Sequential Contracting with Multiple Principals*

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November 19, 2008

Abstract

This paper considers dynamic games in which multiple principals contract sequentially and non-cooperatively with the same agent. We first show that when contracting is private, i.e. when downstream principals observe neither the mechanisms offered upstream nor the decisions taken in these mechanisms, then all PBE outcomes can be characterized through pure-strategy profiles in which the principals offer menus of contracts and delegate to the agent the choice of the contractual terms. We then show that, in most cases of interest for applications, the characterization of the equilibrium outcomes is further facilitated by the fact that the principals can be restricted to offer incentive-compatible extended direct mechanisms in which the agent reports the endogenous payoff-relevant decisions contracted upstream in addition to his exogenous private information. Finally we show how the aforementioned results must be adjusted to accommodate alternative assumptions about the observability of upstream histories and/or the timing of contracting examined in the literature.

Keywords: Sequential common agency, mechanism design, contracts, endogenous types.

JEL Classification Numbers: D89, C72.

*This paper builds on a previous working paper that circulated under the title "A Markovian Revelation Principle for Common Agency Games." For discussions and useful comments, we thank seminar participants at various institutions and conferences where this paper has been presented. A special thank is for Eddie Dekel, Mike Peters, Marciano Siniscalchi, Jean Tirole, an associate editor and four anonymous referees for suggestions that helped us improve the paper. We are also thankful to Itai Sher for excellent research assistance.

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1 Introduction

There are many environments in which multiple principals contract sequentially and non-cooperatively with the same agent.[1] In politics, for example, a ruling administration (upstream principal) that signs a procurement contract with a defense supplier (agent) expects its counterpart to contract also with the next appointed administration (downstream principal). In organizations, an employer who hires a worker anticipates that the latter will leave after a while and be hired by other employers. In a regulatory environment, a multinational firm typically contracts first with domestic authorities (upstream principals) and then with foreign ones. In commerce, a seller (upstream principal) who sets up a menu of contract offers usually expects her buyers (agents) to purchase complementary products and services also from other vendors. In corporate finance, a venture capitalist (upstream principal) who offers a contract to an entrepreneur (agent) anticipates that the latter will borrow also from other lenders and then contract with suppliers, retailers and, perhaps, government agencies (downstream principals).[2]

Characterizing equilibrium outcomes in environments with competing principals is known to be difficult. Contrary to environments with a single principal, the agent can not be assumed to select a contract by simply reporting his "type" i.e. his exogenous private information. First, the agent may have private information also about the mechanisms offered by other principals as well as, in the case of sequential contracting, the decisions taken in these mechanisms. Second, a principal may need to include in her mechanism also contracts that are selected by the agent only off-equilibrium to punish deviations by other principals.[3]

Despite a fast growing literature on sequential common agency, no general characterization results have been established for these games. This is what we aim to do in this paper by providing results that facilitate the construction of equilibrium outcomes in environments in which contracting is sequential.

We build our analysis on a benchmark model of *private contracting* in which principals do not observe other principals' mechanisms, nor the decisions taken in these mechanisms. We also assume that the sequence of bilateral relationships is exogenous in that the agent cannot choose which principal to contract with at each date. Finally, we assume that some irreversible decisions

[1] In what follows, we refer to a principal (equivalently, a mechanism designer) as the party who offers the contract. We also adopt the convention of using masculine pronouns for the agent and feminine pronouns for the principals.

[2] Models of sequential contracting have been used in political economy by [13], [29], and [5]; in sequential trade, by [2], [20], [31], [1], and [8]; in regualtion, by [3], [17], [16] and [12]; in labor relationships, by [15] and [23]; in financial contracting and insurance, by [7], [25], and [32].

[3] See [22], [27], [18], [19], [11], [28], and [26] for a discussion of the problems with standard direct revelation mechanisms under simultaneous common agency and for possible solutions.

are committed at each period.

Later in the paper, we discuss the intricacies that arise under alternative assumptions about the sequence of contracting and/or the observability of upstream histories and show how our results must be adapted to accommodate various alternative extensive forms examined in the literature.

Our first result shows that, when contracting is private, any social choice function[4] that can be sustained with any ad-hoc strategy space for the principals can also be sustained by restricting the principals to offer menus of contracts and delegating to the agent the choice of the contractual terms. We also show that any equilibrium outcome in the game in which the principals can offer all possible menus is robust in the sense that it remains an equilibrium outcome in any game in which the principals' strategy space is enlarged. These results show that the *menu theorems* of simultaneous common agency (cfr [28], and [19]) extend to sequential common agency, provided contracting is private.

Next, we prove the following two theorems, which are specific to sequential common agency. First, when lotteries are feasible, then all social choice functions can be sustained through *pure-strategy* profiles, i.e. by restricting the principals not to mix over their menus and the agent not to mix over the choice of a contract. Second, when information is complete (i.e. when the agent's type is common knowledge as typically assumed in menu auctions and moral hazard settings), then all deterministic social choice functions can be sustained through Markov-perfect equilibria, i.e. by restricting the agent's strategy to depend on any given history only through its payoff-relevant component. Because deterministic social choice functions naturally arise when one restricts attention to pure-strategy profiles and deterministic contracts, these results provide a possible justification for focusing on Markov-perfect equilibria in certain applications.[5]

Based on the aforementioned results, we then show how the characterization of all Markov-perfect equilibrium outcomes can be further simplified by restricting the principals to offer menus that can be conveniently described as *extended direct mechanisms*. In these mechanisms, the agent reports his *extended type*, i.e. his exogenous private information along with the endogenous payoff–relevant decisions contracted upstream. An extended direct mechanism is thus a collection of contracts, one for each extended type.

[4] In the jargon of mechanism design and implementation theory, a social choice function (SCF) or, equivalently, an outcome function, is a mapping from states (here the agent's exogenous private information) to decisions.

[5] Note that a SCF can be deterministic even in the presence of stochastic outcomes. For example, in a moral hazard setting, a principal's decision coincides with the choice of a payment scheme that specifies a reward for the agent as a function of some (typically stochastic) performance measures. In these models, that a SCF is deterministic simply means that (a) the agent does not mix over his choice of effort and (b) the principals do not mix over their choice of a reward scheme. The final outcome, i.e. the payment to the agent, may however be stochastic.

Describing a menu as a direct mechanism has proved useful in games with a single mechanism designer. The same can be done in sequential common agency by extending the notion of type to account for the fact that the agent's payoff-relevant information also includes endogenous decisions. However, there are two important differences with respect to games with a single principal. First, incentive-compatibility is endogenous: the agent's incentives to reveal his extended type depend on the mechanisms offered downstream. Incentive-compatibility must thus be established by backward induction. Second, an incentive-compatible mechanism must specify contracts also for extended types that have zero measure on the equilibrium path. This is because a mechanism must include also contracts that are used only off-equilibrium to punish deviations by upstream principals. In an extended direct mechanism, such out-of-equilibrium contracts are simply those designed for out-of-equilibrium extended types.

Another important difference with respect to the single-principal case is that, when the agent's strategy is not Markov, certain social choice functions cannot be sustained with extended direct mechanisms. With non-Markov strategies, a principal may need to give the same extended type of the agent a menu of contracts to choose from. This is because she may need the agent to punish deviations by upstream principals that altered the distribution of upstream payoff-relevant decisions but nevertheless led to equilibrium extended types. We find such a possibility intriguing from a theoretical viewpoint, but not particularly significant for the type of applications mentioned above.

Below, we conclude the introduction with a road map for our results and the related literature. The rest of the paper is then organized as follows. Section 2 describes the benchmark model of private contracting. Section 3 illustrates with an example why standard direct revelation mechanisms may fail to sustain certain outcomes and introduces the solution offered by our extended direct mechanisms. Section 4 contains the results for menus, while Section 5 contains the results for extended direct mechanisms and shows how to put these mechanisms to work in applications. Section 6 examines alternative extensive forms. Section 7 concludes. All proofs are either in the Appendix or in the Supplementary Material.

1.1 A road map for the results and the related literature

Contrary to simultaneous common agency, when contracting is sequential there is no single model that fits all applications: sequential common agency can in fact be characterized by many different extensive forms depending on the assumptions one makes about the observability of upstream histories and the timing of the relevant decisions. Although it is difficult to have a single unified framework, it is however possible to have results that help characterize equilibria in various dynamic settings considered in applications. In what follows, we present a "road map" for our results and the related literature. This road map is organized around two dimensions that play a key role in most sequential contracting models: *history observability* and *timing*. Each of these two dimensions is associated with specific issues/difficulties with equilibrium characterization. Furthermore, different combinations of these two dimensions correspond to different environments considered in the literature.

Observability of upstream histories. First, consider an environment in which downstream principals observe upstream mechanisms, as for example in [17], [5], [21], [31], and [1]. Contrary to private contracting, in these environments, restricting the principals to offer menus can be with loss of generality. The reason is that payoff-irrelevant details of the mechanisms can be used as correlation devices for the principals' decisions. In the absence of alternative correlation devices, restricting the principals to offer menus then precludes the sustainability of certain outcomes. However, as discussed in Section 6, if one is interested only in pure-strategy profiles (the case considered in all applications that assume mechanism observability), then, not only can one restrict the principals to compete in menus, it is actually safe to restrict them to offer menus that can be described as extended direct mechanisms. This is so irrespective of whether the agent's strategy is Markov. In fact, because the principals can observe the upstream mechanisms, they can punish upstream deviations by offering the agent a different mechanism. There is thus no need to offer the same extended type of the agent multiple contracts to choose from as a function of who deviated upstream. All pure-strategy profiles can thus be sustained with extended direct mechanisms.

Next, consider an environment in which downstream principals do not observe the mechanisms offered upstream, but observe the payoff-relevant decisions taken in these mechanisms. In this environment, restricting the principals' strategy space may mean restricting the extent to which different principals can have different out-of-equilibrium beliefs about the mechanisms used upstream. When the agent's strategy is non-Markov, this means restricting the principals' beliefs about the agent's behavior downstream. Such restrictions may preclude the possibility of sustaining certain outcomes, as shown in [9]. Assuming the principals compete in menus is thus not without loss of generality. However, as discussed in Section 6, the problems with out-of-equilibrium beliefs disappear if one restricts attention to Markov-perfect equilibria. This assumption is standard in the literature that assumes that upstream payoff-relevant decisions are observable (cfr [13], [5], [4]). Furthermore, because in these environments the principals directly observe the payoff-relevant decisions contracted upstream, there is no need for them to ask the agent to communicate such information. By implication, all Markov-perfect equilibrium outcomes can be sustained with standard direct revelation mechanisms in which the agent simply reports his exogenous type.

Finally, consider an environment in which the downstream principals observe the entire upstream history, including the messages sent by the agent upstream. Because the only information that is private to the agent is his exogenous type, one may restrict the principals to offer standard direct revelation mechanisms. However, as shown in [6] for the single-principal case, what is not without loss of generality is restricting the agent's strategy to be truthful. The same is true with multiple principals.

While the observability of upstream histories is exogenous in the literature discussed above, a few recent papers consider environments in which the decision to disclose information to downstream principals is endogenous. [8], for example, derive general conditions for the optimality of privacy in sequential contracting, while [23] examines the interaction between information disclosure and career concerns in a common agency model of the labor market.

Timing of contracting. The timing of contracting can also vary significantly from one application to another. First, the sequence of bilateral relationships can be endogenously determined by the agent's participation decisions, instead of being exogenously fixed as in our benchmark model. This is the case, for example, when a buyer chooses in each period which seller to purchase from. The sequence of bilateral relationships is endogenous, for example, in [2], [30], and [20]. In this setting, principals necessarily observe part of the upstream history. In fact, they observe at least the agent's upstream participation decisions. Furthermore, if in period t the agent decides to participate in principal j's mechanism, then at any subsequent date, principal j necessarily knows the payoff-relevant decisions determined in period t. The problems with menus are then the same as in the case of observable upstream payoff-relevant decisions. However, because these problems emerge only when the players' strategies are non-Markov, it remains possible to use menus to characterize the entire set of social choice functions that can be sustained as Markov-perfect equilibria, as discussed in Section 6. As for direct mechanisms, restricting attention to truthful equilibria is not in general without loss of generality in these environments. The reason is that the agent contracts with the same principal multiple times. Unless the principals can commit to long-term mechanisms, having the agent report truthfully in each period precludes the possibility of sustaining certain outcomes. The problems with truthtelling are the same discussed above for fully observable histories. These problems however vanish if one assumes the agent contracts with each principal at most once, or if one assumes that preferences are common knowledge and restricts attention to pure-strategy equilibria, as in [5].

The sequentiality in the principals-agent decisions may also vary across applications. In our benchmark model, a pair of irreversible decisions, one for the principal, the other for the agent, is committed in each period. There are environments in which instead the principals offer their mechanisms sequentially but where the agent takes decisions only after having observed all mechanisms. Clearly, if the principals do not observe the mechanisms offered by the other principals, these environments are strategically equivalent to simultaneous common agency. In this case, we know from [28] and [19] that all equilibrium outcomes can be sustained with menus. In [26], we show that if one restricts attention to equilibria in which the agent's strategy satisfies the analog of the Markov property described above[6], then all equilibrium outcomes can be characterized by (i) restricting the principals to offer menus that can be conveniently described as incentive-compatible extended direct mechanisms and (ii) by restricting the agent to follow a truthful strategy. The definition of extended direct mechanisms is adjusted to take into account that all decisions are determined simultaneously: the agent is thus asked to report his type along with the decisions he is inducing (through the messages he is sending) with any of the other principals. In the Supplementary Material, we show that the same mechanisms sustain all pure-strategy Markov equilibrium outcomes in sequential games in which the principals observe the mechanisms offered upstream before offering their own mechanism and in which the agent takes decisions with each principal at the end, after having observed the mechanisms offered by all principals.[7] This extensive form corresponds to the environments examined, for example, in [3], [7], [17], [29], and [32].

The extensive forms discussed above are clearly only a subset of the many extensive forms considered in the literature. For example, the repeated common agency games of [5] and [24] do not belong to any of these cases.[8] Nor does the "agenda setting" game of [5].[9] However, most applications combine elements from the various extensive forms discussed above. We thus expect our results to be of guidance in other settings as well.

2 The private contracting model

Players, actions and contracts. There are $n \in \mathbb{N}$ principals who contract sequentially and non-cooperatively with the same agent, A. Each principal P_i is indexed by the date $i \in \mathcal{N} \equiv \{1, ..., n\}$ at which she contracts with the agent. Each P_i must select a contract $x_i : E_i \to \mathcal{A}_i$ from a set X_i of feasible contracts. A contract specifies the action $a_i \in \mathcal{A}_i$ that P_i will take in response to the agent's

[6] In simultaneous games, the agent's strategy is said to be Markov if the decisions the agent selects from each menu depend on his type and the decisions he selects with the other principals, but not on the menus offered by the latter.

[7] The reasons why menus or extended direct mechanisms may fail to sustain certain mixed-strategy equilibrium outcomes are the same as in environments with observable upstream mechanisms.

[8] In these papers, the agent simultaneously contracts with multiple principals at each period.

[9] In this game, upstream decisions determine the sets of feasible decisions downstream.

choice of action/effort $e_i \in E_i$. Both E_i and $\mathcal{A}_i$ may have different interpretations depending on the application under examination. In the relationship between a buyer and a seller, a_i may represent the price that the seller charges to the buyer when the latter chooses quantity/quality e_i. Similarly, when A performs a task on behalf of P_i, a_i may represent the payment that P_i promises as a function of the agent's performance. Depending on the environment, the set of feasible contracts X_i may also be more or less restricted. For example, in standard moral hazard models, e_i is assumed to be the agent's private information, in which case x_i is constant over E_i while the decision a_i should be interpreted as a payment scheme that rewards the agent as a function of some performance measure correlated with the agent's effort. In contrast, in menu auctions, e_i is assumed to be verifiable in which case x_i specifies the transfer that P_i pays to the agent as a function of the "policy" that the agent selects.[10] A profile of contracts will be denoted by $x \equiv (x_1, ..., x_n)$. Similarly, profiles of principals' actions and of agent's efforts will be denoted by $a \equiv (a_1, ..., a_n)$ and $e \equiv (e_1, ..., e_n)$ respectively. We assume the sets E_i and $\mathcal{A}_i$ do not depend on upstream decisions.

Payoffs. All players have expected utility preferences. A principal's payoff is represented by the function $u_i(\theta, e, a)$. Similarly, the agent's payoff is described by the function $v(\theta, e, a)$. The variable θ denotes the agent's exogenous private information. Principals share a common prior about θ represented by the distribution F with support Θ. To avoid measure-theoretic considerations, the sets Θ, $E \equiv \prod_i E_i$ and $\mathcal{A} \equiv \prod_i \mathcal{A}_i$ will be assumed to be finite.

Mechanisms. Principals compete in mechanisms. A mechanism for P_i consists of a message space $\mathcal{M}_i$, a set of signals $\mathcal{S}_i$, and a mapping $\phi_i : \mathcal{M}_i \to \Delta(X_i \times \mathcal{S}_i)$; when A sends a message $m_i \in \mathcal{M}_i$, P_i responds by selecting a contract from X_i and sending the agent a signal $s_i \in \mathcal{S}_i$.[11] The role of these signals is to control the agent's posterior beliefs over X_i so as to fashion his effort decisions.[12] Indeed, the mechanism ϕ_i can also be seen as a mapping $\phi_i' : \mathcal{M}_i \to \Delta(X_i \times \Delta(X_i))$, where $\Delta(X_i)$ denotes the set of the agent's posterior beliefs over X_i. In turn, such a mechanism is equivalent to one where P_i randomizes over $\Delta(X_i)$, then informs the agent of the result of such randomization—i.e. of the particular lottery $d_i \in \Delta(X_i)$ selected—and finally picks a contract x_i from X_i according to the lottery d_i after the agent chooses e_i. Letting Y_i denote the set of (feasible) *stochastic* contracts $y_i : E_i \to \Delta(\mathcal{A}_i)$, we can then suppress the signals s_i and describe a mechanism as a mapping $\phi_i : \mathcal{M}_i \to D_i$ such that, when A selects a message $m_i \in \mathcal{M}_i$, P_i randomizes over Y_i

[10] We assume that x_i does not depend on the agent's effort at dates $t \neq i$. The results can however easily accommodate the case where $x_i : \prod_{i=1}^n E_i \to \mathcal{A}_i$ and where the agent chooses $e = (e_1, ..., e_n)$ only at $t = n + 1$. See Section 6 for a discussion.

[11] Throughout, for any measurable set Z, $\Delta(Z)$ will denote the set of probability measures over Z. Furthermore, given any $\delta \in \Delta(Z)$, we will denote by $Supp[\delta]$ the support of δ.

[12] See also [28] for a discussion of the role of signals in a mechanism.

according to the lottery $\delta_i = \phi_i(m_i) \in \Delta(Y_i)$ and then informs A of the contract $y_i : E_i \to \Delta(\mathcal{A}_i)$ selected by the lottery δ_i before the agent chooses effort e_i. We denote by $D_i \subseteq \Delta(Y_i)$ the set of feasible lotteries over Y_i and by $\text{Im}(\phi_i) \equiv \{\delta_i \in \Delta(Y_i) : \exists\ m_i \in \mathcal{M}_i \text{ s.t. } \phi_i(m_i) = \delta_i\}$ the set of lotteries in the range of ϕ_i. Once again, depending on the application of interest, the set of feasible lotteries D_i may be more or less restricted. For example, in certain applications, it is customary to restrict the principals to offer deterministic mechanisms: this can be accommodated by restricting D_i to contain only degenerate lotteries over deterministic contracts $x_i : E_i \to \mathcal{A}_i$. More generally, the set of feasible lotteries D_i incorporates all sorts of exogenous restrictions dictated by the environment under examination. What is important to us, is that this set is a primitive of the environment, not a choice of P_i.

To save on notation, in the sequel we will often denote a mechanism by ϕ_i, thus dropping the specification of the message space $\mathcal{M}_i$, when this does not create confusion. We then let Φ_i denote the set of feasible mechanisms for P_i and $\phi \equiv (\phi_1, ..., \phi_n)$ and $\phi_{-i} \equiv (\phi_1, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n)$ denote respectively a profile of mechanisms for the n principals and a collection of mechanisms for all P_j with $j \neq i$.[13] As is standard, we assume that principals can fully commit to their mechanisms and that each principal cannot contract directly over the mechanisms, or the contracts, of the other principals.

Timing. The sequence of events is the following:

- At date 0, A privately learns θ.
- At date i, with $i = 1, ..., n$, P_i secretly offers the agent a mechanism $\phi_i \in \Phi_i$. A then chooses a message m_i from $\mathcal{M}_i$, the lottery $\phi_i(m_i) \in \Delta(Y_i)$ determines the contract y_i, and finally given y_i, A chooses effort e_i and P_i's action is determined by the lottery $y_i(e_i) \in \Delta(\mathcal{A}_i)$. None of the principals P_j with $j \neq i$ observes $(\phi_i, m_i, y_i, e_i, a_i)$.
- At date $n + 1$, the game ends.

Although not explicitly modeled, the analysis can easily accommodate the agent's decision (not) to participate in a mechanism. It suffices to add to each mechanism a "null" contract that specifies the default actions that are implemented in case of non-participation, such as no trade at a null price.

[13] Given any collection of sets $\{Z_i\}_{i=1}^n$, the following notation will be used throughout the paper: $z_{-i} \equiv (z_1, ..., z_{i-1}, z_{i+1}, ..., z_n) \in Z_{-i} \equiv \prod_{j \neq i} Z_j$; $z_i^- \equiv (z_k)_{k=1}^{i-1} \in Z_i^- \equiv \prod_{k=1}^{i-1} Z_k$; $z_i^+ \equiv (z_k)_{k=i+1}^{n} \in Z_i^+ \equiv \prod_{k=i+1}^{n} Z_k$; $Z_1^- \equiv Z_n^+ \equiv \varnothing$.

Strategies and Equilibrium. Let Γ summarize the common agency game described above. A strategy for P_i in Γ is simply a probability measure $\sigma_i \in \Delta(\Phi_i)$ over the set of feasible mechanisms Φ_i. The agent's (behavioral) strategy at date i given the history $h_i \equiv \left(h_i^-, \phi_i\right) \in \mathcal{H}_i$, where $h_i^- \equiv \left(\theta, e_i^-, a_i^-, \phi_i^-, m_i^-, y_i^-\right)$ denotes the upstream history, will be denoted by $\sigma_A(h_i)$; the strategy $\sigma_A(h_i)$ consists of a probability measure $\mu(h_i) \in \Delta(\mathcal{M}_i)$ over the messages $\mathcal{M}_i$ along with a probability measure $\xi(h_i, m_i, y_i) \in \Delta(E_i)$ over effort conditional on h_i, m_i and the realized contract y_i. Finally, we denote by σ_A the agent's complete strategy in Γ—with generic behavioral strategy $\sigma_A(h_i)$ at date i—and by $\sigma \equiv (\{\sigma_i,\}_{i=1}^n, \sigma_A)$ an entire strategy profile for the agent and the n principals. Letting $\mathcal{E}(\Gamma)$ denote the set of *perfect Bayesian equilibria* (PBE) for the sequential common agency game Γ, for any $\sigma^* \in \mathcal{E}(\Gamma)$, we then denote by $\pi_{\sigma^*} : \Theta \to \Delta(\mathcal{A} \times E)$ the *social choice function* (SCF) induced by σ^*. Hereafter, unless otherwise specified, when we refer to equilibrium, we mean PBE. However, it is immediate that all the results for the benchmark model of private contracting remain valid even if one considers *sequential equilibrium* as the solution concept.[14]

3 Difficulties with standard direct mechanisms: an example

Consider the following simple environment with no asymmetric information and no effort. Each principal $i = 1, 2$ must select a decision $a_i \in \mathcal{A}_i$, with $\mathcal{A}_1 = \{t, b\}$ and $\mathcal{A}_2 = \{l, r\}$.[15] Payoffs, respectively for P_1, P_2, and A, are given by the triples in the following table:

$a_1 \backslash a_2$	l	r
t	4 0 1	0 0 1
b	1 0 5	-1 0 0

Game 1

Now suppose that principals were restricted to offer standard direct revelation mechanisms. Because in this example the agent does not possess any exogenous private information, a direct revelation mechanism simply coincides with a decision $a_i \in \mathcal{A}_i$. The set of equilibrium outcomes would then coincide with the set of Nash equilibria in the simultaneous game between the principals only: (t, l) and (t, r).

Next consider a game in which the principals' strategy space is the set of all indirect mechanisms with message space $\mathcal{M}_i = \{0, 1\}$. In this game, (b, l) can also be sustained as an equilibrium

[14] Note that a PBE requires to specify also beliefs for each player. However, because with private contracting these beliefs are always pinned down by Bayes' rule, hereafter we denote a PBE simply by its strategy profile σ^*.

[15] In this example, a contract simply coincides with the choice of a payoff-relevant decision $a_i \in \mathcal{A}_i$.

outcome. The equilibrium features P_1 offering the mechanism that responds to both messages with the decision b and P_2 offering the mechanism that responds to $m_2 = 0$ with l and to $m_2 = 1$ with r. In equilibrium, the agent chooses $m_2 = 0$ with P_2 thus implementing the outcome (b, l). The role of the decision r in P_2's mechanism is to block a possible deviation to t by P_1 : if the latter were to deviate to t, the agent would choose $m_2 = 1$ with P_2 giving a payoff of 0 to P_1.

Clearly, the same outcome (b, l) can be sustained in the game in which the principals offer menus, but not in the game in which they offer standard direct revelation mechanisms. The problem with these mechanisms is that they may not be responsive enough to possible deviations in upstream relationships. This problem can be addressed by considering more general direct mechanisms in which the agent reports, in addition to his type, the endogenous payoff-relevant decisions contracted upstream. In this example with no exogenous private information, an extended direct mechanism for P_1 simply coincides with an element of $\mathcal{A}_1$, whereas for P_2 with a mapping $\phi_2^D : \mathcal{A}_1 \to \mathcal{A}_2$. It is then immediate that the following mechanisms sustain (b, l) : P_1 chooses b, whereas P_2 offers the mechanism $\phi_2^{D*}(t) = r$ and $\phi_2^{D*}(b) = l$. Note that ϕ_2^{D*} is incentive-compatible both on and off equilibrium: whatever decision a_1 is taken upstream, the agent has the incentives to report it truthfully to P_2.

More generally, when information is complete, as in the example above, any deterministic SCF that can be sustained with any arbitrary strategy space for the principals can also be sustained by restricting the principals to offer extended direct mechanisms. The same is true with incomplete information and/or stochastic SCFs, provided one restricts the agent's strategy to be Markov.

To see why, with non-Markov strategies, the direct mechanisms described above may fail to sustain certain outcomes, suppose that the principals' strategy space is now the set of all stochastic mechanisms that map $\mathcal{M}_i = \{0, 1, 2\}$ into lotteries over $\mathcal{A}_i$. Continue to assume that the agent observes a_1 before contracting with P_2. The following is then an equilibrium. P_1 offers the mechanism ϕ_1^* that maps each message m_1 into the lottery that gives t and b with equal probabilities. On her part, P_2 offers a mechanism that responds to $m_2 = 0$ with a lottery δ that gives l with probability equal to $1/4$, to $m_2 = 1$ with a lottery λ that gives l with probability $1/2$, and to $m_2 = 1$ with a (degenerate) lottery μ that gives r with certainty. In equilibrium, A chooses $m_2 = 0$ conditional on t and $m_2 = 1$ conditional on b. These mechanisms are sustained by a non-Markov strategy according to which, whenever P_1 deviates from the equilibrium distribution over $\mathcal{A}_1$, the agent chooses $m_2 = 2$ instead of $m_2 = 0$ after t is realized (note that the agent is indifferent about a_2 when $a_1 = t$).

Clearly, this outcome can also be sustained in the game in which the principals offer menus (of lotteries). However, it cannot be sustained in the game in which they offer extended direct mechanisms. In fact, to prevent a deviation from P_1 it is essential that the agent be given the possibility of choosing the (out-of-equilibrium) lottery μ with P_2 in response to a deviation by P_1 that altered the distribution upstream but nonetheless led to an equilibrium decision. This cannot be done with our extended direct mechanisms.[16]

When downstream principals observe part of the upstream history (such as the mechanisms offered upstream and/or the decisions taken in these mechanisms), there are additional reasons why restricting the principals' strategy space may preclude the possibility of sustaining certain outcomes. In these environments, even the restriction to menus is not always without loss of generality, as discussed in Section 6.

4 Menus

In this section we first show that the menu theorems of simultaneous common agency extend to sequential common agency when contracting is private. We then show that, contrary to simultaneous games, all equilibrium outcomes can be characterized by restricting attention to strategy profiles in which the principals do not mix over their menus and in which the agent does not mix over the contracts he selects with each principal. Finally, we show that, when information is complete, all deterministic social choice functions can be sustained through equilibria in which the agent's strategy is Markov. Proving these results is not only useful for applications, but also a key step for the results in the next section.

Definition 1 *A* **menu** *is a mechanism* $\phi_i^M : \mathcal{M}_i^M \to D_i$ *such that (a)* $\mathcal{M}_i^M \subseteq D_i$*, and (b) for any* $\delta_i \in \mathcal{M}_i^M$, $\phi_i^M(\delta_i) = \delta_i$.

A menu is thus a mechanism whose message space is equal to its image and whose mapping is the identity function. In what follows, we denote by Φ_i^M the set of all possible menus for principal i and by Γ^M the "menu game" in which the set of feasible mechanisms for each P_i is Φ_i^M.

Now consider any enlargement of the menu game, that is, a game in which principals have "more" mechanisms than in Γ^M.

Definition 2 *The game* Γ *is an* **enlargement** *of* Γ^M *(*$\Gamma \succcurlyeq \Gamma^M$*) if for all* $i \in \mathcal{N}$,

(i) there exists an embedding $\alpha_i : \Phi_i^M \to \Phi_i$;[17]

[16] Similar issues arise with incomplete information, as discussed in Section 4.

[17] Formally, the embedding $\alpha_i : \Phi_i^M \to \Phi_i$ is an injective mapping such that, for any pair of mechanisms ϕ_i^M, ϕ_i with $\phi_i = \alpha_i(\phi_i^M)$, the following are true: (a) $\mathrm{Im}(\phi_i) = \mathrm{Im}(\phi_i^M)$; (b) there exists an injective function $\tilde{\alpha}_i : \mathcal{M}_i^M \to \mathcal{M}_i$ from the message space of ϕ_i^M to the message space of ϕ_i such that $\phi_i(\tilde{\alpha}_i(\delta_i)) = \delta_i$ for any $\delta_i \in \mathcal{M}_i^M$.

(ii) for any $\phi_i \in \Phi_i$, $\text{Im}(\phi_i)$ is compact.

A simple example of an enlargement of Γ^M is a game in which $\Phi_i \supseteq \Phi_i^M$ for all i. More generally, an enlargement is a game in which every Φ_i is larger than Φ_i^M in the sense that each menu ϕ_i^M is also present in Φ_i, although possibly with a different representation.[18]

Theorem 1 *Let $\Gamma \succcurlyeq \Gamma^M$. A SCF π can be sustained as an equilibrium of Γ if and only if it can be sustained as an equilibrium of Γ^M.*

First consider the "only if" part of the result. The idea behind the proof is simple and can be sketched as follows. Suppose the SCF π can be sustained as an equilibrium of Γ and let σ be the supporting strategy profile. Now suppose in Γ^M each principal's strategy is such that, for any set of menus $R_i \subseteq \Phi_i^M$,

$$\sigma_i^M(R_i) = \sigma_i \left(\bigcup_{\phi_i^M \in R_i} \Phi_i(\phi_i^M) \right) \tag{1}$$

where $\Phi_i(\phi_i^M) \equiv \{\phi_i \in \Phi_i : \text{Im}(\phi_i) = \text{Im}(\phi_i^M)\}$. This strategy consists in offering each menu ϕ_i^M with a probability equal to the total probability assigned by the original strategy σ_i in Γ to the set of all mechanisms in Γ whose image coincides with that of ϕ_i^M. When in the menu game Γ^M all downstream principals are expected to follow the strategy given in (1), in the continuation game that starts after P_i offers the menu ϕ_i^M, it is clearly optimal for A to induce the same outcomes he would have induced in Γ had P_i offered one of the mechanisms in $\Phi_i(\phi_i^M)$. This also implies that after P_i offers the menu ϕ_i^M, it is optimal for the agent to use the conditional distribution

$$\sigma_i(\phi_i \mid \Phi_i(\phi_i^M))$$

to determine his behavior *at any subsequent information set.*[19] Furthermore, when each P_i and A follow the strategies described above, the distribution over $E \times \mathcal{A}$ is the same as in Γ. Starting from σ one can thus construct an equilibrium σ^M for Γ^M that sustains the same outcomes.

Next, consider the "if" part of the result. Suppose the SCF π can be sustained as an equilibrium of Γ^M and let σ^M denote the sustaining strategy profile. Then there always exists an equilibrium σ in Γ that sustains the same outcomes. The agent's strategy σ_A is constructed by "extending" the original strategy σ_A^M over Γ, as follows. For any history $h_t = ((\phi_i, m_i, y_i, e_i, a_i)_{i=1}^{t-1}, \phi_t)$, σ_A induces the same joint distribution over $E_i \times \mathcal{A}_i$ as the strategy σ_A^M in Γ^M after the history $h_t^M =$

[18] The requirement that each mechanism ϕ_i has a compact image guarantees that the agent's best response is well defined.

[19] The existence of such conditional measures as well as the specification of how the agent "translates" his behavior in Γ in his behavior in Γ^M is described in the Appendix.

$((\phi_i^M, \delta_i, y_i, e_i, a_i)_{i=1}^{t-1}, \phi_t^M)$ where the history h_t^M is constructed from the history h_t replacing each mechanism ϕ_i with the menu ϕ_i^M whose image is the same as that of ϕ_i (i.e. $\text{Im}(\phi_i^M) = \text{Im}(\phi_i)$) and each message m_i with the lottery $\delta_i = \phi_i(m_i)$. As for the principals, each strategy σ_i is simply the "translation" of the original strategy σ_i^M in Γ^M using the embedding α_i, i.e. $\sigma_i = \alpha_i(\sigma_i^M)$.[20] Given the principals' strategies $(\sigma_i)_{i=1}^n$, the agent's strategy σ_A is sequentially rational for the agent. Furthermore, given (σ_A, σ_{-i}), no principal has an incentive to deviate from the strategy σ_i.

Any equilibrium σ^M of the menu game is thus *weakly robust* in the sense of [28]: for any enlargement Γ of Γ^M, there is an extension of the agent's strategy σ_A^M over Γ such that it remains optimal for the principals to offer the same equilibrium menus as in the original game and for the agent to induce the same outcomes.[21]

When Γ is not an enlargement of Γ^M, because the environment imposes certain restrictions on the sets Φ_i, there may exist outcomes in Γ that cannot be sustained as equilibrium outcomes in Γ^M and vice-versa. In this case, one can still characterize all equilibrium outcomes of Γ using menus, but it becomes necessary to restrict the principals to offer only those menus that could have been offered in Γ : that is, the set of feasible menus for P_i must be restricted to $\tilde{\Phi}_i^M \equiv \{\phi_i^M : \text{Im}(\phi_i^M) = \text{Im}(\phi_i)$ for some $\phi_i \in \Phi_i\}$. For simplicity, in the sequel we will restrict attention to environments in which the set of feasible menus for each principal is the entire set of *all* possible menus.

The aforementioned results show that the menu theorems of simultaneous common agency extend to sequential common agency when contracting is private. The reason why the results do not follow directly from those theorems is twofold. First, the decisions the agent takes with his upstream principals are irreversible at the time he contracts with the downstream principals. This means that the agent's behavior at period t can be conditioned not only on the mechanisms offered upstream but also on the endogenous payoff-relevant decisions taken in these mechanisms. Second, at the time the agent commits a decision with principal t, he has not seen yet the mechanisms offered by the downstream principals. These differences do not pose serious problems. However, they require an adaptation of the arguments used in simultaneous games.

[20] Formally, for any measurable set $R \subseteq \Phi_i$, $\sigma_i(R) = \sigma_i^M(\tilde{\Phi}_i^M)$, where $\tilde{\Phi}_i^M = \{\phi_i^M : \alpha_i(\phi_i^M) \in R\}$.

[21] Equilibria in Γ^M are weakly robust, but not necessarily *strongly robust*: one may also be able to construct an enlargement Γ and an extension σ_A of σ_A^M over Γ such that $\sigma = (\sigma_A, \{\alpha_i(\sigma_i^M)\}_{i=1}^n)$ is not an equilibrium of Γ. However, what seems important to us is that any equilibrium outcome in Γ^M remains an equilibrium outcome also in Γ. That it can be sustained by *any* strategy profile $(\sigma_A, \{\alpha_i(\sigma_i^M)\}_{i=1}^n)$ in which σ_A is an extension of σ_A^M is not essential.

4.1 Pure strategies

The next result, which is specific to sequential contracting, goes a step further by showing that, in settings in which $D_i = \Delta(Y_i)$, i.e. in environments in which principals can offer *any* lottery over Y_i, the characterization of the equilibrium outcomes is further simplified by the fact that one can restrict attention to pure-strategy profiles.

Definition 3 *A strategy profile $\sigma \in \mathcal{E}(\Gamma)$ is a* **pure-strategy** *equilibrium if and only if it is an equilibrium in which (a) no principal i mixes over Φ_i; (b) after any history h_i, the agent does not mix over $\mathcal{M}_i$, $\forall i \in \mathcal{N}$.*

While in a pure-strategy equilibrium, the agent does not mix over the messages he sends to the principals, he may however mix over effort.

Theorem 2 *Suppose $D_i = \Delta(Y_i)$ for all i. A SCF π can be sustained as an equilibrium of Γ^M only if it can be sustained as a pure-strategy equilibrium.*

Suppose the SCF π is sustained by a mixed-strategy equilibrium σ^M in which P_i randomizes over Φ_i^M according to σ_i^M and in which, given some ϕ_i^M, the agent randomizes over the different lotteries in ϕ_i^M according to $\mu(h_i^-, \phi_i^M)$.[22] The same SCF can be sustained by an equilibrium $\mathring{\sigma}^M$ in which P_i offers with probability one the menu $\mathring{\phi}_i^M$ that contains the compound lotteries (indexed by h_i^-)

$$\int_{\phi_i^M \in \Phi_i^M} \int_{\delta_i \in \mathcal{M}_i^M} \delta_i d\mu(h_i^-, \phi_i^M) d\sigma_i^M \tag{2}$$

that can be obtained by mixing with distribution σ_i^M over the compound lotteries $\int \delta_i d\mu(h_i^-, \phi_i^M)$ that the agent would have induced, for each h_i^-, by following his original strategy $\mu(h_i^-, \phi_i^M)$. Holding constant σ_i^+, when the upstream history is h_i^- and P_i offers the menu $\mathring{\phi}_i^M$, it is then clearly optimal for A to choose with probability one the lottery $\delta_i(h_i^-, \phi_i^M)$ given in (2); at any downstream information set, the agent then adjusts his behavior to induce the same outcomes as in the original equilibrium σ^M.[23]

When instead P_i offers any menu $\phi_i^M \neq \mathring{\phi}_i^M$, let the agent choose with probability one the lottery $\delta_i(h_i^-, \phi_i^M)$ in $Supp[\sigma_A^M(h_i^-, \phi_i^M)]$ that minimizes P_i's expected payoff taking into account that at any subsequent information set the agent's behavior will continue to be determined by the original strategy σ_A^M. When all other principals are expected to follow the same strategy as in σ^M,

[22] Recall that $\mu(h_i, \phi_i^M) \in \Delta(\mathcal{M}_i^M)$ denotes the agent's communication strategy with P_i.

[23] The details of how the agent adjusts his behavior downstream to induce the same distribution over $E \times \mathcal{A}$ as in the original equilibrium σ^M are in the appendix.

the strategy $\mathring{\sigma}_A^M$ constructed this way is clearly sequentially optimal for the agent. Furthermore, given this strategy, no principal has an incentive to deviate. Iterating across all i, it is then possible to construct a pure-strategy equilibrium that implements the same outcomes as σ^M.

The point here is that any randomization induced by a principal mixing over her mechanisms or by the agent mixing over his messages can be replicated by having the principal offer a single menu of lotteries over contracts and having the agent selecting deterministically a lottery as a function of his upstream contractual history. For the transformation described above to work, it is however essential that any principal can offer any lottery in $\Delta(Y_i)$, which explains the qualification in the theorem. This is an assumption that we will maintain in most of the subsequent analysis. We will however be careful to clarify how our results extend to environments in which $D_i \subsetneqq \Delta(Y_i)$ for some i, a restriction that is common, for example, in applications that assume that only (degenerate lotteries over) deterministic contracts are feasible.

The possibility of restricting attention to pure-strategy profiles is appealing for two reasons. First, it is reminiscent of a similar result for games with a single mechanism designer. Second, it is common practice in applications to restrict attention to pure-strategy equilibria—Theorem 2 provides a possible justification for such a practice.

It is also important to note that such a result does not have a counterpart in simultaneous common agency. In fact it is essential that the agent takes decisions with each principal after having committed irreversible decisions upstream and before having seen the mechanisms offered downstream. To see this, consider a simple environment with only two principals, with no adverse selection and with no effort, so that $|\Theta| = |E_i| = 1$, $i = 1, 2$. Suppose in addition that the sets of primitive decisions are $\mathcal{A}_1 = \{t, b\}$ and $\mathcal{A}_2 = \{l, r\}$. The payoffs, respectively, for P_1, P_2 and A, are as in the following table:

$a_1 \backslash a_2$	l			r		
t	2	2	1	0	0	0
b	1	0	1	2	1	2

Game 2

The social choice function that selects (t, l) and (b, r) respectively with probability $q \in (0, 1)$ and $1 - q$ can be sustained as an equilibrium both in our sequential game (with private contracting) and in the canonical simultaneous version of the game considered in the literature. In both games, with probability $q \in (0, 1)$, P_1 offers the degenerate menu that contains the lottery that selects t with certainty, whereas with probability $1 - q$ she offers the degenerate menu that contains the lottery that selects b with certainty. On her part, P_2 offers the menu that contains both the lottery that gives l with certainty and the lottery that gives r with certainty. The agent selects l with P_2 if $a_1 = t$ and r if $a_1 = b$. It is immediate to see that, in the sequential game, the same outcome can be sustained by a pure-strategy equilibrium in which P_1 offers a degenerate menu containing the lottery that gives t and b respectively with probability q and $1 - q$. This is not true in the simultaneous version of the game. Indeed, to sustain the same outcomes with a pure-strategy profile, it is necessary that P_1 offers a menu that contains both the degenerate lottery that gives t with certainty and the degenerate lottery that gives b with certainty. But then A strictly prefers to induce (b, r) rather than randomizing over (b, r) and (t, l).

4.2 Markov-perfect equilibria

In applications, it is also customary to restrict attention to equilibria in which a player's strategy depends on the upstream history only through its payoff relevant component.[24] With private contracting, this is necessarily the case for principals. In what follows, we examine the implications of assuming such a property holds also for the agent's strategy.

Formally, for any upstream history $h_i^- = \left(\theta, \phi_i^-, m_i^-, y_i^-, e_i^-, a_i^-\right)$, let

$$\theta_i^E \equiv \left(\theta, e_i^-, a_i^-\right) \in \Theta_i^E \equiv \Theta \times E_i^- \times \mathcal{A}_i^-$$

denote the payoff-relevant component of h_i^-. For a reason that will become clear in the next section, hereafter we refer to θ_i^E as the agent's *extended type*.

Definition 4 *The agent's strategy σ_A is* **Markov** *at $t = i$ if and only if, for any ϕ_i and any pair of upstream histories $h_i^-, \hat{h}_i^-$ with $h_i^- = (\theta_i^E, \phi_i^-, m_i^-, y_i^-)$ and $\hat{h}_i^- = (\theta_i^E, \hat{\phi}_i^-, \hat{m}_i^-, \hat{y}_i^-)$,*

$$\sigma_A(h_i^-, \phi_i) = \sigma_A(\hat{h}_i^-, \phi_i).$$

A **Markov-perfect equilibrium** *(hereafter, MPE) is an equilibrium in which σ_A is Markov at any t.*

Assuming the agent's strategy is Markov seems appealing. However, it is important to understand what social choice functions cannot be sustained with these strategies. The next result provides an answer and can be seen as a possible justification for restricting attention to Markov perfect equilibria in certain environments.

Theorem 3 *Suppose information is complete (i.e. $|\Theta| = 1$). Then any deterministic SCF π that can be sustained as an equilibrium of Γ^M can also be sustained as a pure-strategy MPE.*

[24] See, among others, [13] and [5].

The idea behind the proof is the following. When preferences are common knowledge and the SCF π is deterministic, there is a unique sequence of equilibrium decisions and hence a unique sequence of equilibrium extended types. Now, assume the agent's strategy in the equilibrium σ^{M*} that supports π is not Markov. Then consider the alternative strategy profile $\tilde{\sigma}^M$ in which all principals offer the same menus as in σ^{M*} and in which the agent behaves as follows. At any period t, if the extended type θ_t^E is the equilibrium one, the agent implements the equilibrium decisions for that extended type, independently of which particular upstream history h_t^- led to θ_t^E. If instead, θ_t^E is not the equilibrium extended type, then let $j \leq t-1$ be the first date at which a departure from the sequence of equilibrium decisions occurred. Starting from period t, at any downstream information set, the agent then chooses among the decisions that are sequentially optimal for him, those that minimize the payoff of principal j. Given this Markov strategy for the agent, no principal has a profitable deviation. The strategy profile $\tilde{\sigma}^M$ is thus an equilibrium for Γ^M and sustains the same outcomes as σ^{M*}.

In the argument sketched above, it is essential that the agent be able to identify (and punish) an upstream principal who deviated from equilibrium play simply by looking at the extended type. This is always possible when information is complete and the SCF π is deterministic. When instead $|\Theta| > 1$ and/or the SCF π is stochastic, it may be necessary to have the agent condition his behavior not only on θ_t^E, but also on payoff-irrelevant information such as the mechanisms offered upstream. This permits the agent to punish deviations that altered the *distribution* of upstream payoff-relevant decisions but nevertheless led to equilibrium extended types.

It is also worth noticing that even if one is interested in characterizing only deterministic SCFs, it may be necessary to allow for menus that contain (non-degenerate) lotteries. To see this, assume that $\mathcal{A}_1 = \{t, b\}$ and $\mathcal{A}_2 = \{l, m, r\}$. The payoffs, respectively for P_1, P_2 and A are given by the triples (u_1, u_2, v) in the following table:

$a_1 \backslash a_2$	l	m	r
t	1 2 1	0 1 0	0 0 0
b	2 3 1	−2 4 0	3 0 2

Game 3

The outcome (t, l) cannot be sustained by restricting attention to menus that contain only degenerate lotteries. Indeed, to punish a possible deviation by P_1 to b, the equilibrium menu offered by P_2 should also contain m. However, conditional on b, A necessarily prefers l over m, which implies that it is impossible to punish P_1's deviation while satisfying the agent's rationality. On the other hand, the outcome (t, l) can be sustained by having P_2 offer any menu that together with the equilibrium contract l contains a lottery that gives l with probability p, m with probability q, and r with probability $1-p-q$, where (p,q) is any pair of positive real numbers that satisfies $q \leq (1-p)/2$ and $q \geq (2-p)/5$. The first bound on q guarantees that, choosing this lottery is incentive-compatible for the agent given b, while the second ensures that P_1 does not find it profitable to deviate.

5 Extended Direct Mechanisms

Building on the results in the previous section, we now show that, in most cases of interest for applications, the characterization of the equilibrium outcomes can be further simplified by restricting the principals to offer menus that can be conveniently described as extended direct mechanisms.

Definition 5 *An* **extended direct mechanism** *is a mapping* $\phi_i^D : \Theta_i^E \to D_i$. *A* **revelation game** Γ^D *is a game in which the principals' strategy space is* $\Delta(\Phi_i^D)$, *where* Φ_i^D *denotes the set of all possible extended direct mechanisms for* P_i, $i = 1, ..., n$.

Extended direct mechanisms can thus be thought of as menus whose allocations are indexed by the payoff-relevant component of the agent's upstream history. Extended direct mechanisms are thus the analogue of standard direct revelation mechanisms with the only difference being that they may specify contracts also for extended types that have zero measure on the equilibrium path.

Next, let $\tilde{V}(h_i^-, \delta_i, \sigma_i^+)$ denote the maximal payoff that A can obtain in the continuation game that starts at $t = i$ when the upstream history is $h_i^- = \left(\theta_i^E, \phi_i^-, m_i^-, y_i^-\right)$, he chooses a lottery δ_i with P_i, and the downstream principals' strategy profile is σ_i^+. Clearly, $\tilde{V}(h_i^-, \delta_i, \sigma_i^+)$ depends on h_i^- only through θ_i^E; in the following, we thus denote the agent's continuation payoff by $V(\theta_i^E, \delta_i, \sigma_i^+)$.

Definition 6 *(i) Fix a strategy profile* σ_i^+ *for the downstream principals. A mechanism* ϕ_i^D *is* **incentive-compatible** *if and only if, for any* $\theta_i^E \in \Theta_i^E$ *and any* $\delta_i' \in \mathrm{Im}(\phi_i^D)$,

$$V(\theta_i^E, \phi_i^D(\theta_i^E), \sigma_i^+) \geq V(\theta_i^E, \delta_i', \sigma_i^+).$$

(ii) The agent's strategy is **truthful** *in* ϕ_i^D *if and only if, for any upstream history* h_i^-, *the agent truthfully reports* θ_i^E.[25] *Given a strategy profile* $\sigma^D \equiv (\{\sigma_i^D\}_{i=1}^n, \sigma_A^D)$ *for* Γ^D, σ_A^D *is said to be truthful if and only if it is truthful in every* $\phi_i^D \in Supp[\sigma_i^D]$, $\forall i \in \mathcal{N}$.

(iii) A **truthful equilibrium** *for* Γ^D *is an equilibrium in which each mechanism* $\phi_i^D \in Supp[\sigma_i^D]$ *is incentive compatible and* σ_A^D *is truthful.*

[25] Formally, for any $h_i^- = (\theta_i^E, \phi_i^-, m_i^-, y_i^-)$, $\hat{\theta}_i^E \in Supp[\mu(h_i^-, \phi_i^D)] \Longrightarrow \hat{\theta}_i^E = \theta_i^E$.

Whether the mechanism ϕ_i^D is incentive-compatible depends on the downstream principals' strategy profile σ_i^+. Contrary to games with a single mechanism designer, incentive-compatibility must thus be established by backward induction. Also note, given a mechanism ϕ_i^D, the agent's strategy is truthful in ϕ_i^D if and only if the agent truthfully reports *any* extended type. This implies that when a deviation from equilibrium occurred at some date $t < i$, the agent still reports his extended type truthfully to P_i.

The following result relates the set of equilibrium outcomes that can be sustained with extended direct mechanisms to the set of outcomes that can be sustained with menus. In virtue of the results in the previous section, it should be clear that this is the relevant comparison.

Theorem 4 *Any SCF π that can be sustained as a MPE of Γ^M can also be sustained as a pure-strategy truthful MPE of Γ^D. Furthermore, any SCF that can be sustained as an equilibrium of Γ^D can also be sustained as an equilibrium of Γ^M.*

Consider the first part. Suppose there exists a MPE σ^{M*} in the menu game that sustains the SCF π. By Theorem 2, without loss of generality one can assume that σ^{M*} is a pure-strategy profile. That the agent's strategy in σ^{M*} is Markov in turn implies that, for any θ_t^E and for any menu ϕ_t^M, there exists a unique lottery $\delta_t(\theta_t^E, \phi_t^M)$ such that the agent selects the lottery $\delta_t(\theta_t^E, \phi_t^M)$ from the menu ϕ_t^M when his extended type is θ_t^E, irrespective of the upstream history h_t^- that has conducted to θ_t^E. The equilibrium that sustains π in Γ^D is then constructed by having each principal offer the direct mechanism ϕ_t^{D*} that responds to each θ_t^E with the lottery $\delta_t(\theta_t^E, \phi_t^{M*})$, where ϕ_t^{M*} is the equilibrium menu in Γ^M. When offered the equilibrium mechanism ϕ_t^{D*}, the agent then responds by reporting θ_t^E truthfully and then selecting effort according to σ_A^M as if the game were Γ^M, the menu offered by P_t were ϕ_t^{M*} and the message sent to P_t were $\delta_t(\theta_t^E, \phi_t^{M*})$. When instead the agent is offered a mechanism $\phi_t^D \neq \phi_t^{D*}$, the agent behaves according to σ_A^M as if the game were Γ^M and the menu offered by P_t were ϕ_t^M, where $\text{Im}(\phi_t^M) = \text{Im}(\phi_t^D)$. Given the aforementioned strategies for the principals, the agent's (Markov) strategy described above is clearly sequentially optimal. Furthermore, given $(\sigma_A^{D*}, \sigma_{-i}^{D*})$, no principal has an incentive to deviate and offer a mechanism $\phi_t^D \neq \phi_t^{D*}$. We conclude that σ^{D*} is a (pure-strategy) MPE of Γ^D and sustains the same outcomes as σ^{M*} in Γ^M.

Note that this result presumes $D_i = \Delta(Y_i)$ for all i. As discussed in the previous section, this is necessary to guarantee that any SCF π that can be sustained as a MPE of Γ^M can also be sustained as a pure-strategy MPE. However, what matters for the possibility of sustaining π as a MPE of Γ^D is only the purity of the agent's strategy in Γ^M; that the principals' strategies are also pure is not important. To understand this, note that when certain lotteries are not feasible, i.e.

when $D_i \subsetneqq \Delta(Y_i)$ for some i, then it may be impossible to replicate the outcomes induced in Γ^M by the agent mixing over the different contracts in a menu with a direct mechanism that simply asks the agent to report his extended type.[26] In environments in which there are restrictions on the sets of feasible lotteries, the result in the first part of the theorem must thus be replaced with the following: Any SCF π that can be sustained as a MPE of Γ^M *in which the agent's strategy is pure* can also be sustained as a truthful MPE of Γ^D.

Next, consider the second part of the theorem. Take any $\sigma^D \in \mathcal{E}(\Gamma^D)$. Irrespective of whether the agent's strategy in σ^D is Markov and of whether σ^D is a pure- or mixed-strategy equilibrium, there always exists a $\sigma^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes. Note that this result does not follow from Theorem 1: in fact, in general, Γ^D is not an enlargement of Γ^M (nor is Γ^M an enlargement of Γ^D). The equilibrium σ^M that sustains π in Γ^M is constructed by having each principal offer each menu ϕ_t^M with the same probability she would have offered all direct mechanisms with the same image as ϕ_t^M. That is, σ_t^M is constructed from σ_t^D using the same transformation as in (1). As for the agent, if P_t offered a menu ϕ_t^M whose image coincides with the image of one of the direct mechanisms in Φ_t^D, then A induces the same outcomes he would have induced in Γ^D had P_t offered[27]

$$\Phi_t^D(\phi_t^M) \equiv \{\phi_t^D \in \Phi_t^D : \text{Im}(\phi_t^D) = \text{Im}(\phi_t^M)\}.$$

If instead, P_t offered a menu ϕ_t^M that is not in the image of any of the mechanisms in Φ_t^D, then A behaves according to σ_A^D as if the game were Γ^D and the mechanism offered by P_t were ϕ_t^D, where ϕ_t^D is such that

$$\phi_t^D(\theta_t^E) \in \arg \max_{\delta_t \in \text{Im}(\phi_t^M)} V(\theta_t^E, \delta_t, \sigma_t^+) \quad \forall \theta_t^E \in \Theta_t^E.$$

These strategies for the principals and the agent clearly constitute an equilibrium for Γ^M that induces the same outcomes of σ^D.

That any equilibrium outcome of the revelation game is also an equilibrium outcome of the menu game is important because it guarantees that the outcomes that one obtains by restricting the principals to offer extended direct mechanisms are not artificially sustained by the impossibility for the principals to offer certain menus that are not available in the revelation game.

Describing menus as direct revelation mechanisms has proved very convenient in games with a single mechanism designer. The same approach can be used to characterize MPE outcomes in

[26] The impossibility of sustaining all outcomes with direct mechanisms when $D \subset \Delta(Y)$ is clearly not specific to common agency; it also applies to settings with a single principal.

[27] That is, A uses $\sigma_t^D(\phi_t^D \mid \Phi_t^D(\phi_t^M))$ to determine his behavior at any subsequent information set.

sequential common agency. We illustrate such a possibility in a canonical buyer-sellers example with incomplete information in the Supplementary Material.

6 Alternative extensive forms

The model considered so far assumes that contracting is *private*: the mechanism offered by principal i, the contract selected by the agent and the decisions taken in response to this contract, are all information that is private to A and P_i, in the sense that it cannot be observed by any of the other principals. It was also assumed that a pair of payoff-relevant decisions (one for the agent, the other for the principal) is committed at each date instead of some decisions being taken only after all principals have offered their mechanisms. Finally, the sequence of bilateral relationships was exogenous in the sense that the agent could not choose which principal to contract with at each date.

Although these assumptions seem reasonable for many applications, it is important to understand how the preceding results must be adapted to accommodate alternative extensive forms examined in the literature. This is what we do in this section. Hereafter, we summarize the key insights. The formal statements of the results and their corresponding proofs can be found in the Supplementary Material.

Observability of upstream payoff-relevant decisions. Consider an environment in which principals observe upstream payoff-relevant decisions before choosing their mechanisms.[28]

As long as one restricts attention to Markov-perfect equilibria, this extension poses no problems to our characterization results: all Markov-perfect equilibrium outcomes can be characterized by restricting the principals to offer either menus or extended direct revelation mechanisms. Furthermore, because the choice of a mechanism is now contingent on the observable upstream payoff-relevant decisions (e_i^-, a_i^-), when using direct mechanisms, there is no need to ask the agent to report such information. One can thus drop (e_i^-, a_i^-) from the agent's message and restrict attention to standard direct revelation mechanisms in which the agent simply reports his exogenous type θ.

If, instead, one is also interested in equilibrium outcomes sustained by non-Markov strategies, then restricting the principals to offer menus (or direct mechanisms) may preclude a complete characterization. The reason is that restricting the principals' strategy space means restricting

[28] Upstream decisions are observable in [17], [5], [21], [31], and [1]. In these papers, the observability of upstream decisions is exogenous. In contrast, [8] and [23] examine models in which the observability of upstream decisions is controlled by upstream principals.

the extent to which principals' out-of-equilibrium beliefs can differ one from the other.[29] The fact that, in an indirect game, the same menu can be offered with different mechanisms may permit downstream principals to have different beliefs about the particular mechanism used upstream to select an out-of-equilibrium decision. When the agent's strategy is non-Markov, this means allowing the principals to have different expectations about the agent's behavior in downstream relationships, a property that may be essential to sustain certain outcomes (see [9] for an example that illustrates such a possibility). For the same reason, the set of equilibrium outcomes of the menu game is no longer a superset of the set of equilibrium outcomes of the revelation game—the same menu can in fact be offered through multiple direct mechanisms; what remains true is that the set of *Markov-perfect* equilibrium outcomes is the same in the two games, as shown in Theorem 5 in the Supplementary Material.

Observability of upstream mechanisms. Next, consider an environment in which every P_i, $i = 2, ..., n$, observes the mechanisms ϕ_i^- offered upstream before choosing her own mechanism. As in the benchmark model, P_i does not observe $(m_i^-, y_i^-, e_i^-, a_i^-)$.

In this setting, restricting the principals' strategy space may mean restricting the extent to which payoff-irrelevant distinctions among mechanisms can be used as correlation devices for the principals' decisions. This may preclude a complete characterization of the equilibrium outcomes (see [9] for an illustration). However, one can safely restrict the principals to offer menus if one is interested only in equilibria in which the principals do not mix—to the best of our knowledge, the case considered in all papers that assume observability of upstream mechanisms.

One way of restoring the possibility of using menus to sustain also mixed-strategy equilibria is to allow for public randomizing devices, as in the case of correlated equilibria.[30] Alternatively, one may allow the principals to send to each other cheap talk messages whose role is to replicate the role of payoff-irrelevant distinctions among mechanisms that are used as correlation devices.

Furthermore, all equilibrium outcomes that can be sustained in the menu game by restricting the principals' strategies to be pure can also be sustained in the revelation game. Contrary to the case of private contracting, this is true irrespective of whether the agent's strategy is Markov. This is because, in this setting, downstream principals can observe directly a deviation by an upstream principal. They can thereby respond to such a deviation by changing the contract offered to each extended type. There is thus no need to offer the same extended type of the agent a menu of contracts to choose from as a function of what happened upstream—see Theorem 6 in the Supplementary Material.

[29] Note that there are no out-of-equilibrium beliefs in the benchmark model of private contracting.

[30] A similar idea has been explored in [14] in the context of a simultaneous multi-agent-multi-principal setting.

However, note that, in this environment, equilibria in direct mechanisms may fail to be robust, in the sense of [28]: there may exist SCFs that can be sustained as equilibria of the revelation game that cannot be sustained as equilibria of the menu game. The reason is that direct mechanisms restrict a principal's ability to offer the agent out-of-equilibrium allocations. These allocations may be essential to induce a certain behavior by the agent and the downstream principals in the continuation game that starts after a mechanism has been announced. By implication, a principal may have a profitable deviation in the menu game even if she did not have it in the revelation game (see the Supplementary Material for an example). Such out-of-equilibrium allocations play no role under private contracting, for they do not affect the behavior of the downstream principals. As discussed after Theorem 4, in this case, any deviation to a menu that could not be offered in the revelation game can be punished by having the agent behave as if the principal offered a direct mechanism that gives to each extended type a contract that is optimal for that extended type among those that are in the menu, *holding fixed the mechanisms offered downstream*. The possibility of holding fixed the mechanisms offered downstream however disappears in environments in which upstream mechanisms are observable. By implication, certain deviations to menus that could not be offered as direct mechanisms may be impossible to punish, thus making equilibrium outcomes in the revelation game not robust. However, this problem with direct mechanisms disappears when direct mechanisms "span" all possible menus[31]—a case that often arises in settings with a continuum of types. Because in this case $\Gamma^D \succcurlyeq \Gamma^M$, then all equilibrium outcomes of Γ^D are also equilibrium outcomes of Γ^M.

Endogenous sequence of bilateral relationships. Consider the following variant of our benchmark contracting game. There are $T > 2$ periods. In each period, *all* principals simultaneously offer the agent a mechanism and the agent chooses at most one mechanism to participate in. The agent may participate in a mechanism offered by the same principal multiple times. As in the benchmark model, contracting is private so that the principals do not observe other principals' mechanisms, nor the messages, the contracts, or the decisions taken in these mechanisms.

This setting may correspond, for example, to an environment in which at each period, a consumer decides which seller to use for the provision of a certain good or service.[32]

The difficulties with this extensive form come from the fact that the agent's continuation payoff at date t may now depend not only on the upstream payoff-relevant decisions but also on the mechanisms, the messages and the contracts selected upstream. In fact, such payoff-irrelevant information may determine which mechanisms will be offered downstream. Furthermore, because the

[31] That is, when for any $\phi_i^M \in \Phi_i^M$, there exists a $\phi_i^D \in \Phi_i^D$ such that $\mathrm{Im}(\phi_i^D) = \mathrm{Im}(\phi_i^M)$.

[32] This is the environment examined, for example, in [2], [30], and [20].

principals observe some of the decisions taken upstream (e.g. the agent's participation decisions), the problems with out-of-equilibrium beliefs discussed above for the case of observable actions carry over to this environment.

These problems, however, disappear if one restricts attention to equilibria in which not only the agent's strategy but also the principals' strategies are Markov. All MPE outcomes can in fact be characterized with menus. They can also be characterized as truthful equilibria, but only if the agent is restricted to contract with each principal at most once. If instead the agent can contract with the same principal multiple times, then having the agent report truthfully is in general not without loss of generality. The reasons are the same as in the literature that assumes a single principal who lacks commitment. As shown in [6], one can safely restrict the principal to offer direct mechanisms but should not restrict the agent to report truthfully in each period. The same is true with multiple principals. One can characterize all MPE outcomes by having the principals compete in extended direct mechanisms, but should not assume the agent reports truthfully to each principal. Restricting attention to truthful equilibria is however fine if the agent does not possess any private information vis a vis the principal; this is the case when preferences are common knowledge (i.e. $|\Theta| = 1$) and when all players' strategies are pure as, for example, in [5].

Sequential offering as opposed to sequential contracting. Sequential offering refers to an environment in which principals offer their mechanisms sequentially, but the agent takes payoff-relevant decisions only after observing the mechanisms offered by all principals. This is in contrast to sequential contracting where some payoff-relevant decisions are committed at each date.[33]

First, consider an environment similar to the one examined in the benchmark model but where the agent's effort is chosen only at the very end, say at $t = n + 1$. At $t = i$, for $i = 1, ..., n$, P_i offers a mechanism $\phi_i : \mathcal{M}_i \to D_i$, where D_i is the set of feasible lotteries over the contracts $y_i : E \to \Delta(\mathcal{A}_i)$, with $e \in E$ now denoting some common effort.[34] After sending the message m_i, the agent may or may not observe the realization y_i of the lottery $\delta_i = \phi_i(m_i)$. As in the benchmark model, principals do not observe other principals' mechanisms, nor the messages the agent sends to other principals, nor the contracts selected in response to these messages.

It is immediate that this extensive form poses no problems to our characterization results. It

[33] Sequential offering has been examined, among others, in [2], [7], [30],[17], [20], [29], [32].

[34] One may think of e as a vector of decisions $e = (e_1, ..., e_n)$. Depending on the application of interest, the set of feasible contracts Y_i may be more or less restricted. For example, when e_j stands for a decision specific to the relationship between A and P_j, it may be reasonable to assume that y_i must not depend on e_j. This could be the case, for example, if e_j represents the quantity/quality of the product of seller j and if the latter could not be observed by P_i.

suffices to adjust the definition of extended type to take into account that, because the decisions (e, a) are now chosen only at $t = n + 1$, at $t = i$, the component of the upstream history that is relevant for the agent's continuation payoff becomes the profile of contracts selected upstream along with the agent's exogenous type. That is, all the results in the benchmark model extend to this environment by letting $\theta_i^E \equiv (\theta, y_i^-)$, or $\theta_i^E \equiv (\theta, \delta_i^-)$, depending on whether at every $t < i$, the agent observes the contract y_t or only the lottery $\delta_t = \phi_t(m_t)$.

Next, consider an environment in which principals offer their mechanisms sequentially, but where the agent sends the messages $(m_1, ..., m_n)$ simultaneously at $t = n + 1$. Given the contracts $(y_i)_{i=1}^n$ selected by the lotteries $(\delta_i)_{i=1}^n = (\phi_i(m_i))_{i=1}^n$, the agent then chooses an action $e \in E$ and finally the principals' decisions $(a_i)_{i=1}^n$ are determined by the contracts $(y_i)_{i=1}^n$. If the principals do not observe other principals' mechanisms, this setting is strategically equivalent to simultaneous common agency; the menu theorems of [28] and [19] then guarantee that the entire set of equilibrium outcomes can be characterized by restricting the principals to offer menus. In many applications, it is however appealing to restrict attention to equilibria in which the decisions the agent induces with each principal P_i depend on the menu offered by P_i, the agent's type θ and the decisions δ_{-i} the agent is inducing with the other principals (but not the menus, or more generally the mechanisms, offered by the latter). Imposing this property is analog to restricting the agent's strategy to be Markov in a sequential contracting game. In [26], we show that any SCF that can be sustained by an equilibrium in which the agent's strategy satisfies the aforementioned Markov property can also be sustained as a truthful equilibrium of a game in which the principals offer incentive-compatible extended direct mechanisms. The definition of these mechanisms in simultaneous games is adjusted to take into account that decisions are determined simultaneously: instead of reporting the payoff-relevant decisions (e_i^-, a_i^-) contracted upstream, the agent is asked to report (in addition to θ) the lotteries δ_{-i} he is inducing with the other principals. The agent's strategy is then truthful if the decisions δ_{-i} he reports to P_i coincide with the true decisions $(\phi_j(m_j))_{j \neq i}$ induced with all principals other than i by sending the messages $m_{-i} = (m_j)_{j \neq i}$.[35]

In the Supplementary Material, we show that the same mechanisms also permit one to sustain all *pure-strategy* MPE outcomes in sequential games in which downstream principals observe the mechanisms offered upstream before choosing their own mechanism.[36] The reasons why extended

[35] When the agent's strategy is not Markov, we show that (i) pure-strategy equilibrium outcomes can be characterized by having the agent report the identity of a deviating principal (if any) in addition to his type and the decisions he is inducing with the other principals, and (ii) that all mixed-strategy equilibrium outcomes can be characterized through incentive-compatible set-valued direct mechanisms in which the agent is offered multiple (payoff-equivalent) decisions to choose from as a function of his type and the decisions he is inducing with the other principals.

[36] Clearly, the same outcomes can also be characterized assuming the principals compete in menus.

direct mechanisms, or menus, fail to sustain certain mixed-strategy equilibrium outcomes are the same discussed for environments in which upstream mechanisms are publicly observable.

7 Concluding remarks

We discussed the intricacies that emerge in environments in which multiple principals contract sequentially and non-cooperatively with the same agent and provided characterization results useful for applications. Our benchmark model was one of private contracting in which downstream principals do not observe the mechanisms offered upstream, nor the decisions taken in these mechanisms. We also assumed that the sequence of bilateral relationships was exogenous and that a pair of payoff-relevant decisions, one for the principal, the other for the agent, was committed at each period.

For this environment, we first showed that all equilibrium outcomes can be characterized by assuming the principals compete in menus, thus proving that the menu theorems of simultaneous common agency extend to this environment. We then proceed by showing that, when lotteries are feasible, then all equilibrium outcomes can be sustained through pure-strategy profiles, as in games with a single mechanism designer (but not in simultaneous common agency). We also showed that when information is complete, any deterministic social choice function (that is, any outcome sustained by the agent not mixing over effort and the principals not mixing over their contracts) can always be sustained as a Markov-perfect equilibrium (that is, by restricting each player's strategy to depend only on payoff-relevant information).

Starting from these results, we then introduced a class of direct mechanisms in which the agent is asked to report his exogenous type along with the payoff-relevant decisions contracted upstream. We showed that all MPE outcomes of the menu game are also MPE outcomes in the game in which principals offer these direct mechanisms. The advantage of these mechanisms is that they permit one to use techniques from standard mechanism design (i.e. incentive-compatibility) to identify necessary and sufficient conditions for equilibrium outcomes. There are however two differences with respect to standard mechanism design. First, incentive compatibility must be established by backward induction: whether a mechanism is incentive compatible or not depends on the mechanisms offered downstream. Second, a mechanism must specify incentive-compatible allocations also for extended types that have zero measure on the equilibrium path: this is because out-of-equilibrium allocations may be necessary to punish upstream deviations.

Finally, we discussed the problems with menus and extended direct mechanisms that emerge in environments in which downstream principals observe the mechanisms and/or the payoff-relevant decisions selected upstream, or when the sequence of bilateral relationships is endogenously determined by the agent's participation decisions. Building on the results for the benchmark model, we proposed solutions that, although do not always permit a complete equilibrium characterization, allow one to characterize the outcomes that are typically of interest for applications. While the various extensive forms considered here do not exhaust all the cases examined in the literature, we expect our results to be useful for many applications of sequential common agency.

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Appendix

Proof of Theorem 1. The proof is in two parts. Part 1 proves that for any equilibrium σ of Γ, there exists an equilibrium σ^M of Γ^M that implements the same outcomes. Part 2 proves the converse.

Part 1. Let $\mathcal{Q}_i$ be a generic partition of Φ_i and denote by $Q_i \in \mathcal{Q}_i$ an element of $\mathcal{Q}_i$. Consider now a partition game $\Gamma^{\mathcal{Q}_i}$ in which at $t = i$, P_i chooses an element Q_i of $\mathcal{Q}_i$, then A selects a mechanism ϕ_i from Q_i, sends a message $m_i \in \mathcal{M}_i$ to P_i and finally, given the realization y_i of the lottery $\delta_i = \phi_i(m_i)$, chooses effort e_i. At any other date $j \neq i$, both P_j and A have exactly the same choice sets as in Γ. Now, let Γ_i^M be a game with the same structure as Γ, except that at $t = i$, the strategy space for P_i is $\Delta(\Phi_i^M)$.

The proof of Part 1 is in three steps. Step 1 identifies a partition of Φ_i that makes the agent indifferent between any two mechanisms in the same cell $Q_i \in \mathcal{Q}_i$ and then constructs an equilibrium $\hat{\sigma}$ of $\Gamma^{\mathcal{Q}_i}$ that implements the same outcomes as σ. Step 2 uses the construction in Step 1 to derive an equilibrium $\mathring{\sigma}$ in Γ_i^M which also implements the same outcomes as σ. Finally, Step 3 shows how the previous two steps can be applied recursively to construct an equilibrium σ^M for Γ^M that implements the same outcomes as σ.

Step 1. Start by considering a generic partition $\mathcal{Q}_i$ of Φ_i that consists of measurable sets[37] and consider the following strategy profile for $\Gamma^{\mathcal{Q}_i}$. For P_i, let $\hat{\sigma}_i \in \Delta(\mathcal{Q}_i)$ be the probability measure

[37] In the sequel, we assume that Φ_i is a Polish space and whenever we talk about measurability, we mean with respect to the Borel σ-algebra Σ on Φ_i.

over $\mathcal{Q}_i$ generated by the original strategy σ_i in Γ. That is, for any subset R of $\mathcal{Q}_i$ whose union is measurable,

$$\widehat{\sigma}_i(R) = \sigma_i(\bigcup R).$$

For any P_j, with $j \neq i$, simply let $\hat{\sigma}_j = \sigma_j$. Next consider the agent. For any $t < i$ and for any h_t, let A's strategy be the same as in σ, that is, $\hat{\sigma}_A(h_t) = \sigma_A(h_t)$. At $t = i$, given any history $h_i = \left(\phi_i^-, m_i^-, y_i^-, e_i^-, a_i^-, Q_i\right)$, A selects each mechanism from Q_i using the regular conditional probability distribution $\sigma_i(\cdot|Q_i)$.[38] At any subsequent information set, A then simply behaves according to the original strategy σ_A, as if at the beginning of $t = i$, the history had been $h_i = \left(\phi_i^-, m_i^-, y_i^-, e_i^-, a_i^-, \phi_i\right)$.

Now, fix the agent's strategy $\hat{\sigma}_A$. Whatever the partition $\mathcal{Q}_i$, the principals' strategies described above constitute an equilibrium for the game $\Gamma^{\mathcal{Q}_i}(\hat{\sigma}_A)$ among the principals.

In the following, we identify a partition $\mathcal{Q}_i$ that makes the strategy $\hat{\sigma}_A$ described above sequentially optimal for the agent. To this purpose, let $\mathcal{Q}_i$ be the partition defined by the following equivalence relation. For any two mechanisms $\phi_i, \phi_i' \in \Phi_i$,

$$\phi_i \sim_i \phi_i' \Longleftrightarrow \mathrm{Im}(\phi_i) = \mathrm{Im}(\phi_i). \tag{3}$$

Clearly, the partition generated by (3) consists of measurable sets. It is also immediate that, in the partition game $\Gamma^{\mathcal{Q}_i}$, $\hat{\sigma}_A$ is a sequentially rational best response to the principals' strategy profile $(\hat{\sigma}_1, ..., \hat{\sigma}_n)$. We conclude that, for any equilibrium σ of Γ, the partition game $\Gamma^{\mathcal{Q}_i}$ — where $\mathcal{Q}_i$ is the partition generated by the equivalence relation $\sim_i$ as given in (3) — admits an equilibrium $\hat{\sigma}$ that implements the same outcomes as σ.

Step 2. We now prove that starting from $\hat{\sigma}$, one can construct an equilibrium $\mathring{\sigma}$ for the game Γ_i^M that implements the same outcomes. Start with P_i. Now let $\mathrm{Im}(Q_i)$ denote the image of any of the mechanisms in Q_i, and for any $\phi_i^M \in \Phi_i^M$, let $Q_i(\phi_i^M) \in \mathcal{Q}_i$ denote the cell defined by

$$Q_i(\phi_i^M) \equiv \{\phi_i \in \Phi_i : \mathrm{Im}(\phi_i) = \mathrm{Im}(\phi_i^M)\}.$$

Then, for any measurable set $K \subseteq \Phi_i^M$, let

$$\mathring{\sigma}_i(K) = \widehat{\sigma}_i(\tilde{\mathcal{Q}}_i),$$

where $\tilde{\mathcal{Q}}_i \equiv \{Q_i \in \mathcal{Q}_i : \mathrm{Im}(Q_i) = \mathrm{Im}(\phi_i^M)$ for some $\phi_i^M \in K\}$. For all principals P_j with $j \neq i$, simply let $\mathring{\sigma}_j = \widehat{\sigma}_j$. Next, consider the agent. At any $t < i$, $\mathring{\sigma}_A(h_t) = \widehat{\sigma}_A(h_t)$ for any h_t. Starting

[38] Assuming that Φ_i is a Polish space endowed with the Borel σ-algebra Σ, the existence and (almost) uniqueness of such a conditional probability distribution follows from standard results (e.g. in [10] Theorem 10.2.2, p. 345).

from $t=i$, for any $\phi_i^M \in \Phi_i^M$, at any subsequent information set, A then induces the same outcomes he would have induced in $\Gamma^{\mathcal{Q}_i}$ had P_i offered the cell $Q_i(\phi_i^M)$. Formally, let

$$\zeta(h_i^-, Q_i) \equiv \int_{\Phi_i} \int_{\mathcal{M}_i} \phi_i(m_i) d\mu(h_i^-, \phi_i) d\sigma_i(\phi_i | Q_i)$$

denote the distribution over Y_i that A would have induced in $\Gamma^{\mathcal{Q}_i}$ by following the strategy $\hat{\sigma}_A$, given (h_i^-, Q_i). Next, let $\mathcal{M}_i^{\Phi_i}$ be the union of all the messages that A can send to P_i in Γ and for any $y_i \in Supp[\zeta(h_i^-, Q_i)]$, let $\beta(y_i; h_i^-, Q_i) \in \Delta(\Phi_i \times \mathcal{M}_i^{\Phi_i})$ denote the conditional distribution over $\Phi_i \times \mathcal{M}_i^{\Phi_i}$ that is obtained from Bayes' rule using the strategy $\hat{\sigma}_A$, conditioning on the event that the contract selected at $t=i$ is y_i and the upstream history at the beginning of period i is (h_i^-, Q_i). Then in Γ_i^M, for any (h_i^-, ϕ_i^M), the agent's mixed strategy $\mathring{\mu}(h_i^-, \phi_i^M) \in \mathcal{M}_i^M$ over the messages in ϕ_i^M induces the same distribution $\zeta(h_i^-, Q_i(\phi_i^M))$ over the set of contracts Y_i as the strategy $\hat{\sigma}_A(h_i^-, Q_i(\phi_i^M))$ in $\Gamma^{\mathcal{Q}_i}$. In the continuation game that starts after the realization of the contract y_i, A then uses the conditional distribution $\beta(y_i; h_i^-, Q_i(\phi_i^M))$ to determine his downstream behavior. That is, at any downstream information set, A behaves according to the strategy $\hat{\sigma}_A$ as if in $\Gamma^{\mathcal{Q}_i}$, A selected the mechanism ϕ_i from $Q_i(\phi_i^M)$ and the message m_i from $\mathcal{M}_i^{\Phi_i}$.

The strategy profile $\mathring{\sigma}$ constructed this way is clearly an equilibrium for Γ_i^M and induces the same outcomes as σ in Γ.

Step 3. Since in the construction of the equilibrium $\mathring{\sigma}$ for Γ_i^M, σ_i^- is kept fixed, Steps 1 and 2 can be applied recursively starting from $t=1$ and proceeding forward to construct an equilibrium σ^M for Γ^M that implements the same outcomes as σ.

Part 2. We now prove that, given *any* equilibrium σ^M of Γ^M, there exists an equilibrium σ of Γ that implements the same outcomes and such that σ_A is an *extension* of σ_A^M over Γ.

First consider the principals. For any P_i, simply let $\sigma_i = \alpha_i(\sigma_i^M)$, where $\alpha_i(\sigma_i^M)$ is the distribution over Φ_i obtained from σ_i^M using the embedding α_i.

Next consider the agent. After any history $h_t = ((\phi_j, m_j, y_j, e_j, a_j)_{j=1}^{t-1}, \phi_t)$, the agent behaves according to σ_A^M as if the game were Γ^M and the history were $h_t^M = ((\phi_i^M, \delta_i, y_i, e_i, a_i)_{i=1}^{t-1}, \phi_t^M)$ where the history h_t^M is obtained from h_t replacing each pair (ϕ_j, m_j) with (ϕ_j^M, δ_j) where ϕ_j^M is the menu whose image is $\text{Im}(\phi_j^M) = \text{Im}(\phi_j)$ and $\delta_j = \phi_j(m_j)$. That is, for any measurable set of messages $M_t \subseteq \mathcal{M}_t$ in ϕ_t, the strategy $\sigma_A(h_t)$ is such that

$$\mu(M_t \mid h_t) = \mu(\beta(M_t) \mid h_t^M)$$

where $\mu(\cdot \mid h_t)$ and $\mu(\cdot \mid h_t^M)$ denote the distributions over $\mathcal{M}_t$ and $\mathcal{M}_t^M$, respectively in Γ and in Γ^M, and

$$\beta(M_t) \equiv \{\delta_t : \delta_t = \phi_t(m_t), \ m_t \in M_t\}.$$

Given the realization y_t of the lottery $\phi_t(m_t)$, A then induces the same distribution over E_t that he would have induced in Γ^M given $(h_t^M, \phi_t(m_t), y_t)$.

The strategy σ_A constructed this way is an extension of σ_A^M over Γ. Furthermore, when A follows the strategy σ_A, no principal has incentive to deviate from $\sigma_i = \alpha_i(\sigma_i^M)$. We conclude that σ is an equilibrium of Γ and implements the same outcomes as σ^M. ■

Proof of Theorem 2. We want to show that, when $D_i = \Delta(Y_i)$ for all i, then for any $\sigma^M \in \mathcal{E}(\Gamma^M)$ there exists a pure-strategy equilibrium $\mathring{\sigma}^M \in \mathcal{E}(\Gamma^M)$ that implements the same outcomes as σ^M.

Suppose $\sigma^M \in \mathcal{E}(\Gamma^M)$ is such that P_i mixes over different menus and/or, given some menu ϕ_i^M, the agent randomizes over the lotteries in ϕ_i^M. We prove that there exists another $\widehat{\sigma}^M \in \mathcal{E}(\Gamma^M)$ in which all P_j with $j \neq i$ follow the strategy $\widehat{\sigma}_j^M = \sigma_j^M$, P_i offers only one menu, $\hat{\phi}_i^M$, and, after any (h_i^-, ϕ_i^M), there is a single lottery $\delta_i(h_i^-, \phi_i^M)$ that the agent selects from ϕ_i^M.

The equilibrium menu $\hat{\phi}_i^M$ is such that

$$\mathrm{Im}\left(\hat{\phi}_i^M\right) = cl\{\delta_i : \delta_i = \int_{\phi_i^M \in \Phi_i^M} \int_{\delta_i \in \mathcal{M}_i^M} \delta_i d\mu(h_i^-, \phi_i^M) d\sigma_i^M \text{ for some } h_i^- \in \mathcal{H}_i^-\}, \tag{4}$$

where $cl(X)$ denotes the closure of the set X. The menu $\hat{\phi}_i^M$ is thus the (closure of the) set of lotteries that can be constructed by mixing with distribution σ_i^M over the different lotteries that A would have induced in each menu ϕ_i^M, for some upstream history $h_i^- \in \mathcal{H}_i^-$.

Next, consider the following strategy for the agent. For any $t < i$ and any h_t, $\widehat{\sigma}_A^M(h_t) = \sigma_A^M(h_t)$. At $t = i$, given h_i^-, if $\phi_i^M = \hat{\phi}_i^M$, then A selects with probability one the lottery

$$\delta_i(h_i^-, \hat{\phi}_i^M) \equiv \int_{\phi_i^M \in \Phi_i^M} \int_{\delta_i \in \mathcal{M}_i^M} \delta_i d\mu(h_i^-, \phi_i^M) d\sigma_i^M.$$

Note that, for any h_i^-, the distribution over Y_i induced by $\delta_i(h_i^-, \hat{\phi}_i^M)$ is the same as that induced by σ_i^M and σ_A^M in the original equilibrium σ^M. Now for any $y_i \in Supp[\delta_i(h_i^-, \hat{\phi}_i^M)]$, let $\lambda(y_i; h_i^-) \in \Delta(\Phi_i^M \times \Delta(Y_i))$ denote the joint distribution over $\Phi_i^M \times \Delta(Y_i)$ that is obtained from Bayes' rule conditioning on the event that the contract selected by P_i is y_i and using the original strategies σ_i^M and σ_A^M. Then, in the continuation game that starts after the realization of the contract y_i, A uses the distribution $\lambda(y_i; h_i^-)$ to determine his downstream behavior. That is, A behaves in any downstream information set according to the original strategy σ_A^M, as if P_i offered the menu ϕ_i^M and A selected the message $\delta_i \in \Delta(Y_i)$. Conditional on P_i offering $\hat{\phi}_i^M$, the distribution over the payoff-relevant decisions $(e_j, a_j)_{j=1}^n$ is then the same as in the original equilibrium σ^M.

If instead $\phi_i^M \neq \hat{\phi}_i^M$, then the particular lottery $\delta_i(h_i^-, \phi_i^M)$ that A selects from ϕ_i^M is any lottery $\delta_i \in Supp[\sigma_A^M(h_i^-, \phi_i^M)]$ that minimizes P_i's expected payoff taking into account that at any subsequent information set the agent's behavior will be determined by the original strategy σ_A^M. Given $(\hat{\sigma}_i^M)_{i=1}^n$, the strategy $\widehat{\sigma}_A^M$ is clearly sequentially optimal for the agent. Furthermore, given $(\widehat{\sigma}_A^M, \hat{\sigma}_{-i}^M)$, no principal has an incentive to deviate from $\hat{\sigma}_i^M$. We conclude that $\widehat{\sigma}^M$ is an equilibrium for Γ^M and induces the same outcomes as σ^M.

Iterating for all $i = 1, ..., n$, starting from $i = 1$ and proceeding forward then gives the result.
■

Proof of Theorem 3. Assume $|\Theta| = 1$ and let $(e^*, a^*) \equiv (e_i^*, a_i^*)_{i=1}^n \in E \times \mathcal{A}$ denote the equilibrium decisions. We prove that if there exists a $\sigma^{M*} \in \mathcal{E}(\Gamma^M)$ that implements (e^*, a^*), there also exists a pure-strategy equilibrium $\tilde{\sigma}^M \in \mathcal{E}(\Gamma^M)$ which implements the same outcomes and such that the agent's strategy is Markov at any $t \in \mathcal{N}$.

First, note that, because (e^*, a^*) is deterministic, for any mixed-strategy equilibrium that sustains (e^*, a^*) there exists a pure-strategy equilibrium that sustains the same outcomes. This is true even if $D_i \subsetneqq \Delta(Y_i)$ for some i. Hence, without loss of generality, assume σ^{M*} is in pure strategies. Given σ^{M*}, let $\theta_i^{E*} \equiv (e_i^{-*}, a_i^{-*})$ denote the equilibrium extended type for $t = i$ and denote by $\mathcal{H}_i^-(\theta_i^{E*}; \sigma^{M*})$ the set of all possible *equilibrium* upstream histories that lead to θ_i^{E*}. Note that even if the SCF is deterministic and σ^{M*} is a pure-strategy profile, $\mathcal{H}_i^-(\theta_i^{E*}; \sigma^{M*})$ need not be a singleton.

Next, consider the strategy profile $\tilde{\sigma}^M$ in which all principals offer the same equilibrium menus as in σ^{M*} and in which the agent's (Markov) strategy $\tilde{\sigma}_A^M$ is constructed from σ_A^{M*} as follows.

Start from $t = n$. First suppose P_n offers the equilibrium menu ϕ_n^{M*}. If h_n^- is such that $\theta_n^E = \theta_n^{E*}$, that is, if the decisions taken upstream are the equilibrium decisions, then irrespective of which particular upstream history led to (e_n^{-*}, a_n^{-*}), A always selects the same lottery $\delta_n(\theta_n^{E*}, \phi_n^{M*})$, where $\delta_n(\theta_n^{E*}, \phi_n^{M*})$ is any lottery that A would have selected in ϕ_n^{M*} after some history $h_n^- \in \mathcal{H}_n^-(\theta_n^{E*}; \sigma^{M*})$. After any of the contracts $y_n \in Supp[\delta_n(\theta_n^{E*}, \phi_n^{M*})]$ is realized, A then chooses the equilibrium effort e_n^* leading to the equilibrium decision a_n^*. Clearly (e_n^*, a_n^*) is the same no matter which particular equilibrium history $h_n^- \in \mathcal{H}_n^-(\theta_n^{E*}; \sigma^{M*})$ one considers and which particular contract y_n is realized.

If, instead, $\theta_n^E \notin \theta_n^{E*}$, that is, if the decisions taken upstream are different from the equilibrium decisions, then let $j \leq n-1$ be the first date at which a departure from equilibrium occurred—that is, the unique date $j \leq n - 1$ such that $\theta_j^E = \theta_j^{E*}$ and $\theta_{j+1}^E \neq \theta_{j+1}^{E*}$. In this case, A selects a lottery $\delta_n(\theta_n^E, \phi_n^{M*})$, followed by the effort strategy $\xi(j) : Y_n \to \Delta(E_n)$, that minimizes P_j's payoff among those that are sequentially optimal for A given θ_n^E. Neither $\delta_n(\theta_n^E, \phi_n^{M*})$ nor $\xi(j)$ depend on the particular upstream history that led to θ_n^E.

Next, suppose P_n offers a menu $\phi_n^M \neq \phi_n^{M*}$. Then let $\tilde{\sigma}_A^M(\theta_n^E, \phi_n^M)$ be any Markov strategy that is sequentially optimal for A and that minimizes the payoff of P_n.

Next move to $t = n-1$ and construct the agent's Markov strategy $\tilde{\sigma}_A^M(\theta_{n-1}^E, \phi_{n-1}^M)$ following the same steps as for $t = n$ assuming that at $t = n$, P_n offers ϕ_n^{M*} and that A follows $\tilde{\sigma}_A^M(\theta_n^E, \phi_n^M)$.

Iterating backwards up to $t = 1$ gives a strategy $\tilde{\sigma}^M$ in which the agent's strategy is Markov at any date. It is immediate that, given $(\tilde{\sigma}_i^M)_{i=1}^n$, the strategy $\tilde{\sigma}_A^M$ is sequentially optimal for the agent. Furthermore, given $(\tilde{\sigma}_A^M, \tilde{\sigma}_{-i}^M,)$ no principal has an incentive to deviate from $\tilde{\sigma}_i^M$. We conclude that $\tilde{\sigma}^M$ is an equilibrium for Γ^M and induces the same outcomes as σ^{M*}. ■

Proof of Theorem 4. Part I. We want to show that any SCF π that can be sustained as a MPE of Γ^M can also be sustained as a pure-strategy truthful MPE of Γ^D.

Let σ^M denote the equilibrium strategy profile that sustains π in Γ^M; in virtue of Theorem 2, without loss of generality, we can assume that σ^M is a pure-strategy profile and denote by ϕ_t^{M*} the equilibrium menu for P_t.[39] That σ_A^M is Markov implies that, for any (θ_t^E, ϕ_t^M), there is a single lottery $\delta_t(\theta_t^E, \phi_t^M)$ such that, whatever the particular upstream history h_t^- that has conducted to θ_t^E, A always chooses $\delta_t(\theta_t^E, \phi_t^M)$ from ϕ_t^M when his extended type is θ_t^E. We henceforth denote the agent's behavioral strategy in period t by $\sigma_A^M(\theta_t^E, \phi_t^M)$. To prove that there exists a pure-strategy truthful MPE of Γ^D that sustains the same outcomes as σ^M, we then proceed in two steps.

Step 1. Consider the game Γ_t in which $\Phi_i = \Phi_i^M$ for all $i \neq t$, whereas for $i = t$, $\Phi_i = \Phi_i^D$. The following is then an equilibrium for Γ_t. For any $i \neq t$, $\sigma_i = \sigma_i^M$, whereas for $i = t$, σ_i is the (pure) strategy that consists in offering the mechanism ϕ_t^{D*} given by

$$\phi_t^{D*}(\theta_t^E) = \delta_t(\theta_t^E, \phi_t^{M*}) \quad \forall \theta_t^E.$$

At any date $i \neq t$, the agent's (Markov) strategy is the same as in Γ^M, i.e. $\sigma_A(\theta_i^E, \phi_i^M) = \sigma_A^M(\theta_i^E, \phi_i^M)$. In period t, if $\phi_t^D = \phi_t^{D*}$, then A reports θ_t^E truthfully and then selects effort according to σ_A^M as if the game were Γ^M, the menu offered by P_t were ϕ_t^{M*} and the message sent to P_t were $\delta_t(\theta_t^E, \phi_t^{M*})$. If instead $\phi_t^D \neq \phi_t^{D*}$, then the agent behaves according to σ_A^M as if the game were Γ^M and the menu offered by P_t were ϕ_t^M, where $\text{Im}(\phi_t^M) = \text{Im}(\phi_t^D)$. It is immediate that σ is a (pure-strategy) MPE for Γ_t, it sustains the same outcomes as σ^M and is such that (a) ϕ_t^{D*} is incentive-compatible and (b) the agent's strategy is truthful at period t.

Step 2. Following the same arguments as in Step 1, one can easily see that, starting from any game Γ_J in which $\Phi_i = \Phi_i^D$ for any $i \in J \subset \mathcal{N}$ while $\Phi_i = \Phi_i^M$ for $i \in \mathcal{N} \backslash J$, and from any

[39] Note that the Markov property of the agent's strategy is preserved by the replication arguments in the proof of Theorem 2. That is, for any $\sigma^M \in \mathcal{E}(\Gamma^M)$ in which the agent's strategy is Markov, there exists a pure-strategy profile $\hat{\sigma}^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes as σ^M and such that $\hat{\sigma}_A^M$ is Markov.

(pure-strategy) MPE $\sigma \in \mathcal{E}(\Gamma_J)$ that sustains the same outcomes as σ^M and satisfies (a) and (b) for any $i \in J$, there exists a game $\Gamma_{J'}$ in which $\Phi_i = \Phi_i^D$ for any $i \in J'$ and $\Phi_i = \Phi_i^M$ for all $i \in \mathcal{N} \backslash J'$ with $J' = J \cup \{t\}$ for some $t \in \mathcal{N} \backslash J$ and a (pure-strategy) MPE $\sigma' \in \mathcal{E}(\Gamma_{J'})$ that sustains the same outcomes as σ^M and satisfies (a) and (b) for all $i \in J'$. Combining Step 1 and Step 2 gives the result.

Part II. We now show that for any $\sigma^D \in \mathcal{E}(\Gamma^D)$ there exists a $\sigma^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes.

Let Γ_J denote the game in which $\Phi_j = \Phi_j^M$ for all $j \in J$ while $\Phi_j = \Phi_j^D$ for all $j \in \mathcal{N} \backslash J$, for some $J \subset \mathcal{N} \cup \{\varnothing\}$. We want to show that given any equilibrium $\sigma \in \mathcal{E}(\Gamma_J)$, there exists an equilibrium $\tilde{\sigma} \in \mathcal{E}(\Gamma_{J'})$, with $J' = J \cup \{t\}$ for some $t \in \mathcal{N} \backslash J$, that sustains the same outcomes. The strategy profile $\tilde{\sigma}$ is constructed from σ as follows. For any $i \neq t$, $\tilde{\sigma}_i = \sigma_i$. For $i = t$, $\tilde{\sigma}_t$ is such that, for any measurable set $R \subseteq \Phi_t^M$,

$$\tilde{\sigma}_t(R) = \sigma_t \left(\bigcup_{\phi_t^M \in R} \Phi_t^D(\phi_t^M) \right).$$

where $\Phi_t^D(\phi_t^M) \equiv \{\phi_t^D : \mathrm{Im}(\phi_t^D) = \mathrm{Im}(\phi_t^M)\}$. Next, consider the agent. Let

$$\bar{\Phi}_t^M \equiv \{\phi_t^M : \mathrm{Im}(\phi_t^M) = \mathrm{Im}(\phi_t^D) \text{ for some } \phi_t^D \in \Phi_t^D\}$$

At any $i < t$, $\tilde{\sigma}_A(h_i) = \sigma_A(h_i)$ for any h_i. Starting from $i = t$, for any $\phi_t^M \in \bar{\Phi}_t^M$, at any subsequent information set, A induces the same outcomes he would have induced in Γ_J had P_t offered one of the mechanisms in $\Phi_t^D(\phi_t^M)$. Formally, let

$$\zeta(h_t^-, \Phi_t^D(\phi_t^M)) \equiv \int_{\Phi_t^D} \int_{\Theta_t^E} \phi_t^D(\theta_t^E) d\mu(h_t^-, \phi_t^D) d\sigma_t(\phi_t^D | \Phi_t^D(\phi_t^M))$$

denote the distribution over Y_t generated in Γ_J by σ_A and σ_t, conditional on $(h_t^-, \Phi_t^D(\phi_t^M))$. Next, for any $y_t \in Supp[\zeta(h_t^-, \Phi_t^D(\phi_t^M))]$, let $\beta(y_t; h_t^-, \Phi_t^D(\phi_t^M)) \in \Delta(\Phi_t^D \times \Theta_t^E)$ denote the conditional distribution over $\Phi_t^D \times \Theta_t^E$ that is obtained from Bayes' rule using the strategies σ_t and σ_A, conditioning on the event that the contract selected in period t is y_t, that the upstream history at the beginning of period t is h_t^-, and that the mechanism selected by P_t belongs to $\Phi_t^D(\phi_t^M)$. Then in $\Gamma_{J'}$, for any (h_t^-, ϕ_t^M) with $\phi_t^M \in \bar{\Phi}_t^M$, the agent's mixed strategy $\tilde{\mu}(h_t^-, \phi_t^M) \in \mathcal{M}_t^M$ over the messages in ϕ_t^M induces the same distribution $\zeta(h_t^-, \Phi_t^D(\phi_t^M))$ over the set of contracts Y_t as the strategy σ_A in Γ_J given $(h_t^-, \Phi_t^D(\phi_t^M))$. In the continuation game that starts after the realization of the contract y_t, A then uses the conditional distribution $\beta(y_t; h_t^-, \Phi_t^D(\phi_t^M))$ to determine his downstream behavior. That is, at any downstream information set, A behaves according to the strategy σ_A as if the game were Γ_J, P_t selected the mechanism ϕ_t^D from $\Phi_t^D(\phi_t^M)$ and A sent the message θ_t^E to P_t.

If instead $\phi_t^M \notin \bar{\Phi}_t^M$, then starting from $i = t$, at any downstream information set A behaves according to σ_A (in the same sense as defined above) as if the game were Γ_J and the mechanism offered by P_t were ϕ_t^D, where ϕ_t^D is obtained from ϕ_t^M as follows:

$$\phi_t^D(\theta_t^E) \in \arg \max_{\delta_t \in \mathrm{Im}(\phi_t^M)} V(\theta_t^E, \delta_t, \sigma_t^+) \quad \forall \theta_t^E \in \Theta_t^E.$$

Given $(\tilde{\sigma}_i)_{i=1}^n$, the strategy $\tilde{\sigma}_A$ is sequentially optimal for the agent. Furthermore, given $(\tilde{\sigma}_A^M, \tilde{\sigma}_{-i}^M,)$ no principal has an incentive to deviate from $\tilde{\sigma}_i$. It follows that $\tilde{\sigma} \in \mathcal{E}(\Gamma_{J'})$. Because $\tilde{\sigma}$ sustains the same outcomes as σ, the result then follows by iterating across periods, starting from $t = 1$ and proceeding forward by letting $J' = J \cup \{t+1\}$. ■

Supplementary Material for
Sequential Contracting with Multiple Principals

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September 3, 2008

Abstract

This note contains additional material omitted in the paper. Section A1 contains an example that illustrates how extended direct mechanisms can be put to work to identify necessary and sufficient conditions for the sustainability of an outcome as a sequential common agency equilibrium. Section A2 contains the formal statements (with corresponding proofs) of the results discussed in Section 6 in the paper (Alternative extensive forms).

A1. Extended direct mechanisms: A buyer-sellers example

Consider a private contracting environment in which two sellers, P_1 and P_2, contract sequentially with a common buyer, A. The buyer is interested in purchasing two complementary products, one from each seller. An action $a_i = (s_i, t_i) \in \mathcal{A}_i = \{0,1\} \times \mathbb{R}$ for P_i thus consists of a decision to trade s_i along with a transfer t_i, with $s_i = 0$ in the case of no trade and $s_i = 1$ in the case of trade.[1] The buyer's preferences are described by the quasilinear function $v(a,\theta) = \theta(s_1+s_2)+s_1s_2-t_1-t_2$ where $\Theta = \{\underline{\theta}, \overline{\theta}\}$ with $\underline{\theta} > 1$ and $\Delta\theta \equiv (\overline{\theta} - \underline{\theta}) \in (0,1)$. The probability the buyer is a high type is $\Pr(\overline{\theta}) = p$. The sellers' payoffs are given by $u_i(a) = t_i - s_i$. It is common knowledge that the buyer contracts first with P_1 and then with P_2 (think of P_1 as a hardware supplier and of P_2 as a software provider). We assume that the buyer's participation in either relationship is voluntary and that the buyer can contract with P_2 after rejecting P_1's offer. In case the buyer rejects P_i's offer, the default contract $(0,0)$ with no trade and zero transfer is implemented.

In this setting, it seems reasonable to assume that the agent's behavior with P_2 depends only on the payoff-relevant decisions contracted upstream and not on things such as the mechanism used upstream or the message sent in this mechanism. We thus assume the agent's strategy is Markov.

Now suppose one is interested in understanding which SCFs $\pi : \Theta \to \Delta(\{0,1\}^2 \times \mathbb{R}^2)$ can be sustained as MPE when principals can offer any lottery over $\{0,1\} \times \mathbb{R}$. It then suffices to proceed as follows.[2]

First, consider downstream contracting. Because preferences are quasilinear, the transfer t_1 has no effect on the agent's preferences over $\mathcal{A}_2$. Without loss, we can thus simplify and let $\Theta_2^E = \Theta \times \{0,1\}$, with $\theta_2^{E,1} \equiv \underline{\theta}$, $\theta_2^{E,2} \equiv \overline{\theta}$, $\theta_2^{E,3} \equiv \underline{\theta}+1$ and $\theta_2^{E,4} \equiv \overline{\theta}+1$. Furthermore, because P_2 never finds it optimal to introduce randomizations over the decision to trade, we can restrict attention to deterministic extended direct mechanisms $\phi_2^D : \Theta_2^E \to \{0,1\} \times \mathbb{R}$, with $s_2(\theta_2^{E,i}) = s_2^i$

[1] In this representation, there is no effort, i.e. $|E_1| = |E_2| = 1$. Alternatively, one could assume that $E_i = \{0,1\}$ and that $\mathcal{A}_i = \mathbb{R}$ for each i. In this case, a contract $y_i : \{0,1\} \longrightarrow \mathbb{R}$ specifies a price for each decision $s_i \in \{0,1\}$. The two representations are clearly equivalent.

[2] In this example, we are restricing attention to MPE, but we not imposing any restriction on the set of feasible lotteries D_i. The approach illustrated here clearly applies also to environments where principals are restricted to offer deterministic contracts.

and $t_2(\theta_2^{E,i}) = t_2^i$ denoting respectively the decision to trade and the price asked to the extended type $\theta_2^{E,i}$, $i = 1, ..., 4$.

Now let β^i denote the (posterior) probability that P_2 assigns to $\theta_2^{E,i}$, with $\beta \equiv (\beta^1, \beta^2, \beta^3, \beta^4)$. Each β^i is derived from ϕ_1^D using Bayes' rule. With a slight abuse of notation, let $\delta_1(\theta) = \Pr(s_1 = 1 \mid \theta)$. We then have that $\beta^1 = (1-p)[1-\delta_1(\underline{\theta})]$, $\beta^2 = p[1-\delta_1(\overline{\theta})]$, $\beta^3 = (1-p)\delta_1(\underline{\theta})$ and $\beta^4 = p\delta_1(\overline{\theta})$. From standard results in contract theory (e.g. Maskin and Riley, 1986) we know that, in any optimal mechanism for P_2 the decision to trade is monotonic in θ_2^E so that $s_2^i \leq s_2^{i+1}$ $i = 1, 2, 3$, that all downward adjacent incentive compatibility constraints bind so that

$$\theta_2^{E,i} s_2^i - t_2^i = \theta_2^{E,i} s_2^{i-1} - t_2^{i-1}, \; i = 2, 3, 4, \tag{1}$$

and that the participation constraint for the lowest type binds so that $\underline{\theta} s_2^1 - t_2^1 = 0$.[3] Substituting the transfers

$$\begin{array}{l} t_2^1 = \underline{\theta} s_2^1, \quad t_2^2 = \overline{\theta} s_2^2 - \Delta\theta s_2^1, \quad t_2^3 = (\underline{\theta}+1) s_2^3 - (1-\Delta\theta) s_2^2 - \Delta\theta s_2^1 \\ t_2^4 = (\overline{\theta}+1) s_2^4 - \Delta\theta s_2^3 - (1-\Delta\theta) s_2^2 - \Delta\theta s_2^1 \end{array} \tag{2}$$

into P_2's payoff, we have that

$$U_2 = \sum_{i=1}^{4} \mathcal{W}^i(\delta_1) s_2^i \tag{3}$$

where $\mathcal{W}^i(\delta_1)$ denotes the virtual surplus of selling to type i, given the upstream decisions $\delta_1 \equiv (\delta_1(\overline{\theta}), \delta_1(\underline{\theta}))$:

$$\begin{aligned} \mathcal{W}^1 &\equiv \beta^1(\underline{\theta} - 1) - (1 - \beta_1)\,\Delta\theta \\ \mathcal{W}^2 &\equiv \beta^2\left(\overline{\theta} - 1\right) - \left(\beta^3 + \beta^4\right)(1 - \Delta\theta) \\ \mathcal{W}^3 &\equiv \beta^3 \underline{\theta} - \beta^4 \Delta\theta \\ \mathcal{W}^4 &\equiv \beta^4 \overline{\theta}, \end{aligned}$$

with $\beta^i = \beta^i(\delta_1)$. A mechanism $\phi_2^{D^*}$ is thus an incentive-compatible best response to ϕ_1^{D*} if and only if (a) the allocation rule $s_2^i(\cdot)$ maximizes (3) subject to the monotonicity constraint $s_2^i \leq s_2^{i+1}$, $i = 1, 2, 3$ and (b) the transfers t_2^i are given by (2).[4]

Next, consider upstream contracting. When the allocation rule in ϕ_2^{D*} is monotonic and the transfers satisfy (2), the buyer's payoff at $t = 1$ satisfies the single-crossing property with respect to θ and δ_1. This in turn implies that the optimal mechanism $\phi_1^{D*} : \Theta \to \Delta(\{0, 1\}) \times \mathbb{R}$ solves the following program

$$\max \; p[t_1(\overline{\theta}) - \delta_1(\overline{\theta})] + (1-p)[t_1(\underline{\theta}) - \delta_1(\underline{\theta})]$$

[3] Note that, because $\theta s_1(\theta) - t_1(\theta)$ is sunk, from the perspective of P_2, it is as if the buyer's reservation payoff is zero, for all θ_2^E.

[4] That ϕ_2^{D*} must necessarily solve the aforementioned program follows from the fact that P_2 can always make the agent strictly prefer to truthfully reveal his private information by breaking the agent's indifference by an $\varepsilon > 0$, for ε arbitrarily small.

subject to

$$[\overline{\theta} + (\overline{\theta}+1)s_2^4 - t_2^4]\delta_1(\overline{\theta}) + [\overline{\theta}s_2^2 - t_2^2][1-\delta_1(\overline{\theta})] - t_1(\overline{\theta}) = \\ [\overline{\theta} + (\overline{\theta}+1)s_2^4 - t_2^4]\delta_1(\underline{\theta}) + [\overline{\theta}s_2^2 - t_2^2][1-\delta_1(\underline{\theta})] - t_1(\underline{\theta}) \tag{4}$$

$$[\underline{\theta} + (\underline{\theta}+1)s_2^3 - t_2^3]\delta_1(\underline{\theta}) + [\underline{\theta}s_2^1 - t_2^1][1-\delta_1(\underline{\theta})] - t_1(\underline{\theta}) = \underline{\theta}s_2^1 - t_2^1 \tag{5}$$

and

$$\delta_1(\bar{\theta}) \geq \delta_1(\underline{\theta}). \tag{6}$$

Condition (4) guarantees that $\bar{\theta}$ is indifferent between $(\delta_1(\bar{\theta}), t_1(\bar{\theta}))$ and $(\delta_1(\underline{\theta}), t_1(\underline{\theta}))$, while condition (5) guarantees that $\underline{\theta}$ is indifferent between $(\delta_1(\underline{\theta}), t_1(\underline{\theta}))$ and the null contract $(0,0)$. The high type's participation is then guaranteed by (4) and (5) while incentive-compatibility for the low type is guaranteed by the monotonicity condition (6).

Equivalently, ϕ_1^{D*} maximizes

$$U_1 = p\left[\delta_1(\overline{\theta})\bar{\mathcal{V}}(\delta_1)\right] + (1-p)\left[\delta_1(\underline{\theta})\underline{\mathcal{V}}(\delta_1)\right] \tag{7}$$

subject to $\delta_1(\bar{\theta}) \geq \delta_1(\underline{\theta})$, where

$$\begin{aligned} &\bar{\mathcal{V}}(\delta_1) \equiv \overline{\theta} + \left[(\overline{\theta}+1)s_2^4 - t_2^4\right] - \left[(\overline{\theta}s_2^2 - t_2^2)\right] - 1 \\ &\underline{\mathcal{V}}(\delta_1) \equiv \underline{\theta} + \left[(\underline{\theta}+1)s_2^3 - t_2^3\right] - \left[\underline{\theta}s_2^1 - t_2^1\right] - 1 \\ &-\tfrac{p}{1-p}\{\Delta\theta + \left[(\overline{\theta}+1)s_2^4 - t_2^4 - (\overline{\theta}s_2^2 - t_2^2)\right] - \left[(\underline{\theta}+1)s_2^3 - t_2^3 - (\underline{\theta}s_2^1 - t_2^1)\right]\} \end{aligned}$$

Two observations are in order. First note that ϕ_2^{D*} must specify allocations also for extended types that may have zero measure on the equilibrium path (this is the case, for example, when $\delta_1(\underline{\theta}) = 0$ so that $\beta^3 = 0$). Second note that whether ϕ_1^{D*} is incentive-compatible or not depends on the mechanism ϕ_2^{D*} offered downstream. We thus have the following result.

Example A1. *The outcome $\pi^* = (\delta_1^*(\cdot), t_1^*(\cdot), s_2^*(\cdot), t_2^*(\cdot))$ can be sustained as a MPE of Γ^D (equivalently, of Γ^M) if and only if:*

(I) given $\delta_1^*(\cdot)$, $s_2^*(\cdot)$ maximizes (3) subject to the monotonicity condition $s_2^i \leq s_2^{i+1}$, $i = 1,2,3$, while $t_2^*(\cdot)$ solves (2);

(II) given $s_2^*(\cdot)$ and $t_2^*(\cdot)$, $\delta_1^*(\cdot)$ maximizes (7) subject to the monotonicity condition $\delta_1(\bar{\theta}) \geq \delta_1(\underline{\theta})$ while $t_1^*(\cdot)$ solves (4) and (5).

Extended direct mechanisms thus offer the possibility of using familiar techniques from games with a single mechanism designer to characterize necessary and sufficient conditions for equilibrium outcomes in sequential common agency. The preceding example illustrated such a possibility in a very simple way. In certain applications, the characterization of these conditions may require the use of the techniques from the multi-dimensional screening literature. This need not always be simple. However, when this is the case, assuming the principals offer menus instead of direct mechanisms does not simplify the analysis. In fact, the difficulties with multi-dimensional screening simply stem from the difficulty of controlling for the optimality of the agent's behavior. This is something one has to deal with, irrespective of how the menus are described.

A2. Alternative extensive forms: Formal results

A2-1. Observability of upstream payoff-relevant decisions

Consider an environment in which principals observe upstream payoff-relevant decisions before choosing their mechanisms. Let $\bar{\Gamma}^D$ denote the game in which the principals offer ***standard direct revelation mechanisms*** $\phi_i^D : \Theta \to D_i$ as opposed to extended direct mechanisms. We then have the following result.

Theorem 5 (Observable decisions). *(Part I: Menus) Let* $\Gamma \succcurlyeq \Gamma^M$*. Any SCF that can be sustained as a MPE of* Γ *can also be sustained as a MPE of* Γ^M*. Furthermore, any SCF that can be sustained as an equilibrium of* Γ^M *(not necessarily in Markov strategies) can also be sustained as an equilibrium of* Γ.

(Part II: Direct Mechanisms) Any SCF that can be sustained as a MPE of Γ^M *can be sustained as a pure-strategy truthful MPE of* $\bar{\Gamma}^D$*. Furthermore, any SCF that can be sustained as a MPE of* $\bar{\Gamma}^D$ *can also be sustained as a MPE of* Γ^M.

As with Theorem 4 in the main text, the result in Part (II) presumes that $D_i = \Delta(Y_i)$ for all i, which guarantees that outcomes in Γ^M sustained by mixed strategies can be sustained in $\bar{\Gamma}^D$ through mechanisms that respond to θ with lotteries over contracts. In environments in which not all possible lotteries are feasible, i.e. $D_i \subsetneqq \Delta(Y_i)$ for some i, the result in Part (II) must be replaced by the following: Any SCF that can be sustained as a MPE of Γ^M in which the *agent's strategy is pure* can also be sustained as a truthful MPE of $\bar{\Gamma}^D$.

Proof of Theorem 5. Part I: Menus. First, consider the claim that any SCF π that can be sustained as a MPE of Γ can also be sustained as a MPE of Γ^M. The proof follows from the same steps used to establish Part 1 of Theorem 1 in the paper, with the following two (minor) adjustments. (i) The transformation of the principals' strategies indicated in that proof must now be done for any (e_t^-, a_t^-). (ii) The principals' strategies are now sustained by beliefs λ^M over upstream histories that satisfy Bayes' rule on the equilibrium path, whereas for any out-of-equilibrium (e_t^-, a_t^-), $t = 2, ..., n$, satisfy

$$\hat{\lambda}(e_t^-, a_t^-) = \hat{\lambda}^M(e_t^-, a_t^-) \tag{8}$$

where $\hat{\lambda}(e_t^-, a_t^-)$ and $\hat{\lambda}^M(e_t^-, a_t^-)$ denote the marginal distribution of λ and λ^M over Θ, respectively in the original game Γ and in the menu game Γ^M. Because the agent's strategy is Markov and (e_t^-, a_t^-) is public information, any profile of beliefs with these properties makes the principals' strategies sequentially optimal.

Next, consider the claim that any SCF that can be sustained as an equilibrium of Γ^M (not necessarily in Markov strategies) can also be sustained as an equilibrium of Γ.. The proof parallels that of Part 2 in Theorem 1. In the following, we simply construct a profile of beliefs that sustains the principals' strategies.

For any $i = 1, ..., n$, let $\mathcal{H}_i^-$ and $\mathcal{H}_i^{M-}$ denote the sets of all possible upstream histories, respectively in Γ and in Γ^M, and $\Sigma(\mathcal{H}_i^-)$ and $\Sigma(\mathcal{H}_i^{M-})$ the corresponding Borel sigma algebras. For any (e_i^-, a_i^-), let $\varkappa_i(e_i^-, a_i^-) \in \Delta(\mathcal{H}_i^-)$ and $\varkappa_i^M(e_i^-, a_i^-) \in \Delta(\mathcal{H}_i^{M-})$ denote P_i's beliefs about upstream histories, respectively in Γ and in Γ^M. If (e_i^-, a_i^-) is on the equilibrium path, then $\varkappa_i(e_i^-, a_i^-)$ is obtained from Bayes' rule using the equilibrium strategy profile σ. If instead (e_i^-, a_i^-) is an out-of-equilibrium observation, then $\varkappa_i(e_i^-, a_i^-)$ is constructed as follows. For any measurable set of histories $H_i^{M-} \in \Sigma(\mathcal{H}_i^{M-})$ in Γ^M, let $\Xi_i(H_i^{M-}) \in \Sigma(\mathcal{H}_i^-)$ denote the measurable set of histories in Γ that are obtained by substituting each history

$$h_i^{M-} = \left(\theta, e_i^-, a_i^-, \phi_i^{M-}, \delta_i^-, y_i^-\right)$$

in H_i^{M-} with the history

$$f_i(h_i^{M-}) \equiv \left(\theta, e_i^-, a_i^-, (\alpha_l(\phi_l^M))_{l=1}^{i-1}, (\tilde{\alpha}_l(\delta_l))_{l=1}^{i-1}, y_i^-\right).$$

The history $f_i(h_i^{M-})$ is simply the "translation" of the history h_i^{M-} using the embedding α_i. For any out-of-equilibrium (e_i^-, a_i^-), then let $\varkappa_i(e_i^-, a_i^-)$ be the unique beliefs that satisfy

$$\varkappa_i(\Xi_i(H_i^{M-})|e_i^-, a_i^-) = \varkappa_i^M(H_i^{M-}|e_i^-, a_i^-) \ \forall H_i^{M-} \in \Sigma(\mathcal{H}_i^{M-}).$$

Together with these beliefs, the strategy profile σ constructed from σ^M following the steps indicated in the proof of Theorem 1 constitutes an equilibrium for Γ which sustains the same outcomes as σ^M.

Part II: Direct Mechanisms. The proof parallels that of Theorem 4. The equilibrium strategy profiles σ^D and σ^M are sustained by any beliefs that are consistent with Bayes' rule on the equilibrium path, whereas for any out-of-equilibrium (e_t^-, a_t^-), satisfy

$$\hat{\lambda}^D(e_t^-, a_t^-) = \hat{\lambda}^M(e_t^-, a_t^-)$$

where $\hat{\lambda}^D(e_t^-, a_t^-)$ and $\hat{\lambda}^M(e_t^-, a_t^-)$ denote the marginal distributions of λ^D and λ^M over Θ, respectively in the revelation game $\bar{\Gamma}^D$ and in the menu game Γ^M. ■

A2-2. Observability of upstream mechanisms

Consider an environment in which every P_i, $i = 2, ..., n$, observes the mechanisms ϕ_i^- offered upstream before choosing her own mechanism. As in the benchmark model, P_i does not observe $(m_i^-, y_i^-, e_i^-, a_i^-)$.

Theorem 6 (Observable mechanisms). *(Part I: Menus) Let $\Gamma \succcurlyeq \Gamma^M$. For any $\sigma \in \mathcal{E}(\Gamma)$ in which all principals' strategies are pure, there exists a $\sigma^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes. Furthermore, any SCF that can be sustained as an equilibrium of Γ^M can be sustained as an equilibrium of Γ.*

(Part II: Direct Mechanisms) For any $\sigma^M \in \mathcal{E}(\Gamma^M)$ in which all principals' strategies are pure, there exists a pure-strategy truthful MPE $\sigma^D \in \mathcal{E}(\Gamma^D)$ that sustains the same outcomes.

Once again, the result in Part (II) presumes $D_i = \Delta(Y_i)$ for all i. When this is not the case, Part (II) must be replaced by the following: For any $\sigma^M \in \mathcal{E}(\Gamma^M)$ in which both the principals' and the agent's strategies are pure, there exists a pure-strategy truthful MPE $\sigma^D \in \mathcal{E}(\Gamma^D)$ that sustains the same outcomes.

Proof of Theorem 6. Part I: Menus. The proof is in two steps.

Step 1. We want to prove that, for any $\sigma \in \mathcal{E}(\Gamma)$ in which all principals' strategies are pure, there exists a $\sigma^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes.

Let $\sigma_i(\phi_i^-)$ denote the unique mechanism offered by P_i when the profile of upstream mechanisms is ϕ_i^-. Next, consider the game Γ_i in which P_i is restricted to offer menus, whereas all other principals have the same strategy space as in Γ. Now consider the following strategy profile $\mathring{\sigma}$ for Γ_i. For all principals P_j with $j < i$, simply let $\mathring{\sigma}_j = \sigma_j$. For P_i, let $\mathring{\sigma}_i$ be the strategy that maps each ϕ_i^- into the menu ϕ_i^M whose image is $\mathrm{Im}(\phi_i^M) = \mathrm{Im}(\sigma_i(\phi_i^-))$. Finally, for any P_j with $j > i$, $\mathring{\sigma}_j$ is as follows. If ϕ_j^- is such that at $t = i$, $\phi_i^M = \mathring{\sigma}_i(\phi_i^-)$, then

$$\mathring{\sigma}_j(\phi_i^-, \phi_i^M, \phi_{i+1}, ..., \phi_{j-1}) = \sigma_j(\phi_i^-, \sigma_i(\phi_i^-), \phi_{i+1}, ..., \phi_{j-1}).$$

If instead, $\phi_i^M \neq \mathring{\sigma}_i(\phi_i^-)$, then

$$\mathring{\sigma}_j(\phi_i^-, \phi_i^M, \phi_{i+1}, ..., \phi_{j-1}) = \sigma_j(\phi_i^-, \alpha_i(\phi_i^M), \phi_{i+1}, ..., \phi_{j-1}),$$

where $\alpha_i(\phi_i^M)$ is the embedding of ϕ_i^M into Φ_i.

Next, consider the agent. At any $t < i$, $\mathring{\sigma}_A(h_t) = \sigma_A(h_t)$ for any h_t. If at $t = i$, P_i offers the menu $\phi_i^M = \mathring{\sigma}_i(\phi_i^-)$, then at any downstream information set A induces the same outcomes that he would have induced in Γ had P_i offered the mechanism $\sigma_i(\phi_i^-)$, in the sense defined in the proof of Theorem 1. If, instead, P_i offers a mechanism $\phi_i^M \neq \mathring{\sigma}_i(\phi_i^-)$, then starting from $t = i$, at any subsequent information set, A behaves according to σ_A as if the game were Γ and the mechanism offered by P_i were $\alpha_i(\phi_i^M)$.

This completes the description of $\mathring{\sigma}_A$ at the information sets which are relevant for equilibrium. For all other information sets (i.e. those associated to upstream deviations by the agent), simply let $\mathring{\sigma}_A$ specify any behavior that is sequentially optimal for A given the payoff-relevant variables θ_t^E and the downstream principals' strategy profile $\mathring{\sigma}_t^+$. Given $(\mathring{\sigma}_i^+)_{i=1}^n$, the strategy $\mathring{\sigma}_A$ is clearly sequentially optimal for the agent at any information set. Thus consider the optimality of the principals' strategies. After any ϕ_j^-, $j = 1, ..., n$, beliefs about upstream histories are necessarily pinned down by Bayes' rule using the agent's strategy $\mathring{\sigma}_A$. This follows from the "no signal what you do not know" property of PBE: the observation of ϕ_j^- conveys no information about the agent's behavior in these mechanisms which hence must be assumed to have been consistent with what prescribed by the equilibrium strategy. Given these beliefs, the principals' strategies are sequentially rational. We conclude that the strategy profile $\mathring{\sigma}$ with the associated beliefs described above is an equilibrium for Γ_i and induces the same outcomes as σ in Γ.

Starting from $t = 1$ and proceeding forward, one can then apply the arguments described above to any $i = 1, ..., n$ to construct a pure-strategy equilibrium of Γ^M that implements the same outcomes as σ.

Step 2. We now prove that for any $\sigma^M \in \mathcal{E}(\Gamma^M)$ there exists a $\sigma \in \mathcal{E}(\Gamma)$ that sustains the same outcomes.

First consider the agent. The strategy σ_A is constructed by extending the strategy σ_A^M over Γ exactly as in the proof of Theorem 1. Next, consider the principals. For any t, let $\sigma_t(\phi_t^-) = \alpha_t(\sigma_t^M(\phi_t^{M-}))$, where $\alpha_t(\sigma_t^M(\cdot))$ denotes the mixed strategy over Φ_t obtained from the mixed strategy σ_t^M using the embedding α_t, while ϕ_t^{M-} denotes the profile of upstream menus that is obtained from ϕ_t^- by letting each ϕ_j^M be the menu whose image is $\text{Im}(\phi_j^M) = \text{Im}(\phi_j)$, $j = 1, ..., t-1$. The strategy profile σ constructed this way, along with the beliefs for the principals that are obtained from Bayes' rule using σ_A, is an equilibrium of Γ and sustains the same outcomes as σ^M.

Part II: Direct Mechanisms. We show that, for any $\sigma^M \in \mathcal{E}(\Gamma^M)$ in which all principals' strategies are pure, there exists a pure-strategy truthful MPE $\sigma^D \in \mathcal{E}(\Gamma^D)$ that sustains the same outcomes. Note that the agent's strategy in Γ^M need not be Markov—which explains why the proof does not follow directly from the same arguments used to establish Theorem 4.

Consider a game Γ_J in which $\Phi_j = \Phi_j^D$ for all $j \in J$ while $\Phi_j = \Phi_j^M$ for all $j \in \mathcal{N} \backslash J$, for some $J \subset \mathcal{N} \cup \{\varnothing\}$. We prove the result by showing that given any equilibrium $\sigma \in \mathcal{E}(\Gamma_J)$ in which all principals' strategies are pure, there exists an equilibrium $\mathring{\sigma} \in \mathcal{E}(\Gamma_{J'})$ that also has the property that all principals' strategies are pure and that sustains the same outcomes as σ, for any $J' = J \cup \{t\}$ with $t \in \mathcal{N} \backslash J$.

For any ϕ_t^-, let $\Theta_t^E(\phi_t^-) \subseteq \Theta_t^E$ denote the set of extended types that are consistent with σ_A (i.e. that can be generated by using σ_A recursively in Γ_J starting from $i = 1$ and proceeding forward). For any $\theta_t^E \in \Theta_t^E(\phi_t^-)$, then let $\eta(\theta_t^E, \phi_t^-) \in \Delta(\mathcal{H}_t^-)$ denote the conditional distribution over $\mathcal{H}_t^-$ that is obtained from Bayes' rule using the agent's strategy σ_A in Γ_J and conditioning on the event that the extended type in period t is θ_t^E and the mechanisms offered upstream are ϕ_t^-.

Now consider the following (pure) strategy for P_t in $\Gamma_{J'}$. For any profile of upstream mechanisms ϕ_t^-, let $\phi_t^M = \sigma_t(\phi_t^-)$ denote the equilibrium menu offered by P_t in Γ_J in response to ϕ_t^-. Then the extended direct mechanism $\phi_t^D = \mathring{\sigma}_t(\phi_t^-)$ that P_t offers in $\Gamma_{J'}$ in response to ϕ_t^- is such that

$$\phi_t^D(\theta_t^E) = \begin{cases} \int_{h_t^- \in \mathcal{H}_t^-} \int_{\delta_t \in \mathcal{M}_t^M} \delta_t d\mu(h_t^-, \sigma_t(\phi_t^-)) d\eta(\theta_t^E, \phi_t^-) & \text{if } \theta_t^E \in \Theta_t^E(\phi_t^-) \\ \delta_t \in \arg\max_{\delta_t' \in \text{Im}(\sigma_t(\phi_t^-))} \mathring{V}(\theta_t^E, \phi_t^-, \sigma_t(\phi_t^-), \delta_t', \sigma_t^+) & \text{if } \theta_t^E \notin \Theta_t^E(\phi_t^-) \end{cases} \tag{9}$$

where $\mathring{V}(\theta_t^E, \phi_t^-, \sigma_t(\phi_t^-), \delta_t', \sigma_t^+)$ denotes the agent's continuation payoff in Γ given $(\theta_t^E, \phi_t^-, \sigma_t(\phi_t^-), \delta_t', \sigma_t^+)$. Note that the agent's continuation payoff now depends also on upstream mechanisms; this is because the latter now determine which mechanisms will be offered downstream. The mechanism ϕ_t^D described in (9) thus responds to each $\theta_t^E \in \Theta_t^E(\phi_t^-)$ with the same distribution over Y_t that A would have induced in the menu $\sigma_t(\phi_t^-)$ when his extended type is θ_t^E and the mechanisms offered upstream are ϕ_t^-. For any other $\theta_t^E \notin \Theta_t^E(\phi_t^-)$, the mechanism simply responds by giving the agent one of the lotteries in the menu $\phi_t^M = \sigma_t(\phi_t^-)$ that would have been optimal for θ_t^E given the mechanisms $(\phi_t^-, \sigma_t(\phi_t^-))$ and the profile of strategies σ_t^+ for the downstream principals in Γ_J.

Now consider the following strategy profile $\mathring{\sigma}$ for $\Gamma_{J'}$. For all principals P_j with $j < t$, simply let $\mathring{\sigma}_j = \sigma_j$. For P_t, let $\mathring{\sigma}_t$ be the strategy described above. Finally, for any P_j with $j > t$, $\mathring{\sigma}_j$ is as follows. If ϕ_j^- is such that in period t, P_t offered the mechanism $\phi_t^D = \mathring{\sigma}_t(\phi_t^-)$, then

$$\mathring{\sigma}_j(\phi_t^-, \phi_t^D, \phi_{t+1}, ..., \phi_{j-1}) = \sigma_j(\phi_t^-, \sigma_t(\phi_t^-), \phi_{t+1}, ..., \phi_{j-1}).$$

If instead, $\phi_t^D \neq \mathring{\sigma}_t(\phi_t^-)$, then

$$\mathring{\sigma}_j(\phi_t^-, \phi_t^D, \phi_{t+1}, ..., \phi_{j-1}) = \sigma_j(\phi_t^-, \phi_t^M, \phi_{t+1}, ..., \phi_{j-1}).$$

where ϕ_t^M is the menu whose image is $\text{Im}(\phi_t^M) = \text{Im}(\phi_t^D)$.

Next, consider the agent. At any $j < t$, $\mathring{\sigma}_A(h_j) = \sigma_A(h_j)$ for any h_j. If in period t, P_t offers the mechanism $\phi_t^D = \mathring{\sigma}_t(\phi_t^-)$, A truthfully reports his extended type and then at any subsequent information set, he induces the same outcomes that he would have induced in Γ_J had P_t offered the menu $\sigma_t(\phi_t^-)$. Formally, for any $y_t \in Supp[\phi_t^D(\theta_t^E)]$, let $\zeta(y_t; \theta_t^E, \phi_t^-, \sigma_t(\phi_t^-)) \in \Delta(\mathcal{H}_t^- \times \Delta(Y_t))$ denote the conditional distribution over the profiles $(h_t^-, \delta_t) \in \mathcal{H}_t^- \times \Delta(Y_t)$ in Γ_J that is obtained from Bayes' rule using the strategy σ_A, conditioning on the event that the contract selected in period t is y_t, that the agent's extended type is θ_t^E and that the mechanisms offered at $t = 1, ..., t$ are $(\phi_t^-, \sigma_t(\phi_t^-))$. In the continuation game that starts after the realization of the contract y_t, A then uses the conditional distribution $\zeta(y_t; \theta_t^E, \phi_t^-, \sigma_t(\phi_t^-))$ to determine his downstream behavior. That is, at any downstream information set, A behaves according to the strategy σ_A as if the game were Γ_J, and before choosing e_t, the history had been $(h_t^-, \sigma_t(\phi_t^-), \delta_t)$.

Finally, consider the continuation game that starts after P_t offers a mechanism $\phi_t^D \neq \mathring{\sigma}_t(\phi_t^-)$. Starting from period t, at any subsequent information set, A behaves according to σ_A as if the game were Γ_J and the menu offered by P_t were ϕ_t^M, where ϕ_t^M is the menu whose image is $\text{Im}(\phi_t^M) = \text{Im}(\phi_t^D)$.

This completes the description of $\mathring{\sigma}_A$ at the information sets which are relevant for equilibrium. For all other information sets (i.e. those associated to upstream deviations by the agent), simply let $\mathring{\sigma}_A$ specify any behavior that is sequentially optimal for A given the payoff-relevant variables θ_t^E and the downstream principals' strategy profile $\mathring{\sigma}_t^+$. Given the principals' strategies, the strategy $\mathring{\sigma}_A$ is sequentially optimal for the agent at any information set.

Next, consider the optimality of the principals' strategies. Given $(\mathring{\sigma}_A, \mathring{\sigma}_{-i})$, the optimality of $\mathring{\sigma}_i$ follows from the same arguments as in the proof of Part I–Step 1. We conclude that the strategy profile $\mathring{\sigma}$ with the associated beliefs $\mathring{\lambda}$ obtained from $\mathring{\sigma}$ using Bayes' rule, is an equilibrium for $\Gamma_{J'}$ and induces the same outcomes as σ in Γ_J.

Iterating across all periods, starting from $t = 1$ and proceeding forward by letting $J' = J \cup \{t+1\}$, then gives a pure-strategy truthful equilibrium of Γ^D that implements the same outcomes as σ^M. ■

Note that, contrary to the benchmark model of private contracting and to the case of observable decisions considered above, the result in Part (II) in Theorem 6 does not have a converse. There may exist SCFs that can be sustained as equilibria of Γ^D and that cannot be sustained as equilibria of Γ^M. To see this, consider the following example where $n = 2$, $|\Theta| = |E_i| = 1$, $i = 1, 2$, $\mathcal{A}_1 = \{t, b\}$ and $\mathcal{A}_2 = \{l, r\}$. The payoffs, respectively for P_1, P_2 and A are given by the triples (u_1, u_2, v) in the following table:

$a_1 \backslash a_2$	l			r		
t	1	3	0	3	3	4
b	2	0	5	2	2	3

Game A1

For simplicity, assume that only deterministic mechanisms are feasible so that $D_i = \mathcal{A}_i$, $i = 1, 2$.

Now consider the revelation game Γ^D. Here a direct mechanism for P_1 coincides with the choice of an element of $\mathcal{A}_1$ whereas a direct mechanism for P_2 is a mapping $\phi_2^D : \mathcal{A}_1 \to \mathcal{A}_2$. The following is then a pure-strategy equilibrium for Γ^D. P_2 reacts to the direct mechanism of P_1 that selects t with the mechanism that responds to both t and b with l and to the mechanism that selects b with the mechanism that responds to both t and b with r. Given this reaction, P_1 in equilibrium chooses the mechanism that selects b. The equilibrium outcome is thus (b, r).

Next consider the menu game Γ^M. Suppose P_1 offers the menu $\{t, b\}$. Because l is weakly dominated for P_2, there are only two possible outcomes in the continuation game between A and P_2 that starts after P_1 offers $\{t, b\}$. In the first one, A selects t and P_2 selects r. In the second, A selects t and P_2 randomizes over l and r, respectively with probability 1/6 and 5/6. In both cases, P_1 obtains a payoff of $16/6 > 2$. It follows that the SCF that selects (b, r) with certainty cannot be sustained as an equilibrium in the menu game because P_1 has a profitable deviation.

A2-3. Endogenous sequence of bilateral relationships

Consider the following game with endogenous sequence of contractual relationships. There are $T < \infty$ periods. In each period, *all* principals simultaneously offer the agent a mechanism $\phi_{i,t}$ from a set $\Phi_{i,t}$. The agent chooses at most one mechanism, say $\phi_{i,t}$, to participate in, then chooses a message $m_{i,t}$ from $\mathcal{M}_{i,t}$ and a contract $y_{i,t}$ is selected by the lottery $\phi_{i,t}(m_{i,t}) \in \Delta(Y_{i,t})$. Given $y_{i,t}$, the agent then chooses an action $e_{i,t}$ from $E_{i,t}$ and finally the contract $y_{i,t} : E_{i,t} \to \Delta(\mathcal{A}_{i,t})$ determines P_i's decision. The agent may, or may not, participate in a mechanism offered by the same principal multiple times. For those principals who are not selected in period t, simply let $e_{j,t} = e_{j,t}^{\varnothing}$ and $a_{j,t} = a_{j,t}^{\varnothing}$, where $(e_{j,t}^{\varnothing}, a_{j,t}^{\varnothing})$ are the exogenous default decisions that are implemented in the absence of contracting, such as no trade at a null price.

Payoffs, respectively for P_i, $i = 1, ..., n$, and for A continue to be denoted by $u_i(\theta, e, a)$ and $v(\theta, e, a)$, with $e_\tau \equiv (e_{1,\tau}, ..., e_{n,\tau})$ and $a_\tau \equiv (a_{1,\tau}, ..., a_{n,\tau})$ now denoting an entire profile of payoff-relevant decisions for period τ, one for each possible bilateral relationship, and $e \equiv (e_1, ..., e_T)$ and $a \equiv (a_1, ..., a_T)$.

For any $t = 1, ..., T$, any $i = 1, ..., n$ and any upstream history h_t^-, let $z_{i,t} = f_{i,t}(h_t^-)$ denote the elements of h_t^- that are observed by P_i in period t.[5] The function $f_{i,t} : \mathcal{H}_t^- \to Z_{i,t}$ maps each possible upstream history $h_t^- \in \mathcal{H}_t^-$ into an observation $z_{i,t} \in Z_{i,t}$, where $Z_{i,t} \equiv \{z_{i,t} : z_{i,t} = f_{i,t}(h_t^-), h_t^- \in \mathcal{H}_t^-\}$. As in the benchmark model, contracting is *private* in the sense that principals do not observe other principals' mechanisms, nor the messages, the contracts, or the decisions taken in these mechanisms. These restrictions are embedded in the mappings $f_{i,t}$.

For any $z_{i,t} \in Z_{i,t}$, let $\psi(z_{i,t})$ denote the payoff-relevant component of $z_{i,t}$, that is, the part of the agent's extended type $\theta_t^E = (\theta, e_t^-, a_t^-)$ that is observed by P_i at date t. Note that the agent's extended type now contains profiles of payoff-relevant decisions $e_\tau \equiv (e_{1,\tau}, ..., e_{n,\tau})$ and $a_\tau \equiv (a_{1,\tau}, ..., a_{n,\tau})$, one for each bilateral relationship, with $e_t^- \equiv (e_\tau)_{\tau=1}^{t-1}$ and $a_t^- \equiv (a_\tau)_{\tau=1}^{t-1}$.

Principal i's behavioral strategy in period t is now described by the distribution $\sigma_{i,t}(z_{i,t}) \in \Delta(\Phi_{i,t})$ over the mechanisms in $\Phi_{i,t}$. The agent's behavioral strategy $\sigma_A(h_t^-, \phi_t)$ given the upstream history $h_t^- \in \mathcal{H}_t^-$ and the profile of mechanisms $\phi_t \equiv (\phi_{1,t}, ..., \phi_{n,t})$ offered in period t, is decomposed as follows: $w^t(h_t^-, \phi_t) \in \Delta(\mathcal{N} \cup \varnothing)$ denotes the agent's participation strategy; $\mu_t(h_t^-, \phi_t, I_t) \in \Delta(\tilde{\mathcal{M}}_t)$ denotes the agent's message strategy after he chooses to participate in principal I_t's mechanism, where $I_t \in \mathcal{N} \cup \varnothing$ denotes the identity of the principal selected in period t and $\tilde{\mathcal{M}}_t \equiv \prod_i (\mathcal{M}_{i,t} \cup \varnothing)$; finally, $\xi(h_t^-, \phi_t, I_t, m_t, y_t) \in \Delta(\tilde{E}_t)$ denotes the agent's effort strategy, with $\tilde{E}_t \equiv \prod_i (E_{i,t} \cup \varnothing)$.[6]

Definition A1. *Principal i's strategy in period t is Markov if and only if, for any $z_{i,t}, z_{i,t}' \in Z_{i,t}$ such that $\psi(z_{i,t}) = \psi(z_{i,t}')$, $\sigma_{i,t}(z_{i,t}) = \sigma_{i,t}(z_{i,t}')$.*

[5] A history h_t^- now also includes the agent's upstream participation decisions.

[6] The vector $m_t \equiv (m_{1,t}, ..., m_{n,t})$ denotes the profile of messages sent by the agent in period t, with $m_{j,t} = \varnothing$ for any $j \neq I_t$. Similarly, $y_t \equiv (y_{1,t}, ..., y_{n,t})$ and $e_t \equiv (e_{1,t}, ..., e_{n,t})$ denote, respectively, the vector of contracts and the vector of effort choices, for period t, with $y_{j,t}, e_{j,t} = \varnothing$ for any $j \neq I_t$.

The agent's strategy in period t is Markov if and only if the following are true:

(a) for any (h_t^-, ϕ_t) and $(\tilde{h}_t^-, \phi_t)$ such that θ_t^E is the same in h_t^- and $\tilde{h}_t^-$, $w^t(h_t^-, \phi_t) = w^t(\tilde{h}_t^-, \phi_t)$;

(b) for any (h_t^-, ϕ_t, I_t) and $(\tilde{h}_t^-, \tilde{\phi}_t, I_t)$ such that θ_t^E is the same in h_t^- and $\tilde{h}_t^-$ and $\phi_{I_t,t}$ is the same in ϕ_t and $\tilde{\phi}_t$, $\mu_t(h_t^-, \phi_t, I_t) = \mu_t(\tilde{h}_t^-, \tilde{\phi}_t, I_t)$;[7]

(c) for any $(h_t^-, \phi_t, I_t, m_t, y_t)$ and $(\tilde{h}_t^-, \tilde{\phi}_t, I_t, \tilde{m}_t, y_t)$ such that θ_t^E is the same in h_t^- and $\tilde{h}_t^-$, $\xi(h_t^-, \phi_t, I_t, m_t, y_t) = \xi(\tilde{h}_t^-, \tilde{\phi}_t, I_t, \tilde{m}_t, y_t)$.

An equilibrium $\sigma \in \mathcal{E}(\Gamma)$ is a MPE if and only if all players' strategies are Markov at any $t = 1, ..., T$.

Theorem 7 (Endogenous sequence). *(Part I: Menus) Let $\Gamma \succcurlyeq \Gamma^M$.*[8] *Any SCF that can be sustained as a MPE of Γ can also be sustained as a MPE of Γ^M. Furthermore, any SCF that can be sustained as an equilibrium of Γ^M (not necessarily in Markov strategies) can also be sustained as an equilibrium of Γ.*

(Part II: Direct Mechanisms) Suppose the agent can contract with each principal at most once. Then any SCF that can be sustained as a MPE of Γ^M can also be sustained as a truthful MPE of Γ^D. Furthermore, any SCF that can be sustained as a MPE of Γ^D can also be sustained as a MPE of Γ^M.

Proof of Theorem 7. Part (I). The proof is in two steps and combines arguments from the proofs of Theorems 1 and 5.

Step 1. We want to show that given any MPE $\sigma \in \mathcal{E}(\Gamma)$, there exists a MPE $\sigma^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes as σ. The arguments here are similar to those in the proof of Theorem 1. The only differences come from the fact that (a) one has to adjust the replication arguments to take into account that the principals' strategies are now contingent on what they have observed upstream and (b) that one must specify supporting beliefs for the principals' strategies.

Consider the partition game $\Gamma^{\mathcal{Q}_{i,t}}$ in which, in period t, P_i chooses a cell $Q_{i,t}$ from the partition $\mathcal{Q}_{i,t}$ of $\Phi_{i,t}$ simultaneously with the other principals choosing their mechanisms $\phi_{j,t}$ from $\Phi_{j,t}$, $j \neq i$. Given $(Q_{i,t}, (\phi_{j,t})_{j \neq i})$, A first selects a mechanism $\phi_{i,t}$ from $Q_{i,t}$ and then, given the profile $(\phi_{i,t}, (\phi_{j,t})_{j \neq i})$, he chooses which mechanism to participate in. The choice of $\phi_{i,t}$ is observed by P_i, but not by the other principals. For any other principal and any other date, the choice set in $\Gamma^{\mathcal{Q}_{i,t}}$ is the same as in Γ; that is, for any $(j, \tau) \neq (i, t)$, the strategy space for P_j at date τ remains $\Phi_{j,\tau}$.

[7] If $I_t = \varnothing$, then $\phi_{I,t} = \varnothing$.

[8] The game Γ is an enlargement of Γ^M if, for any $i = 1, ..., n$, and any $t = 1, ..., T$, the following are true:

(a) $\text{Im}(\phi_{i,t})$ is compact, for any $\phi_{i,t} \in \Phi_{i,t}$;

(b) there exists a injective mapping $\alpha_{i,t} : \Phi_{i,t}^M \to \Phi_{i,t}$ such that, for any pair of mechanisms $\phi_{i,t}^M, \phi_{i,t}$ with $\phi_{i,t} = \alpha_{i,t}(\phi_{i,t}^M)$,(i) $\text{Im}(\phi_{i,t}^M) = \text{Im}(\phi_{i,t})$, and (ii) there exists an injective function $\tilde{\alpha}_{i,t} : \mathcal{M}_{i,t}^M \to \mathcal{M}_{i,t}$ such that $\phi_{i,t}^M(\delta_{i,t}) = \delta_{i,t} = \phi_i(\tilde{\alpha}_i(\delta_{i,t}))$ for any $\delta_{i,t} \in \mathcal{M}_{i,t}^M$;

(c) there exists an injective mapping $\mathring{\alpha}_{i,t} : Z_{i,,t}^M \to Z_{i,t}$ from the set of possible signals $Z_{i,t}^M$ in Γ^M to the set of possible signals $Z_{i,t}$ in Γ.

Now let $\mathcal{Q}_{i,t}$be the partition of $\Phi_{i,t}$ given by the equivalence relation

$$\phi_{i,t} \sim_{i,t} \phi'_{i,t} \Longleftrightarrow \text{Im}(\phi_{i,t}) = \text{Im}(\phi'_{i,t}). \tag{10}$$

Following the same construction as in Step 1 in the proof of Theorem 1, it is easy to see that there exists an equilibrium $\hat{\sigma}$ for $\Gamma^{\mathcal{Q}_{i,t}}$ which sustains the same outcomes as σ in Γ. In this equilibrium, all P_j with $j \neq i$, follow the same strategy as in Γ, i.e. $\widehat{\sigma}_j = \sigma_j$. As for P_i, at any $\tau \neq t$ and for any $z_{i,\tau} \in Z_{i,\tau}$, $\widehat{\sigma}_i(z_{i,\tau}) = \sigma_i(z_{i,\tau})$.[9] In period t, for any $z_{i,t} \in Z_{i,t}$, P_i randomizes over any subset R of $\mathcal{Q}_{i,t}$ whose union is measurable with probability

$$\widehat{\sigma}_i(R\ ;\ z_{i,t}) = \sigma_i(\bigcup R\ ;\ z_{i,t}).$$

The agent's strategy is such that at any $\tau < t$, $\widehat{\sigma}_A(h_\tau) = \sigma_A(h_\tau)$ for any $h_\tau \in \mathcal{H}_\tau$. In period t, given any $(Q_{i,t}, (\phi_{j,t})_{j\neq i})$, A uses the conditional probability distribution $\sigma_{i,t}(\cdot|Q_{i,t}; z_{i,t})$ to select a mechanism $\phi_{i,t}$ from $Q_{i,t}$. At any subsequent informational set, A then behaves as if the game were Γ and the mechanism offered by P_i were $\phi_{i,t}$. As far as beliefs are concerned, at any information set, all principals have the same marginal beliefs over upstream payoff-relevant information as in Γ (note that, on the equilibrium path, this is consistent with principals' beliefs be obtained from Bayes rule). Because all players' strategies in $\hat{\sigma}$ are Markov, given these beliefs, all principals' strategies are sequentially rational.

Next, consider the game $\Gamma^M_{i,t}$ in which, in period t, P_i's choice set is $\Phi^M_{i,t}$, whereas for any $(j,\tau) \neq (i,t)$, P_j's choice set in period τ is the same as in Γ. Now, for any $\tau = 1, ..., T$, let $Z^M_{j,\tau}$ denote the set of possible signals that P_j can receive in $\Gamma^M_{i,t}$ in period τ, with $Z^M_{j,\tau} = Z_{j,\tau}$ for any (j,τ) such that either $j \neq i$, or $\tau \leq t$.

Because all players' strategies are Markov in $\hat{\sigma}$, starting from $\hat{\sigma}$ and following essentially the same construction as in Step 2 in the proof of Theorem 1, one can show that there exists a MPE $\mathring{\sigma} \in \mathcal{E}(\Gamma^M_{i,t})$ that sustains the same outcomes as σ. We refer the reader to that proof for the details of how to construct the strategies in $\mathring{\sigma}$ from the strategies in $\hat{\sigma}$. The only important observation is that, given the menu $\phi^M_{i,t}$ offered by P_i in period t, the agent uses the conditional distribution $\sigma_{i,t}(\cdot|Q_{i,t}(\phi^M_{i,t}); z_{i,t})$ to determine not only the messages to send to P_i in case he decides to participate in $\phi^M_{i,t}$ but also his participation decision. That is, given any profile of mechanisms $(\phi^M_{i,t}, (\phi_{j,t})_{j\neq i})$, A uses the conditional probability distribution $\sigma_{i,t}(\cdot|Q_{i,t}(\phi^M_{i,t}); z_{i,t})$ to select in his mind a mechanism $\phi_{i,t}$ from $Q_{i,t}(\phi^M_{i,t}) \equiv \{\phi_{i,t} : \text{Im}(\phi_{i,t}) = \text{Im}(\phi^M_{i,t})\}$ and then uses the original strategy $w^t(h^-_t, (\phi_{i,t}, (\phi_{j,t})_{j\neq i}))$ for Γ to determine his participation decision. At all subsequent information sets, the construction of $\mathring{\sigma}$ parallels that of $\hat{\sigma}$ in $\Gamma^{\mathcal{Q}_{i,t}}$.

The principals' strategies in $\mathring{\sigma}$ can be sustained by beliefs $\mathring{\lambda}_{j,\tau}(z^M_{j,\tau}) \in \Delta(\mathcal{H}^-_\tau)$ over upstream histories that satisfy the following properties.

[9] Formally, for any $\tau > t$, $z_{i,\tau}$ now includes the cell $Q_{i,t}$. However, because to any $\phi_{i,t}$ corresponds a unique cell $Q_{i,t}$, we can drop $Q_{i,t}$ from $z_{i,\tau}$.

Case (i). If $z^M_{j,\tau}$ is such that, given the mechanisms $(\phi_{j,l})_{l=1}^{\tau-1}$ offered by P_j upstream, the decisions in $z^M_{j,\tau}$ are consistent with $\mathring{\sigma}_A$ and $(\mathring{\sigma}_k)_{k\neq j}$, then $\mathring{\lambda}_{j,\tau}(z^M_{j,\tau})$ are obtained from Bayes' rule using $\mathring{\sigma}_A$ and $(\mathring{\sigma}_k)_{k\neq j}$. For all P_j with $j \neq i$, these beliefs necessarily have the same marginal distribution over Θ^E_τ as in $\Gamma^{\mathcal{Q}_{i,t}}$ given $z_{j,\tau} = z^M_{j,\tau}$. Clearly, the same is true for P_i if $\tau \leq t$, but not necessarily if $\tau > t$. In fact, if $\tau > t$, then P_i's posterior beliefs about θ^E_τ in $\Gamma^M_{i,t}$ after P_i offered the menu $\phi^M_{i,t}$ in period t are a convex combination of the beliefs she would have had in $\Gamma^{\mathcal{Q}_{i,t}}$ had she offered $Q_{i,t}(\phi^m_{i,t})$ in period t. More precisely, let $z_{i,\tau} = (\left(z^M_{i,\tau}\backslash\phi^m_{i,t}\right) \wedge \phi_{i,t}) \in Z_{i,\tau}$ denote the observation that is obtained from $z^M_{i,\tau}$ by substituting the mechanism $\phi^m_{i,t}$ with $\phi_{i,t}$. Similarly, let $z_{i,\tau} = (\left(z^M_{i,\tau}\backslash\phi^m_{i,t}, \delta_{i,t}\right) \wedge \phi_{i,t}, m_{i,t}) \in Z_{i,\tau}$ denote the observation that is obtained from $z^M_{i,\tau}$ by substituting the mechanism $\phi^m_{i,t}$ and the message $\delta_{i,t}$ with $\phi_{i,t}$ and $m_{i,t}$. Now let $\mathring{\Lambda}_{i,\tau}$ and $\Lambda_{i,\tau}$ denote P_i's marginal beliefs over Θ^E_τ, respectively in $\Gamma^M_{i,t}$ in $\Gamma^{\mathcal{Q}_{i,t}}$. First, suppose the agent did not participate in P_i's mechanism in period t, so that $I_t \neq i$. Then P_i's posterior beliefs over Θ^E_τ in period $\tau > t$ satisfy

$$\mathring{\Lambda}_{i,\tau}(z^M_{i,\tau}) = \int_{\Phi_{i,t}(\phi^m_{i,t})} \Lambda_{i,t}(\left(z^M_{i,\tau}\backslash\phi^m_{i,t}\right) \wedge \phi_{i,t}) d\eta_{i,t}(\phi_{i,t}|z^M_{i,\tau})$$

where $\eta_{i,t}(\phi_{i,t}|z^M_{i,\tau})$ denote P_i's beliefs that the agent in period t behaved as if the game were $\Gamma^{\mathcal{Q}_{i,t}}$ and selected $\phi_{i,t}$ from $\Phi_{i,t}(\phi^m_{i,t})$, given $z^M_{i,\tau}$. Next, suppose $I_t = i$ and let $\mathcal{M}_{i,t}(\delta_{i,t})$ denote the set of messages in $\Phi_{i,t}(\phi^m_{i,t})$ that lead to the lottery $\delta_{i,t}$. Then P_i's posterior beliefs over Θ^E_τ in period $\tau > t$ satisfy

$$\mathring{\Lambda}_{i,\tau}(z^M_{i,\tau}) = \int_{\Phi_{i,t}(\phi^m_{i,t})} \int_{\mathcal{M}_{i,t}(\delta_{i,t})} \Lambda_{i,t}(\left(z^M_{i,\tau}\backslash\phi^m_{i,t}, \delta_{i,t}\right) \wedge \phi_{i,t}, m_{i,t}) d\gamma_{i,t}(\phi_{i,t}, m_{i,t}|z^M_{i,\tau})$$

where $\gamma_{i,t}(\phi_{i,t}, m_{i,t}|\phi^M_{i,t}, \delta_{i,t})$ denote P_i's beliefs that the agent in period t behaved as if the game were $\Gamma^{\mathcal{Q}_{i,t}}$, he selected $\phi_{i,t}$ from $\Phi_{i,t}(\phi^m_{i,t})$, and then sent the message $m_{i,t}$. This difference in beliefs with respect to $\Gamma^{\mathcal{Q}_{i,t}}$ is due to the fact that the choice of the mechanism $\phi_{i,t}$ from $\Phi_{i,t}(\phi^m_{i,t})$ and of the message $m_{i,t}$ from $\mathcal{M}_{i,t}(\delta_{i,t})$ is now only in the agent's mind and is thus not directly observed by P_i.

Given the aforementioned beliefs, the (behavioral) strategies $\mathring{\sigma}_{j,\tau}(z^M_{j,\tau}) = \hat{\sigma}_{j,\tau}(z^M_{j,\tau})$ for all (j,τ) such that either $j \neq i$ or $\tau < t$ are clearly sequentially optimal.[10] Thus consider $j = i$ and $\tau > t$. Because the strategy $\hat{\sigma}_{i,\tau}$ was Markov in $\Gamma^{\mathcal{Q}_{i,t}}$, then $\hat{\sigma}_{i,\tau}(z_{i,\tau}) = \hat{\sigma}_{i,\tau}(z'_{i,\tau})$ for any $z_{i,\tau}$ and $z'_{i,\tau}$ that contain the same payoff-relevant information, i.e. such that $\psi(z_{i,\tau}) = \psi(z'_{i,\tau})$. Now, suppose $z^M_{i,\tau}$ is such that $I_t \neq i$ and let $Z_{i,\tau}(z^M_{i,\tau})$ denote the set of observations $z_{i,\tau} \in Z_{i,\tau}$ such that $z_{i,\tau} = (\left(z^M_{i,\tau}\backslash\phi^m_{i,t}\right) \wedge \phi_{i,t}) \in Z_{i,\tau}$, with $\phi_{i,t} \in \Phi_{i,t}(\phi^m_{i,t})$. Clearly $\psi(z_{i,\tau}) = \psi(z'_{i,\tau})$ for any pair $z_{i,\tau}, z'_{i,\tau} \in Z_{i,\tau}(z^M_{i,\tau})$. That in $\Gamma^{\mathcal{Q}_{i,t}}$ the strategy $\hat{\sigma}_{i,\tau}$ was Markov implies that $\hat{\sigma}_{i,\tau}(z_{i,\tau})$ was optimal for any $z_{i,\tau} \in Z_{i,\tau}(z^M_{i,\tau})$ and hence for any beliefs $\Lambda_{i,t}(\left(z^M_{i,\tau}\backslash\phi^m_{i,t}\right) \wedge \phi_{i,t})$, with $\phi_{i,t} \in \Phi_{i,t}(\phi^m_{i,t})$. Because $\mathring{\Lambda}_{i,\tau}(z^M_{i,\tau})$ is a convex combination of $\Lambda_{i,t}(\left(z^M_{i,\tau}\backslash\phi^m_{i,t}\right) \wedge \phi_{i,t})$, with $\phi_{i,t} \in \Phi_{i,t}(\phi^m_{i,t})$, this necessarily implies that $\mathring{\sigma}_{i,\tau}(z^M_{i,\tau}) = \hat{\sigma}_{i,\tau}(z_{i,\tau})$, with $z_{i,\tau} \in Z_{i,\tau}(z^M_{i,\tau})$, is sequentially optimal.

[10] Recall that for these (j,τ), $Z^M_{j,\tau} = Z_{j,\tau}$.

Next, suppose that $z^M_{i,\tau}$ is such that $I_t = i$ and let $Z_{i,\tau}(z^M_{i,\tau})$ denote the set of observations $z_{i,\tau} \in Z_{i,\tau}$ such that $z_{i,\tau} = (\left(z^M_{i,\tau} \backslash \phi^m_{i,t}, \delta_{i,t}\right) \wedge \phi_{i,t}, m_{i,t})$, with $\phi_{i,t} \in \Phi_{i,t}(\phi^m_{i,t})$ and $\phi_{i,t}(m_{i,t}) = \delta_{i,t}$. The same arguments as for $I_t \neq i$ imply that the strategy $\mathring{\sigma}_{i,\tau}(z^M_{i,\tau}) = \hat{\sigma}_{i,\tau}(z_{i,\tau})$, with $z_{i,\tau} \in Z_{i,\tau}(z^M_{i,\tau})$, is sequentially optimal.

Case (ii). Next, suppose the observation $z^M_{j,\tau}$ indicates that a departure from equilibrium play occurred by either A or some P_j, $j \neq i$. Then let $\mathring{\lambda}_{j,\tau}(z^M_{j,\tau})$ be any beliefs that are consistent with[11] $z^M_{j,\tau}$ and satisfy $\mathring{\Lambda}_{j,\tau}(z^M_{j,\tau}) = \Lambda_{j,\tau}(z_{j,\tau})$, where $\mathring{\Lambda}_{j,\tau}$ and $\Lambda_{j,\tau}$ denote P_j's marginal beliefs over Θ^E_τ, respectively in $\Gamma^M_{i,t}$ conditional on $z^M_{j,\tau}$ and in $\Gamma^{\mathcal{Q}_{i,t}}$ conditional on $z_{j,\tau}$, where $z_{j,\tau}$ is any signal that contains the same payoff-relevant information as $z^M_{j,\tau}$.

Because $\Gamma^M_{i,t} \succcurlyeq \Gamma^M$ and because $\mathring{\sigma}$ is a MPE of $\Gamma^M_{i,t}$, one can keep iterating the same construction described above across all i and all t, starting from $t = 1$ and proceeding forward. This gives a MPE $\sigma^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes as σ.

Step 2. Next, we prove that, given any $\sigma^M \in \mathcal{E}(\Gamma^M)$ (not necessarily in Markov strategies) there exists a $\sigma \in \mathcal{E}(\Gamma)$ that sustains the same outcomes as σ^M. The construction parallels that in the proof of Theorems 1 and 5.

First, consider the agent. The strategy σ_A is constructed from σ^M_A as in the proof of Theorem 1. After any history $h_t = (\theta, (\phi_\tau, I_\tau, m_\tau, y_\tau, e_\tau, a_\tau)^{t-1}_{\tau=1}, \phi_t)$, the agent behaves according to σ^M_A (in the same sense as in the proof of Theorem 1 in the main text) as if the game were Γ^M and the history were $h^M_t = (\theta, (\phi^M_\tau, I_\tau, \delta_\tau, y_\tau, e_\tau, a_\tau)^{t-1}_{\tau=1}, \phi^M_t)$ where the history h^M_t is obtained from h_t replacing $((\phi_\tau)^{t-1}_{\tau=1}, \phi_t)$ with $((\phi^M_\tau)^{t-1}_{\tau=1}, \phi_t)$ and $(m_\tau)^{t-1}_{\tau=1}$ with $(\delta_\tau)^{t-1}_{\tau=1}$, where each $\phi^M_{j,\tau}$ in h^M_t is the menu whose image is $\text{Im}(\phi^M_{j,\tau}) = \text{Im}(\phi_{j,\tau})$ and where $\delta_{j,\tau} = \phi_{j,\tau}(m_{j,\tau})$.[12]

Next, consider the principals. For any t, any i and any $z_{i,t} \in Z_{i,t}$, let $\sigma_i(z_{i,t}) = \alpha_i(\sigma^M_i(\beta(z_{i,t})))$, where $\alpha_i(\sigma^M_i)$ is the distribution over Φ_i obtained from σ^M_i using the embedding α_i and where $z^M_{i,t} = \beta(z_{i,t})$ is the observation obtained from $z_{i,t}$, using the same transformation of $\phi_{i,\tau}$ and $m_{i,\tau}$ indicated above for the agent.

The principals' strategies are supported by the following beliefs. For any t, let $\mathcal{H}^-_t$ and $\mathcal{H}^{M-}_t$ denote the sets of all possible upstream histories, respectively in Γ and in Γ^M, and $\Sigma(\mathcal{H}^-_t)$ and $\Sigma(\mathcal{H}^{M-}_t)$ denote the corresponding Borel sigma algebras. For any $z_{i,t}$ and $z^M_{i,t}$, let $\varkappa_{i,t}(z_{i,t}) \in \Delta(\mathcal{H}^-_t)$ and $\varkappa^M_{i,t}(z^M_{i,t}) \in \Delta(\mathcal{H}^{M-}_t)$ denote P_i's period-t beliefs about upstream histories, respectively in Γ and in Γ^M. If $z_{i,t}$ is such that, given the mechanisms $(\phi_{i,\tau})^{t-1}_{\tau=1}$ offered by P_i upstream, the decisions in $z_{i,t}$ are consistent with σ_A and $(\sigma_k)_{k\neq i}$, then $\varkappa_{i,t}(z_{i,t})$ is obtained from Bayes' rule using σ_A and $(\sigma_k)_{k\neq i}$. Otherwise, $\varkappa_{i,t}(z_{i,t})$ are constructed as follows. For any measurable set of upstream histories $H^{M-}_t \in \Sigma(\mathcal{H}^{M-}_t)$ in Γ^M, let $\Xi_t(H^{M-}_t) \in \Sigma(\mathcal{H}^-_t)$ denote the measurable set of histories in

[11] The beliefs $\mathring{\lambda}_{j,\tau}(z^M_{j,\tau}) \in \Delta(\mathcal{H}^-_\tau)$ are consistent with $z^M_{i,t}$ if they assign positive measure only to upstream histories h^-_t such that $f_{i,t}(h^-_t) = z_{i,t}$.

[12] For any principal i not selected in period τ, $\delta_{i,\tau}, y_{i,\tau} = \varnothing$.

Γ that are obtained by substituting each history

$$h_t^M = (\theta, (\phi_\tau^M, I_\tau, \delta_\tau, y_\tau, e_\tau, a_\tau)_{\tau=1}^{t-1})$$

in H_t^{M-} with the family of histories $f_t(h_t^{M-}) \in \Sigma(\mathcal{H}_t^-)$ such that, each history

$$h_t^- = (\theta, (\phi_\tau, I_\tau, m_\tau, y_\tau, e_\tau, a_\tau)_{\tau=1}^{t-1})$$

in $f_t(h_t^{M-})$ has the following properties: (a) $(\theta, (I_\tau, y_\tau, e_\tau, a_\tau)_{\tau=1}^{t-1})$ is the same as in h_t^M; (b) each $\phi_{i,\tau}$ is such that $\text{Im}(\phi_{i,\tau}) = \text{Im}(\phi_{i,\tau}^M)$; each $m_{i,\tau}$ is such that $m_{i,\tau} = \varnothing$ if $\delta_{i,\tau} = \varnothing$ and $\phi_{i,\tau}(m_{i,\tau}) = \delta_{i,\tau}$ if $\delta_{i,\tau} \neq \varnothing$. For any out-of-equilibrium $z_{i,t}$, then let $\varkappa_i(z_{i,t})$ be the unique beliefs that are consistent with $z_{i,t}$ and satisfy

$$\varkappa_{i,t}(\Xi_t(H_t^{M-}) \mid z_{i,t}) = \varkappa_{i,t}^M(H_t^{M-} \mid \beta(z_{i,t}))\ \forall H_t^{M-} \in \Sigma(\mathcal{H}_t^{M-})$$

where $z_{i,t}^M = \beta(z_{i,t})$ is obtained from $z_{i,t}$, using the transformation of $\phi_{i,\tau}$ and $m_{i,\tau}$ indicated above for the agent. With these beliefs, the strategy σ_i given by $\sigma_i(z_{i,t}) = \alpha_i(\sigma_i^M(\beta(z_{i,t})))$ for any $z_{i,t}$ is sequentially rational for P_i, given σ_A and $(\sigma_k)_{k\neq i}$.

Furthermore, given the principals' strategies $(\sigma_i)_{i=1}^n$ constructed above, the agent's strategy σ_A is clearly sequentially rational. We conclude that $\sigma \in \mathcal{E}(\Gamma)$. That σ implements the same SCF as σ^M is then immediate.

Proof of Part (II). The proof is in two steps.

Step 1. Consider an environment in which the agent contracts with each principal at most once. We want to show that given any MPE $\sigma^M \in \mathcal{E}(\Gamma^M)$, there exists a MPE $\sigma^D \in \mathcal{E}(\Gamma^D)$ that sustains the same SCF as σ^M. To ease the exposition, hereafter we allow the principals to offer mechanisms also in periods subsequent to the one they contracted with the agent. This is clearly inconsequential for the arguments below.

Let Γ_J denote a game in which $\Phi_{j,\tau} = \Phi_{j,\tau}^D$ for all $(j,\tau) \in J$, while $\Phi_{j,\tau} = \Phi_{j,\tau}^M$ for all $(j,\tau) \in \mathcal{R} \backslash J$, for some $J \subset \mathcal{R} \cup \{\varnothing\}$, where $\mathcal{T} \equiv \{1, ..., T\}$ and $\mathcal{R} \equiv (\mathcal{N} \times \mathcal{T})$. We prove the result by showing that, given any MPE $\sigma \in \mathcal{E}(\Gamma_J)$, there exists an MPE $\tilde{\sigma} \in \mathcal{E}(\Gamma_{J'})$, with $J' = J \cup \{i,t\}$ for some $\{i,t\} \in \mathcal{R} \backslash J$, that sustains the same outcomes.

That the agent's strategy in σ is Markov implies that, for any $\phi_{i,t}^M \in \Phi_{i,t}^M$, there is a single probability distribution $\delta_{i,t}(\theta_t^E, \phi_{i,t}^M) \in \Delta(Y_{i,t})$ over $Y_{i,t}$ such that, conditional on having decided to participate in $\phi_{i,t}^M$, whatever the particular upstream history h_t^- that conducted to θ_t^E, A always induces the distribution $\delta_{i,t}(\theta_t^E, \phi_{i,t}^M)$ when his extended type is θ_t^E.

The MPE $\tilde{\sigma}$ that sustains π in $\Gamma_{J'}$ is obtained from σ as follows. For any $\tau \neq t$, all players' (Markov) strategies are the same as in σ. For $\tau = t$, if $j \neq i$, then $\tilde{\sigma}_{j,t} = \sigma_{j,t}$. If instead $j = i$, then $\tilde{\sigma}_{i,t}$ is obtained from $\sigma_{i,t}$ as follows. For any menu $\phi_{i,t}^M$, let $\phi_{i,t}^D = g_{i,t}(\phi_{i,t}^M)$ be the direct mechanism given by

$$\phi_{i,t}^D(\theta_t^E) = \delta_{i,t}(\theta_t^E, \phi_{i,t}^M) \quad \forall \theta_t^E \in \Theta_t^E.$$[13]

Now, let $\Phi_{i,t}^D(g_{i,t}) \equiv \{\phi_{i,t}^D : \phi_{i,t}^D = g_{i,t}(\phi_{i,t}^M),\ \phi_{i,t}^M \in \Phi_{i,t}^M\}$. After any $z_{i,t} \in Z_{i,t}^{J'}$, P_i uses his original behavioral strategy $\sigma_i(z_{i,t})$ to randomize over $\Phi_{i,t}^D$; formally, for any measurable subset $K \subseteq \Phi_{i,t}^D$

$$\tilde{\sigma}_i(K; z_{i,t}) = \sigma_i(B_K; z_{i,t})$$

where $B_K \equiv \{\phi_{i,t}^M \in \Phi_i^M : g_{i,t}(\phi_{i,t}^M) \in K\}$. Clearly, any menu in B_K is payoff-equivalent for the agent. Given any profile of mechanisms $(\phi_{i,t}^D, (\phi_{j,t})_{j \neq i})$ with $\phi_{i,t}^D \in \Phi_{i,t}^D(g_{i,t})$, A then uses the conditional distribution $\sigma_i(\cdot \mid B_{\phi_{i,t}^D})$ to determine his participation decision. That is, with probability $\sigma_i(\phi_{i,t}^M \mid B_{\phi_{i,t}^D})$, A behaves according to the participation strategy $w^t(\theta_t^E, \phi_{i,t}^M, (\phi_{j,t})_{j \neq i})) \in \Delta(\mathcal{N} \cup \varnothing)$ as if the game were Γ_J and the mechanisms offered by the principals were $\phi_t = (\phi_{i,t}^M, (\phi_{j,t})_{j \neq i})$. If the lottery $w^t(\theta_t^E, \phi_{i,t}^M, (\phi_{j,t})_{j \neq i}))$ selects P_i, A reports his extended type truthfully to P_i. If instead, $w^t(\theta_t^E, \phi_{i,t}^M, (\phi_{j,t})_{j \neq i}))$ selects a P_j with $j \neq i$, then A uses the same Markov strategy as in Γ_J to select which messages to send to P_j. In either case, the agent's choice of effort is governed by the same Markov strategy as in Γ_J.

Next, consider a $(\phi_{i,t}^D, (\phi_{j,t})_{j \neq i})$ such that $\phi_{i,t}^D \notin \Phi_{i,t}^D(g_{i,t})$.Then, at any downstream information set A behaves as if the game were Γ_J and the menu offered by P_i were $\phi_{i,t}^M$ where $\phi_{i,t}^M$ is the menu whose image is $\mathrm{Im}(\phi_{i,t}^M) = \mathrm{Im}(\phi_{i,t}^D)$.

The principals' strategies in $\tilde{\sigma}$ can be sustained by beliefs over upstream histories that satisfy the (analog of the) properties described in the proof of Part 1—Step 1.[14] Along with these beliefs, the strategy profile $\tilde{\sigma}$ is a MPE for $\Gamma_{J'}$ and sustains the same outcomes as σ in Γ_J.

Iterating across all i, t gives the result.

Step 2. We now prove that for any MPE $\sigma^D \in \mathcal{E}(\Gamma^D)$, there exists a MPE $\sigma^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes. The proof parallels that of Theorems 4 and 5.

Let Γ_J denote a game in which $\Phi_{j,\tau} = \Phi_{j,\tau}^M$ for all $(j,\tau) \in J$, while $\Phi_{j,\tau} = \Phi_{j,\tau}^D$ for all $(j,\tau) \in \mathcal{R} \backslash J$, for some $J \subset \mathcal{R} \cup \{\varnothing\}$ with $\mathcal{R} \equiv \mathcal{N} \times \mathcal{T}$. We prove the result by showing that, given any MPE $\sigma \in \mathcal{E}(\Gamma_J)$, there exists an MPE $\tilde{\sigma} \in \mathcal{E}(\Gamma_{J'})$, with $J' = J \cup \{i,t\}$ for some $\{i,t\} \in \mathcal{R} \backslash J$, that sustains the same outcomes.

The (Markov) strategy profile $\tilde{\sigma}$ is constructed from σ as follows. For any $(j,\tau) \neq (i,t)$, $\tilde{\sigma}_{j,\tau} = \sigma_{j,\tau}$. For $(j,\tau) = (i,t)$, the strategy $\tilde{\sigma}_{i,t}$ is such that, for any measurable set $R \subseteq \Phi_{i,t}^M$ and any $z_{i,t} \in Z_{i,t}$

$$\tilde{\sigma}_{i,t}(R \mid z_{i,t}) = \sigma_{i,t}\left(\bigcup_{\phi_{i,t}^M \in R} \{\phi_{i,t}^D : \mathrm{Im}(\phi_{i,t}^D) = \mathrm{Im}(\phi_{i,t}^M)\} \mid z_{i,t} \right).$$

Next, consider the agent. Let

$$\bar{\Phi}_{i,t}^M \equiv \{\phi_{i,t}^M : \mathrm{Im}(\phi_{i,t}^M) = \mathrm{Im}(\phi_{i,t}^D) \text{ for some } \phi_{i,t}^D \in \Phi_{i,t}^D\}$$

[14] Take a $z_{i,\tau}$ such that, given $(\phi_{i,l})_{l=1}^{\tau-1}$, $z_{i,\tau}$ is consistent with σ_A and σ_{-i}. If $\tau > t$ and $I_t = i$, then it is no longer true that P_i's marginal beliefs over Θ_τ^E are a convex combination of her beliefs in Γ^J. However, because in this case A will never contract again with P_i, this is irrelevant for the result.

and for any $\phi_{i,t}^M \in \bar{\Phi}_{i,t}^M$, let $\Phi_{i,i}^D(\phi_{i,t}^M) \equiv \{\phi_{i,t}^D : \text{Im}(\phi_{i,t}^D) = \text{Im}(\phi_{i,t}^M)\}$. At any $\tau \neq t$, $\tilde{\sigma}_A$ induces the same behavior as σ_A in Γ_J (recall that σ_A is Markov). At $\tau = t$, for any $(\phi_{i,t}^M, (\phi_{j,t})_{j\neq i})$ such that $\phi_{i,t}^M \in \bar{\Phi}_{i,t}^M$, A uses the conditional distribution $\sigma_i(\cdot \mid \Phi_{i,i}^D(\phi_{i,t}^M))$ to determine his participation decision. That is, with probability $\sigma_i(\phi_{i,t}^D \mid \Phi_{i,i}^D(\phi_{i,t}^M))$, A behaves according to the participation strategy $w^t(\theta_t^E, \phi_{i,t}^D, (\phi_{j,t})_{j\neq i})) \in \Delta(\mathcal{N} \cup \varnothing)$ as if the game were Γ_J and the mechanisms offered by the principals were $(\phi_{i,t}^D, (\phi_{j,t})_{j\neq i})$. In case the lottery $w^t(\theta_t^E, \phi_{i,t}^M, (\phi_{j,t})_{j\neq i}))$ selects P_i, A then also induces the same distribution over $Y_{i,t}$ as in Γ_J given $(\theta_t^E, \phi_{i,t}^D)$, where $\phi_{i,t}^D$ is the same mechanism selected by the distribution $\sigma_i(\cdot \mid \Phi_{i,i}^D(\phi_{i,t}^M))$. If instead, $w^t(\theta_t^E, \phi_{i,t}^M, (\phi_{j,t})_{j\neq i}))$ selects a P_j with $j \neq i$, then A uses the same Markov strategy as in Γ_J to select which messages to send to P_j. In either case, the agent's choice of effort is governed by the same Markov strategy as in Γ_J.

Next, consider a $(\phi_{i,t}^M, (\phi_{j,t})_{j\neq i})$ such that $\phi_{i,t}^M \notin \bar{\Phi}_{i,t}^M$. At any downstream information set A behaves as if the game were Γ_J and the direct mechanism offered by P_i were $\phi_{i,t}^D$ where $\phi_{i,t}^D$ is obtained from $\phi_{i,t}^M$ as follows:

$$\phi_{i,t}^D(\theta_t^E) \in \arg \max_{\delta_{i,t} \in \text{Im}(\phi_{i,t}^M)} V(\theta_t^E, \delta_{i,t}, \sigma_t^+) \quad \forall \theta_t^E \in \Theta_t^E$$

where $V(\theta_t^E, \delta_{i,t}, \sigma_t^{D+})$ denotes the agent's continuation payoff in $\Gamma_{J'}$ when his extended type is θ_t^E, he chooses to participate in P_i's mechanism and the principals' downstream strategies are σ_t^+.[15]

Because all players' strategies are Markov, the principals' strategies in $\tilde{\sigma}$ can be sustained by beliefs over upstream histories that satisfy the analog of the properties in Part 1—Step 1. Together with these beliefs, the strategy profile $\tilde{\sigma}$ is a MPE for $\Gamma_{J'}$ and sustains the same outcomes as σ in Γ_J. ■

A2-4. Sequential offering as opposed to sequential contracting

Finally, consider an environment in which principals offer their mechanisms sequentially, but where the agent sends the messages $(m_1, ..., m_n)$ simultaneously at $t = n + 1$. Assume that any P_t, $t = 2, ..., n$, observes the mechanisms ϕ_t^- selected upstream before choosing her own mechanism. A (pure) strategy for P_i thus consists of a function $\sigma_i : \Phi_i^- \to \Phi_i$ such that $\sigma_i(\phi_i^-)$ is the mechanism offered by P_i when the profile of upstream mechanisms is ϕ_i^-.

Since the agent's decisions are now taken only at the end of the game, the definition of extended type must be modified as follows. For any $i = 1, ..., n$, let $\theta_i^E \equiv (\theta, \delta_{-i})$ with $\delta_{-i} \equiv (\delta_j)_{j\neq i}$. From the perspective of P_i, the agent's extended type thus consists of his exogenous type θ along with the lotteries δ_{-i} he is inducing at $t = n + 1$ with the other principals. An extended direct mechanism $\phi_i^D : \Theta_i^E \to D_i$ is then defined as in the benchmark model. The definition of incentive-compatibility and truthful equilibrium must however be adjusted as follows. Let $V(\theta, \delta)$ denote the maximal payoff that type θ can obtain by choosing the lotteries δ.

[15] Because all principals' strategies are Markov, V depends on any upstream history only through θ_t^E.

Definition A2. *(i) A mechanism ϕ_i^D is incentive-compatible if and only if, for any $\theta_i^E \in \Theta_i^E$,*

$$\phi_i^D(\theta_i^E) \in \arg \max_{\delta_i \in \text{Im}(\phi_i^D)} V(\theta_i^E, \delta_i)$$

(ii) Given a profile of mechanisms $\phi^D \in \Phi^D$, the agent's strategy is truthful in ϕ_i^D if and only if, for any $\theta \in \Theta$ and any $(m_i^D, m_{-i}^D) \in Supp[\mu(\theta, \phi^D)]$,

$$m_i^D = (\theta, (\phi_j^D(m_j^D))_{j \neq i})$$

(iii) A strategy profile $\sigma^D \in \mathcal{E}(\Gamma^D)$ is a pure-strategy truthful equilibrium of Γ^D if and only if it is a pure-strategy equilibrium in which, given any profile of mechanisms ϕ^D such that $|\{j \in \mathcal{N} : \phi_j^D \neq \sigma_j^D(\phi_j^{D-})\}| \leq 1$, the agent's strategy is truthful in every mechanism ϕ_i^D for which $\phi_i^D = \sigma_j^D(\phi_j^{D-})$.

A mechanism ϕ_i^D is thus incentive-compatible if and only if, conditional on being a type θ and choosing the lotteries δ_{-i} with all principals other than i, the lottery $\delta_i = \phi_i^D(\theta_i^E)$ that the agent obtains by reporting $\theta_i^E \equiv (\theta, \delta_{-i})$ truthfully to P_i leads to an expected payoff for the agent that is at least as high as the one that he obtains by reporting any other $\hat{\theta}_i^E \in \Theta_i^E$. Given a profile ϕ^D of extended direct mechanisms, the agent's strategy is then truthful in ϕ_i^D if the message each type θ sends to P_i coincides with his true type along with the true decisions $\delta_{-i} = \phi_j^D(m_j^D))_{j \neq i}$ that he induces (by sending the messages m_{-i}^D) to the other principals. A strategy profile $\sigma^D \in \mathcal{E}(\Gamma^D)$ is a pure-strategy truthful equilibrium of Γ^D if and only if, whenever at most one principal deviated from her equilibrium strategy (i.e. offered a mechanism $\phi_j^D \neq \sigma_j^D(\phi_j^{D-})$), the agent's strategy at $t = n + 1$ is truthful in the mechanisms of any of the principals who conformed to the equilibrium strategy.

The following is then a natural adaptation of the notion of Markov strategies to this setting.

Definition A3. *Let Γ be a game with arbitrary choice sets for the principals. Given any pure-strategy profile $\sigma \in \mathcal{E}(\Gamma)$, we say that the agent's strategy σ_A is Markov with P_i if and only if, for any $\theta \in \Theta$, $\delta_{-i} \in D_{-i}$ and $\phi_i \in \Phi_i$, there exists a unique lottery $\delta_i(\theta, \delta_{-i}; \phi_i) \in \text{Im}(\phi_i)$ such that A always selects $\delta_i(\theta, \delta_{-i}; \phi_i)$ with P_i when the latter offers the mechanism ϕ_i, the agent's type is θ and the decisions A induces with the other principals are δ_{-i}. We then say that the agent's strategy is Markov if and only if it is Markov with all P_i, $i \in \mathcal{N}$.*

We then have the following result.

Theorem 8 (Sequential offering). *(Part I: Menus) Let $\Gamma \succcurlyeq \Gamma^M$. For any $\sigma \in \mathcal{E}(\Gamma)$ in which all principals' strategies are pure, there exists a $\sigma^M \in \mathcal{E}(\Gamma^M)$ that sustains the same outcomes. Furthermore, any SCF π that can be sustained as an equilibrium of Γ^M can be sustained as an equilibrium of Γ.*

(Part II: Direct Mechanisms) For any pure-strategy equilibrium $\sigma^M \in \mathcal{E}(\Gamma^M)$ in which the agent's strategy is Markov, there exists a pure-strategy truthful equilibrium $\sigma^D \in \mathcal{E}(\Gamma^D)$ that sustains the same outcomes.

Proof of Theorem 8. Part I: Menus. The proof parallels that of Part I in Theorem 6 and is thus omitted (one can easily verify that the proof is actually simpler when the agent takes decisions only at $t = n + 1$).

Part II: Direct Mechanisms. We show that, for any pure-strategy $\sigma^M \in \mathcal{E}(\Gamma^M)$ in which the agent's strategy is Markov, there exists a pure-strategy truthful equilibrium $\sigma^D \in \mathcal{E}(\Gamma^D)$ that sustains the same outcomes.

Consider a game Γ_J in which $\Phi_j = \Phi_j^D$ for all $j \in J$ while $\Phi_j = \Phi_j^M$ for all $j \in \mathcal{N} \backslash J$, for some $J \subset \mathcal{N} \cup \{\varnothing\}$. We prove the result by showing that given any pure-strategy equilibrium $\sigma \in \mathcal{E}(\Gamma_J)$ in which the agent's strategy is Markov there exists a pure-strategy equilibrium $\mathring{\sigma} \in \mathcal{E}(\Gamma_{J'})$ in which the agent's strategy is also Markov that sustains the same outcomes as σ, for any $J' = J \cup \{t\}$ with $t \in \mathcal{N} \backslash J$. The construction of $\mathring{\sigma}$ will also reveal that the strategy profile σ^D obtained from σ^M by iterating across all t,starting from $t = 1$ and moving forward, is such that σ_A^D is truthful.

Consider the following (pure) strategy for P_t in $\Gamma_{J'}$. For any profile of upstream mechanisms ϕ_t^-, let $\phi_t^M = \sigma_t(\phi_t^-)$ denote the equilibrium menu that P_t would have offered in Γ_J in response to ϕ_t^-. The extended direct mechanism $\phi_t^D = \mathring{\sigma}_t(\phi_t^-)$ that P_t offers in $\Gamma_{J'}$ in response to ϕ_t^- is such that, for any $\theta_t^E \in \Theta_t^E$,

$$\phi_t^D(\theta_t^E) = \delta_t(\theta, \delta_{-t}; \sigma_t(\phi_t^-))$$

Clearly, $\phi_t^D = \mathring{\sigma}_t(\phi_t^-)$ is incentive-compatible. Now consider the following strategy profile $\mathring{\sigma}$ for $\Gamma_{J'}$. For all principals P_j with $j < t$, simply let $\mathring{\sigma}_j = \sigma_j$. For P_t, let $\mathring{\sigma}_t$ be the strategy described above. Finally, for any P_j with $j > t$, $\mathring{\sigma}_j$ is constructed from σ_j as follows. If ϕ_j^- is such that in period t, P_t offered the mechanism $\phi_t^D = \mathring{\sigma}_t(\phi_t^-)$, then

$$\mathring{\sigma}_j(\phi_t^-, \phi_t^D, \phi_{t+1}, ..., \phi_{j-1}) = \sigma_j(\phi_t^-, \sigma_t(\phi_t^-), \phi_{t+1}, ..., \phi_{j-1}).$$

If instead, $\phi_t^D \neq \mathring{\sigma}_t(\phi_t^-)$, then

$$\mathring{\sigma}_j(\phi_t^-, \phi_t^D, \phi_{t+1}, ..., \phi_{j-1}) = \sigma_j(\phi_t^-, \phi_t^M, \phi_{t+1}, ..., \phi_{j-1}).$$

where ϕ_t^M is the menu whose image is $\text{Im}(\phi_t^M) = \text{Im}(\phi_t^D)$.

Next, consider the agent. Given any profile of mechanisms $(\phi_t^-, \phi_t^D, \phi_{t+1}, ..., \phi_n)$ such that $\phi_t^D = \mathring{\sigma}_t(\phi_t^-)$, at $t = n + 1$ each type θ of the agent induces the same outcomes he would have induced in Γ_J had the mechanisms offered been $(\phi_t^-, \sigma_t(\phi_t^-), \phi_{t+1}, ..., \phi_n)$. Note that this can be achieved by reporting $(\theta, (\phi_j(m_j))_{j \neq t})$ truthfully to P_t. If, instead, $\phi_t^D \neq \mathring{\sigma}_t(\phi_t^-)$, then A induces the same outcomes he would have induced in Γ_J had the mechanisms offered been $(\phi_t^-, \phi_t^M, \phi_{t+1}, ..., \phi_n)$, where ϕ_t^M is the menu whose image is $\text{Im}(\phi_t^M) = \text{Im}(\phi_t^D)$. Clearly, this strategy is sequentially optimal for the agent. Furthermore, given $(\mathring{\sigma}_A, \mathring{\sigma}_{-i})$, no principal has a profitable deviation. We conclude that the strategy profile $\mathring{\sigma}$ constructed this way is an equilibrium for $\Gamma_{J'}$ and induces the same outcomes as σ in Γ_J.

Iterating across all periods, starting from $t = 1$ and letting $J = \{\varnothing\}$ and proceeding forward by letting $J' = J \cup \{t+1\}$, gives a pure-strategy truthful equilibrium of Γ^D that sustains the same outcomes as σ^M. ■

References

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