Manipulation through political endorsements*

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Abstract

We study elections with three candidates under plurality voting. A candidate is a Condorcet loser if the majority of the voters place that candidate at the bottom of their preference rankings. We first show that a Condorcet loser might win the election in a three-way race. Next we introduce to the model an endorser who has private information about the true probability distribution of the preferences of the voters. Observable endorsements facilitate coordination among voters who may otherwise split their votes and lead to the victory of the condorcet loser. When the endorser has an ideological bias towards one of the candidates, the coordination impact of endorsements remains unaltered, moreover the endorser successfully manipulates the outcome of the election in favor of his bias, even if his ideological bias is known by the voters. The results are true for any endorsement cost and any magnitude of bias as long as the electorate is large enough.

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Introduction

Consider an election with three candidates under plurality voting. A candidate is a Condorcet loser if it loses to any other candidate in a two-way race. In particular, a candidate that the majority of the voters place at the bottom of their preference ranking is a Condorcet loser. However a Condorcet loser might win the election in a three-way race.

As Fey (1997) states, a situation where a Condorcet loser wins arose in the 1970 New York senatorial election. There were two liberal candidates, Richard L. Ottinger and Charles E. Goodell, who received more than 60% of the votes cast but split this share in such a way that the conservative, James R. Buckley, won the election with just 39% of the votes. Both Ottinger, who ended up with 37% of the vote, and Goodell, with 24% of the vote, were credible candidates. The liberal majority in the electorate was unable to coordinate its support behind one of the two liberals in the race, and the result was a conservative victory.

In our model there is a Condorcet loser (labeled C), which creates the coordination problem among the majority of the voters that find the Condorcet loser the least preferable candidate. There is heterogeneity among these voters about their preferences over the non-Condorcet loser (mainstream) candidates (labeled A and B). We assume that candidate C gets a commonly known fraction of the vote totals. This fraction is more than one third to allow for the possibility that C wins, and is less than one half to allow for the possibility that C loses if the majority can coordinate their votes on one of the mainstream candidates, A or B.

This model is equivalent to a model where there is a least preferred status-quo, and there are 2 alternatives competing to replace the status-quo. The decision to replace the status-quo is determined by a supermajority voting among N voters. For example, suppose there is an incumbent in office, and there is voting among a group of decision makers (for example stockholders) either to keep the incumbent or replace it with one of the two alternative candidates. If the voting rule is q-rule, that is if the number of voters is N and at least $\lceil qN \rceil$ votes are needed for an alternative to replace the incumbent, and none of the voters prefer the incumbent to any of the alternatives, then this model turns out to be identical to ours.

Pre-election activities may help voters coordinate their votes on one of two similar candidates. During the 1987 general election in the U.K., a group called TV87 formed, whose sole purpose was to instruct voters how to best vote strategically to prevent a Conservative victory. In Turkey, where there are multiple parties, the media groups tend to favor one of the many central candidates in order to coordinate the votes of the mainstream voters. The media groups get rewards in return for their support when the party they support gets elected.

Parties attract a certain number of voters regardless of their electoral situation. Every party has loyal voters who stay with their party through periods of boom and bust. Moreover, there are "protest" voters who are not interested in the winner of the election, but are casting their ballots for a particular party to protest an alternative mainstream party. The number of such "loyal" voters and "protest" voters may depend on the current political and economic situations.

In this paper, we study the coordination problem that a group of voters face when there are "loyal voters" (extremists) for each candidate, and the expected fraction of each extremist voter type is uncertain. We model pre-election activities by political candidate endorsements, where the endorser has some pre-election information about the distribution of the preferences (in particular the expected fraction of the extremist voters) across the electorate. We have in mind situations where the endorser has more detailed information from polls, or the endorser is a media group that has more access to public opinion. Endorsements are modeled as investment opportunities;

endorsing a candidate is costly, but if the candidate gets elected, the rewards are big enough to cover the cost of endorsement.

Due to the private information of the endorser, endorsing a candidate might inform voters about the degree to which others support specific candidates. Therefore endorsements can help coordination between groups of voters who might otherwise split their votes among several similar candidates, allowing the election of another, who is much less preferred. With this pre-election information, voters can form expectations about which candidates are likely to win elections and they can cast optimal votes. In this respect, this paper is investigating similar situations as Rietz, Myerson and Weber (1998) do.

We show in theorem 2 that an endorser who is motivated only by monetary rewards is able to coordinate non-extremist voters on a mainstream candidate who is stronger against the Condorcet loser. In our main theorem (theorem 3) we show that if the endorser has also an ideological bias towards one of the mainstream candidates then he always (in *all* voting equilibria) manages to successfully manipulate the outcome of the election in his favor even if his bias is common knowledge across the electorate. The possibility of miscoordination leads the voters to vote for the candidate that is endorsed even if the cost of endorsement is low and the magnitute of the political bias is large provided that the electorate is large enough.

The coordination impact of endorsements continue in some equilibria when there are multiple endorsers even if they have different biases. In such equilibria the endorsers endorse the same candidate at all states of nature. However there is only one endorser whose recommendation the voters follow. If one of the endorsers doesn't have any bias, then there is an equilibrium where none of the other endorsers has any manipulation power. There are also equilibria where the coordination impact of endorsements is absent. The reason is that when there are multiple endorsers, there may be coordination problems among the endorsers that is akin to the coordination problem of the voters in the absence of an endorser.

In the final section we discuss some of our assumptions and how they are related to our results. We emphasize that even a very small cost of endorsement, and a very small return on endorsement activity when the endorsed candidate gets elected is enough to generate our results. However without any cost of endorsement or any incentive for the endorser to correctly predict the winner of the election, voters may still fail to coordinate their votes and candidate C may win the election in some voting equilibria. Our results are robust to small imperfection of the endorser's information about the true state of nature, provided that the imperfection decreases at an exponential rate with the size of the electorate. In particular in a model where the endorser observes the preferences of a tiny fraction of the electorate, all our results would be true.

Proposition 1 is proven in the main text, the proofs of all other results are in the appendix.

Related Literature

The problem we are analyzing is one where a group of decision makers should coordinate their votes to prevent the victory of a least preferred option. Rietz, Myerson and Weber (1998) do an experimental study where voters have the option of sending a costly message to the other voters revealing their types, and the costly message is interpreted as campaign contributions.

The methodology and techniques of our paper closely resemble to those used in information agregation in voting games. In particular Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1997) analyzed how voters with different private signals about the candidates' characteristics make their decisions when they have to choose one of two candidates. The uncertainty in our model is not on a particular characteristic of a candidate, but on the distribution of preference intensities across the electorate.

Other papers that analyze voting equilibria in multi-candidate elections are Palfrey (1989), Myerson and Weber (1993) and Fey (1997).

There has been a recent interest in models where a group of decision makers need to coordinate their votes in order to prevent an unpreferred third option. In these models, a group of voters need to agree (supermajority rule) on one of 2 available options in order to defeat a status quo (or disagreement). The voters differ in the intensity of their preferences for the 2 options, and the preferences are drawn from a distribution as in our model.

In Eliaz, Ray and Razin (2007) voters can either cast a vote for one of the options, or declare neutrality which counts a vote for each of the alternatives. They show that there is an equilibrium where the option that a minority of the voters prefers gets chosen more frequently.

In another recent paper Myatt (2007) studies the same problem where each voter has a single vote. Each voter knows her own preference and a signal both of which imperfectly informs her about a common value parameter that affects the preferences of all voters in the electorate.

A class of Pre-election activities, in particular campaign contributions have been modeled in Myerson and Morton (1992). In their model, there are two candidates both of whom choose their policy platforms and campaign levels. Campaigns in favor of a candidate i help candidate i win the election through changing the preferences of voters directly towards candidate i.

The Model

3 candidates compete in an election. The winner of the election is the candidate who gets most of the votes (plurality voting). Candidates are denoted $j \in \{A, B, C\}$. Candidate C gets a given fraction of vote totals with probability 1. This fraction is denoted α. The remaining $(1-\alpha)$ fraction of voters may vote for candidate A, B or C. [1]The total number of voters is a finite number $\frac{n+1}{1-\alpha}$, $\frac{\alpha(n+1)}{1-\alpha}$ of which cast votes for C, and $n+1$ of which weakly prefers A and B to C. Preferences for this $(1-\alpha)$ fraction of voters (i.e. $n+1$ voters) depend on a preference parameter $x \in [0,1] = X$ and the outcome of the election. For a voter type x, let $u(j, x)$ denote the utility difference between candidate j and candidate C for $j \in \{A, B\}$. A voter is an A(or B) extremist if he prefers candidate A(or B) to both B(or A) and C, and is indifferent between B(or A) and C.[2]

Each voter knows his preference type but is uncertain about the types of other voters. Nature first selects a parameter (called the state of nature) $\theta \in \Theta \equiv \{(z_1, z_2, z_3) \in \mathbb{R}^3 | z_1 + z_2 + z_3 = 1, z_1 \geq 0, z_2 \geq 0, z_3 \geq 0\}$ according to a probability distribution G over Θ. Throughout the paper we assume that G has full support with a continous density function g. The preference type of each of the $n+1$ voters is realized according to the realization of $\theta = (z_1^\theta, z_2^\theta, z_3^\theta)$. At a state of nature θ, a voter is an A extremist with probability z_1^θ, a B extremist with probability z_2^θ and not an extremist with probability z_3^θ. When a voter is not an extremist, his preference type $x \in (0,1)$ is chosen according to a probability distribution F with support X.

Definition 1 *Candidate A is called viable at a state of nature θ if $z_1^\theta + z_3^\theta > \frac{\alpha}{1-\alpha}$. Similarly*

[1]Throughout the paper we assume that the number of voters and the number of votes are integers, but we don't explicitly use the correct notation. The reader should think of the closest integer to the referred number whenever appropriate.

[2]We model the preference types of voters by a one dimensional parameter, however this is without loss of generality. One could think of $p = u(A, x)/u(B, x)$ for when $p \in [0, 1]$ to be the probability that makes a voter type x indifferent between a lottery where A wins for sure, and a lottery where B wins with probability p and C wins with probability $1-p$. If $p > 1$, then $1/p \in [0, 1]$ is the probability that makes a voter type x indifferent between a lottery where B wins for sure, and a lottery where A wins with probability $1/p$ and C wins with probability $1 - 1/p$.

Candidate B is called viable at a state of nature θ if $z_2^\theta + z_3^\theta > \frac{\alpha}{1-\alpha}$.

A candidate j is viable at a state of nature θ, if the expected vote fraction of that candidate would exceed that of candidate C if all non-extreme voters voted for j. We make the following structural assumptions that characterize a voting game with a Condorcet loser:

Assumption 1 $1/3 < \alpha < 1/2$.

Assumption 1 says that candidate C might both win and lose the elections. Without this assumption candidate C would not be a Condorcet loser and there would not be a coordination problem.

Assumption 2 $u(A, 0) = 1$, $u(A, 1) = 0$, $u(B, 0) = 0$, $u(B, 1) = 1$.

Assumption 2 is a normalization of the utilities of the extreme preference types. It follows from the definition of an extremist and this normalization that preference types $x = 0$ are extremists of candidate A, and types $x = 1$ are extremists of candidate B.

Assumption 3 *$u(A, \cdot)$ is a strictly decreasing continous function and $u(B, \cdot)$ is a strictly increasing continuous function.*

Assumption 3 is a description of the structure of the utilities. The bigger the preference type is, the more that voter enjoys the victory of B; and the smaller the preference type is, the more that voter enjoys the victory of A.

Assumption 4 *$F(\cdot)$ is a continous probability distribution function with $F(0) = 0$.*

Assumption 4 says that the probability distribution function of non-extreme voter types is continous. The probability that a non-extremist voter has a preference type 0 is set to 0. Note that $F(\cdot)$

is the probability distribution function of voter types conditional on a voter not being an extremist. We allow for the possibility of a probability mass for the extreme preference types at the first stage of the resolution of uncertainty and these are denoted by z_1^θ and z_2^θ. In the rest of the paper we maintain assumptions 1-4.

Since the probability of being an extremist is state dependent, we define the probability distribution of voter types at each state of nature θ as follows:

$F_\theta^-(x) \equiv z_1^\theta + z_3^\theta \cdot F(x)$ for each $x \in [0,1]$; $F_\theta(x) \equiv F_\theta^-(x)$ for $x < 1$, $F_\theta(1) \equiv F_\theta^-(1) + z_2^\theta$.

Note that F_θ is weakly increasing, right-continous, $F_\theta(0) \geq 0$ and $F_\theta(1) = 1$; therefore F_θ is a probability distribution function and there are potentially two mass points at $x = 0$ and $x = 1$. We denote a voting game by $V(g, F, n+1)$ where g is the density function over the states of nature, F is the c.d.f. over preference types and $n+1$ is the number of voters who put candidate C at the bottom of their preference rankings.

Strategies and Equilibrium

A mixed strategy for voter i, π_i, is a measurable function from a voter's type to the probability of voting for candidate A, i.e, $\pi_i : X \to [0,1]$

We define a *voting equilibrium* π^* to be a symmetric Nash equilibrium in which no voter uses a weakly dominated strategy. The implication of this is that, voters with $x = 0$ always vote for A, and voters with type $x = 1$ always vote for B.

The only times a voter can influence the outcome of the election is if his vote is pivotal, i.e., exactly $(\frac{a}{1-a})n$ of the other n voters voted for A or for B. Hereon q denotes $\frac{\alpha}{1-\alpha}$. As $q > 1/2$ (by assumption 1), the two events can't realize together.

Given a symmetric strategy profile π, one can compute the probability that a vote is pivotal in a race between A and C, or between B and C as a function of the parameter θ. Let

$$t(\theta, \pi) = \int_X \pi(x) dF_\theta(x) \tag{1}$$

denote the probability that a randomly selected voter votes for A when the state of nature is θ.

When a voter who is not an extreme type learns his type $x \in (0, 1)$, he updates his beliefs about θ using Bayes' rule. In particular, the density of the posterior probability that the true state θ is calculated (by Bayes' rule) as:

$$g'(\theta, x) = \frac{z_3^\theta g(\theta)}{\int\limits_\Theta z_3^\theta dG(\theta)} \tag{2}$$

Note that the expression for g' doesn't depend on x, so hereon we drop this dependence.

Let $pivA$, $pivB$ denote the events that a vote is pivotal between A and C, and B and C respectively. These events are mutually exclusive, and the probabilities assigned to these events by a voter who is not an extreme type are given by:

$$Pr(pivA|\pi) = E_{g'}\left[\binom{n}{qn} \cdot t(\theta, \pi)^{qn} \cdot (1 - t(\theta, \pi))^{n-qn}\right] \tag{3}$$

$$Pr(pivB|\pi) = E_{g'}\left[\binom{n}{qn} \cdot (1 - t(\theta, \pi))^{qn} \cdot t(\theta, \pi)^{n-qn}\right] \tag{4}$$

Where the expectation is taken using the density function g'.

A strategy is characterized by a cutpoint if there is a cutpoint x such that a voter votes for A whenever his preference type is smaller than x and for B whenever his preference type is bigger than x.

Definition 2 *A strategy π has a cutpoint structure if there is a cutpoint x^* with the property that $0 < x^* < 1$ and $\pi(x) = 1$ for $x < x^*$, $\pi(x) = 0$ for $x > x^*$.*

In proposition 1 we show that the set of voting equilibria is non-empty and all voting equilibria have a cutpoint structure.

Proposition 1 *There is at least one voting equilibrium π. Every voting equilibrium π has a cutpoint structure with a cutpoint x^* such that $Pr(pivA|\pi) \cdot u(A, x^*) = Pr(pivB|\pi) \cdot u(B, x^*)$.*

Proof. : First we demonstrate that any best response to a weakly undominated strategy has a cutpoint structure.

Note that the implication of weakly undominated strategies is that voters with preference types $x = 0$ always vote for A and those with preference types $x = 1$ always vote for B. Since G has full support, for any $\nu \in (0, 1)$ we have $Pr(F_\theta(0) > \nu) > 0$ and $Pr(F_\theta^-(1) < 1 - \nu) > 0$ and the probability that a voter votes for A and that a voter votes for B are strictly positive for any weakly undominated strategy π. In particular $Pr(pivA|\pi)$ and $Pr(pivB|\pi)$ are both positive. A voter with a preference type x votes for $A(B)$ whenever $Pr(pivA|\pi) \cdot u(A, x) > (<)\ Pr(pivB|\pi) \cdot u(B, x)$. By assumption 2 and 3, there is a unique cutpoint $0 < x^* < 1$ such that $Pr(pivA|\pi) \cdot u(A, x^*) = Pr(pivB|\pi) \cdot u(B, x^*)$, and voters with preference types $x < x^*$ vote for A, voters with preference types $x > x^*$ vote for B. Since $Pr(pivA|\pi)$ and $Pr(pivB|\pi)$ are bounded below by a positive number, and bounded above by a number less than 1, any cutpoint strategy that is a best response to some weakly undominated strategy is bounded away from the boundaries.

Let a be the cutpoint of a strategy π and x^* be the cutpoint of the best response. Putting this into equation (1) we get:

$$t(\theta, \pi) = F_{\theta}^{-}(a) \tag{5}$$

where $F_{\theta}^{-}(x)$ was defined in the text such that $F_{\theta}^{-}(x) = z_1^{\theta} + z_3^{\theta} F(x)$. Then,

$$\frac{u(B, x^*)}{u(A, x^*)} = \frac{Pr(pivA|\pi)}{Pr(pivB|\pi)} = \frac{E_{g'}[(F_{\theta_i}^{-}(a))^{nq} \cdot (1 - F_{\theta_i}^{-}(a))^{n-nq}]}{E_{g'}[(F_{\theta_i}^{-}(a))^{n-nq} \cdot (1 - F_{\theta_i}^{-}(a))^{nq}]}$$

We showed that any best response to a weakly undominated strategy has a cutpoint structure and the cutpoint is away from 0 and 1, in particular in some interval $[\varepsilon, 1 - \varepsilon]$. To demonstrate existence, consider the following function:

$$\begin{aligned} \Psi &: [\varepsilon, 1 - \varepsilon] \to [\varepsilon, 1 - \varepsilon] \\ \Psi(a) &: \frac{u(B, \Psi(a))}{u(A, \Psi(a))} = \frac{E_{g'}[(F_{\theta_i}^{-}(a))^{nq} \cdot (1 - F_{\theta_i}^{-}(a))^{n-nq}]}{E_{g'}[(F_{\theta_i}^{-}(a))^{n-nq} \cdot (1 - F_{\theta_i}^{-}(a))^{nq}]} \end{aligned}$$

For any cutpoint $a \in [\varepsilon, 1 - \varepsilon]$, let $\Psi(a)$ be the unique cutpoint of the best response to the strategy π characterized by cutpoint a. $F_{\theta}^{-}(a)$ is continuous in a, and thus $\dfrac{Pr(pivA|\pi)}{Pr(pivB|\pi)}$ is continuous in a and the continuity and monotonicity of $\dfrac{u(B, .)}{u(A, .)}$ implies that $\Psi(a)$ is a continuous function. Thus, by Brouwer fixed point theorem, the map Ψ has a fixed point, and hence the game has a voting equilibrium. ■

Coordination Failures

In this section we study the equilibrium behavior of cutpoints in large electorates. A point $x \in [0, 1]$ is a limit equilibrium cutpoint if there is a sequence x_n converging to x where x_n is an equilibrium cutpoint of the game $V(g, F, n)$. We say that there is full coordination if 0 or 1 is a limit equilibrium cutpoint.

In theorem 1 we provide a necessary condition for a point to be the limit of a sequence of cut-points as the electorate gets large. Using this partial characterization, we show that full coordination is impossible.

Theorem 1 *A necessary condition for a point x to be a limit equilibrium cut point is:*

$$\frac{u(B,x)}{u(A,x)} = \frac{\int_{\max\{\frac{q-F(x)}{1-F(x)},0\}}^{q} g(z, 1-z-\frac{q-z}{F(x)}, \frac{q-z}{F(x)})dz}{\int_{\max\{\frac{1-q-F(x)}{1-F(x)},0\}}^{1-q} g(z, 1-z-\frac{1-q-z}{F(x)}, \frac{1-q-z}{F(x)})dz}$$

Proof. In the appendix. ■

Corollary 1 *Full coordination is impossible when G has a continous density function g.*

Observe that the term

$$\frac{\int_{\max\{\frac{q-F(x)}{1-F(x)},0\}}^{q} g(z, 1-z-\frac{q-z}{F(x)}, \frac{q-z}{F(x)})dz}{\int_{\max\{\frac{1-q-F(x)}{1-F(x)},0\}}^{1-q} g(z, 1-z-\frac{1-q-z}{F(x)}, \frac{1-q-z}{F(x)})dz}$$

is the ratio of the areas over the lines $line_1 = \{\theta \in \Theta | z_1^\theta + z_3^\theta F(x) = q\}$ and $line_2 = \{\theta \in \Theta | z_1^\theta + z_3^\theta F(x) = 1-q\}$. An immediate consequence of theorem 1 is that as long as G has a continous density function (a smoothness condition), full coordination is never achieved in the limit as the electorate size gets large.

Endorsements and Coordination

In this section we assume that there is an endorser who can support any of the candidates A or B, or choose to remain silent. As a benchmark to highlight how an endorsement activity can achieve full coordination, we first assume that the endorser does not have a strict preference over the candidates in this section. The endorser observes the true state of nature θ.[3] After observing θ, the endorser endorses any of the candidates A or B at a cost l, or remains silent without any cost. If he endorses a candidate and that candidate wins the election, the endorser receives an amount of money $r > l$, otherwise receives 0. The endorser is assumed to be an expected utility maximizer.

We model this scenario by a two-stage game where in the first stage the endorser observes the true state of nature θ, and takes an action from the set $A_I = \{A, B, \oslash\}$. Action A corresponds to endorsing candidate A, action B corresponds to endorsing candidate B, and action $\oslash$ corresponds to remaining silent. At the second stage of the game, each voter observes the endorsement activity and his own preference type, and then all voters simultaneously cast their votes. At the end of the second stage, the winner is announced, and the endorser earns r if the candidate he endorsed wins the election, and earns 0 otherwise. There are three possible outcomes for the endorser; $-l, 0$ and $r-l$. Therefore we need to consider only preferences over lotteries over these outcomes. We denote utilities by $u(\cdot)$ and assume that it is increasing in the outcomes. We normalize the utilities so that $u(0) = 0$.

For any finite set R, let $\Delta(R)$ denote the set of all probability distributions over R. A mixed strategy for the endorser is a map $s : \Theta \to \Delta(A_I)$. A mixed strategy for voter i, π_i, is a measurable function from a voter's type and the endorser's action set to a probability of voting for candidate A, i.e, $\pi_i : X \times A_I \to [0, 1]$.

We define a voting equilibrium (s, π^*) to be a Bayesian Nash Equilibrium of the above game in which no voter uses weakly dominated strategies, and each voter uses the same strategy. The implication of this is that, voters with a type $x = 0$ always vote for A, and voters with type $x = 1$

[3] We discuss how to relax perfect observability of the true state of nature later.

always vote for B.

A strategy for voters has a cutpoint structure in this game if there are cutpoints $x_A, x_B, x_\oslash$ such that the voter votes for A after observing the action k of the endorser whenever the preference type is smaller than x_k and for B whenever the preference type is bigger than x_k.

Definition 3 *A strategy for the voters π has a cutpoint structure in this game if there are cutpoints x_k with the property that $0 < x_k < 1$ and $\pi(x,k) = 1$ for $x < x_k$, $\pi(x,k) = 0$ for $x > x_k$.*

Given a symmetric strategy profile π for the voters, let

$$t(\theta, \pi, k) = \int_X \pi(x,k) dF_\theta(x) \tag{6}$$

denote the probability that a randomly selected voter votes for A when the state of nature is θ and the investor's action is k. Then the probability that A wins the election after the action k of the investor and the strategy π of the voters is calculated as:

$$pr(Awins|\theta, \pi, k) = \sum_{r=nq+1}^{n} \binom{n}{r} \cdot t(\theta, \pi, k)^r \cdot (1 - t(\theta, \pi, k))^{n-r} \tag{7}$$

The right hand side of the above equation is the probability with which at least $nq+1$ voters vote for A. We will use the notation $m^n(t(\theta, \pi, k))$ to express the probability that at least $nq+1$ voters out of n voters vote for A when the probability that a randomly selected voter votes for A is $t(\theta, \pi, k)$.

Proposition 2 *The game with endorser has a voting equilibrium, and in all voting equilibria the strategies of the voters have a cutpoint structure.*

This proposition first establishes an existence result for a voting equilibrium, and says that every voting equilibrium has a cutpoint structure. Having characterized the equilibrium structure, now we study the behavior of the cutpoints when the electorate is large. In the next proposition we show that asymptotically all non-extreme type voters vote for the candidate the endorser endorses.

Proposition 3 *Let x_A^n, x_B^n, be the equilibrium cutpoints after observing support for A and B in the voting game with $\frac{n+1}{1-a}$ voters. For each $\varepsilon > 0$, there is a N_ε such that for $n > N_\varepsilon$, in any voting equilibrium, $x_A^n > 1 - \varepsilon$ and $x_B^n < \varepsilon$.*

Proposition 3 says that endorsements coordinate the voters to vote for the endorsed candidate. The intuition for the result is as follows. Note that since the fraction of C votes is more than 1/3, for a fixed voter strategy either the probability that A wins or B wins should converge to 0 in large elections. Also in equilibrium the endorser endorses a candidate only if the probability that the candidate wins given the equilibrium voter behavior is strictly positive when the electorate is large. Suppose at some state of nature θ endorser endorses candidate A. Then the probability that B wins the elections at θ gets very close to 0. Since the endorser endorses candidate A, voters infer from the endorsement activity that the probability that B may win is much less than the probability with which A may win. This in turn implies that the probability that their vote is pivotal in a close race between B and C approaches to 0 much faster than the probability that their vote is pivotal in a close race between A and C. Hence for any non-extremist voter, voting for A yields a better payoff than voting for B.

We emphasize that there is a tension between the probability of winning and the pivotal probabilities in other voting models in large elections. In our model the two probabilities move in the same direction. The reason is that the required fraction of votes of non-extremist voters for coordination exceeds 1/2. Hence if A is winning with a positive proability (in the limit), the pivotal probability in a race between A and C goes to zero at a much slower rate than the pivotal probability in a race between B and C.

For an outside observer the equilibrium behavior of the voters looks like as if they are willing to vote for a winning candidate. This behavior emerges in equilibrium since the probability that a weaker candidate wins the elections converges to zero.

Our result would lose its strength if the endorser could verify the state of nature to the voters (either costlessly or by paying a cost and getting some positive returns after he verifies it). Because then there would always be an equilibrium where candidate C wins[4] when both candidates A and B are viable. There would still be other equilibria where coordination is achieved, however equilibria where coordination failure is a possibility would remain.

It is critical for our result that the endorsement activity is rewarded conditional on the outcome of the elections. This gives the right incentives to the endorser for undertaking the appropriate endorsement activity after observing the state of nature. We only require that the reward is more than the cost of endorsement.

Typically in costly signalling models, there is also a babbling equilibrium where the voters ignore the message and the sender knowing this doesn't send any costly messages. In our model this is not true. The reason is that, by the full support assumption, the state space is rich enough to include at least a state for each candidate where the expected fraction of the extremists for that candidate is big enough to ensure the victory of that candidate. At such a state of nature, it is a dominant action for the endorser to support that candidate. There is also a state where both types

[4] In an earlier version of this paper we studied the case when the state of nature is known by the voters. We proved that there is a sequence of equilibria where the cutpoints converge to the median voter, and hence the voters split their votes and C wins with a probability that approaches 1.

of extremists are big enough that a victory of C is inevitable. In such states, the endorser remains silent. Therefore there is no babbling equilibrium in our model when the electorate is large enough.

The information that the endorser provides to the voters is coarse. In particular the voters don't know the state of nature after observing the endorsement activity. This enables the endorser to pool states where the victory of A is inevitable with those states where A is more viable than B. Hence coarseness of information delivers the uniqueness (in the limit) of equilibrium cutpoints by eliminating babbling equilibrium.

The result doesn't hinge on the fact that the endorser's action set doesn't include an option to support candidate C. All our results would be still true when we include such an option.

Definition 4 *Candidate A (B) is said to be more viable than B (A) at state $\theta \in \Theta$ if A is viable at θ and $z_1^\theta > z_2^\theta$ ($z_1^\theta < z_2^\theta$).*

A candidate $j \in \{A, B\}$ is more viable than the candidate $j' = \{A, B\} \backslash j$ if j is viable, and the expected fraction of j extremists is more than that of j'. In theorem 2 we show that endorsements achieve full coordination on the more viable candidate.

Theorem 2 *For any $\theta \in \Theta$ and for n large enough:*

1. If A is more viable than B then the endorser endorses A w.p.1, and candidate A wins the election with a probability close to 1.

2. If B is more viable than A, then the endorser endorses B w.p.1, and candidate B wins the election with a probability close to 1.

3. If neither A nor B is viable, then the endorser remains silent, and candidate C wins the election with a probability close to 1.

Theorem 2 says that, the endorser endorses a candidate if and only if it is viable, and the expected fraction of extremists for that candidate is more than the other candidate. The endorser remains silent if none of the mainstream candidates is viable. When theorem 2 is combined with proposition 3 it follows immediately that the endorsed candidate wins the elections with a probability close to 1. The intuition is as follows. By proposition 3, in all voting equilibria almost all of the non-extremist voters vote for the endorsed candidate. So the endorser has more chances of getting the reward if he endorses the candidate that has more expected number of extremist voters. And if neither of the candidates is viable, the victory of C is inevitable, so the endorser remains silent.

Endorsements and Manipulation

In this section we introduce an ideological bias to the preferences of the endorser. We assume that the endorser prefers candidate B to candidates A and C all other things being equal. All our results would go through if the roles of candidates B and A were reversed. The preferences of the endorser are represented by the following Von-Neumann utility function.

$U(x, y) : \{A, B, \oslash\} \times \{A, B, C\} \to R$, where the first component is the endorsed candidate, and the second one is the winner of the election. In particular,

$$U(x, y) = 1_{\{x=y\}} \cdot u(r - l) + 1_{\{y=B\}} \cdot v + 1_{\{x \neq y \& x \neq \oslash\}} \cdot u(-l).$$

The endorser's preferences are common knowledge and he is an expected utility maximizer. Endorser has a positive bias towards candidate B and its magnitude is v. Note that we normalize the utility functions so that $u(0) = 0$. We assume that $l > 0$, $r > l$ and $v > 0$.

Proposition 4 *The game with a biased endorser has a voting equilibrium, and in all voting equi-*

libria the strategies of the voters have a cutpoint structure.

This proposition characterizes the equilibrium behavior of the voters. Using this result, in proposition 5 we show that the endorser achieves full coordination across the electorate.

Proposition 5 *Let x_A^n, x_B^n, be the equilibrium cut points after observing support for A and B in this voting game with $\frac{n+1}{1-a}$ voters. For each $\varepsilon > 0$, there is a natural number N_ε such that for $n > N_\varepsilon$, in any symmetric equilibrium, cutpoint $x_A^n > 1 - \varepsilon$ and cutpoint $x_B^n < \varepsilon$.*

Proposition 5 says that voters continue to follow the endorser, that is almost all non-extreme preference types vote for the endorsed candidate. The next theorem is the main result of our paper:

Theorem 3 *For any $\theta \in \Theta$, and for large enough n*:

1. If B is viable then the endorser endorses B w.p.1, and candidate B wins the election with a probability close to 1.

2. If A is viable and B is not viable then the endorser endorses A w.p.1, and candidate A wins the election with a probability close to 1.

3. If neither A nor B is viable then the endorser stays silent w.p.1, and candidate C wins the election with a probability close to 1.

This theorem says that, as long as candidate B is viable, the endorser endorses B, and B wins the elections with a probability close to 1. Candidate A is supported only if B is not viable and A is viable. The reason is that, as the number of voters gets large, the endorser cannot change the outcome of the election in favor of B, because B has no chance of winning no matter what the non-extreme preference types do.

Manipulation with Imperfect Information

In this section, we answer the question: Can the endorser manipulate the elections when he observes the state of nature only imperfectly? If the precision of the information doesn't increase fast enough as the electorate size increases, then coordination problems remain in some equilibria, and the manipulation power of the endorser might be gone. The reason is the following: In the game without the endorser, the impossibility of full coordination doesn't depend on the prior probability distribution of the states of nature. When the endorser has imperfect information about the true state of nature, the endorsement activity can at best transmit endorser's (imperfect) information. But this additional information changes the distribution over states of nature without changing the support of the distribution. Therefore full coordination would be impossible similar to the corollary of theorem 1.

However, if the precision of the endorser's information also increases fast enough with the electorate size, all our results continue to hold[5]. For the following analysis we assume that the endorser has acces to the results of a poll that informs him/her about the number of A extremists and number of B extremists in a group of voters of size $s \cdot n$ where $s \in (0, 1)$ is a fixed number. Let $p = (p_1, p_2, p_3)$ denote a vector that contains the frequencies of A extremists, B extremists and non-extremists in the poll. We assume that the voters are truthful in polls. As the number of voters n increases, p becomes a very accurate estimate of the state of nature θ. The only difference we bring to the model is that the endorser doesn't observe the state of nature θ, instead observes the outcome of the poll p.

[5] In particular suppose for each electorate size of n, the endorser observes a signal from a set Y_n. A sufficient condition on the precision of the signals for our results to hold is the following: For each $\epsilon > 0$, $\exists \delta < 1$ and an integer N such that whenever $n > N$: for each $y \in Y_n$, $\exists \theta_y \in \Theta$ satisfying $Pr(|\theta - \theta_y| > \epsilon \mid y) < \delta^n$. In lemma 2 in the appendix we show that this is satisfied when the endorser observes the poll results.

Theorem 4 *When the endorser observes the summary statistic p, theorem 2 and theorem 3 continue to hold.*

Theorem 4 states that our results (both coordination and manipulation effects of endorsements) are robust to imperfect information provided that endorser's information gets more precise with the electorate size. However if the endorser's information structure remains fixed (and imperfect) as the electorate size gets larger, our results (in particular the uniqueness of equilibrium behavior) don't hold anymore.

Robustness to Model Assumptions

In this section we analyze different scenarios that link the assumptions of our model to the manipulation result.

Costless observation: If each voter observes the realization of the state of the nature, then there is always an equilibrium where the votes of the non-extremist voters are split among candidates A and B, and candidate C wins the elections. This result covers situations where the state of nature is revealed to the public exogeneously, or by a sender who has no cost of revealing information, and doesn't have any incentives to predict the outcome of the election.

Rewards: Suppose we assumed that the endorser is only motivated with the outcome of the election, and there is no monetary reward for when the endorsed candidate wins the elections. In other words, the endorser's reward is independent of his endorsement. No matter how costly (costless) the endorsement activity is, there is a babbling equilibrium where the voters ignore the endorser's messages if there is any. In such equilibria any equilibrium in the original game without endorsements remain to be equilibria.

Constant fraction of C supporters: We assume that α, the fraction of C votes, is constant across all states of nature. None of our results would change if α was also part of the description of the state of nature, and the endorser observed α before the endorsement decision as long as $\alpha > 1/3$ (part of Assumption 1) at all states of nature. That is we require that the existence (or lack) of coordination problem is common knowledge across the electorate. When α is allowed to be less than $1/3$, it becomes possible that a voter can be pivotal between a race between A and B (a possibility that never occurs when $\alpha > 1/3$), and the manipulation power of the endorser may be gone.

Constant rewards across states of nature: We assume that the amount of rewards that the endorser receive is constant (and bigger than the cost of endorsement) across the states of nature. If the rewards were less than or equal to the cost of endorsement when say A is viable and B is not (a situation where the endorser is not needed for coordination), then the uniqueness result doesn't hold. However we could interpret the rewards as follows: The endorser may be hurt if he doesn't take sides with the winning candidate. In this interpretation, $-r_i$ becomes the cost of not endorsing the winning candidate. As long as the cost of not endorsing the candidate is more than the cost of endorsements, that is as long as $r_i - l_i > 0$, all of our results continue to hold.

Endorsement is a binary decision: We assume that there are only two levels of endorsement, for example endorsing at an intermediate level is not allowed. Let $c(e) : [0, 1] \to \mathbb{R}_+$ and $r(e) : [0, 1] \to \mathbb{R}_+$ be increasing, continous and bounded functions interpreted as the cost of endorsing a candidate and the reward of endorsing a winning candidate an intensity level e respectively. If $c(0) = r(0) = 0$ and there exists an $e^* > 0$ with $r(e^*) - c(e^*) > 0$, then theorems 2 and 3 would continue to hold. The endorser would choose an intensity level $e^* = \max r(e) - c(e)$ whenever he endorses a candidate.

Multiple Endorsers:

If there are many endorsers with similar or divergent ideological preferences, there exist equilibria where all endorsers end up endorsing the same candidate. In such equilibria voters follow the recommendation of only one endorser. In other words the manipulation power is given to only one endorser. If this endorser has a bias he manipulates the outcome as in theorem 3 and if he doesn't have a bias then he manages to coordinate the electorate. In all such equilibria full coordination is achieved, and the probability that candidate C wins gets close to 0 at any state of nature where either A or B is viable.

In the presence of multiple endorsers there is less scope for manipulation because a purely money motivated endorser is sufficient for coordination across the electorate. In this equilibrium all other endorsers lose their manipulation power. Note that all else equal, endorsers prefer to endorse the winning candidate. When the electorate coordinates on one endorser, then other endorsers enjoy the opportunity to collect the profits by endorsing the same candidate.

However there may also be equilibria where the voters coordinate on a candidate only if both endorsers endorse the same candidate, but they split the votes when the endorsers endorse different candidates. In this case there is a mixed strategy equilibrium where the endorsers randomize between endorsing A and B when both candidates are viable[6].

Summary

We study elections with 3 candidates where the majority of the voters prefer to avoid the victory of candidate C. However the existence of 2 other candidates (A and B) creates a coordination problem, that is if the voters split their votes among A and B, C gets elected.

[6]For example when there are 2 endorsers with biases in different directions, a mixed strategy equilibrium that resembles the mixed strategy equilibrium of the battle of the sexes game exists. If the endorsers have biases in the same direction then a mixed strategy equilibrium that resembles the mixed strategy equilibrium of a coordination game exists.

When there is uncertainty about the preferences across the electorate, voters never fully manage to coordinate their votes on one of the mainstream candidates. Even when the uncertainty about the distribution of the preferences is resolved exogeneously there is an equilibrium where the voters split their votes in such a way that C wins again.

We model political endorsements as investment opportunities by media groups, or people who are better informed about the probability distribution of the preferences across the electorate. Endorsing a candidate is a costly and publicly observable activity, and the endorser gets private benefits if the endorsed party wins the elections.

We show that political endorsements always (in all voting equilibria) manage to prevent the victory of C by fully coordinating the voters on one of the mainstream candidates A or B. If the endorser is only money motivated coordination is achieved on the stronger mainstream candidate. If the endorser also has an ideological bias towards one of the mainstream candidates then coordination is achieved on his more preferred candidate instead of the stronger candidate resulting in a manipulation.

We view our results as suggesting that pre-election activities by media goups in large elections may coordinate voters when disagreement is extremely undesirable. Coordination failures disappear most prominently when the parties are allowed to give private benefits to the endorser if they win the elections.

Enabling coordination gives the endorser manipulation power over the election outcome. However we would like to emphasize that our model doesn't leave out the possibility that candidate C wins the elections even in the presence of an endorser. First, there is uncertainty about the states of nature. In all scenarios we analyze the probability that neither candidate A nor B is viable is strictly positive. Second, our results are true for large elections. For instance, even when A is viable, and the endorser endorses A, the exact number of each preference type is determined from a probability distribution. Therefore, our model allows the endorser to make mistakes, and candidate C may win the elections, however the probability of this approaches to zero as the electorate gets larger.

In the paper we don't allow the endorser to support the Condorcet loser, however all results would hold if we allowed this. In all voting equilibria of this scenario, the endorser would endorse the Condorcet loser whenever neither candidate A nor B is viable.

APPENDIX

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Proof of Theorem 1:

We start by proving the following lemma:

Lemma 1 *Let* $\{f_n(.)\}_{n=1,2,...}$ *be a sequence of continous functions and* $f_n : [0,1] \to [a,b]$ *for some positive* a *and* b*, and the sequence of functions* f_n *converge to* f *(in the sup norm). Let* $S_n = \int_0^1 (k^q(1-k)^{1-q})^n f_n(k)dk$*. Then* $\lim_{n\to\infty} \frac{\sqrt{n}S_n}{f_n(q)(q^q(1-q)^{1-q})}$ *exists, and is independent of* f*.*

Proof.

$$\begin{aligned}\frac{\sqrt{n}S_n}{(q^q(1-q)^{1-q})^n} &= \sqrt{n}\int_0^1 (\frac{k^q(1-k)^{1-q}}{(q^q(1-q)^{1-q})})^n f_n(k)dk \\ &= \sqrt{n}\int_0^q (\frac{k^q(1-k)^{1-q}}{(q^q(1-q)^{1-q})})^n f_n(k)dk + \sqrt{n}\int_q^1 (\frac{k^q(1-k)^{1-q}}{(q^q(1-q)^{1-q})})^n f_n(k)dk\end{aligned}$$

Let $L_n = \sqrt{n}\int_0^q (\frac{k^q(1-k)^{1-q}}{(q^q(1-q)^{1-q})})^n f_n(k)dk$, $U_n = \sqrt{n}\int_q^1 (\frac{k^q(1-k)^{1-q}}{(q^q(1-q)^{1-q})})^n f_n(k)dk$.

Fix $\varepsilon > 0$. Let $r(t) = (\frac{t^q(1-t)^{1-q}}{(q^q(1-q)^{1-q})})$. Let $t_1, t_2, ..., t_{n/\varepsilon+1}$ be numbers between 0 and q such that $r(t_i) = (1 - \frac{(i-1)\varepsilon}{n})$.

Observation: r is an increasing and concave function in the interval $[0, q]$. Therefore $t_i - t_{i+1}$ is decreasing in i.

Step 1: $L_n \leq \sqrt{n}(\sum_{i=1}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1}) \max_{k\in[t_{i+1},t_i]} f_n(k))$

Step 2: $\sqrt{n}(\sum_{i=\lfloor\sqrt{n}\rfloor}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1}) \max_{k\in[t_{i+1},t_i]} f_n(k)) \to 0$ because $\sqrt{n}(1 - \frac{\sqrt{n}\varepsilon}{n})^n \to 0$

Step 3:$\sqrt{n}(t_1 - t_2) \to \sqrt{\frac{\varepsilon}{q(1-q)}}$, $r(t_i)^n \to e^{-(i-1)\varepsilon}$.

Step 4: $\sum_{i=1}^{\infty} e^{-(i-1)\varepsilon} < \infty$ therefore $\sqrt{n}(\sum_{i=1}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1}))$ has a limit independent of f_n. Each f_n is bounded above, and $t_{\lfloor\sqrt{n}\rfloor} \to t_1$ hence $\max_{k\in[t_1,t_{\lfloor\sqrt{n}\rfloor}]} f_n(k) \to f(q)$ and $\lim\sqrt{n}(\sum_{i=1}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1}) \max_{k\in[t_{i+1},t_i]} f_n(k)) = f(q)\lim\sqrt{n}(\sum_{i=1}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1}))$

Step 5: $L_n \geq \sqrt{n}(\sum_{i=2}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1}) \min_{k\in[t_i,t_{i-1}]} f_n(k))$, similar to the previous steps, we have $\lim\sqrt{n}(\sum_{i=2}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1}) \min_{k\in[t_i,t_{i-1}]} f_n(k)) = f(q)\lim\sqrt{n}(\sum_{i=1}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1}))$

Step 6: $f(q)\lim\sqrt{n}(\sum_{i=1}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1})) - f(q)\lim\sqrt{n}(\sum_{i=1}^{n/\varepsilon} r(t_i)^n (t_i - t_{i+1})) = \sqrt{\frac{\varepsilon}{q(1-q)}}$

Step 7: Since the choice of ε is arbitrary, $\frac{\lim L_n}{f(q)}$ exists and is independent of f.

step 8: Similarly $\frac{\lim U_n}{f(q)}$ exists and is independent of f, therefore $\frac{\lim S_n}{f(q)}$ exists and is independent of f. ■

Proof of theorem 1. If $x_n \to x$, then $\frac{u(B,x_n)}{u(A,x_n)} \to \frac{u(B,x)}{u(A,x)}$. The equilibrium condition is that

$$\frac{u(B,x_n)}{u(A,x_n)} = \frac{E_{g'}[(F_\theta^-(x_n))^{nq} \cdot (1-F_\theta^-(x_n))^{n-nq}]}{E_{g'}[(F_\theta^-(x_n))^{n-nq} \cdot (1-F_\theta^-(x_n))^{nq}]}.$$

Equivalently,

$$\frac{u(B,x_n)}{u(A,x_n)} = \frac{\int_0^{1-z_1^\theta}\int_0^1 (z_1^\theta + z_3^\theta F(x_n))^{nq} \cdot (1-z_1^\theta - z_3^\theta F(x_n))^{n-nq} \cdot g(z_1^\theta, 1-z_1^\theta - z_3^\theta, z_3^\theta) \cdot d(z_1^\theta) d(z_3^\theta)}{\int_0^{1-z_1^\theta}\int_0^1 (z_1^\theta + z_3^\theta F(x_n))^{n-nq} \cdot (1-z_1^\theta - z_3^\theta F(x_n))^{nq} \cdot g(z_1^\theta, 1-z_1^\theta - z_3^\theta, z_3^\theta) \cdot d(z_1^\theta) d(z_3^\theta)}$$

For each x_n, let $f_n(k) = \int_{\max\{\frac{k-F(x_n)}{1-F(x_n)},0\}}^k g(z_1, 1-z_1-\frac{k-z_1}{F(x_n)}, \frac{k-z_1}{F(x_n)})dz_1$ for $k \in [0,1]$, and $f(k) = \int_{\max\{\frac{k-F(x)}{1-F(x)},0\}}^k g(z_1, 1-z_1-\frac{k-z_1}{F(x)}, \frac{k-z_1}{F(x)})dz_1$. Then we have:

$$\frac{u(B,x_n)}{u(A,x_n)} = \frac{\int_0^1 (k^q(1-k)^{1-q})^n f_n(k)dk}{\int_0^1 (k^{1-q}(1-k)^q)^n f_n(k)dk}$$

Since $x_n \to x$, f_n converges to f. Since g is bounded above, there is a uniform upper bound for each f_n. By lemma 1 and by a change of variables for the denominator ($u = 1-k$), we have $\lim \frac{\int_0^1 (k^q(1-k)^{1-q})^n f_n(k)dk}{\int_0^1 (k^{1-q}(1-k)^q)^n f_n(k)dk} = \frac{f(q)}{f(1-q)}$. We also have $\lim \frac{u(B,x_n)}{u(A,x_n)} = \frac{u(B,x)}{u(A,x)}$, hence $\frac{u(B,x)}{u(A,x)} = \frac{f(q)}{f(1-q)}$. ■

Proof of Proposition 2:

Proof. The action set for the endorser is to support A, to support B, or stay silent after observing the true state of nature, θ. We denote this by $A_I = \{A, B, \oslash\}$. The action set for a voter is to vote for A or B. We allow mixed strategies for the endorser. Then a strategy for the endorser is a collection $\{s_\theta\}$, where s_θ is a probability distribution over the set A_I. Let's define $pr(A|\theta_i, \pi, j)$ as the probability that A wins the election when the state of nature is θ_i, voters use strategy π and the endorser chooses the action $j \in A_I$.

The endorser's best response to any symmetric strategy π of the voters is as follows: Observing θ,

the endorser endorses A ($s_\theta(A) > 0$) only if:

$pr(A|\theta, \pi, A) \cdot u(r - l) + (1 - pr(A|\theta, \pi, A)) \cdot u(-l) \geq u(0) = 0$ or;

$$pr(A|\theta, \pi, A) \geq \frac{-u(-l)}{u(r-l) - u(-l)}, \text{ and } pr(A|\theta, \pi, A) \geq pr(B|\theta, \pi, B)$$

the endorser endorses B ($s_\theta(B) > 0$) only if:

$$pr(B|\theta, \pi, B) \geq \frac{-u(-l)}{u(r-l) - u(-l)}, \text{ and } pr(B|\theta, \pi, B) \geq pr(A|\theta, \pi, A)$$

the endorser stays silent ($s_\theta(\oslash) > 0$) only if:

$$pr(A|\theta, \pi, A) \leq \frac{-u(-l)}{u(r-l) - u(-l)} \text{ and } pr(B|\theta, \pi, B) \leq \frac{-u(-l)}{u(r-l) - u(-l)}$$

Fix a positiveinteger n. Let $A^* \equiv \{\theta \in \Theta | m^n(z_1^\theta) \geq \frac{-u(-l)}{u(r-l) - u(-l)}$ and $m^n(z_1^\theta) \geq m^n(1 - z_1^\theta)\}$, A^* is the set of states at which the endorser endorses A irrespective of the strategy of the non-extremist voters. By the full support assumption $G(A^*) > 0$ and $s_\theta(A) = 1$ for every $\theta \in A^*$ and any equilibrium strategy s. Similarly there is a positive measure subset of Θ called B^* such that $s_\theta(B) = 1$ for every $\theta \in B^*$

Now, we will show that any best response of the voters to a strategy which is weakly undominated for the voters has a cutpoint structure.

The pivotal probabilities after observing action k of the endorser are as follows:

$$Pr(pivA|\pi,k) = \int_\Theta \left(\frac{g'(\theta)\cdot s_\theta(k)}{\int_\Theta g'(\theta_j)\cdot s_{\theta_j}(k)d\theta_j}\right)\cdot\binom{n}{qn}\cdot t(\theta_i,\pi,k)^{qn}\cdot(1-t(\theta_i,\pi,k))^{n-qn}\cdot d\theta$$

$$Pr(pivB|\pi,k) = \int_\Theta \left(\frac{g'(\theta)\cdot s_\theta(k)}{\int_\Theta g'(\theta_j)\cdot s_{\theta_j}(k)d\theta_j}\right)\cdot\binom{n}{qn}\cdot t(\theta_i,\pi,k)^{n-qn}\cdot(1-t(\theta_i,\pi,k))^{qn}\cdot d\theta$$

Note that all the above probabilities are non zero for $k\in\{A,B\}$. A voter votes for $A(B)$ if $Pr(pivA|\pi,k)\cdot u(A,x) > (<)\ Pr(pivB|\pi,k)\cdot u(B,x)$. As the pivotal probabilities are non zero, there are unique numbers x_A and x_B each strictly between 0 and 1, and each satisfy $Pr(pivA|\pi,k)\cdot u(A,x_k) = Pr(pivB|\pi,k)\cdot u(B,x_k)$ for $k\in\{A,B\}$. By assumption 3, after observing the action k of the investor, voter types with $x<x_k$ vote for A and those types with $x>x_k$ vote for B. These cutpoints are bounded away from 0 and 1 (say between ε and $1-\varepsilon$). This follows the same lines in the proof of proposition 1.

For any 2 cutpoints $x_A,x_B\in[\varepsilon,1-\varepsilon]$, $\theta_A(x_A,x_B)=\{\theta|m^n(F_\theta(x_A)) > \dfrac{-u(-l)}{u(r-l)-u(-l)}$ and $F_\theta(x_A) > F_\theta(x_A)\}$, and similarly define $\theta_B(x_A,x_B)$ and $\theta_\oslash(x_A,x_B)$. $\theta_A(x_A,x_B)$ and $\theta_B(x_A,x_B)$ are non-empty and open sets, and $\theta_\oslash(x_A,x_B)$ may be an empty set. Observe that the best response of the endorser when the state is in the set $\theta_A(x_A,x_B)$ ($\theta_B(x_A,x_B)$) is to endorse candidate A (B) w.p.1. Combining this with the full support assumption and the assumption that there are no jumps, we have the best response Ψ of the voter cutpoints as:

$$\begin{aligned}\Psi(x_A,x_B,A) &: \frac{u(B,\Psi(x_A,x_B,A))}{u(A,\Psi(x_A,x_B,A))}=\frac{\int_{\theta_A(x_A,x_B)} g'(\theta)\cdot\binom{n}{qn}\cdot F_\theta(x_A)^{qn}\cdot(1-F_\theta(x_A))^{n-qn}\cdot d\theta}{\int_{\theta_A(x_A,x_B)} g'(\theta)\cdot\binom{n}{qn}\cdot(1-F_\theta(x_A))^{qn}\cdot F_\theta(x_A)^{n-qn}\cdot d\theta} \\ \Psi(x_A,x_B,B) &: \frac{u(B,\Psi(x_A,x_B,B))}{u(A,\Psi(x_A,x_B,B))}=\frac{\int_{\theta_B(x_A,x_B)} g'(\theta)\cdot\binom{n}{qn}\cdot F_\theta(x_B)^{qn}\cdot(1-F_\theta(x_B))^{n-qn}\cdot d\theta}{\int_{\theta_B(x_A,x_B)} g'(\theta)\cdot\binom{n}{qn}\cdot(1-F_\theta(x_B))^{qn}\cdot F_\theta(x_B)^{n-qn}\cdot d\theta}\end{aligned}$$

Note that the expressions on the right hand side of the equalities are continous in x_A and x_B, hence $\Psi(x_A,x_B,A)$ and $\Psi(x_A,x_B,B)$ are continous functions of x_A and x_B. To show the existence of the equilibrium consider the function:

$$\begin{aligned}\Gamma &: [\varepsilon,1-\varepsilon]^2\rightarrow[\varepsilon,1-\varepsilon]^2 \text{ such that:} \\ \Gamma(x,y) &= (\Psi(x,y,A),\Psi(x,y,B))\end{aligned}$$

Γ is a continous function, hence by Brouwer fixed point theorem, the map Γ has a fixed point. Let x_A^*, x_B^* be a fixed point. If $\theta_\oslash(x_A^*,x_B^*)$ is non-empty, then $x_\oslash^*$ is the solution to

$$\frac{u(B,x_\oslash^*)}{u(A,x_\oslash^*)}=\frac{\int_{\theta_\oslash(x_A^*,x_B^*)} g'(\theta)\cdot\binom{n}{qn}\cdot F_\theta(x_\oslash^*)^{qn}\cdot(1-F_\theta(x_\oslash^*))^{n-qn}\cdot d\theta}{\int_{\theta_\oslash(x_A^*,x_B^*)} g'(\theta)\cdot\binom{n}{qn}\cdot(1-F_\theta(x_\oslash^*))^{qn}\cdot F_\theta(x_\oslash^*)^{n-qn}\cdot d\theta}$$

if $\theta_\oslash(x_A^*,x_B^*)$ is empty, then we set $x_\oslash^*$ any equilibrium cutpoint if the voters believed w.p 1 that the state of nature is $\theta=(1/3,1/3,1/3)$. Clearly these cutpoints constitute an equilibrium. ■

Proof of Proposition 3:

Proof. By large sample properties of binomial distribution, we know that for each $\delta>0$, $\exists N_\delta$ such that for $n>N_\delta$, $m^n(p)\geq\dfrac{u(0)-u(-l)}{u(r-l)-u(-l)}$ implies $p>q-\delta$. Let δ be such that $q-\delta>1/2$. In any voting equilibrium of the game with $n > N_\delta$, endorser endorses A only if the randomly chosen voter votes for A with probability more than $q - \delta$.

Also the equilibrium cutpoint x_A^n satisfies:

$$\frac{u(B, x_A^n)}{u(A, x_A^n)} = \frac{\int_{\theta_A(x_A^n, x_B^n)} g'(\theta) \cdot \binom{n}{qn} \cdot F_\theta(x_A^n)^{qn} \cdot (1 - F_\theta(x_A^n))^{n-qn} \cdot d\theta}{\int_{\theta_A(x_A^n, x_B^n)} g'(\theta) \cdot \binom{n}{qn} \cdot (1 - F_\theta(x_A^n))^{qn} \cdot F_\theta(x_A^n)^{n-qn} \cdot d\theta}$$

Note that $\theta \in \theta_A(x_A^n, x_B^n)$ implies $F_\theta(x_A^n) > q - \delta > 1/2$ for $n > N_\delta$. But this implies,

$$\frac{u(B, x_A^n)}{u(A, x_A^n)} > \left(\frac{q - \delta}{1 - (q - \delta)}\right)^{(2q-1)n}$$

As right hand side goes to infinity as $n \to \infty$, $x_A^n \to 1$. A very similar analysis shows that $x_B^n \to 0$. ■

Proof of Theorem 2:

Proof. 1. At θ, A is more viable than B implies that $F_\theta^-(1) > 1 - F_\theta(0)$. Since F_θ is continuous in the interior, there is an $\epsilon > 0$ such that $F_\theta(1 - \epsilon) > 1 - F_\theta(0)$. By proposition 3, we know that for n large enough x_A^n, x_B^n are in ϵ neighborhood of 1 and 0. So $F_\theta(x_A^n) \geq F_\theta(1 - \epsilon) > 1 - F_\theta(0) \geq 1 - F_\theta(x_B^n)$, that is supporting A has a better chance of getting the payoff r. Since A is also viable, $F_\theta^-(1) > q$, and there is a δ such that $F_\theta(1 - \delta) > q$. For large enough n, $x_A^n > 1 - \delta$, and $F_\theta(x_A^n) > q$. For large enough n, $m^n(F_\theta(x_A^n)) > \dfrac{-u(-l)}{u(r-l) - u(-l)}$ by large sample properties of binomial distribution. This with the inequality $F_\theta(x_A^n) > 1 - F_\theta(x_B^n)$ implies that endorser endorses A w.p.1 after observing θ for large enough n.

2. Same as 1.

3. In this case, $F_\theta(x_A^n) \leq F_\theta^-(1) < q$, and $1 - F_\theta(x_B^n) \leq 1 - F_\theta(0) < q$, so $m^n(F_\theta(x_A^n)) < \frac{-u(-l)}{u(r-l) - u(-l)}$, and $m^n(1 - F_\theta(x_B^n)) < \frac{-u(-l)}{u(r-l) - u(-l)}$ for n large enough, and the probability that C wins goes to 1, so the endorser stays silent. ■

Proof of Proposition 4:

Proof. We first characterize the best responses of the endorser to weakly undominated strategies of voters, π. Let $v(j|\theta_i, \pi)$ be the expected payoff of the endorser if he endorses candidate j, when the voters use the strategy π. Then,

$v(A|\theta_i, \pi) = u(-l) \cdot (1 - pr(A|\theta_i, \pi, A)) + u(r - l) \cdot pr(A|\theta_i, \pi, A) + v \cdot pr(B|\theta_i, \pi, A)$

$v(B|\theta_i, \pi) = u(-l) \cdot (1 - pr(B|\theta_i, \pi, B)) + (v + u(r - l)) \cdot pr(B|\theta_i, \pi, B)$

$v(\oslash|\theta_i, \pi) = v \cdot pr(B|\theta_i, \pi, \oslash)$

$s_{\theta_i}(j) > 0$ only if $v(j|\theta_i, \pi) = \max\{v(A|\theta_i, \pi), v(B|\theta_i, \pi), v(\oslash|\theta_i, \pi)\}$

Fix $n > 1$. Let π be a cutoff strategy with the cutoffs $x_A = 0, x_B = 1, x_\oslash = 1$. Let $A^* \equiv \{\theta \in \Theta | v(A|\theta, \pi) > v(B|\theta, \pi)$ and $v(A|\theta, \pi) > v(\oslash|\theta, \pi)\}$, A^* is the set of states at which the endorser endorses A irrespective of the strategy of the non-extremist voters. By the full support assumption $G(A^*) > 0$ (note that for $\theta = (1, 0, 0)$, $v(A|\theta, \pi) > v(B|\theta, \pi)$ and $v(A|\theta, \pi) > v(\oslash|\theta, \pi)$, since the function v is continous in θ, $G(A^*)$ has positive measure) and $s_\theta(A) = 1$ for every $\theta \in A^*$ and any equilibrium strategy s. Similarly there is a positive measure subset of Θ called B^* such that $s_\theta(B) = 1$ for every $\theta \in B^*$

The rest of the proof follows similar lines as proposition 2 hence we skip it and refer the reader to the working paper version of this paper for details. ■

Proof of Proposition 5 and Theorem 3:

Proof. Step 1. $x_B^n \to 0$.

If the endorser endorses B at a state of nature θ, then:

$u(-l) \cdot (1 - pr(B|\theta, \pi, B)) + (v + u(r-l)) \cdot pr(B|\theta, \pi, B) \geq v \cdot pr(B|\ \theta, \pi, \oslash)$. Since v is non-negative, we obtain:

$$u(-l) \cdot (1 - pr(B|\theta, \pi, B)) + (v + u(r-l)) \cdot pr(B|\theta, \pi, B) \geq 0$$

which is true only if $m^n(1 - F_\theta(x_B^n)) \geq \dfrac{-u(-l)}{v + u(r-l) - u(-l)}$. When n is large, this is true only if $1 - F_\theta(x_B^n) > q - \delta$ for each $\delta > 0$. The rest of the proof for $x_B^n \to 0$ is identical to that of proposition 3.

Step 2. The endorser endorses B whenever B is viable.

At any θ_i where candidate B is viable $pr(B|\theta_i, \pi, B)$ gets close to 1 as n increases since $x_B^n \to 0$. Then optimality of endorsing candidate B follows.

Step 3. $x_A^n \to 1$.

At a state of nature θ the endorser endorses A only if:

i) $u(-l) \cdot (1 - pr(A|\theta, \pi, A)) + u(r-l) \cdot pr(A|\theta, \pi, A) + v \cdot pr(B|\theta, \pi, A) \geq v \cdot pr(B|\ \theta, \pi, \oslash)$,

and ii) candidate B is not viable (otherwise s/he would endorse candidate B from step 1). If B is not viable, then $pr(B|\theta, \pi, A) \to 0$ and $pr(B|\ \theta_i, \pi, \oslash) \to 0$. Then we have the following inequality, $m^n(F_\theta(x_A^n)) \geq \dfrac{-u(-l)}{u(r-l) - u(-l)} > 0$. For n big enough, this is true only if $F_\theta(x_A^n) > q - \delta$ for each $\delta > 0$. Again the rest of the proof is identical to that of proposition 3.

Step 4. The endorser endorses A whenever A is viable and B is not viable, and doesn't endorse any candidate if neither of them is viable.

Step 2 established that endorser endorses B whenever it is viable. At any θ_i where candidate A is viable but B is not, since $x_A^n \to 1$, $pr(A|\theta_i, \pi, A) \to 1$ and optimality of endorsing candidate A follows. When neither A nor B is viable, both $pr(A|\theta_i, \pi, \cdot)$ and $pr(B|\theta_i, \pi, \cdot)$ go to 0, hence staying silent becomes optimal. ■

Proof of Theorem 4:

We start by some definitions. Let $T^n = \{(\frac{a_1}{sn}, \frac{a_2}{sn}, \frac{a_3}{sn})|a_1+a_2+a_3 = sn, a_i \in \mathbb{Z}_+ \text{ for } i \in \{1,2,3\}\}$ be the set of all possible poll outcomes in the game when there are n voters. Let $t_n = (t_{1n}, t_{2n}, t_{3n})$ denote a generic element of T^n and let P^n be the probability distribution over outcomes $W = \Theta \times T^n$.

A strategy for the endorser σ is a mapping from the set of all poll outcomes T^n to the set of all probability distributions over A_I. We use the following lemma about P^n.

Lemma 2 *$\forall\varepsilon > 0$, $\theta = (\theta_1, \theta_2, \theta_3) \in \Theta$, $t_n \in T^n$, $\exists\delta < 1$ and a N s.t. for n>N, $P^n(|\theta_i - t_{in}| > \varepsilon) < \delta^n$ for each $i \in \{1,2,3\}$.*

Proof of Lemma 2. By the central limit theorem, $\sqrt{sn}(\theta_i - t_{in}) \to N(0, \theta_i(1-\theta_i))$ where $N(0,\theta_i(1-\theta_i))$ is the normal distribution with mean 0 and variance $\theta_i(1-\theta_i)$. Let $\varepsilon^* = \frac{\varepsilon}{\max_{\theta_i\in[0,1]}\sqrt{\theta_i(1-\theta_i)}} = 2\varepsilon$, then $P^n(|\theta_i - t_i| > \varepsilon) = P^n(\frac{\sqrt{sn}|\theta_i - t_i|}{\sqrt{\theta_i(1-\theta_i)}} > \frac{\sqrt{sn}\varepsilon}{\sqrt{\theta_i(1-\theta_i)}}) \to 2\int_{\frac{\varepsilon\sqrt{sn}}{\sigma_\theta}}^{\infty} \frac{1}{2\sqrt{\pi}} e^{-y^2} dy \leq \int_{\varepsilon^*\sqrt{sn}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy.$

Using the following well known inequality for normally distributed random variables,

$$\int_x^\infty \frac{1}{2\sqrt{\pi}} e^{-y^2} dy < \frac{1}{2\sqrt{\pi}} e^{-x^2}/x$$

we obtain:

$$\int_{\varepsilon^*\sqrt{sn}}^\infty \frac{1}{\sqrt{\pi}} e^{-y^2} dy < \frac{e^{-(\varepsilon^*)^2 sn}}{\sqrt{\pi}\cdot\varepsilon^*\sqrt{sn}} < (e^{-(\varepsilon^*)^2 s})^n$$

for n large enough (for n satisfying $\sqrt{\pi} \cdot \varepsilon^* \sqrt{sn} > 1$). Since $e^{-(\varepsilon^*)^2 s} < 1$, picking $\delta = e^{-(\varepsilon^*)^2 s}$ proves the assertion. ■

Proof of theorem 4. step 1: There is a N such that for any $n > N$, $P^n(t \in T^n | \, \sigma(t)(A) = 1) > 0$, $P^n(t \in T^n | \, \sigma(t)(B) = 1) > 0$ and $P^n(t \in T^n | \, \sigma(t)(\varnothing) = 1) > 0$. This follows from Lemma 2 and the full support assumption on Θ. $G(\theta \in \Theta | \, z_1^\theta > q + \phi) > 0$ for some $\phi > 0$. Consequently, there is an integer N_1 such that for $n > N_1$, $P^n(t_1 > q + \frac{\phi}{2} | \, z_1^\theta > q + \phi) > 1 - \delta_1^n$ for some $\delta_1 < 1$. Using Lemma 2 again, there is an integer N_2 such that for $n > N_2$, $P^n(\theta_1 > q + \frac{\phi}{4} | \, t_1 > q + \frac{\phi}{4}) > 1 - \delta_2^n$ for some $\delta_2 < 1$. Therefore, by the law of large numbers, there is an integer N_3 such that for $n > N_3$, $\sigma(t)(A) = 1$ for t such that $t_1 > q + \frac{\phi}{2}$. Choosing $N = \max\{N_1, N_2, N_3\}$, for $n > N$, $P^n(t \in T^n | \, \sigma(t)(A) = 1) > G(\theta \in \Theta | \, z_1^\theta > q + \phi)(1 - \delta_1^n) > 0$. The same argument can be used to prove $P^n(t \in T^n | \, \sigma(t)(B) = 1) > 0$ and $P^n(t \in T^n | \, \sigma(t)(\varnothing) = 1) > 0$.

step 2: Let x_B^n denote an equilibrium cutpoint for when B is endorsed and the number of voters is n. Let x_B be a limit point of a sequence of cutpoints. For each $\phi > 0$ there is an integer $N(\phi)$ such that if the endorser endorses B after observing some poll outcome $t \in T^n$, then $t_2 + t_3(1 - F(x_B)) \geq q - \phi$ for $n > N(\phi)$. This follows from the law of large numbers, because otherwise if $t_2 + t_3(1 - F(x_B)) < q - \phi$, then $t_2 + t_3(1 - F(x_B^n)) < q - \phi$ for n large enough, and $P^n(z_2^\theta + z_3^\theta F(x_B^n) \geq q - \phi | t) \to 0$ (since by the law of large numbers the posterior probability that θ is ϵ away from t goes to zero for any positive ϵ), hence the endorser wouldn't endorse B whether he has a bias for B or not.

step 3: Let $P^n(t|\theta)$ be the conditional probability that poll outcome is t at the state of nature θ. Let $p(t, \theta) = \left(\dfrac{g(\theta) \cdot P^n(t|\theta) \cdot \sigma(t)(B)}{\sum_{t \in T^n} \int_{\theta_j \in \Theta} g(\theta_j) \cdot P^n(t|\theta_j) \cdot \sigma(t)(B) d\theta_j} \right)$ be the joint probability that the sate of nature is θ and the poll outcome is t conditional on the endorser endorsing candidate B. Then the cutpoint x_B^n satisfies:

$$\frac{u(B,x_B^n)}{u(A,x_B^n)} = \frac{\sum_{t\in T^n}\int_{\theta\in\Theta} p(t,\theta)\cdot\binom{n}{qn}\cdot(F_\theta(x_B^n))^{qn}\cdot(1-F_\theta(x_B^n))^{n-qn}d\theta}{\sum_{t\in T^n}\int_{\theta\in\Theta} p(t,\theta)\cdot\binom{n}{qn}\cdot(F_\theta(x_B^n))^{n-qn}\cdot(1-F_\theta(x_B^n))^{qn}d\theta}$$

Pick $\varepsilon = \frac{2q-1}{4}$, let δ be the number that satisfies the assertion in lemma 2. Choose ϕ such that $i)$ $\dfrac{\min_{k\in(q-\phi,q+\phi)}(k)^q\cdot(1-k)^{1-q}}{(1-q)^{1-q}\cdot(q)^q} > \delta$, $ii)$ $\min_{k\in(q-\phi,q+\phi)}(k)^q\cdot(1-k)^{1-q} > 1/2$ and $iii)$ $\phi < \frac{2q-1}{4}$.Take a convergent subsequence of x_B^n converging to some number x_B. Let θ_B be the set of states θ for which $1-F_\theta(x_B)\in(q-\phi,q+\phi)$. The following inequality follows:

$$\frac{u(B,x_B^n)}{u(A,x_B^n)} \le \frac{(1-\delta^n)(1/2)^n+\delta^n(q)^{qn}(1-q)^{qn}}{\sum_{t\in T^n}\int_{\theta\in\theta_B} p(t,\theta)\cdot d\theta\min_{k\in(q-\phi,q+\phi)}(k)^{qn}\cdot(1-k)^{n-qn}}$$

This is because: if $\sigma(t)(B) > 0$, $t_2+t_3(1-F(x_B^n)) \ge q-\phi$. But then for any θ such that $F_\theta(x_B^n) \ge 1/2$, $|\theta_i - t_i| > \varepsilon$ for some $i\in\{1,2,3\}$, and by Lemma 2, the probability that $F_\theta(x_B^n)\ge 1/2$ is less than δ^n conditional on $\sigma(t)(B) > 0$. For $F_\theta(x_B^n)\le 1/2$, $F_\theta(x_B^n)^q(1-F_\theta(x_B^n))^{1-q}\le 1/2$. $F_\theta(x_B^n)^q(1-F_\theta(x_B^n))^{1-q}$ is maximized at $F_\theta(x_B^n) = q > 1/2$, so an upper bound for the pivotal probability in a race between B and C is $(1-\delta^n)(1/2)^n+\delta^n(q)^{qn}(1-q)^{qn}$. The probability that the state of nature θ is in θ_B conditional on the endorser endorsing candidate B is $\sum_{t\in T^n}\int_{\theta\in\theta_B} p(t,\theta)d\theta$. Since $\phi > 0$, as n gets large, this probability stays bounded above some lower bound b strictly between 0 and 1.

Since ϕ was chosen such that $\dfrac{\min_{k\in(q-\phi,q+\phi)}(k)^q\cdot(1-k)^{1-q}}{(1-q)^{1-q}\cdot(q)^q} > \delta$, and $\min_{k\in(q-\phi,q+\phi)}(k)^q\cdot(1-k)^{1-q} > 1/2$, $\dfrac{(1-\delta^n)(1/2)^n+\delta^n(q)^{qn}(1-q)^{qn}}{\sum_{t\in T^n}\int_{\theta\in\theta_A} p(t,\theta)d\theta\cdot\min_{k\in(q-\phi,q+\phi)}(k)^{qn}\cdot(1-k)^{n-qn}}$ goes to 0.

$\dfrac{u(B,x_B^n)}{u(A,x_B^n)}\to 0$ gives us $x_B^n\to 0$.

Similarly $\dfrac{u(B,x_A^n)}{u(A,x_A^n)}\to 1$, hence $x_A^n\to 1$.

step 4: Steps 1-3 established that $x_A^n \to 1$ and $x_B^n \to 0$. In words, non-exremist voters follow the recommendation of the edorser. To show that theorem 2 holds, we make the following observations:

$\forall \epsilon > 0 \; \exists N_\epsilon$ s.t if the number of voters is more than N_ϵ then:

i) if $t_1 > t_2 + \epsilon$ and $t_1 + t_3 > q + \epsilon$ then $t = (t_1, t_2, t_3) \in \theta_A$

ii) if $t_2 > t_1 + \epsilon$ and $t_2 + t_3 > q + \epsilon$ then $t = (t_1, t_2, t_3) \in \theta_B$

iii) if $t_1 + t_3 < q - \epsilon$ and $t_2 + t_3 < q - \epsilon$ then $t = (t_1, t_2, t_3) \notin \theta_A \cup \theta_B$

proof of i) : $P^n(\theta_1 > \theta_2$ and $\theta_1 + \theta_3 > q + \epsilon/2 \mid t_1 > t_2 + \epsilon$ and $t_1 + t_3 > q + \epsilon) \to 1$ by the law of large numbers. Since $x_A^n \to 1$, the probability with which A wins the elections if the endorser endorses A after observing t goes to 1, and the probability with which A wins if A is endorsed becomes strictly larger than the probability with which B wins if B is endorsed. Therefore $t \in \theta_A$. Proofs of ii) and iii) are similar to the proof of i).

Since ϵ is arbitrary, the probability that $t \in \theta_A$ for when A is more viable than B gets arbitrarily close to 1 by the law of large numbers. This proves that the probability with which A wins the elections when A is more viable than B gets close to 1. Theorem 3 follows from steps 1-3 in a similar way. ■