among all the claimants.

We model a *bankruptcy situation* by a triple (N, E, c), where $N = \{1, \ldots, n\}$ is the set of players/creditors, $E \in \mathbb{R}_+$ represents the estate (the available resources of the debtor) and $c = (c_1, \ldots, c_n) \in \mathbb{R}_+^N$ is the vector of claims of the creditors. We assume $\sum_{i \in N} c_i \geq E$, so the estate is insufficient to meet all the claims.

By B^N we denote the set of all bankruptcy problems with creditor set N. A bankruptcy rule is a function $f : B^N \to \mathbb{R}^N$ that allocates to every bankruptcy problem (N, E, c) a vector $f(N, E, c) \in \mathbb{R}^N$ such that for all $i \in N$, $0 \leq f_i(N, E, c) \leq c_i$ (f is reasonable) and $\sum_{i \in N} f_i(N, E, c) = E$ (f is efficient).

A *cooperative game with transferable utility* (or *TU game*) is a pair (N, v), where $N = \{1, \cdots, n\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subset N$ its worth $v(S)$. By convention, $v(\emptyset) = 0$. We denote the class of TU games with player set N by TU^N. A solution concept is a function $f : TU^N \to \mathbb{R}^N$ that assigns to every TU game $(N, v) \in TU^N$ an allocation $f(N, v) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(N, v) = v(N)$.

For every bankruptcy problem (N, E, c), O'Neill (1982) defines an associated *bankruptcy game* $(N, v_{E,c})$. In this game, the value of a coalition S is the part of the estate that remains after paying the creditors in $N \backslash S$ all their claims, that is, $v_{E,c}(S) = \max\{E - \sum_{i \in N \backslash S} c_i, 0\}$ for all $S \subset N$.

Curiel et al. (1987) study this class of games. They call a bankruptcy rule *game-theoretic* if the solution of a situation only depends on the game. So, for a game-theoretic $f : B^N \to \mathbb{R}^N$, we can find a function $F : TU^N \to \mathbb{R}^N$ such that $f(N, E, c) = F(N, v_{E,c})$ for all bankruptcy problems $(N, E, c) \in B^N$. Note that the corresponding F is only uniquely determined on the class of bankruptcy games. In this paper, we only consider game-theoretic bankruptcy rules.

We represent a *bankruptcy problem with a priori unions* by $(N, E, c, \mathcal{P})$ where (N, E, c) is a standard bankruptcy problem and $\mathcal{P} = \{P_k\}_{k \in R}$ is a partition of the set of players. We denote by BU^N the set of all bankruptcy problems with a priori unions and player set N.

Our aim is to define bankruptcy with a priori unions rules, that is, functions $\varphi : BU^N \to \mathbb{R}^N$ that assign to each bankruptcy problem with a priori unions $(N, E, c, \mathcal{P})$

a vector $\varphi(N, E, c, \mathcal{P}) \in \mathbb{R}^N$ such that for all $i \in N$, $0 \leq \varphi_i(N, E, c, \mathcal{P}) \leq c_i$ and $\sum_{i \in N} \varphi_i(N, E, c, \mathcal{P}) = E$.

If $(N, E, c, \mathcal{P}) \in BU^N$ is a bankruptcy problem with unions, we can define the corresponding bankruptcy problem among the unions $(R, E, c^{\mathcal{P}})$, the so-called *quotient problem*, where $c^{\mathcal{P}} = (c_k^{\mathcal{P}})_{k \in R}$ is the vector of total claims of the unions, so $c_k^{\mathcal{P}} = \sum_{i \in P_k} c_i$ for each union P_k of creditors. Note that $(R, E, c^{\mathcal{P}})$ is a well defined bankruptcy problem.

Next, we introduce disjoint issue allocation situations. These are a special case of multi-issue allocation situations as introduced in Calleja et al. (2001). The basic idea behind this class of problems is that the agents do not simply have just a single claim on the estate, as in the standard bankruptcy model, but a number of claims, each of which results from a particular *issue*. The basic assumption is that these issues are dealt with in turn: as soon as money is distributed according to one particular issue, this issue must first be completed before the next one is considered.

A *multi-issue allocation situation* is a triple (N, E, C), where $N = \{1, \ldots, n\}$ is the set of players, $E \in \mathbb{R}_+$ is the estate and $C \in \mathbb{R}_+^{R \times N}$ is the matrix of claims. Every row in C represents an issue and the set of issues is denoted by $R = \{1, \ldots, r\}$. An element $c_{ki} \geq 0$ represents the amount that player $i \in N$ claims according to issue $k \in R$. If a player is not involved in a particular issue, his claim corresponding to that issue equals zero.

The claim matrix C is assumed to satisfy the following properties:

- Every issue gives rise to a claim: $\sum_{i \in N} c_{ki} > 0$ for all $k \in R$.
- Every player is involved in at least one issue: $\sum_{k \in R} c_{ki} > 0$ for all $i \in N$.
- The allocation problem is nontrivial: $\sum_{k \in R} \sum_{i \in N} c_{ki} \geq E$.

A *disjoint issue allocation situation* is a multi-issue allocation situation in which every player is involved in exactly one issue.

Every bankruptcy situation with a priori unions $(N, E, c, \mathcal{P})$ gives rise to a disjoint issue allocation situation $(N, E, C^{c,\mathcal{P}})$, where the issues correspond to the unions and the matrix $C^{c,\mathcal{P}} = (C_{kj}^{c,\mathcal{P}})_{k \in R, j \in N}$ is defined by

$$C_{kj}^{c,\mathcal{P}} = \begin{cases} c_j & \text{if } j \in P_k \\ 0 & \text{if } j \notin P_k \end{cases}$$

for all $k \in R, j \in N$.

For ease of notation, we define $c_k = \sum_{i \in N} c_{ki}$ to be the total of claims according to issue $k \in R$. Similarly, we define $c_{kS} = \sum_{i \in S} c_{ki}$ for every coalition $S \subset N$. An *ordering* of the players in N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(i)$ denotes which player in N is at position i. The set of all $n!$ permutations of N is denoted by $\Pi(N)$. Similarly, the set of permutations of the set of issues R is denoted by $\Pi(R)$.

In order to analyse disjoint issue allocation situations, we define a corresponding game. As explained before, a bankruptcy game is defined from a pessimistic point of view: in order to compute $v(S)$, it is assumed that the claims of the players in $N \backslash S$ are satisfied first. In the case of disjoint issue allocation situations, this can be established by assuming that the issues are handled in that order which gives S the lowest payoff. But for this, we first need to specify how the claims *within* each issue are dealt with.

If a certain amount of money $E' \leq c_k$ is given to an issue k, then the resulting allocation problem within issue k is a standard bankruptcy problem. So, let f be a bankruptcy rule. Then the f-game, v^f, corresponding to the disjoint issue allocation game (N, E, C) is defined as follows.[1] Let $\tau \in \Pi(R)$ be an order on the issues. Now the players in S first address the first t issues completely, where $t = \max\{t' \mid \sum_{s=1}^{t'} c_{\tau(s)} \leq E\}$. The part of the estate that is left, $E' = E - \sum_{s=1}^{t} c_{\tau(s)}$, is divided by applying f to the claims in issue $\tau(t+1)$. So in total, the players in S receive

$$g_S^f(\tau) = \sum_{s=1}^{t} c_{\tau(s),S} + \sum_{j \in S} f_j(N, E', (c_{\tau(t+1),i})_{i \in N}).$$

The value of coalition $S \subset N$ is the amount of money they get when the worst order on the issues is chosen:

$$v^f(S) = \min_{\tau \in \Pi(R)} g_S^f(\tau).$$

The core of a game (N, v) is defined by

$$C(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \forall S \subset N : \sum_{i \in S} x_i \geq v(S)\}.$$

A game (N, v) is called *exact* if for each $S \subset N, S \neq \emptyset$, there exists an $x^S \in C(v)$ such that $\sum_{i \in S} x_i^S = v(S)$. A game is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subset N$.

[1] Because the underlying situation is usually clear from the context, we denote the game by v^f rather than $v_{E,C}^f$.

Proposition 2.1 *Let (N, E, C) be a disjoint issue allocation situation and let f be a bankruptcy rule. Then (N, v^f) is exact.*

Proof: Let $S \subset N$ and let $\tau^\circ \in \Pi(R)$ be such that $g_S^f(\tau^\circ)$ is minimal. Define $x = (g_i^f(\tau^\circ))_{i\in N}$. Then $\sum_{i\in N} x_i = E = v^f(N)$ and $\sum_{i\in T} x_i = g_T^f(\tau^\circ) \geq \min_{\tau\in\Pi(R)} g_T^f(\tau) = v^f(T)$ for every coalition $T \subset N$. So, $x \in C(v^f)$. Furthermore, $\sum_{i\in S} x_i = g_S^f(\tau^\circ) = v^f(S)$. Hence, (N, v^f) is exact. □

It follows immediately from the proof of Theorem 3.3 in Calleja et al. (2001) that given a rule f, for every nonnegative exact game v, one can find a multi-issue allocation situation such that the corresponding game v^f coincides with v. However, the class of games corresponding to disjoint issue allocation situations is a strict subclass.

As stated before, every bankruptcy situation with a priori gives rise to a disjoint issue allocation situation in a natural way. This is illustrated in Example 2.1, in which the constrained equal award rule is used. For $(N, E, c) \in B^N$, the *constrained equal award rule* is defined for all $i \in N$ by $CEA_i(N, E, c) = \min\{\lambda, c_i\}$, where λ is such that $\sum_{i\in N} \min\{\lambda, c_i\} = E$. This rule awards the same amount to all claimants with the restriction that no player can get more than his claim. The constrained equal award is used by different authors, among others Dagan (1996) and Herrero and Villar (2001), who provide different axiomatic characterisations.

Example 2.1 Consider the 4-creditor bankruptcy problem (N, E, c) with $E = 10$ and $c = (6, 2, 8, 5)$. Suppose that creditors 1 and 2 form a union and creditors 3 and 4 another one, that is, $\mathcal{P} = \{\{1, 2\}, \{3, 4\}\}$.

This situation gives rise to the 4-player disjoint issue allocation problem $(N, E, C^{c,\mathcal{P}})$ with $E = 10$ and the following claim matrix:

$$C^{c,\mathcal{P}} = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 0 & 0 & 8 & 5 \end{bmatrix}.$$

Take $S = \{1, 3\}$. In order to determine $v^{CEA}(S)$, we first compute $g_S^{CEA}(\tau)$ for both possible $\tau \in \Pi(R)$:

τ	$g_S^{CEA}(\tau)$
$1, 2$	$6 + CEA_3(\{3, 4\}, 2, (8, 5)) = 7$
$2, 1$	$CEA_3(\{3, 4\}, 10, (8, 5)) = 5$

So, $v^{CEA}(S) = \min_{\tau\in\Pi(R)} g_S^{CEA}(\tau) = 5$. Similarly, taking $T = \{1, 4\}$, we obtain $v^{CEA}(T) = 5$, $v^{CEA}(S \cup T) = 8$ and $v^{CEA}(S \cap T) = 0$. Hence, $v^{CEA}(S) + v^{CEA}(T) >$

$v^{CEA}(S \cup T) + v^{CEA}(S \cap T)$. So, although v^{CEA} is exact (cf. Proposition 2.1), it is not convex. $\triangleleft$

If $\mathcal{P}$ is the discrete partition $\mathcal{P}^n = \{\{1\}, \ldots, \{n\}\}$, then the game v^f equals the bankruptcy game $v_{E,c}$ for all f. If $\mathcal{P}$ is the trivial partition $\mathcal{P}^N = \{N\}$, then for all f the game v^f is additive with $v^f(\{i\}) = f_i(N, E, c)$ for all $i \in N$.

3 Extending bankruptcy rules

In this section, we consider various ways to extend a bankruptcy rule to a rule for bankruptcy situations with a priori unions. We use the CEA rule to illustrate these extensions.

Before extending bankruptcy rules, we first present the relationship between a game-theoretic bankruptcy rule and its corresponding solution for TU games. Recall that a rule $f : B^N \to \mathbb{R}^N$ is called game-theoretic if there exists a function $F : TU^N \to \mathbb{R}^N$ such that $f(N, E, c) = F(N, v_{E,c})$ for all $(N, E, c) \in B^N$. To construct this F, we use the utopia vector. The *utopia vector* of a game (N, v), $M(v)$, is defined by

$$M_i(v) = v(N) - v(N \backslash \{i\})$$

for all $i \in N$. For a bankruptcy game $(N, v_{E,c})$, the utopia point $M_i(v_{E,c})$ equals $c_i^E = \min\{c_i, E\}$.

It readily follows that if $f(N, E, c) = f(N, E, c^E)$ for all $(N, E, c) \in B^N$, ie, if the solution is not changed by truncating claims that are larger than the estate, then

$$F(N, v_{E,c}) = f(N, v_{E,c}(N), M(v_{E,c}))$$

does the trick and f is game-theoretic. In fact, Curiel et al. (1987) show that this truncation property is not only sufficient, but also necessary for f to be game-theoretic. It is easily seen that the CEA rule is a game-theoretic rule.

If we want to divide the total estate among the creditors, one approach is to divide the estate among the unions first and second to divide the allocation of each union among the creditors of this union. Let $f : B^N \to \mathbb{R}^N$. We define the two-stage extension $\bar{f} : BU^N \to \mathbb{R}^N$ as follows. Let $(N, E, c, \mathcal{P}) \in BU^N$ be a bankruptcy problem with a priori unions. First, define $E_k^f = f_k(R, E, c^{\mathcal{P}})$ for all $k \in R$ and second, for $i \in P_k$, $\bar{f}_i(N, E, c, \mathcal{P}) = f_i(P_k, E_k^f, (c_j)_{j \in P_k})$.

The following example illustrates the two-stage constrained equal award rule with unions.

Example 3.1 Consider the bankruptcy problem with a priori unions as described in Example 2.1. To obtain the constrained equal award of the bankruptcy problem with unions $(N, E, c, \mathcal{P})$, we first consider the quotient bankruptcy problem $(R, E, c^{\mathcal{P}})$ involving the unions. In this problem, union $\{1, 2\}$ has a claim of 8 and union $\{3, 4\}$ has a claim of 13. Then, each union obtains half the estate: $E_1 = E_2 = 5$. If we divide the allocation of each union among its creditors, we obtain $\overline{CEA}(N, E, c, \mathcal{P}) = (3, 2, 5/2, 5/2)$. The constrained equal award of the bankruptcy problem without a priori unions is $CEA(N, E, c) = (8/3, 2, 8/3, 8/3)$. $\triangleleft$

The $\overline{CEA}$ rule for bankruptcy situations with a priori unions generalises the standard CEA rule for bankruptcy situations, in the sense that both $\overline{CEA}(N, E, c, \mathcal{P}^N)$ and $\overline{CEA}(N, E, c, \mathcal{P}^n)$ coincide with $CEA(N, E, c)$. Also note that by construction, $\overline{CEA}_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k$ for all $k \in R$.

The $\overline{CEA}$ solution of a bankruptcy situation with a priori unions coincides with the CEA solution for a corresponding TU game with a priori unions, which we are going to define next.

A cooperative game with transferable utility with a priori unions is a triple $(N, v, \mathcal{P})$ where (N, v) is a standard TU game and $\mathcal{P} = \{P_k\}_{k \in R}$ is a partition of the set of players, R being the set of unions. For $(N, v, \mathcal{P})$, we define the corresponding TU game among the unions $(R, v^{\mathcal{P}})$, the quotient game, where $v^{\mathcal{P}}(L) = v(\cup_{k \in L} P_k)$ for all $L \subset R$.

Let $(N, v, \mathcal{P})$ be a TU-game with a priori unions. The constrained equal award solution of this game, $CEA(N, v, \mathcal{P})$ is defined in two steps.[2] First, the payoff to each union $P_k \in \mathcal{P}$ equals $CEA(R, v^{\mathcal{P}})$, ie, the constrained equal award solution of the quotient game.

In the second step, the payoff to each union is divided among its players. To do this, we consider for every player $i \in N$ his cooperation possibilities with the players outside i's union. We should note that a similar idea is used in Owen (1977), where a modification of the Shapley value for TU games with a priori unions is defined.

[2]The CEA rule for TU games with a priori unions is only well-defined for a subclass of such games. If the underlying game is exact, then the CEA rule is well-defined.

Let $P_k \in \mathcal{P}$ and let $i \in P_k$. The "claim" of player i is defined as his contribution to the coalition $\cup_{\ell \in R\setminus\{k\}} P_\ell \cup \{i\}$, that is, $M_i(v,\mathcal{P}) = v(\cup_{\ell \in R\setminus\{k\}} P_\ell \cup \{i\}) - v(\cup_{\ell \in R\setminus\{k\}} P_\ell)$. The constrained equal award solution of the game $(N,v,\mathcal{P})$ for player $i \in P_k$ is then defined by

$$CEA_i(N,v,\mathcal{P}) = CEA_i(P_k, CEA_k(R,v^{\mathcal{P}}), (M_j(v,\mathcal{P}))_{j\in P_k}).$$

In general, $\overline{CEA}(N,E,c,\mathcal{P}) \neq CEA(N,v_{E,c},\mathcal{P})$. However, $\overline{CEA}$ does coincide with the CEA of the game $(N,v^{CEA},\mathcal{P})$, as is shown in the following proposition.

Proposition 3.1 *For every bankruptcy problem with a priori unions $(N,E,c,\mathcal{P})$ we have that $\overline{CEA}(N,E,c,\mathcal{P}) = CEA(N,v^{CEA},\mathcal{P})$.*

Proof: Let $(N,E,c,\mathcal{P})$ be a bankruptcy problem with a priori unions. First, it is easy to see that

$$v^{CEA}(\cup_{k\in L} P_k) = \max\{E - \sum_{i\in N\setminus\cup_{k\in L}P_k} c_i, 0\}$$

for all $L \subset R$ and hence, the games $(R,(v^{CEA})^{\mathcal{P}})$ and $(R,v_{E,c^{\mathcal{P}}})$ coincide. So,

$$CEA_k(R,(v^{CEA})^{\mathcal{P}}) = CEA_k(R,v_{E,c^{\mathcal{P}}}) = E_k^{CEA}$$

for all $k \in R$.
Next, for $i \in P_k$,

$$M_i(v^{CEA},\mathcal{P}) = \begin{cases} CEA_i(P_k,E,(c_j)_{j\in P_k}) & \text{if } E \leq c_k^{\mathcal{P}}, \\ c_i & \text{if } E > c_k^{\mathcal{P}}. \end{cases}$$

From this, one can easily see that

$$CEA_i(P_k,E_k^{CEA},(c_j)_{j\in P_k}) = CEA_i(P_k,CEA_k(R,v^{\mathcal{P}}),(M_j(v,\mathcal{P}))_{j\in P_k})$$

for all $i \in P_k$. □

Instead of a two-stage procedure, bankruptcy rules can also be extended by means of the bankruptcy cover. For this, we define TU_+^N to be the class of games in which the utopia vector is nonnegative and adds up to at least the value of the grand coalition, ie,

$$TU_+^N = \{v \in TU^N \mid M(v) \geq 0, \sum_{i\in N} M_i(v) \geq v(N) \geq 0\}.$$

For all $v \in TU_+^N$, we define the corresponding *bankruptcy cover* game $(N,\hat{v})$ by

$$\hat{v}(S) = \max\{v(N) - \sum_{i \in N \setminus S} M_i(v), 0\}$$

for all $S \subset N$. The term cover results from the following lemma.

Lemma 3.2 *For all $v \in TU^N_+$, $\hat{\hat{v}} = \hat{v}$.*

The bankruptcy cover $\hat{v}$ is the bankruptcy game with estate $v(N)$ and claim vector $M(v)$. Indeed, $\hat{v}$ equals v if and only if v is a bankruptcy game itself.

Lemma 3.3 *Let $v \in TU^N_+$. Then $\hat{v} = v$ if and only if v is a bankruptcy game.*

We extend a bankruptcy rule f to a rule $\hat{f}$ on the class of all games in TU^N_+ by defining

$$\hat{f}(N, v) = F(N, \hat{v}).$$

Note that on the subclass of bankruptcy games, $\hat{f}$ coincides with F by definition.

Using the concept of disjoint issue allocation situations, we now extend f to a rule $\tilde{f}$ on the class of bankruptcy situations with a priori unions in the following way:

$$\tilde{f}(N, E, c, \mathcal{P}) = \hat{f}(N, v^f)$$

for all $(N, E, c, \mathcal{P}) \in BU^N$. The right hand side is well defined, because the corresponding disjoint issue allocation game v^f belongs to TU^N_+.

Example 3.2 Consider again the bankruptcy situation with a priori unions as described in Example 2.1. To compute $\widetilde{CEA}$, we need the following values:

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$v^{CEA}(S)$	5	5	8	4	10

Hence, $M(v^{CEA}) = (6, 2, 5, 5)$ and $\widetilde{CEA}(N, E, c, \mathcal{P}) = (8/3, 2, 8/3, 8/3)$. ◁

Note that in the previous example, $\widetilde{CEA}(N, E, c, \mathcal{P})$ coincides with $CEA(N, E, c)$. This holds for every bankruptcy situation with a priori unions, as is shown in the following proposition.

Proposition 3.4 *Let $(N, E, c, \mathcal{P})$ be a bankruptcy situation with a priori unions. Then $\widetilde{CEA}(N, E, c, \mathcal{P}) = CEA(N, E, c)$.*

Proof: Let $k \in R$ and let $i \in P_k$. Then

$$v^{CEA}(N\backslash\{i\}) = \begin{cases} E - c_i & \text{if } c_k \leq E, \\ E - CEA_i(P_k, E, (c_j)_{j\in P_k}) & \text{if } c_k > E. \end{cases}$$

Since $v^{CEA}(N) = E$, we have

$$M_i(v^{CEA}) = \begin{cases} c_i & \text{if } c_k \leq E, \\ CEA_i(P_k, E, (c_j)_{j\in P_k}) & \text{if } c_k > E. \end{cases} \tag{3.1}$$

Next,

$$\widetilde{CEA}(N, E, c, \mathcal{P}) = \widehat{CEA}(v^{CEA}) = CEA(N, v^{CEA}(N), M(v^{CEA})) \\ = CEA(N, E, M(v^{CEA}))$$

and since the truncation in (3.1) has no effect on the outcome, we have $\widetilde{CEA}(N, E, c, \mathcal{P}) = CEA(N, E, c)$. □

Note that for an arbitrary bankruptcy rule f, $\tilde{f}(N, E, c, \mathcal{P}) = f(N, E, c)$ does not hold in general.

4 Characterisations of the two-step constrained equal awards rule

In this section, we provide two characterisations of the $\overline{CEA}$ rule as defined in section 3. Consider the following properties for a rule $\varphi : BU^N \to \mathbb{R}^N$.

Composition (COMP): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$, $\varphi(N, E, c, \mathcal{P}) = \varphi(N, E', c, \mathcal{P}) + \varphi(N, E - E', c - \varphi(N, E', c, \mathcal{P}), \mathcal{P})$ for all $0 \leq E' \leq E$.

This property considers the situation in which after the estate (E') has been divided among the agents, this estate is reevaluated and turns out to be a bigger amount (E). In these cases, we have two options. We can cancel the initial division and apply the rule to the new problem, or we can preserve the initial division and apply the rule to the increment of the estate by considering a new vector of claims, taking into account the quantities already received. The composition property says that both options should lead to the same result.

Path independence (PI): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$, $\varphi(N, E, c, \mathcal{P}) = \varphi(N, E, \varphi(N, E', c, \mathcal{P}), \mathcal{P})$ for all $E' \geq E$.

Here, the opposite situation is considered, one where the estate (E) is actually smaller than the one initially considered (E'). Then, we can apply the rule to the new problem or divide the new value by taking the initial divisions as claim vector. Path independence states that both ways of proceeding should result in the same vector of allocations.

Equal treatment within the unions (ET): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$ and for each two agents i, j of a union $P_k \in \mathcal{P}$ such that $c_i = c_j$, $\varphi_i(N, E, c, \mathcal{P}) = \varphi_j(N, E, c, \mathcal{P})$.

This property requires that agents of the same union with equal claims obtain equal payoffs.

Quotient problem property (QPP): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$ and for each union $P_k \in \mathcal{P}$, $\sum_{i \in P_k} \varphi_i(N, E, c, \mathcal{P}) = \varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$.

In a bankruptcy problem with unions we can consider the associated quotient problem where the unions negotiate about the division of the estate. After this, a negotiation within every union takes place. The quotient problem property states that the total gains of the agents of a union in the initial problem equal the gains of this union in the quotient problem. Note that if φ is the two-step extension $\bar{f}$ of a bankruptcy rule f, then $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^f$. (Recall that $E_k^f = f_k(R, E, c^{\mathcal{P}})$ is the amount that union $k \in R$ gets in the quotient problem according to f.)

Stability within the unions (STA): For each bankruptcy problem with unions $(N, E, c, \mathcal{P})$, for each union $P_k \in \mathcal{P}$ and for each agent $i \in P_k$,

$$\varphi_i(N, E, c, \mathcal{P}) = \varphi_i(P_k, \sum_{i \in P_k} \varphi_i(N, E, c, \mathcal{P}), (c_j)_{j \in P_k}, \mathcal{P}^{P_k}),$$

where $\mathcal{P}^{P_k} = \{P_k\}$ is the trivial partition of P_k into one union.

Consider a bankruptcy problem with a priori unions $(N, E, c, \mathcal{P})$ and its solution $\varphi(N, E, c, \mathcal{P})$. The agents of a union P_k can renegotiate their awards,

$\sum_{i\in P_k}\varphi_i(N,E,c,\mathcal{P})$, using the initial claims. Then a new bankruptcy problem appears with a boundary structure of unions, $(P_k,\sum_{i\in P_k}\varphi_i(N,E,c,\mathcal{P}),(c_j)_{j\in P_k},\mathcal{P}^{P_k})$. We say that φ is stable within the unions if the agents in P_k obtain the same quantity in the initial problem as in the new problem. It is easy to see that the adjusted proportional rule for bankruptcy problems with a priori unions as introduced in Casas-Méndez et al. (2000) satisfies the quotient problem property, but it does not satisfy stability within the unions.

Invariance under claims truncation within the unions (ICT): For each bankruptcy problem with unions $(N,E,c,\mathcal{P})$ and for every player i of a union $P_k\in\mathcal{P}$ such that $c_i>\sum_{j\in P_k}\varphi_j(N,E,c,\mathcal{P})$, we have $\varphi(N,E,c,\mathcal{P})=\varphi(N,E,c',\mathcal{P})$, where $c'_j=c_j$ for all $j\in N\setminus\{i\}$ and $c'_i=\sum_{j\in P_k}\varphi_j(N,E,c,\mathcal{P})$.

Suppose that the claim of an agent is greater than the total quantity that his union gets. Then ICT states that the awards of the agents are not affected if we replace the claim of this agent by the total payoff of his union.

Sustainability of creditors within the unions (SUS): For each bankruptcy problem with unions $(N,E,c,\mathcal{P})$ and for every player i who is *sustainable* within his union $P_k\in\mathcal{P}$, ie, $\sum_{j\in P_k}\min\{c_i,c_j\}\leq\varphi_k(R,E,c^{\mathcal{P}},\mathcal{P}^R)$, we have $\varphi_i(N,E,c,\mathcal{P})=c_i$.

This property establishes a protective criterion within the unions in the sense that small claims should be completely satisfied. The claim of agent i is considered sustainable within his union if the worth of this union in the quotient problem is enough to pay each agent in this union his claim, truncated by the claim of agent i.

Composition and path independence are in essence identical to the corresponding properties for bankruptcy rules. Equal treatment within the unions is a weak version of equal treatment of bankruptcy rules. Stability within the unions, invariance under claims truncation within the unions and sustainability of creditors within the unions are natural extensions of other properties for bankruptcy rules to this context of a priori unions. Note that the quotient problem property implies that the rule involves some two-step procedure to obtain the solution.

Proposition 4.1 *The $\overline{CEA}$-rule satisfies composition, path independence, equal*

treatment within the unions, stability within the unions, invariance under claims truncation within the unions, sustainability of creditors within the unions and the quotient problem property.

Proof: Equal treatment within the unions, stability within the unions, sustainability of creditors within the unions and the quotient problem property are straightforward to show. We only prove composition. The proof of path independence and invariance under claims truncation within the unions follows similar lines.
Let $P_k \in \mathcal{P}$ and let $i \in P_k$. By definition of $\overline{CEA}$ we have that

$$\overline{CEA}_i(N, E, c, \mathcal{P}) = CEA_i(P_k, E_k^{CEA}, (c_j)_{j \in P_k}).$$

Consider now $0 \leq E' \leq E$. Then

$$\overline{CEA}_i(N, E', c, \mathcal{P}) = CEA_i(P_k, E_k^{CEA'}, (c_j)_{j \in P_k}),$$

with $E_k^{CEA'} = CEA_k(R, E', c^{\mathcal{P}})$. Define $c' = c - \overline{CEA}(N, E', c, \mathcal{P})$. Then we have

$$\overline{CEA}_i(N, E - E', c', \mathcal{P}) =$$
$$CEA_i(P_k, CEA_k(R, E - E', (c')^{\mathcal{P}}), (c'_j)_{j \in P_k}).$$

Because the constrained equal award rule for bankruptcy problems satisfies composition (Dagan (1996)), we have that

$$\begin{aligned} E_k^{CEA} - E_k^{CEA'} &= CEA_k(R, E, c^{\mathcal{P}}) - CEA_k(R, E', c^{\mathcal{P}}) \\ &= CEA_k(R, E - E', c^{\mathcal{P}} - CEA(R, E', c^{\mathcal{P}})) \\ &= CEA_k(R, E - E', (c')^{\mathcal{P}}). \end{aligned}$$

From the previous, it follows that

$$\begin{aligned} &\overline{CEA}_i(N, E, c, \mathcal{P}) = CEA_i(P_k, E_k^{CEA}, (c_j)_{j \in P_k}) \\ &= CEA_i(P_k, E_k^{CEA'}, (c_j)_{j \in P_k}) + CEA_i(P_k, E_k^{CEA} - E_k^{CEA'}, (c'_j)_{j \in P_k}) \\ &= \overline{CEA}_i(N, E', c, \mathcal{P}) + CEA_i(P_k, CEA_k(R, E - E', (c')^{\mathcal{P}}), (c'_j)_{j \in P_k}) \\ &= \overline{CEA}_i(N, E', c, \mathcal{P}) + \overline{CEA}_i(N, E - E', c', \mathcal{P}). \end{aligned}$$

Hence, we have that $\overline{CEA}$ satisfies composition. □

In the following theorem we axiomatically characterise the $\overline{CEA}$ rule. This theorem is inspired by a similar result for the CEA rule for bankruptcy games in Dagan (1996).

Theorem 4.2 *The $\overline{CEA}$ rule is the unique rule for bankruptcy problems with a priori unions that satisfies equal treatment within the unions, composition, the quotient problem property and invariance under claims truncation within the unions.*

Proof: In view of Proposition 4.1 we only need to prove that the rule $\overline{CEA}$ is the only rule satisfying the four properties.
Let φ be a rule for BU^N satisfying ET, QPP, COMP and ICT. Let $(N, E, c, \mathcal{P})$ be a bankruptcy problem with unions and consider the quotient problem $(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$. Without loss of generality, suppose that $0 \leq c_1^{\mathcal{P}} \leq \ldots \leq c_r^{\mathcal{P}}$. In Proposition 1 of Dagan (1996) it is established that the constrained equal award rule is the only rule for bankruptcy problems that satisfies the bankruptcy equivalents of ET, COMP and ICT. Since the quotient problem with $\mathcal{P}^R$ is basically a bankruptcy problem, it follows that $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^{CEA}$ for all $k \in R$.
Now, we consider the first union $P_1 \in \mathcal{P}$. Suppose without loss of generality that $P_1 = \{1, \ldots, n_1\}$ and that $c_{11} \leq \ldots \leq c_{1n_1}$.

Step 1. If $0 \leq E \leq rc_{11}$, then $E_1^{CEA} \leq c_{11}$ and because of ICT, QPP and ET, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$.

If $rc_{11} < E \leq rc_{11} + rc_{11}(1 - \frac{1}{n_1})$, then equality is established using COMP. Repeating the same construction, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11}$.

Step 2. If $rn_1c_{11} < E \leq rn_1c_{11} + r(c_{12} - c_{11})$, by COMP and Step 1 we have $\varphi(N, E, c, \mathcal{P}) = x + \varphi(N, E - rn_1c_{11}, c - x, \mathcal{P})$, where $x_i = \varphi_i(N, rn_1c_{11}, c, \mathcal{P}) = \overline{CEA}_i(N, rn_1c_{11}, c, \mathcal{P}) = c_{11}$ for all $i \in P_1$. Furthermore, $E - rn_1c_{11} \leq r(c_{12} - c_{11})$. So because of ICT and ET we have $\varphi_i(N, E - rn_1c_{11}, c - x, \mathcal{P}) = \overline{CEA}_i(N, E - rn_1c_{11},\ c - x, \mathcal{P})$ for all $i \in P_1$ and hence, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$.

Repeating the same argument one can prove that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11} + r(n_1 - 1)(c_{12} - c_{11})$.

Using the same arguments, we obtain that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_1$ if $0 \leq E \leq rn_1c_{11} + r(n_1 - 1)(c_{12} - c_{11}) + \ldots + r(c_{1n_1} - c_{1,n_1-1}) = r(c_{11} + c_{12} + \ldots + c_{1n_1}) = rc_1^{\mathcal{P}}$.

Now, we consider the second union. We distinguish between two cases. If $E \leq rc_1^{\mathcal{P}}$, we can use the same arguments as in the first union to obtain $\varphi_i(N, E, c, \mathcal{P}) =$

$\overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_2$.
So, suppose that $E > rc_1^{\mathcal{P}}$. Because φ satisfies COMP, we have that

$$\varphi(N, E, c, \mathcal{P}) = \varphi(N, rc_1^{\mathcal{P}}, c, \mathcal{P}) + \varphi(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P}),$$

where $x = \varphi(N, rc_1^{\mathcal{P}}, c, \mathcal{P})$. By the previous case, $\varphi_i(N, rc_1^{\mathcal{P}}, c, \mathcal{P}) = \overline{CEA}_i(N, rc_1^{\mathcal{P}}, c, \mathcal{P})$ for all $i \in P_2$. With the second term, $\varphi(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$, we proceed as with the first union with estate $E - rc_1^{\mathcal{P}}$ and claims $c - x$ and we obtain $\varphi_i(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P}) = \overline{CEA}_i(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$ for all $i \in P_2$. Note that in the problem $(N, E - rc_1^{\mathcal{P}}, c - x, \mathcal{P})$ all the members of P_1 obtain zero. Because $\overline{CEA}$ satisfies COMP, we have $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_2$.
Repeating the same arguments with all the unions, we conclude the statement. □

Our second characterisation is based on Herrero and Villar (2001). In order to give this result, we first present some lemmas.

Lemma 4.3 *If φ is a rule for bankruptcy problems with unions that satisfies path independence and sustainability of creditors within the unions then for every bankruptcy problem with unions $(N, E, c, \mathcal{P})$ we have that $\varphi_k(R, E, c^{\mathcal{P}}, \mathcal{P}^R) = E_k^{CEA}$ for all $k \in R$.*

Proof: Let $\varphi : BU^N \to \mathbb{R}^N$ be a rule satisfying PI and SUS and let $(N, E, c, \mathcal{P}) \in BU^N$. Consider the associated quotient problem $(R, E, c^{\mathcal{P}}, \mathcal{P}^R)$. Theorem 1 of Herrero and Villar (2001) states that the constrained equal award rule is the only rule for bankruptcy problems that satisfies the bankruptcy equivalents of path independence and sustainability. From this, the statement readily follows. □

Lemma 1 of Herrero and Villar (2001) states that if a bankruptcy rule satisfies path independence and sustainability, then it satisfies equal treatment of equals. In a similar way we can establish the next result for a rule for bankruptcy problems with a priori unions.

Lemma 4.4 *If a rule for bankruptcy problems with a priori unions satisfies the quotient problem property, path independence and sustainability within the unions then it satisfies equal treatment within the unions.*

Now we can give our second axiomatic characterisation of the $\overline{CEA}$ rule.

Theorem 4.5 *The $\overline{CEA}$ rule is the unique rule for bankruptcy problems with a priori unions that satisfies path independence, sustainability of creditors within the unions and the quotient problem property.*

Proof: In view of Proposition 4.1 we only need to prove that the rule $\overline{CEA}$ is the only one satisfying the three properties.
Let φ be a rule for BU^N satisfying QPP, PI and SUS and let $(N, E, c, \mathcal{P}) \in BU^N$. Let $P_k \in \mathcal{P}$. We have to show that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$. We use the following notation: $n_1^k = \max_{i \in P_k} c_i$, $N_1^k = \{i \in P_k \,|\, c_i = n_1^k\}$, $n_2^k = \max_{i \in P_k \setminus N_1^k} c_i$, $N_2^k = \{i \in P_k \,|\, c_i = n_2^k\}$.

Step 1. Suppose that, in the union P_k, the claims of the agents in $P_k \setminus N_1^k$ are sustainable. Then $\varphi_i(N, E, c, \mathcal{P}) = c_i$ for all $i \in P_k \setminus N_1^k$ because φ satisfies SUS. Now, we have that $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$ because φ satisfies ET (by Lemma 4.4) and by QPP and Lemma 4.3, $\sum_{i \in P_k} \varphi_i(N, E, c, \mathcal{P}) = E_k^{CEA}$.

Step 2. Suppose now that, in the union P_k, the claims of the agents in $P_k \setminus (N_1^k \cup N_2^k)$ are sustainable. Let $E' > E$ be such that $\varphi_k(R, E', c^{\mathcal{P}}, \mathcal{P}^R)$ is the minimum quantity that sustains the claims of $P_k \setminus N_1^k$ within union P_k, which is possible because of Lemma 4.3 and the basic properties of CEA. Let $c' = \varphi(N, E', c, \mathcal{P})$. By step 1, $c'_i = c_i$ for all $i \in P_k \setminus N_1^k$ and $c'_i = c'_j$ for all $i, j \in N_1^k \cup N_2^k$. Because φ and $\overline{CEA}$ satisfy PI, we have that $\varphi_i(N, E, c, \mathcal{P}) = \varphi_i(N, E, c', \mathcal{P})$ and $\overline{CEA}_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c', \mathcal{P})$ for all $i \in N$. By step 1, $\varphi_i(N, E, c', \mathcal{P}) = \overline{CEA}_i(N, E, c', \mathcal{P})$ for all $i \in P_k$ and hence, $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$.

Repeating this procedure, we obtain $\varphi_i(N, E, c, \mathcal{P}) = \overline{CEA}_i(N, E, c, \mathcal{P})$ for all $i \in P_k$. □

5 Consistency

In this section we define two further extensions of bankruptcy rules to bankruptcy situations with a priori unions. We also introduce a property of consistency that we subsequently use to characterise one of these extensions. We should mention that the rules and properties in this section are clearly inspired by concepts that

appear in O'Neill (1982). These concepts are the method of recursive completion, the random arrival rule and the property of consistency.

Our third extension of a bankruptcy rule f to bankruptcy situations with a priori unions is the recursive completion of f. Let f be a bankruptcy rule and let $(N, E, c, \mathcal{P})$ be a bankruptcy situation with a priori unions. Then we define the recursive completion of f, RC^f, in the following way:

$$RC_i^f(N, E, c, \mathcal{P}) = \frac{1}{r}\left[f_i(P_k, E', (c_j)_{j\in P_k}) + \sum_{\ell\in R, \ell\neq k} f_i(P_k, E_k^{f,\ell}, (c_j)_{j\in P_k})\right]$$

for all $i \in P_k$, where $E' = \min\{E, c_k^{\mathcal{P}}\}$, $E_k^{f,\ell} = f_k(R\backslash\{\ell\}, \max\{E - c_\ell^{\mathcal{P}}, 0\}, c_{-\ell}^{\mathcal{P}})$ and $c_{-\ell}^{\mathcal{P}} = (c_t^{\mathcal{P}})_{t\in R\backslash\{\ell\}}$.

The first term in the sum in the definition of the RC_i^f is the quantity that agent $i \in P_k$ obtains according f when union P_k receives its maximum. The second term is the amount that agent $i \in P_k$ obtains according f, when union $\ell \neq k$ gets its maximum and the remaining estate is divided among the other unions in the quotient problem according to f. Hence, we are computing the average amount that agent i obtains according to f over the r situations where one of the unions gets its maximum.

In the next example, we illustrate the recursive completion of the CEA-rule.

Example 5.1 We compute RC^{CEA} in the bankruptcy situation with a priori unions of Example 2.1. In the associated disjoint issue allocation problem, $E = 10$ and

$$C^{c,\mathcal{P}} = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 0 & 0 & 8 & 5 \end{bmatrix}.$$

We complete and modify the claim matrix by considering the two situations where one of the unions obtains its maximum:

$$\begin{bmatrix} 6 & 2 & 1 & 1 \\ 0 & 0 & 5 & 5 \end{bmatrix},$$

so if we average over the two unions, we obtain $RC^{CEA} = (3, 1, 3, 3)$. Note that $RC^{CEA}(N, E, c, \mathcal{P})$ differs from both $\overline{CEA}(N, E, c, \mathcal{P})$ and $\widetilde{CEA}(N, E, c, \mathcal{P})$ ◁

If $\mathcal{P} = \mathcal{P}^N$, then RC^f coincides with the f rule, that is, $RC^f(N, E, c, \mathcal{P}^N) = f(N, E, c)$ for every bankruptcy problem (N, E, c). If $\mathcal{P} = \mathcal{P}^n$, the formula for RC^f corresponds to the definition of O'Neill consistency for bankruptcy rules. Since the random arrival rule for bankruptcy situations (which we denote by RA) is the

only O'Neill consistent rule we have $RC^{RA}(N,E,c,\mathcal{P}^n) = RA(N,E,c)$ for every bankruptcy problem (N,E,c). For arbitrary f, $RC^f(N,E,c,\mathcal{P}^n) = f(N,E,c)$ does not hold in general.

Because of this special property, we think that RC^{RA} is an interesting rule for bankruptcy situations with a priori unions. Nevertheless, RC^{RA}, and in general the RC^f rules, can not be extended easily to rules for multi-issue allocation situations that are not disjoint. Recall from section 2 that a bankruptcy situation with a priori unions can be seen as a special case of a multi-issue situation in which the unions are the issues. If these issues are not disjoint, ie, if a player can have a claim according to more than one issue, the recursive completion procedure does not work. In order to solve this difficulty, we introduce a new extension of bankruptcy rules.

Let f be a bankruptcy rule and let $(N,E,c,\mathcal{P})$ be a bankruptcy problem with a priori unions. Then we define the f-random arrival rule in the following way:

$$RA_i^f(N,E,c,\mathcal{P}) = \frac{1}{r!}\left[\sum_{\sigma\in\Pi(R)} f_i(P_k, E_\sigma, (c_j)_{j\in P_k})\right]$$

for all $i \in P_k$, where $E_\sigma = \max\{0, E - \sum_{\ell\in R,\sigma(\ell)<\sigma(k)} c_\ell^{\mathcal{P}}\}$.

The interpretation of this rule is similar to that of other solutions inspired by ideas of random arrival. Here, we suppose that the claims of the different unions are satisfied following a fixed order. If at the moment to allocate money to a particular union, the remaining estate is not enough to satisfy its total claim, we use the rule f to distribute within this union. So, the f-random arrival rule allocates to an agent the average of the amounts he obtains according to the previous procedure over all the possible orders on the unions.

Note that if $\mathcal{P} = \mathcal{P}^n$ we have $RA^f(N,E,c,\mathcal{P}^n) = RA(N,E,c)$, that is, in this boundary case, RA^f coincides with the random arrival rule for bankruptcy problems for every bankruptcy rule f. If $\mathcal{P} = \mathcal{P}^N$, the f-random arrival rule coincides with the rule f. Note that in the case of two unions, both extensions of f, RC^f and RA^f, coincide.

Now, we define the property of consistency for bankruptcy with a priori unions rules. A bankruptcy with a priori unions rule φ is *consistent* if for every $(N,E,c,\mathcal{P})$, for each union $P_k \in \mathcal{P}$ and for each agent $i \in P_k$ we have

$$\varphi_i(N,E,c,\mathcal{P}) =$$
$$\tfrac{1}{r}\left[\varphi_i(P_k, E', (c_j)_{j\in P_k}, \mathcal{P}^{P_k}) + \textstyle\sum_{\ell\in R,\ell\neq k} \varphi_i(N\backslash P_\ell, E_{-\ell}, c_{-\ell}, \mathcal{P}_{-\ell})\right],$$

where $E' = \min\{E, c_k^{\mathcal{P}}\}$, $c_{-\ell} = (c_j)_{j \in N \setminus P_\ell}$, $E_{-\ell} = \max\{E - c_\ell^{\mathcal{P}}, 0\}$ and $\mathcal{P}_{-\ell}$ is the partition of the set $N \setminus P_\ell$ induced by $\mathcal{P}$.

So, a rule is consistent if in a bankruptcy problem with a priori unions it allocates to an agent the average of what he gets when the rule is applied to the problem restricted to his own union and the solutions of the $r-1$ bankruptcy situations in which the estate is the amount that remains when each of the other unions gets its maximum. Note that if $\mathcal{P} = \mathcal{P}^n$, this definition of consistency corresponds to O'Neill consistency.

Let f be a bankruptcy rule. We say that a consistent rule φ for bankruptcy problems with a priori unions is *f-consistent* if for every bankruptcy problem (N, E, c) we have that $\varphi(N, E, c, \mathcal{P}^N) = f(N, E, c)$. That is, φ is f-consistent if φ is consistent and it coincides with f when the a priori unions structure $\mathcal{P}$ is the boundary system $\mathcal{P}^N$.

The next theorem establishes, for a fixed bankruptcy rule f, the existence and uniqueness of an f-consistent rule. This result extends the O'Neill result of existence and uniqueness of a bankruptcy consistent rule; this unique rule is the random arrival rule.

Theorem 5.1 *The f-random arrival rule RA^f is the unique f-consistent rule for bankruptcy problems with a priori unions.*

Proof: Let f be a bankruptcy rule.
Existence: First we show that the f-random arrival rule, RA^f, is f-consistent. We know that for every bankruptcy problem (N, E, c), $RA^f(N, E, c, \mathcal{P}^N) = f(N, E, c)$. So, it remains to be shown that RA^f is consistent. Let $(N, E, c, \mathcal{P})$ a bankruptcy situation with a priori unions. Let $i \in P_k$. Define E_σ, E' and $E_{-\ell}$ as before define and $E_{-\ell,\sigma} = \max\{E_{-\ell} - \sum_{t \in R \setminus \{\ell\}: \sigma(t) < \sigma(k)} c_t^{\mathcal{P}}, 0\}$ for all $\sigma \in \Pi(R), \ell \in R$. Then,

$$\begin{aligned}
RA_i^f(N, E, c, \mathcal{P}) &= \frac{1}{r!} \sum_{\sigma \in \Pi(R)} f_i(P_k, E_\sigma, (c_j)_{j \in P_k}) \\
&= \frac{1}{r!} \Big[(r-1)! f_i(P_k, E', (c_j)_{j \in P_k}) + \\
&\quad \sum_{\ell \in R, \ell \neq k} \sum_{\sigma \in \Pi(R \setminus \{\ell\})} f_i(P_k, E_{-\ell,\sigma}, (c_j)_{j \in P_k}) \Big] \\
&= \frac{1}{r} \Big[f_i(P_k, E', (c_j)_{j \in P_k}) +
\end{aligned}$$

$$\sum_{\ell\in R,\ell\neq k}\frac{1}{(r-1)!}\sum_{\sigma\in\Pi(R\setminus\{\ell\})}f_i(P_k,E_{-\ell,\sigma},(c_j)_{j\in P_k})\Big]$$

$$= \frac{1}{r}\Big[RA_i^f(P_k,E',(c_j)_{j\in P_k},\mathcal{P}^{P_k})+ \sum_{\ell\in R,\ \ell\neq k}RA_i^f(N\setminus P_\ell,\max\{E-c_\ell^{\mathcal{P}},0\},c_{-\ell},\mathcal{P}_{-\ell})\Big].$$

Hence, RA^f is consistent and therefore f-consistent.
Uniqueness: Now we show that if φ is an f- consistent rule for bankruptcy problems with a priori unions then φ coincides with the f-random arrival rule RA^f. We show this by induction on the number of unions. If $r = 1$ then $\varphi(N,E,c,\mathcal{P}^N) = f(N,E,c) = RA^f(N,E,c)$ by the definition of f-consistency. Suppose that this holds for $r = m-1$. For $r = m$, f-consistency implies that $\varphi(N\setminus P_\ell,\max\{E - c_\ell^{\mathcal{P}},0\},c_{-\ell},\mathcal{P}_{-\ell})$ is completely determined and hence we conclude that there is a unique f-consistent rule, which is the f-random arrival rule. □

Calleja et al. (2001) characterise two random arrival rules for multi-issue allocation situations by using consistency properties that also extend O'Neill consistency. Whereas the consistency properties in Calleja et al. (2001) consider an average over the agents, our consistency property considers an average over the unions. It is possible to extend our consistency property to multi-issue allocation situations by considering an average over the number of issues.

O'Neill (1982) also shows that the random arrival rule of a bankruptcy problem coincides with the Shapley value of the bankruptcy game associated. Owen (1977) extends the Shapley value to the context of cooperative games with a priori unions, resulting in the Owen value, Ow. The next theorem extends the previous result by O'Neill . We omit the proof that follows a similar line to the proof of the preceding theorem.

Theorem 5.2 *If $(N,E,c,\mathcal{P})$ is a bankruptcy problem with a priori unions, then its RA-random arrival rule coincides with the Owen value of the associated bankruptcy game with a priori unions, that is,*

$$RA^{RA}(N,E,c,\mathcal{P}) = Ow(N,v_{E,c},\mathcal{P}).$$

Now, from the previous two theorems, we immediately obtain the next result.

Theorem 5.3 *The only rule for bankruptcy problems with a priori unions satisfying random arrival-consistency is the Owen value of the associated bankruptcy games with a priori unions.*

In Winter (1992) and Hamiache (1999), the Owen value is axiomatically characterised on the class of cooperative games with a priori unions by using two different properties of consistency. Note that in the current paper, we characterise the Owen value on the class of bankruptcy situations with a priori unions, using another consistency property that extends the O'Neill consistency property for bankruptcy problems.

To finish this section, we provide an example that shows that the two new extensions of the CEA rule need not coincide. We compare these two rules with the two extensions introduced in section 4.

Example 5.2 Consider the bankruptcy problem with a priori unions $(N, E, c, \mathcal{P})$, where the set of agents is $N = \{1, \ldots, 6\}$, the estate is $E = 10$, the vector of claims is $c = (6, 3, 11, 2, 3, 2)$ and the structure of a priori unions is $\mathcal{P} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. For this situation, we have

$$RC^{CEA}(N, E, c, \mathcal{P}) = (29/12, 17/12, 38/12, 14/12, 13/12, 9/12),$$
$$RA^{CEA}(N, E, c, \mathcal{P}) = (29/12, 17/12, 39/12, 13/12, 13/12, 9/12),$$
$$\overline{CEA}(N, E, c, \mathcal{P}) = \widetilde{CEA}(N, E, c, \mathcal{P}) = (5/3, 5/3, 5/3, 5/3, 5/3, 5/3).$$

◁

6 An application

In this section we apply the various extensions of the CEA rule to one particular bankruptcy situation, the Pacific Gas and Electric Company, a fully owned subsidiary of PG&E Corporation and one of the largest combined natural gas and electricity utilities in the United States. Due to negative stocktaking they filed for reorganisation under Chapter 11 of the US Bankruptcy Code in a San Francisco bankruptcy court in 2001.

The debtor's 20 largest unsecured creditors are listed in the following table, which is taken from *www.bankruptcydata.com*.

#	Nature of claim	Claim ($)
1	Bank bonds	2,207,250,000
2	Power purchases	1,966,000,000
3	Bank bonds	1,302,100,000
4	Power purchases	1,228,800,000
5	Bank bonds	938,461,000
6	Bank bonds	310,000,000
7	Power purchases	57,928,385
8	Power purchases	49,452,611
9	Power purchases	48,400,572
10	Power purchases	45,706,378
11	Power purchases	40,147,245
12	Power purchases	40,122,073
13	Power purchases	32,867,878
14	Gas purchases	29,523,530
15	Gas purchases	28,210,551
16	Gas purchases	24,718,334
17	Gas purchases	23,849,455
18	Power purchases	22,576,506
19	Power purchases	21,506,087
20	Power purchases	19,800,248

According to this table, the creditors claim money on the basis of three issues (nature of claims). So we can analyse this as a bankruptcy situation with three unions of creditors: $P_1 = \{1, 3, 5, 6\}$ related to bank bonds, $P_2 = \{2, 4, 7, 8, 9, 10, 11, 12, 13, 18, 19, 20\}$ related to power purchases and $P_3 = \{14, 15, 16, 17\}$ related to gas purchases. The total estate (E) to be allocated to unsecured creditors equals \$1,060,000,000.

We compute the $\overline{CEA}$, RC^{CEA} and RA^{CEA} solutions for the bankruptcy situation with the three unions and compare them with the solution obtained by applying the CEA rule to the same situation without the unions (which coincides with the $\widetilde{CEA}$ solution). The next table shows the results, where all amounts have been rounded to the nearest integer.

#	CEA	$\overline{CEA}$	RC^{CEA}	RA^{CEA}	Union
1	95,865,025	119,212,266	128,070,755	128,070,755	P_1
2	95,865,025	52,089,822	130,945,277	161,514,515	P_2
3	95,865,025	119,212,266	128,070,755	128,070,755	P_1
4	95,865,025	52,089,822	130,945,277	161,514,515	P_2
5	95,865,025	119,212,266	128,070,755	128,070,755	P_1
6	95,865,025	119,212,266	128,070,755	128,070,755	P_1
7	57,928,385	52,089,822	36,672,736	28,964,193	P_2
8	49,452,611	49,452,611	32,968,407	24,726,306	P_2
9	48,400,572	48,400,572	32,267,048	24,200,286	P_2
10	45,706,378	45,706,378	30,470,919	22,853,189	P_2
11	40,147,245	40,147,245	26,764,830	20,073,623	P_2
12	40,122,073	40,122,073	26,748,049	20,061,037	P_2
13	32,867,878	32,867,878	21,911,919	16,433,939	P_2
14	29,523,530	29,523,530	9,841,177	9,841,177	P_3
15	28,210,551	28,210,551	9,403,517	9,403,517	P_3
16	24,718,334	24,718,334	8,239,445	8,239,445	P_3
17	23,849,455	23,849,455	7,949,818	7,949,818	P_3
18	22,576,506	22,576,506	15,051,004	11,288,253	P_2
19	21,506,087	21,506,087	14,337,391	10,753,044	P_2
20	19,800,248	19,800,248	13,200,165	9,900,124	P_2

The first conclusion of these results is that all three rules that take the unions into account are more favourable for P_1 and less favourable for P_2 than the CEA rule without unions. Since the idea behind constrained equal award is that the smaller creditors are protected, it is better for the (smaller) claimants in P_2 to be considered as separate creditors than as one big group.

The RC^{CEA} rule and the RA^{CEA} are worst for P_3, which contains only small claimants. The protective aspect of constrained equal award is partly neutralised by taking averages over a number of extreme outcomes.

The only difference between the RC^{CEA} and RA^{CEA} solutions is that the larger claimants in P_2 are better off when the last rule is applied, at the expense of the smaller claimants in the same union. The RA^{CEA} solution is an average over more extreme outcomes than the RC^{CEA} solution, and hence, the smaller creditors are again less protected.

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