Wharton

Financial
Institutions
Center

Empirical Tests of Models of Catastrophe Insurance Futures

by
Knut K. Aase
Bernt Arne Ødegaard

96-18

The Wharton School
University of Pennsylvania

THE WHARTON FINANCIAL INSTITUTIONS CENTER

The Wharton Financial Institutions Center provides a multi-disciplinary research approach to the problems and opportunities facing the financial services industry in its search for competitive excellence. The Center's research focuses on the issues related to managing risk at the firm level as well as ways to improve productivity and performance.

The Center fosters the development of a community of faculty, visiting scholars and Ph.D. candidates whose research interests complement and support the mission of the Center. The Center works closely with industry executives and practitioners to ensure that its research is informed by the operating realities and competitive demands facing industry participants as they pursue competitive excellence.

Copies of the working papers summarized here are available from the Center. If you would like to learn more about the Center or become a member of our research community, please let us know of your interest.

Anthony M. Santomero
Director

The Working Paper Series is made possible by a generous grant from the Alfred P. Sloan Foundation

Empirical Tests of Models of Catastrophe Insurance Futures [1]

April 29, 1996

Abstract : In this paper we empirically investigate models of insurance futures derivatives contracts. In the fall of 1993 the Chicago Board of Trade (CBOT) started trading a contract designed to scrutinize catastrophic risk, which is currently done in the reinsurance markets. There are obvious advantages to trading on organized exchanges (standardization, liquidity, much reduced credit risk, etc.) as opposed to OTC markets. There has so far been little academic on these contracts. In this paper we look at the price history for the first two years within the context of a pricing model of Aase [1995].

---

[1] Knut K. Aase and Bernt Arne Ødegaard are at the Norwegian School of Economics and Business Administration and the Norwegian School of Management.

We thank the Chicago Board of Trade for providing the data for this study.

Empirical tests of models of catastrophe insurance futures.

Knut K Aase and Bernt Arne Ødegaard*

29 April 1996

Abstract

In this paper we empirically investigate models of insurance futures derivative contracts.

1 Introduction.

In the fall of 1993 the Chicago Board of Trade (CBOT) started trading a contract designed to securitize catastrophe risk, which is currently done in the reinsurance markets. There are obvious advantages to trading on organized exchanges (standarization, liquidity, much reduced credit risk, etc) as opposed to OTC markets.

There has so far been little academic work on these contracts. In this paper we look at the price history for the first two years within the context of a pricing model of Aase [1995]. The questions we investigate are

- Does the model seem to be able to explain the data?
- Are the estimated parameters meaningful. (E.g: Does the risk parameters seem meaningful?)
- In what dimensions do the model have problems?

2 Contracts.

2.1 Futures.

2.2 Options.

2.3 Spreads.

3 Data.

Our data consists of the observed prices for various insurance futures derivatives. The time period is the first two years of trading. Appendix B lists the contracts in more detail.

Figure 3 gives an example of the typical price development for the SEP 94 40/60 spread contract.

Figure 1 Price observations for the SEP 94 40/60 spread contract.

We are also getting data on the underlying claims process, which will be incorporated in the estimations in a later version of the paper.

*Norwegian School of Economics and Business Administration and Norwegian School of Management. We thank the Chicago Board of Trade for providing the data for this study.

4 Pricing model

There has so far been little academic interest in the pricing of these contracts. In the estimations we do in this paper we investigate the model described in Aase [1995].

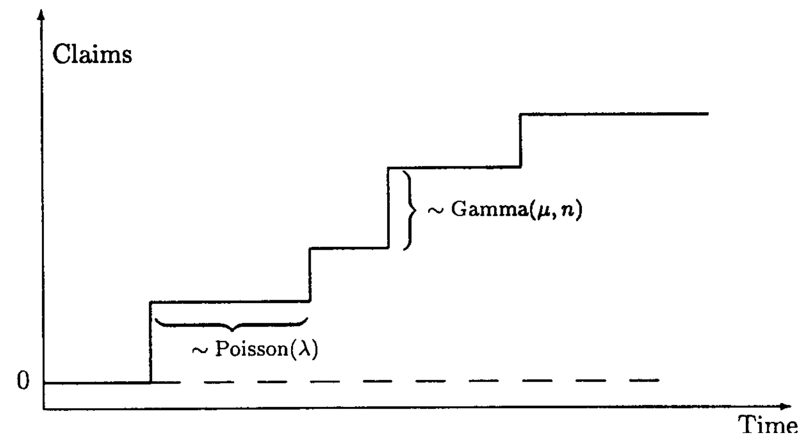
The only other published work we are aware of is Cummins and Geman [1995]. (Describe model)

Although we will want to compare these models at a later stage, the Cummins and Geman [1995] does not have a closed form solution, which makes it hard to estimate without going to simulation methods, as e.g. used in Bossaerts and Hillion [1993]

Let us now describe the Aase model. The important feature of the model is an assumed stochastic process for the underlying claims process.

Figure 2 illustrates the assumed process. The process is assumed to be a compound jump process. The waiting time till the next jump is distributed as Poisson with parameter λ. The size of the jump has a gamma distribution with parameters (μ, n).

Figure 2 The underlying stochastic process for claims.



5 Results

6 Conclusion

A Notation.

This appendix summarizes the notation that is used in the paper.

c	cap/call parameter
c_1	spread parameter
c_2	spread parameter
F_t	Futures price relative
n	Jump size parameter (Gamma)
T	Settlement date
$T-t$	Time to settlement
Z_t	Loss ratio
α	Risk aversion parameter
λ	Jump waiting time parameter (Poisson).
μ	Jump size parameter (Gamma)
Π	Total insurance premium paid
$\pi^{(x\wedge c)}(F_t,t)$	Price for a futures cap
$\pi^{(x\wedge c)^+}(F_t,t)$	Price for a futures option
$\pi^{\phi(x)}(F_t,t)$	Price for a futures spread
$\phi(\cdot)$	Spread payoff description

B Description of the price data.

The price data has been provided by the CBOT, and consists of daily price observations.

Table ?? gives an overview of the various contracts available to us, with the time intervals we have price observations for.

References.

Knut K Aase. An equilibrium model of catastrophe insurance futures and spreads. Unpublished working paper, The Norwegian School of Economics and Business Administration, Bergen, Norway, 1995.

Peter Bossaerts and Pierre Hillion. A test of a general equilibrium stock option pricing model. *Mathematical Finance*, 3(4):311-47, 1993.

J David Cummins and Hélyette Geman. Pricing catastrophe insurance futures and call spreads: An arbitrage approach. *Journal of Fixed Income*, pages 46–58, 1995.

An equilibrium model of catastrophe insurance futures and spreads

by

Knut K. Aase[1]

January 1995
This version: March 1995

Abstract

This paper presents a valuation theory of futures contracts and derivatives on such contracts when the underlying delivery value follows a stochastic process containing jumps of random claim sizes at random time points of accident occurrence. Applications of the theory are made on insurance futures, a new type of instrument for risk management launched by the Chicago Board of Trade in 1992, anticipated to start soon in Europe, and perhaps also in other parts of the world in the near future.

The welfare loss in ordinary insurance markets due to adverse selection is likely to be reduced due to the introduction of this new market.

Several closed pricing formulas are derived, both for futures contracts and for futures derivatives, such as caps, call options and spreads. The framework is that of general, and partial, economic equilibrium theory under uncertainty.

Key words: Insurance futures, futures derivatives, claims processes, reinsurance

[1]Norwegian School of Economics
and Business Administration
Helleveien 30, 5035 Bergen, Norway

1. Introduction

The newly founded market for insurance derivatives is the motivation behind this paper. This market was established in December 1992 by the Chicago Board of Trade (CBoT), and offers insurers an alternative to reinsurance as a hedging device for underwriting risks. The terminal cash flow is related to the aggregate claims incurred during a calender quarter, or more precisely, to moves in a loss ratio based on figures for claims and premiums compiled by the Insurance Services Office (ISO). The settlement price for each futures contract increases by $250 for each percentage point upwards movement in the ratio.

Another interesting innovation is a set of new agricultural insurance contracts approved by the CBoT on October 18, 1994, expected to begin trading in the first half of 1995. These contracts - known as area yield options - provide a means for hedging against shortfalls in the harvest of particular crops. An advantage of the crop yield contracts is that there is already an OTC derivatives market in this area. More ambitious OTC deals are on the drawing board. For example it would be possible to devise instruments which would effectively swap hurricane risk for earthquake risk. Another possibility is “act of God” bonds with coupons which decline as the number of catastrophe insurance claims rises. Insurers would be natural issuers of such products. With all these new and old instruments, markets are getting more and more complete, and we are indeed getting close to Arrow-Debreu securities in the end.

The present paper presents a situation where the underlying stochastic dynamics is assumed to allow for unpredictable jumps at random time points. In particular we have in mind claims caused by accidents in an insurance framework, as the loss ratio index in the CBoT exchange, and we intend to model an index of such claims by a random, marked point process. An arbitrage pricing model based on this assumption usually contains many equivalent martingale measures, so this approach does not lead to a unique pricing rule. Although some progress has been made in this direction (see e.g., Føllmer and Sondermann (1986), Schweitzer (1991)), we choose to stay within the framework of equilibrium theory, and derive prices of forward contracts and relevant derivatives within this setting. There is, however, an arbitrage type approach given by Cummins and Geman (1995) for catastrophe insurance futures. They use the time integral of geometric Brownian motion as a model for the accumulated claims, and use an Asian options approach.

Unlike reinsurance, hedging through futures has the advantage of reversibility since any position may be closed before the maturity of the contract. In principle also a traditional reinsurance contract may be reversed, however in practice reversing a reinsurance transaction exposes the insurer to relatively high transaction costs presumably to protect the reinsurer against adverse selection.

Adverse selection is also present in traditional reinsurance even if transactions are supposed to take place under *umberrimae fidei.* The rating system for insurance companies by e.g., Insurance Solvency International may be taken as an indication of this. Traditional insurance against shortfalls in the harvest of crops would be practically impossible because of the adverse incentives this would create for the farmers. Insurance through area yield options or catastrophe insurance futures are not subject to these kinds of objections since the contracts relate to anonymous indexes rather than individual crops or contracts, so it may seem like the problem with moral hazard is essentially eliminated with these new instruments. Because of this and the absence of adverse selection, the low transactions costs that are common to traditional futures contracts could be presumed to prevail also for these new types of contracts. Furthermore there should be the advantages of liquidity associated with ordinary futures markets. Put together this new market is likely to improve welfare.

The paper deals with the delicate problem of pricing *catastrophe risk*, as such, risk which is priced in this model and not treated as unsystematic risk. In the representation of the loss ratio index, we follow insurance tradition by using a standard actuarial approach. The presented model thus combines economic theory with actuarial practice and theory.

The paper is organized as follows. In section 2 the underlying economic model is presented as well as the pricing results from general equilibrium theory under the kind of uncertainty indicated above. In section 3 the theory is applied to futures contracts on indexes represented by random marked point processes, and in section 4 some applications are made to the CBoT market, where a simple futures pricing formula is derived. In section 5 derivatives on the futures index are discussed, and as some particularly important applications we analyze in detail a futures cap, a futures call option and a futures spread. In section 6 we offer some concluding remarks.

2. The economic model

We consider a pure exchange economy along the lines of Aase (1992, 1993a, b).

2.1 The model of uncertainty.

A complete probability space $(\Omega, \mathcal{F}, P)$ is given, where Ω is the set of states of nature with generic element ω, $\mathcal{F}$ is the σ-algebra of possible events on $(\Omega, \mathcal{F})$. A random measure $\nu(\omega, A; t)$ is defined on $(\Omega, \mathcal{F}, P)$, where $\nu(A; t)$ is the number of jumps some fundamental stochastic process makes in the time interval $(0, t]$ with values falling in the set A, $A \in \mathcal{B}^d$, where $\mathcal{B}^d$ equals the Borel measurable subsets in R^d. The interpretation is that at random time points $\tau_1, \tau_2, \cdots$ events happen and a corresponding sequence of jumps $U^{(1)}, U^{(2)}, \cdots$ with values in R^d are realized by the fundamental stochastic process. The economy has a finite horizon $\mathcal{T} = [0, T]$. The flow of information is given by a natural filtration, i.e. the augmented filtration $\{\mathcal{F}_t; t \in \mathcal{T}\}$ of σ-algebras of $\mathcal{F}$ generated by ν. Let L be the set of adapted processes satisfying the integrability constraint $E\left(\int_0^T Y(t)^2 dt\right) < \infty$.

We assume that the preferences of I agents, $I = \{1,2, \ldots, I\}$, can be represented by utility functions U^i which are additively separable, that is

$$(2.1) \qquad U^i(X^{(i)}) = E\left\{\int_0^T u_i(X^{(i)}(t),t)dt\right\}, \quad i \in I,$$

where u_i: $L \times [0, T] \to R$ is sufficiently smooth. Conditions are known on the endowments, or accumulated portfolios $X^{(i)}$ of the agents, and the utility functions U^i for a contingent-capital market equilibrium of the Arrow-Debreu type to exist (see e.g. Araujo and Monteiro (1989) and Duffie and Zame (1989)). Given an equilibrium, a utility function representing the *market is* a function U_k of the form

$$(2.2) \qquad U_k(X) = \max_{x_i \in X_+} \sum_{i=1}^{I} k_i U^i(X^{(i)}) \quad \text{subject to } \textstyle\sum_{i=1}^{I} X^{(i)} \leq X,$$

such that (φ, X) is the no trade equilibrium for the single agent economy (U_k, X), where φ is the pricing functional. The function U_k is also known as the utility function of the representative agent (see e.g. Lucas (1978)). Under assumptions known in the literature it has been shown that there exists a contingent capital market equilibrium $(\varphi, X^{(i)})$ with a function U_k for some $k \in R_+^I$. Let u_k be defined by

$$(2.3) \qquad u_k(X,t) = \max_{X^{(i)}} \sum_{i=1}^{I} k_i u_i(X^{(i)}, t) \quad \text{subject to } \textstyle\sum_{i=1}^{I} X^{(i)} \leq X.$$

Then U_k is additively separable, i.e. $U_k(Y) = E\left\{\int_0^T u_k(Y(t),t)dt\right\}$ for any $Y \in L$, and $k \in R_+^I$ can be picked so that, for any $Y \in L$, $\varphi(Y) = E\left\{\int_0^T u_k'(X(t),t)Y(t)dt\right\}$, where $X(t) = \sum_i X^{(i)}(t)$ is the aggregate endowment in the market, and where prime stands for differentiation with respect to the first argument. Our underlying assumption is that the vector of dividend processes in the market are marked point processes driven of random measures explained in the beginning. We may change to real security price processes $\hat{S}$ defined by $\hat{S}(t) = S(t)/p(t)$, where p(t) is the spot price of the aggregate endowment X. With these conventions, the following result can be proved (see e.g., See Aase (1993b))

Proposition 1

Let $\{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P), (U^i, X^{(i)}), D\}$ be a security-spot market economy such that:

(a) U^i is additively separable and "sufficiently" regular, $i = 1, 2, \cdots I$.

(b) The aggregate endowment $X = \sum_i X^{(i)}$ is bounded away from zero.

Then there is a capital market equilibrium with a set of Pareto optimal allocations $\{Y^{(i)}(t); 0 \leq t \leq T\}$ and a market representation U_k, additively separable and "sufficiently" regular, for which the real price process $\hat{S}$ of the real dividend process $\hat{D}$ satisfies

(2.4) $$\hat{S}(t) = \frac{1}{u_k'(X_t,t)} E\left\{\int_t^T u_k'(X_s,s)d\hat{D}(s) \mid \mathcal{F}_t\right\}, \qquad P\text{-a.s., } t \leq T.$$

2.2 An insurance index of the CBoT-type

We now think of the model above as an insurance syndicate, where each member is characterized by a utility function U^i and net reserves $X^{(i)}(t) = w_0^{(i)} + \int_0^t a^{(i)}(s)ds - Z^{(i)}(t)$, where $w_0^{(i)}$ is the initial endowment of insurer i, $a^{(i)}(t)$ is a premium rate, a predictable process, and where

(2.5) $$Z^{(i)}(t) = \int_0^t\int_{R_+} h^{(i)}(u,s)v^{(i)}(du; ds) = \sum_{n=1}^{\infty} h^{(i)}(U_n,\tau_n)1(\tau_n \leq t) = \sum_{n=1}^{N^{(i)}(t)} h^{(i)}(U_n,\tau_n).$$

Here $N^{(i)}(t)$ is the number of claims incurred by time t in the portfolio of insurer i, $R_+ = [0,\infty)$, $v^{(i)}$ is a random measure on R_+, and $h^{(i)}(U_n,\tau_n)$ is the size of the n-th claim occurring at time τ_n.

We now define an index of insurance claims as the aggregate of processes of the above type, i.e. let

(2.6) $$X(t) = W(t) + \int_0^t a(s)ds - Z(t)$$

where W is a stochastic process independent of the insurance claims and where $\int_0^t a(s)ds =$ premiums earned in the market by time t. The length of the calendar quarter (the "event quarter") equals T, and $\int_0^T a(s)ds = \Pi$ is known before the event quarter starts. In general a_t is a (possibly time-varying) premium rate, which may be a bounded variation, predictable stochastic process. The stochastic process Z(t) represents the aggregated claims by time t reported to, say, the ISO pool. Since claims, and in particular catastrophes, can not be considered as infinitesimal, we represent Z by a process of the type

(2.7) $$Z(t) = \int_0^t\int_{R_+} u\nu(du; ds) = \sum_{n=1}^{\infty} U_n 1(\tau_n \le t) = \sum_{n=1}^{N(t)} U_n$$

where N(t) is the number of claims incurred by time t, $R_+ = [0,\infty)$, and where U_n is the size of the n-th claim occurring at time τ_n. This interpretation will be used in the rest of the paper. For more on this model in reinsurance markets, see e.g., Aase (1992), (1993 c).

3. Futures and forward contracts on insurance indexes

3.1 The CBoT-index

In this section we apply the results of the previous section to the pricing of forward and futures contracts on insurance indexes. We start by describing the index introduced by the Chicago Board of Trade in December 1992, but we do not intend to give any detailed explanation of these contracts, only the skeleton necessary to understand the principles.

Before the insurance risk can be securitised, it must be standardized. In the case of the CBoT'S catastrophe insurance contracts, this meant devising an index on which to base derivatives. Unlike the equity, bond or commodity markets, the insurance market has no obvious, continuously updated underlying cash price. The solution chosen by the Board of Trade is the loss ratio index, calculated by the Insurance Service Office, which uses data from at least 25 designated reporting companies. The loss ratio is the dollar value or reported losses incurred in a given quarter (the loss quarter) and reported by the end of the following quarter (the run-off quarter) divided by one fourth of the dollar value of the premiums collected in the previous year. The contract value is $25,000 times the loss ratio.

The premiums in the pool is a known constant throughout the trading period, and price changes are attributable solely to changes in the market's attitude towards risk and expectations of loss liabilities at each time t, given the available information at that time.

The idea for the insurers is to use this market to hedge against unexpected losses in the following quarter. Clearly, if the loss ratio of the pool is not perfectly correlated with that of the insurer, this hedge will not be perfect. The splitting of the index into different regions, with some common pattern of risk exposures within each region, and with the risk inhomogenity being between the regions, is clearly an advantage towards making the hedge more effective.

The opposite side of the market (sellers of futures) consists primarily of investors and speculators. We should add here that the way the loss ratio index is constructed may lead to a moral hazard problem. It would be an advantage if it had been a purely scientific index of some sort.

A perfect hedge can only be obtained using traditional reinsurance, but since insurers' business is precisely that of risk bearing, they will normally not be interested in a "perfect" hedge, since the best they can hope for then is profits close to the riskless rate, which will not satisfy most stock holders of insurance companies. Partly due to adverse selection, a perfect hedge can be expensive, and sometimes traditional insurance is impossible to obtain. This new futures insurance market may therefore be an excellent innovation in the insurance business, and possibly improving welfare at large through better risk sharing and risk distribution, at least when combined with traditional insurance and reinsurance.

Since risk averse insurers are seeking "reinsurance" protection in the CBoT futures index, they should normally be willing to pay a risk premium for this protection. On the other side are the investors requiring compensation for bearing risk.

3.2 Forward (and futures) prices on insurance indexes.

Returning to the general economic theory in section 2, consider now a forward contract written on an index Z. The aggregate endowment process appearing in Proposition 1 is then $X(t) = W(t) + \int_0^t a(s)ds - Z(t)$ for all $0 \le t \le T$. We now set

$$p(t) = \exp\left(-\int_0^t r(s)ds\right), \tag{3.1}$$

where r is the short term world interest-rate in the market, which we assume exogenously given for the moment, making our approach a partial equilibrium analysis. We return to a full equilibrium model below. With an appropriate independence assumption, our price for the forward contract will also coincide with the futures price. Thus we concentrate on forward contracts for the moment. We denote one fourth of the dollar value of the premiums collected in the previous year by Π. To this end, the real accumulated dividend process $\hat{D}$ for the forward contract is given by

(3.2)
$$\hat{D}_s = \begin{cases} 0 & \text{when } t < s < T \\ (\hat{Z}_T - F_t) & \text{when } s \geq T \end{cases}$$

where $\hat{Z}(T) = Z(T)/\Pi$ is the loss ratio. Furthermore, the spot price $\hat{S}(t)$ of a forward contract is zero. We now use the pricing rule of Proposition 1 and get

(3.3)
$$F_t = \frac{E\{u_k'(X_T,T)\hat{Z}_T \mid \mathcal{F}_t\}}{E\{u_k'(X_T,T) \mid \mathcal{F}_t\}} \qquad \text{for } t \leq T.$$

Since our application is to the insurance market, where it seems reasonable to assume that the short interest rate r is conditionally statistically independent of Z, given $\mathcal{F}_t$, we should perhaps anticipate that the theoretical futures price is close to the forward price. More precisely, the following can be shown in the present model:

Proposition 2.

Then the equilibrium real futures price process F(t) is given by

(3.4)
$$F(t) = \frac{E\{u_k'(X_T,T)\hat{Z}(T)\exp\{\int_{(t,T]} r(u)du\} \mid \mathcal{F}_t\}}{E\{u_k'(X_T,T)\exp\{\int_{(t,T]} r(u)du\} \mid \mathcal{F}_t\}}$$

A proof can be constructed using the definition of a futures contract.

In the case that the interest rate process r is conditionally statistically independent of the insurance loss ratio index $\hat{Z}_T$ and endowment process X_T, given $\mathcal{F}_t$, the formula (3.4) reduces to the expression (3.3). Without the independence of r and X, still this may happen for certain u_k-functions, as we shall see below.

3.3 An equilibrium model involving also interest rate

In a full equilibrium model also the interest rate r must be endogenized, and this is possible. Referring to Aase (1993a), it follows from the analysis there that the equilibrium interest rate r(t) is given by the expression

(3.5)
$$r(t) = -\frac{m(t)}{u_k'(X_t,t)},$$

where

(3.6)
$$m(t) = \frac{\partial}{\partial t}u_k'(X(t),t) + u_k''(X(t),t)a(t) +$$
$$\int_{R^d}\{u_k'(X_t - h^X(u,t),t) - u_k'(X_t,t) + u_k''(X_t,t)h^X(u,t)\}\lambda(t)F(t;du),$$

where F(s; du) is the claim size distribution, and where h^x stems from the representation

(3.7) $$dZ(t) = \int_{R^d} h^x(u,t)\nu(du;dt) \quad \text{for } t \le T.$$

In words; there exists an equilibrium under our stipulated conditions, in which the real interest rate process r(t) equals minus the growth rate of the market's marginal utility for aggregate wealth.

If the dimension d is large enough, all the relevant assets could in principle be included in the model. In such a world it would not be a natural assumption that r is independent of Z; in fact the main reason for adopting this more involved model is precisely to take into account a possible dependency here. Such a stochastic association is then motivated from the fact that we are considering catastrophes, which, after having occurred also may influence the values of the interest rate r (consider e.g. a scenario with a major earthquake in Tokyo).

3.4 An interpretation of the market price of insurance risk.

Returning to the model explained in section 2, consider a predictable process h. Stochastic integrals can be defined with respect to such processes as follows:

(3.8) $$h \bullet \nu_t = \int_0^t \int_{R^d} h(s,u)\nu(du;\, ds) = \sum_{n=1}^{\infty} h(\tau_n,\, U^{(n)})1(\tau_n \le t),$$

where 1(B) is the indicator function of the set $B \subseteq \Omega$. We assume that the $(P, \mathcal{F}_t)$ - predictable intensity-transition kernel $\lambda(\omega, t;\, du)$ associated with the random measure ν, the *dual predictable projection* of ν, can be factored into a conditional joint probability transition kernel $F(\omega,t;du)$ and a non-negative $\mathcal{F}_t$ - predictable intensity process $\lambda(\omega, t)$ as follows

(3.9) $$\lambda(t;\, du)dt = \lambda(t)F(t;\, du)dt, \qquad t \in \mathcal{T},\ \omega \in \Omega.$$

For any dividend process D there exists a probability measure Q equivalent to P - an equivalent martingale measure - such that for some predictable $\tilde{h}_D(t,u)$

(3.10) $$D_t = \int_0^t \int_{R^d} \tilde{h}_D(s,u)\tilde{\nu}(du;\, ds),$$

under Q, where $\tilde{\nu}$ is a centered random measure under Q, and (λ^*, F^*) are the local characteristics of ν under Q: The equation (3.10) is a consequence the martingale representation theorem and Girsanov's change of probability theorems for random measures (see e.g., Jacod and Shiryaev (1987)). The Radon-Nikodym derivative $\xi(T) = \frac{dQ}{dP}$ and its associated density process $\xi(t) = E(\xi(T) \mid \mathcal{F}_t)$ is given by

$$(3.11)\quad \xi(t) = \Big(\prod_{n\geq 1}\mu(\tau_n)v(\tau_n,U^{(n)})1(\tau_n \leq t)\Big)\exp\Big\{\int_0^t\int_{R^d}\Big(1-\mu(s)v(s,u)\Big)\lambda(s)F(s;\,du)ds\Big\}$$

for any $t \in [0, T]$. The local characteristics λ^* and F^* of the random measure ν under Q are given by $\lambda^*(t) = \mu(t)\lambda(t)$ and $F^*(t;\,du) = v(t,u)F(t;\,du)$, where

$$(3.12)\qquad \int_0^T \mu(t)\lambda(t)dt < \infty, \quad \text{P- a.s.}$$

$$(3.13)\qquad \int_{R^d} v(t,u)F(t;\,du) = 1, \quad \forall\, t \in \mathcal{T},\ \text{P- a.s.}$$

The two terms $\mu(t)$ and $v(t,u)$ we interpret as the market prices of frequency risk and of claim size risk respectively. Using Itô's formula on the term $\ln(u_k'(X(t),t))$, we may compare terms in the relation $u_k'(X(t),t) = p(t)\xi(t)$, where $\xi(t)$ is interpreted as a state price. After some calculations we obtain

$$(3.14)\qquad \mu(t)v(t,u) = \frac{u_k'(X(t)-h^X(t,u),t)}{u_k'(X(t),t)}, \quad 0 \leq t \leq T;\ u \in R^d.$$

Noticing from (3.7) that h^x is positive, under risk aversion we see that the product $\mu(t)v(t,u) > 1$ for all u and t, and since $v(t,u)$ is a density for each t, (2.16) implies that $\mu(t) > 1$ for all $t \geq 0$. *Thus, in a risk averse insurance market, the risk adjusted frequency $\lambda\mu$ of the claims is larger than the (objective) frequency λ.*

As an illustration, consider the case of power intertemporal utility function and time separation, i.e. $u_k'(x,t) = e^{-\rho t}x^{\gamma-1}$, $x > 0$, $t \geq 0$, where ρ is the subjective time impatience discount rate, and $(1-\gamma)$ is the intertemporal coefficient of *relative* risk aversion. Here we get

$$(3.14a)\qquad \mu(t)v(t,u) = \frac{1}{(1-\frac{h^X(t,u)}{X(t)})^{1-\gamma}}, \quad 0 \leq t \leq T;\ u \in R^d,$$

where $h^X(t,u) < X(t)$. For the exponential intertemporal utility case we have $u_k'(x,t) = e^{-\rho t}e^{-\alpha x}$, $x \in R$, $t \geq 0$, and now we get

$$(3.14b)\qquad \mu(t)v(t,u) = e^{\alpha h^{\wedge}(t,u)}, \quad 0 \leq t \leq T;\ u \in R^d,$$

where α is the intertemporal coefficient of *absolute* risk aversion.

We see that when the insurance claim size $h^X(t,u)$ increases the term $\mu(t)v(t,u)$ increases. This seems reasonable in view of our interpretations of the terms μ and v. Furthermore, in both the above cases we observe that the term μv increases as the respective type of risk aversion increases. We return to these examples later.

4 A model for catastrophe futures contracts

4.1 Introduction

In this section we apply the results of the previous sections to the pricing of futures contracts on insurance indexes for a specific model.

4.2 A basic futures formula

In order to illustrate the ideas we make some simplifying assumptions. For the representation of the claims index $Z(t) = \sum_{i=1}^{N(t)} Y_i$ we assume that Z(t) is a compound Poisson process with gamma (n, μ) distributed claim size amounts, and with frequency λ. In this case $\hat{Y}_i = Y_i/\Pi$ is also gamma (n, $\mu\Pi$) - distributed, and as before $\hat{Z}(t) = Z(t)/\Pi$ is the loss ratio index. The model $X(t) = w_0 + at - Z(t)$ is called the Lundberg model, well known in insurance. Here both w_0, the initial endowments, and the premium rate a are constants. Assume now that the aggregate endowment process $X(t) = W(t) + at - Z(t)$, where W is some stochastic process independent of Z. Furthermore we assume that the marginal utility function of the representative agent is of the form $u_k'(x,t) = e^{-\alpha x} e^{-\rho t}$, where α is the intertemporal coefficient of the absolute risk aversion, a constant here assumed smaller that the parameter μ, and where ρ is the subjective time impatience rate of the market. We now want to show that the following:

Theorem 1

Under the assumptions above, and assuming that the short rate process r is conditionally independent of Z, given $\mathcal{F}_t$, the following futures formula obtains

$$(4.1) \qquad F_t = \$25{,}000\ \left(\hat{Z}_t + \frac{n\mu^n \lambda (T-t)}{\Pi(\mu-\alpha)^{n+1}}\right) \quad \text{for } t \leq T \text{ and } 0 \leq \alpha < \mu.$$

Proof: Direct computation. Alternatively, see section 4.5.

4.3 Discussion of the futures formula.

Notice that the formula (4.1) only depends on parameters that can be estimated from market data. In particular this is true for the process parameters n, μ and λ, but also α may be estimated from available data. The formula (4.1) can be written, modulo \$25,000,

$$F_t = \hat{Z}_t + \frac{n\lambda(T-t)}{\Pi\mu}\left(\frac{\mu}{\mu-\alpha}\right)^{n+1} = \hat{Z}_t + E(\hat{Z}_T - \hat{Z}_t \mid \mathcal{F}_t)\left(\frac{\mu}{\mu-\alpha}\right)^{n+1},$$

the last expression being true since $E(\hat{Z}_T - \hat{Z}_t \mid \mathcal{F}_t) = \lambda(T-t)\frac{n}{\Pi\mu}$, which follows from our remark in section 4.2 concerning the distribution of the loss ratio $\hat{Z}$. We remark the following:

F_t depends upon preferences in the market through the parameter α = intertemporal coefficient of absolute risk aversion in the market. The futures price process F_t is seen to increase as α increases, ceteris paribus, i.e. more risk aversion in the market leads to more expensive "reinsurance" as well as to higher required risk premiums for the investors/speculators on the opposite side of the contracts. This seems to be well in accordance with intuition.

In addition F_t increases, ceteris paribus, with the time to settlement $(T - t)$, with the claims frequency λ, and finally, as the expected sizes of the claims increase through the parameter n ($EY_1 = n/\mu$).

It may be noticed that the expression F_t does not depend on the endowment process W in the market. This is a consequence of the fact that for the exponential utility, the absolute risk aversion is not depending on the level of wealth. This is of course one unrealistic feature of our model, since it would indeed be desirable with some kind of wealth dependence on F_t (even if the probability distribution of W is not really known).

The futures price process F_t is seen to depend upon the premium level $\Pi = aT$, however, a known quantity in this market. More precisely, as the level of premiums increase, ceteris paribus, the futures price F_t decreases.

4.4 Risk premiums

From the information manual of the CBoT - market, it seems as if no economic risk premium is included in the values of F_t calculated so far. Of course, the market will decide the futures prices and hence the risk premiums, not the actuaries. In our model the risk premium equals

$$(4.2) \qquad F_t - \{\hat{Z}_t + E(\hat{Z}_T - \hat{Z}_t \mid \mathcal{F}_t)\} = \frac{n}{aT\mu}\lambda(T-t)\Big\{\Big(\frac{\mu}{\mu-\alpha}\Big)^{n+1} - 1\Big\}$$

Since $\alpha < \mu$, the term $\left(\frac{\mu}{\mu-\alpha}\right)^{n+1} > 1$ for n a positive integer, so the risk premium is positive, in accordance with our earlier remarks. Under risk neutrality $\alpha = 0$, in which case it follows from (4.2) that the risk premium is zero. Thus the term $\left(\frac{\mu}{\mu-\alpha}\right)^{n+1}$ corrects for risk aversion in the market. For a given value of μ, the risk premium is an increasing function of both α and n, and for given values of α and n, the risk premium is a *decreasing* function of μ. Since $EY_1 = n/\mu$ and $\mathrm{var}Y_1 = n/\mu^2$, an increase in μ leads to a decrease in both the expected size of the loss and in the variance of the loss, so as a result a risk averse market would tend to require less compensations for risk bearing, also well in accordance with economic intuition. Finally we notice that the risk premium decreases with the premium level $\Pi = aT$, ceteris paribus.

4.5 Risk adjusted evaluation

Returning to the results of section 3, we may compute $F(t) = E^Q(\hat{Z}_T \mid \mathcal{F}_t)$ by alternatively finding the distribution of Z under the equivalent martingale measure Q, which in the present case corresponds to $\kappa \cdot v(y) = e^{\alpha y}$, where κ here equals the risk adjustment on the frequency λ (called μ_t in section 3). Let h(y) be the probability density of the gamma (n, μ)-distribution. Since $\int v(y)h(y)dy = 1$ from condition (2.16), we see that $\kappa \int v(y)h(y)dy = (\frac{\mu}{\mu - \alpha})^n$, $0 < \alpha < \mu$, so $\kappa = (\frac{\mu}{\mu - \alpha})^n > 1$. *Thus the risk adjusted frequency equals $\lambda\kappa > a$.*

Furthermore the claim sizes $Y_1, Y_2, \cdots$ are all independent and identically distributed with density v(y)h(y), meaning that they are all gamma (n, $\mu - \alpha$) - distributed. From the comments in the above subsection, since $\alpha > 0$, *this risk adjustment amounts to making the claim size distribution more risky under Q than under P.* Thus, in the constructed Q - economy where the "pseudo-agents" are all risk neutral, they agree on a probability distribution for the loss ratio index that is more risky than the objective distribution.

Now the above computation is straight-forward, since Z is a compound Poisson process also under Q (see e.g. Delbaen and Haezendonck (1989), Aase (1993c)) and F(t) = $E^Q(\hat{Z}_T \mid \mathcal{F}_t) = \hat{Z}_t + E^Q(\hat{Z}_T - \hat{Z}_t \mid \mathcal{F}_t) = \hat{Z}_t + \lambda \cdot \kappa (T - t) \frac{n}{\mu - \alpha} \frac{1}{\Pi}$, which is exactly the formula (4.1). Thus we here have an alternative proof of Theorem 4.1.

The condition that $X_t = W(t) + at - Z_t$ is a (Q, $\mathcal{F}_t$)-martingale, gives us a possibility to find endogenously the market value of the premium rate a, or of the premium $\Pi = aT$. Since $X(0) = E^Q(X(T) \mid \mathcal{F}_0)$, and $W(0) = X(0)$, we get that $\Pi = aT = E^Q(Z(T) \mid \mathcal{F}_0) + (W(0) - E^Q W(t))$, or

$$\Pi = \frac{\lambda T n \mu^n}{(\mu - \alpha)^{n+1}} + \left(W(0) - E^Q W(T)\right) \tag{4.3}$$

with an obvious simplification if also W is Q-martingale.

In conclusion the formulas (4.1), (4.2) and (4.3) for the market value of the insurance catastrophe futures contract, the associated economic risk premium and the market value of the insurance claims for the next quarter all seem to capture well some essential economic features of the pricing of the risky instruments under investigation. In the final section of the paper we turn to the pricing of derivatives on the futures index.

Example 1. In order to illustrate the use of catastrophe insurance futures, consider the following simple numerical example. Ins Ltd. expects to earn $1 million in premiums on its insurance policies during the third quarter of 1995. Ins Ltd.'s actuaries have forecasted that $400 000 in catastrophic losses will be incurred by the company for this period. It is also predicted that the third quarter catastrophic losses for the sample companies that report to ISO will be $200 million, with associated premiums amounting to $400 million. Ins Ltd. decides to buy December 1995 Eastern Catastrophe contracts at the beginning of July. Ignoring possible reporting lags, we assume that the key parameters are estimated as follows: $\hat{\lambda} = 10$, $(T - t) = .25$, $\hat{n} = 80$, $\hat{\mu} = 10^{-6}$, $(\hat{\mu}/(\hat{\mu}-\hat{\alpha}))^{81} = 1.10$ corresponding to a 5% risk premium, resulting in $10^6\hat{\alpha} = .00118$. According to the formula (4.1) the futures price equals \$ $25000\cdot(200/400)\cdot 1.10$ = \$ 13 750. The company decides to buy 40 contracts (= 1 million/25 000).

Scenario A: Assume that the weather in August and September was worse than anticipated. Ins Ltd.'s actual catastrophic losses turn out to be $600 000, $200 000 more than anticipated. The ISO reporting companies were similarly affected and the total catastrophic losses incurred by these companies during the third quarter were $300 million, $100 million more than expected. The final settlement price then becomes \$ $25\,000\cdot(300/400)$ = \$ 18 750. The gain from 40 long futures contracts then becomes \$ $(18\,750 - 13\,750)\cdot 40$ = \$ 200 000, offsetting Ins Ltd.´s unexpected catastrophic losses by 100%.

Scenario B: Assume that the weather in the third quarter turned out better than forecasted. Ins Ltd. lost only $200 million ($ 200 million less than anticipated), the ISO reporting companies lost $150 million ($ 50 million less than expected). In this case the final settlement price was \$ $25\,000\cdot(150/400)$ = \$ 9 375. In the case that Ins Ltd. held the futures position long until settlement, the loss would amount to $175 000 on this position, resulting in only $25 000 more than anticipated in the final result. On these contracts the investors/speculators on the opposite side of the contracts made $ 175 000. □

5 Derivatives on the futures index

5.1 Introduction

In practice futures contracts are not the frequently traded instruments in CBoT - market, but rather spreads on the futures index. Usually insurers are accustomed to profit from losses that are less than originally anticipated. As an alternative the insurer can e.g., buy a call option on the futures index.

Furthermore it is likely that a cap is needed to limit the credit risk in the case of unusually large losses, and this will also have the advantage of making the contract behave more like a non-proportional reinsurance policy. The contract is therefore really a futures derivative with price at maturity $\phi(F_T)$, where $F_T = \hat{Z}_T$, and it is characterized in the CBoT-market as follows:

$$(5.1) \qquad \phi(F_T) = \$25{,}000 \min\left(\frac{Z(T)}{\Pi}, 2\right) = \$25{,}000\left\{\hat{Z}(T) - \max\left(\hat{Z}(T) - 2, 0\right)\right\},$$

where $\hat{Z}(T)$ is the loss ratio.

We make a distinction here between a *pure futures option* and a *conventional futures option.* A conventional call option, for example, requires payment of the option premium when purchased, and at exercise pays the buyer any excess of the underlying asset price over the exercise price. A pure futures option, on the other hand, calls for the buyer to receive (or pay) daily any change in the futures option price in order to mark the buyer's margin account to market. The equilibrium market price of a pure futures contract in our model then becomes

$$(5.2) \qquad \pi^{\phi}(F_t,t) = \frac{E\{u_k'(X_T,T)\exp\{\int_{(t,T]} r(u)du\}\phi(F_T) \mid \mathcal{F}_t\}}{E\{u_k'(X_T,T)\exp\{\int_{(t,T]} r(u)du \mid \mathcal{F}_t\}} \qquad \text{for } t \le T$$

while that of a conventional contract is

$$(5.3) \qquad \pi_c^{\phi}(F_t,t) = \frac{E\{u_k'(X_T,T)\phi(F_T) \mid \mathcal{F}_t\}}{u_k'(X_t,t)} \qquad \text{for } t \le T.$$

The connection between the price of the conventional and the pure futures instrument is as follows. Consider the price $\pi_f^{\phi}(F_t,t)$ of a forward contract on the underlying futures instrument. It is

$$(5.4) \qquad \pi_f^{\phi}(F_t,t) = \frac{E\{u_k'(X_T,T)\,\phi(F_T) \mid \mathcal{F}_t\}}{E\{u_k'(X_T,T) \mid \mathcal{F}_t\}} \qquad \text{for } t \le T,$$

and since the price of a zero coupon bond is given by

$$\Lambda_{t,T} = \frac{E\{u_k'(X_T,T) \mid \mathcal{F}_t\}}{u_k'(X_t,t)} \qquad \text{for } t \le T,$$

it follows that

(5.5) $$\pi_f^\phi(F_t,t) = = \frac{1}{\Lambda_{t,T}}\pi_c^\phi(F_t,t)$$

We notice that under our conditions (stated after Proposition 2) $\pi_f^\phi(F_t,t) = \pi^\phi(F_t,t)$. Thus under these assumptions $\pi^\phi(F_t,t) = \frac{1}{\Lambda_{t,T}}\pi_c^\phi(F_t,t)$, and since $\Lambda_{t,T} \leq 1$, a pure futures instrument has a price higher than or equal to that of a conventional futures contract.

5.4 Prices of futures derivatives for a fully specified model

We consider the model of section 4, where Z is a compound Poisson process and the intertemporal representative utility function is exponential. We retain the conditions of Theorem 1. Consider a pure futures instrument with terminal payoff $\phi(x)$ on the futures index x. The market premium is denoted by $\pi^\phi(F_t,t)$, whereas the market premium of the corresponding conventional futures instrument is denoted by $\pi_c^\phi(F_t,t)$. We then have under the above assumptions

Theorem 2

The market premium of the pure futures instrument is given by

(5.6) $$\pi^\phi(F_t,t) = e^{-\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n}\left\{\phi(\hat{Z}_t) + \sum_{k=1}^{\infty}\frac{(\lambda(T-t))(\frac{\mu}{\mu-\alpha})^n)^k}{k!}E\{\phi(\hat{Z}_t+\hat{V}_k) \mid \hat{Z}_t\}\right\},$$

where $\hat{V}_k$ is conditionally gamma $(kn, aT(\mu-\alpha))$ - distributed, given $\hat{Z}_t$. The market value of the conventional futures instrument is

(5.7) $$\pi_c^\phi(F_t,t) = e^{-(T-t)(\rho+\lambda+\alpha a)}\left\{\phi(\hat{Z}_t) + \sum_{k=1}^{\infty}\frac{(\lambda(T-t))(\frac{\mu}{\mu-\alpha})^n)^k}{k!}E\{\phi(\hat{Z}_t+\hat{V}_k) \mid \hat{Z}_t\}\right\}.$$

Proof.

Starting with the pure futures instrument, the market premium is given by

(5.8) $$\pi^\phi(F_t,t) = \frac{E\{e^{-\alpha(W(T)+aT-Z(T))}\phi(\hat{Z}_T) \mid \mathcal{F}_t\}}{E\{\exp[-\alpha(W(T)+aT-\sum_{i=1}^{N(T)}Y_i)] \mid \mathcal{F}_t\}}.$$

The denominator in (5.8) is given by the expression

$$\exp\{-\alpha X(t) - \alpha a(T-t) + \lambda(T-t)(g(\alpha)-1)\}h(\alpha,t),$$

where $g(\alpha) = \left(\frac{\mu}{\mu-\alpha}\right)^n$ and $h(\alpha,t) = E\{\exp\big(\alpha(W(T)\text{-}W(t))\big) \mid \mathcal{F}_t\}$. As for the numerator we can use the same line of reasoning as in Theorem 1. This leads to

$$E\{e^{-\alpha(W(T)+aT-Z(T))}\phi(\hat{Z}_T) \mid \mathcal{F}_t\} =$$

$$\exp\{-\alpha(X(t)-a(T-t)-\lambda(T-t)\}\left\{\phi(\hat{Z}_t) + \sum_{k=1}^{\infty}\frac{(\lambda(T-t))^k}{k!}\int_{[0,\infty)} e^{\alpha v}\phi(\hat{Z}_t+\hat{v})dH^{*k}(v)\right\}h(\alpha,t),$$

where $\hat{v} = v/aT$. The first term in the parenthesis is the contribution if no accidents happen in the time interval (t, T], in which case the size of the additional claims equals zero, an event which happens with probability $\exp\{-\lambda(T-t)\}$. The convolution integral can be written

$$\int_{[0,\infty)} e^{\alpha v}\,\phi(\hat{Z}_t+\hat{v})dH^{*k}(v) = \int_0^\infty e^{\alpha v}\phi(\hat{Z}_t+\hat{v})\frac{\mu^{kn}}{\Gamma(kn)}v^{kn-1}e^{-\mu v}dv =$$

$$\left(\frac{\mu}{\mu-\alpha}\right)^{kn}\int_0^\infty \phi(\hat{Z}_t+\hat{v})\frac{(\mu-\alpha)^{kn}}{\Gamma(kn)}v^{kn-1}e^{-(\mu-\alpha)v}dv.$$

The last integral is the conditional expected value of $\phi(\hat{Z}_t+\hat{V}_k)$ given $\hat{Z}_t$, where $\hat{V}_k = V_k/aT$ and the conditional distribution of V_k given $\hat{Z}_t$ is gamma $(kn, \mu-\alpha)$. Thus we obtain (5.6). The expression (5.7) follows from (5.3) and the above result, assuming the conditional interest rate-type independence from the insurance index X_T. □

We notice in particular that the market price of the conventional instrument depends in addition on the time impatience rate ρ of the representative agent.

The formulas (5.6) and (5.7) may be taken as the starting point for deriving useful pricing formulas for futures derivatives in practice. Usually one would expect that numerical techniques must be employed, but we are indeed able to derive closed form expressions and approximations below.

5.4 A futures cap

As an important illustration of Theorem 2, let us consider a cap in where $\phi(Z_T) = \$25{,}000 \min\left(\frac{Z(T)}{aT}, 2\right)$. In ordinary reinsurance this contract exhibits similar characteristics to a non-proportional reinsurance treaty with an upper limit. We concentrate on the pure futures derivative, since the conventional market price just differs by some multiplicative constant, given the information at time t. Consider the case where $\hat{Z}_t < 2$.

Theorem 3

The real market price at time t of the futures cap with expiration T is given by

(5.9) $$\pi^{(x\wedge 2)}(F_t,t) = \$25{,}000\Big[F_t +$$

$$\exp\left\{-(\mu-\alpha)\Pi(2-\hat{Z}_t) - \lambda(T\text{-}t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right\}\left((2-\hat{Z}_t)\Sigma_1 - \frac{n}{aT\mu}\lambda(T\text{-}t)\left(\frac{\mu}{\mu-\alpha}\right)^{n+1}\Sigma_0\right)\Big],$$

where

(5.10) $$\Sigma_0 = \sum_{k=0}^{\infty} \frac{(\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n)^k}{k!} e_{(k+1)n}((\mu-\alpha)\Pi(2-\hat{Z}_t)),$$

(5.11) $$\Sigma_1 = \sum_{k=1}^{\infty} \frac{(\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n)^k}{k!} e_{kn-1}((\mu-\alpha)\Pi(2-\hat{Z}_t)),$$

and where

(5.12) $$e_{n-1}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}.$$

The market value of $\Pi = aT$ is given in (4.3).

Proof. Let min (x, y) = $(x \wedge y)$. If $\hat{Z}_t \geq 2$, obviously $\pi^{(x\wedge 2)}(F_t,t) = \$50,000$, so let us consider the interesting case where $\hat{Z}_t < 2$. According to Theorem 2 we then have to compute

$$E\{(\hat{Z}_t+\hat{V}_k)\wedge 2 \mid \hat{Z}_t\} = \hat{Z}_t + E\{\hat{V}_k \wedge (2-\hat{Z}_t) \mid \hat{Z}_t\} =$$

$$\hat{Z}_t + \int_0^{\infty} \min(\hat{v}, 2-\hat{Z}_t) \frac{(\mu-\alpha)^{kn}}{\Gamma(kn)} v^{kn-1} e^{-(\mu-\alpha)v} dv =$$

$$\hat{Z}_t + \frac{kn}{aT(\mu-\alpha)} \Gamma(kn+1, (\mu-\alpha)\Pi(2-\hat{Z}_t)) + (2-\hat{Z}_t)(1-\Gamma(kn, (\mu-\alpha)\Pi(2-\hat{Z}_t))$$

where $\Gamma(n, \mu x)$ is the cumulative probability distribution function of the Gamma (n, μ)-distribution, called the incomplete gamma function. Since n is supposed to be an integer, we have the following relation

(5.13) $$\Gamma(n, \mu x) = 1 - e_{n-1}(\mu x) e^{-x\mu}$$

(see e.g., Abramowitz and Stegun (1972)). Using (5.13) we obtain

$$E\{(\hat{Z}_t+\hat{V}_k)\wedge 2 \mid \hat{Z}_t\} = \hat{Z}_t + \frac{kn}{aT(\mu-\alpha)}\left(1 - e_{kn}((\mu-\alpha)\Pi(2-\hat{Z}_t))e^{-(\mu-\alpha)\Pi(2-\hat{Z}_t)}\right) +$$
$$(2-\hat{Z}_t)e_{kn-1}((\mu-\alpha)\Pi(2-\hat{Z}_t))e^{-(\mu-\alpha)\Pi(2-\hat{Z}_t)}.$$

Inserting this into (5.6) we get

$$\pi^{(x\wedge 2)}(F_t,t) = \$25,000\{e^{-\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n}\left\{\hat{Z}_t + \sum_{k=1}^{\infty} \frac{(\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n)^k}{k!}\left(\hat{Z}_t + \right.\right.$$
$$\left.\left.\frac{kn}{aT(\mu-\alpha)}\left(1 - e_{kn}((\mu-\alpha)\Pi(2-\hat{Z}_t))e^{-(\mu-\alpha)\Pi(2-\hat{Z}_t)}\right) + (2-\hat{Z}_t)e_{kn-1}((\mu-\alpha)\Pi(2-\hat{Z}_t))e^{-(\mu-\alpha)\Pi(2-\hat{Z}_t)}\right)\right\}.$$

Consider now the series. The first two terms are:

$$\hat{Z}_t\left(e^{\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n} - 1\right) + \frac{n}{aT(\mu-\alpha)}\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n e^{\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n},$$

and the last two terms can be written

$$-\frac{n}{aT(\mu-\alpha)}\lambda(T-t)\Big(\frac{\mu}{\mu-\alpha}\Big)^n e^{-(\mu-\alpha)\Pi(2-\hat{Z}_t)}\sum_{k=0}^{\infty}\frac{(\lambda(T-t))(\frac{\mu}{\mu-\alpha})^n)^k}{k!}e_{(k+1)n}((\mu-\alpha)\Pi(2-\hat{Z}_t)) +$$

$$(2-\hat{Z}_t)e^{-(\mu-\alpha)\Pi(2-\hat{Z}_t)}\sum_{k=1}^{\infty}\frac{(\lambda(T-t))(\frac{\mu}{\mu-\alpha})^n)^k}{k!}e_{kn-1}((\mu-\alpha)\Pi(2-\hat{Z}_t))$$

Using (5.10) and (5.11) we obtain the conclusion of the theorem. □

The two last terms in (5.9) adjust the futures price F_t given in (4.1) for the capping off at the 200% point of the loss ratio.

The above formula is fairly simple, and can be further simplified by observing that the sum in (5.12) converges quickly to e^x, so that for n large or moderately large we may substitute the exponential for this truncated sum. Assume now that this approximation is reasonable. We shall comment on the error we are doing below. In this case we get the approximations

(5.14) $$\Sigma_0 \approx \exp\Big\{(\mu-\alpha)\Pi(2-\hat{Z}_t)+\lambda(T-t)\Big(\frac{\mu}{\mu-\alpha}\Big)^n\Big\}$$

and

(5.15) $$\Sigma_1 \approx \exp\Big\{(\mu-\alpha)\Pi(2-\hat{Z}_t)+\lambda(T-t)\Big(\frac{\mu}{\mu-\alpha}\Big)^n\Big\}-\exp\Big\{(\mu-\alpha)\Pi(2-\hat{Z}_t)\Big\},$$

where the given formulas are both upper bounds. Inserting these expressions into (5.9) we get the approximation

(5.16) $$\pi^{(x\wedge 2)}(F_t,t) \approx \$25{,}000\Big[2+(\hat{Z}_t-2)\exp\Big\{-\lambda(T-t)\Big(\frac{\mu}{\mu-\alpha}\Big)^n\Big\}\Big].$$

We see that the futures cap price approaches \$50,000 from below as $\hat{Z}_t$ approaches 2 from below. Held together with the result for $\hat{Z}_t \geq 2$, the cap price is seen to converge to \$25,000$\hat{Z}_T$ as time t moves to the expiration time T. Since $\hat{Z}_t < 2$ in the above formula, we observe that as the risk parameter μ increases, the futures cap price decreases. Furthermore the market price in (5.16) is seen to be an increasing function of the claim frequency parameter λ, of the claims size parameter n, of the intertemporal absolute risk aversion parameter α of the market, and of the time to settlement $(T-t)$. In the case of a conventional futures cap, this price is in addition a decreasing function of the subjective time impatience rate ρ.

5.5 The bound of the approximation error

Here we give a bound on the approximations (5.14) and (5.15). Let us use the notation x =

$\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n$ and $y = (\mu-\alpha)(2-\hat{Z}_t)$, where both x and y are seen to be positive. Given positive constants $K > 0$ and $\varepsilon > 0$ there exists an integer n_0 such that if $|x| \le K$, $|y| \le K$ and $n \ge n_0$, then

$$|e^{x+y} - \Sigma_0| < e^K \frac{(K+\varepsilon)^{n+1}}{(n+1)!} \text{ and } |(e^x-1)e^y - \Sigma_1| < e^K \frac{(K+\varepsilon)^{n+1}}{(n+1)!},$$

which tells us that the approximations we have done are good for n even of moderate size, due to the rapid increase of factorials. Since the bounds of the above sums are both upper bounds, and since these sums appear with opposite signs in all our pricing formulas, the final error is further reduced. However, in certain ranges of the parameter values the approximation is poor, for example when the term y gets large, which means when Z_t is small relative to 2, or in general, compared to some cap-off-point c (see (5. 18)-(5. 19) below).

5.6 Futures call options

In this subsection we compute the market price of futures call options. This contract mimics to some extent a standard stop loss reinsurance treaty. Again we only treat the pure futures version. Let us denote the market value of the pure futures call option by $\pi^{(x-c)^+}(F_t,t)$, where c stands for the call exercise price, where $\phi(x) = (x-c)^+ = \max(x-c, 0)$. Assuming the futures call option has the same expiration date as the underlying futures contract, according to Theorem 2 we have to compute

$$\pi^{(x-c)^+}(F_t,t) = \$25{,}000 e^{-\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n}\left\{\phi(\hat{Z}_t) + \sum_{k=1}^{\infty}\frac{(\lambda(T-t))(\frac{\mu}{\mu-\alpha})^n)^k}{k!}E\{(\hat{Z}_t+\hat{V}-c)^+ \mid \hat{Z}_t\}\right\}.$$

This quantity can now readily be found for the interesting case $\hat{Z}_t < c$ from the results in the previous section, Theorem 3 and the use of (5. 1) with c instead of the number 2 for the cap-off point. For the case $\hat{Z}_t > c$ we immediately obtain $\pi^{(x-c)^+}(F_t,t) = \$25{,}000(F_t - c)$. For $\hat{Z}_t < c$ we now get:

$$\text{(5.17)} \quad \pi^{(x-c)^+}(F_t,t) = \$25{,}000\left[\exp\left\{-(\mu-\alpha)\Pi(c-\hat{Z}_t) - \lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right\}\right]\cdot \left(\frac{n}{aT\mu}\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^{n+1}\Sigma_0^c - (c-\hat{Z}_t)\Sigma_1^c\right),$$

where

$$\text{(5.18)} \quad \Sigma_0^c = \sum_{k=0}^{\infty}\frac{(\lambda(T-t))(\frac{\mu}{\mu-\alpha})^n)^k}{k!}e_{(k+1)n}((\mu-\alpha)\Pi(c-\hat{Z}_t))$$

and

$$(5.19) \qquad \Sigma_1^c = \sum_{k=1}^{\infty} \frac{(\lambda(T-t))(\frac{\mu}{\mu-\alpha})^n)^k}{k!} e_{kn-1}((\mu-\alpha)\Pi(c-\hat{Z}_t)).$$

From our two expressions for the market price for the futures call option, we see that $\pi^{(x-c)^+}(F_t,t) \to \$25{,}000(\hat{Z}_T - c)^+$ as $t \to T$, where we have to remember that the process $\hat{Z}$ is a non-decreasing process in the time parameter.

If we can use the above approximation, we get approximately in the interesting case where $\hat{Z}_t < c$ that

$$(5.20) \qquad \pi^{(x-c)^+}(F_t,t) \approx \$25{,}000\Big[(\hat{Z}_t - c)\Big(1 - \exp\{-\lambda(T-t)\Big(\frac{\mu}{\mu-\alpha}\Big)^n\}\Big) + \frac{n}{aT\mu}\lambda(T-t)\Big(\frac{\mu}{\mu-\alpha}\Big)^{n+1}\Big].$$

We notice that the futures call option decreases as the strike price c increases, and it increases as $\hat{Z}_t$ increases, as normally is the case for a call option. Naturally, when c grows large compared to the other parameter values, the approximation becomes poor and eventually invalid, since the price can become negative because of the linearity in c. A closer examination of the proof of Theorem 3 reveals why this is so.

Example 2. Consider again Ins Ltd. in Example 1 in the same market structure. Instead of buying futures contracts, the company decides to buy at-the-money call options on the futures index. Using formula (5.20) with c = .55 (corresponding to the futures contract trading at \$ 25 000 · .55 = \$ 13 750) we obtain

$$\pi^{(x-.55)^+}(F_0,0) = \$\ 25\ 000[(-.55)(1 - \exp\{-10\cdot.25\cdot1.0987\} + .55] = \$\ 881.85.$$

Still buying 40 contracts, the immediate call option cost equals \$35 274.

Scenario A: In this case the final settlement price equals \$18 750, so Ins Ltd.'s total option position settled at a market value equal to \$200 000, an overall market net gain of \$ 164 726, offsetting 82% of its \$200 000 in unexpected incurred losses, a hedge slightly less effective than the futures position of Example 1.

Scenario B: In this case the final settlement price equals \$9 375, and with a strike price of \$ 13 750, the gain equals zero, resulting in an overall market loss of \$35 274. In this case the company made \$200 000 more than anticipated, resulting in net \$164 726 more than expected. This is to be compared to the futures position in Example 1, leading to a net result of \$25 000 more than anticipated, illustrating the advantages to the buyer of call options over that of a buyer of futures contracts in a scenario with losses less than projected. In the present scenario the insurer keeps his upside potential.

Summary of examples 1 and 2:

	Ins Ltd.	ISO	Futures	Options
Scenario A	600 000	300 m	200 000	164 726
Scenario B	200 000	150 m	-175 000	-35 274
Net result			25 000	164 726

□

5.7 A capped futures call option (a spread)

The final contract we consider is a capped futures call option By this we mean a futures instrument with terminal payoff

$$(5.21) \qquad \phi(x) = \$25{,}000 \begin{cases} 0 & \text{if } x \le c_1 \\ (x - c_1) & \text{if } c_1 < x \le c_2 \\ (c_2 - c_1) & \text{if } x > c_2 \end{cases}$$

The capping of the call option will limit the risk of the investors on the opposite side of the insurers at the exchange. This contract looks very much like a conventional non-proportional reinsurance treaty of the X.L.-type, with a "retention" c_1 and an upper limit c_2, where $c_1 < c_2$.

The market price of this contract follows from the above results, since to hold such a contract is equivalent to hold long one futures call option with strike price c_1 and to sell short one futures call option with strike price c_2. This follows since the payoffs at expiration are identical for these two positions. Thus

$$(5.22) \qquad \pi^{\phi}(F_t,t) = \pi^{(x-c_1)^+}(F_t,t) - \pi^{(x-c_2)^+}(F_t,t),$$

where the two expressions to the right are found from equation (5.17) above. This is the market price of a bull spread. For the simplified version in (5.20), we get an approximation of the market price of the contract (5.21) as

$$(5.23) \quad \pi^{\phi}(F_t,t) \approx \$25{,}000 \begin{cases} ((1 - \exp\{-\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n\})(c_2 - c_1)), & \text{if } \hat{Z}_t < c_1 \\ (c_2 - c_1 + (\hat{Z}_t - c_2)\exp\{-\lambda(T-t)(\frac{\mu}{\mu-\alpha})^n\}), & \text{if } c_1 \le \hat{Z}_t < c_2 \\ (c_2 - c_1), & \text{if } \hat{Z}_t \ge c_2 \end{cases}$$

a simple formula. The most interesting case is $\hat{Z}_t < c_1$, where we see that the premium *increases* with the difference $(c_2 - c_1)$ between the upper limit and the "retention" limit, the claim frequency parameter λ, time to settlement $(T - t)$, the market parameter α of risk attitude, and with the risk parameter n of the loss ratio index. The market price is seen to be a *decreasing* function of the risk parameter μ, which seems natural in view of the fact that increasing this parameter decreases the "riskiness", properly interpreted, of the loss ratio index. In the case $c_1 < \hat{Z}_t < c_2$ we notice that the price increases with the value of $\hat{Z}_t$, and otherwise we get the same signs in the comparative statics analysis for the parameters λ, $(T - t)$, n and α as in the first case. It is not likely that this approximation is very accurate, especially in the region $\hat{Z}_t < c_1$, when $\hat{Z}_t$ is small. The approximation improves as the index increases. Otherwise the same remark about the validity of this approximation is valid here, as for the formula (5.20), in terms of the parameter $(c_2 - c_1)$.

The formulas in this section remain valid also in the case where there is no underlying futures index which is traded, since we have used the principle of convergence F(T)= Z(T), and only considered futures instruments with the same expiry date T as the futures contract. Thus only a loss ratio index is really needed.

6. Conclusions

We have presented a valuation theory of forward and futures contracts, and of futures derivatives, when the underlying asset is an accumulated insurance loss ratio. We have taken into account the essentials in modeling such an index; by a relevant non-decreasing stochastic process containing claim size jumps at random time points of accidents.

We have derived partial equilibrium market premiums of futures prices and of futures derivatives. In particular we have presented closed form formulas for the futures price, futures call options, futures caps and capped futures call options - the most important contracts traded on the CBoT-insurance futures exchange. From these the prices of most other futures instruments can readily be derived. We have used an approximation to simplify the formulas for the futures derivatives, making the expressions tractable for comparative statics and analytic treatment.

The theory is in a form where it may easily be tested from market data. This may reveal if we have gone too far in our simplification. Parsimony is important in theoretical analyses, but realism is important in practice. It will be an interesting question to see how well the values derivable from this paper will fit observed prices in this insurance market. We will also get an estimate of the risk attitude in this market. The paper contains the general theory, however, from which more realistic approaches could start. This is not likely to lead to closed form solutions, but numerical techniques can then be employed.

Using statistical inference for stochastic processes on the data from the CBoT - exchange, we intend to test our results empirically, in particular the simple formulas. If more realism is needed, we will try to handle that as well, within the framework of this paper, in a subsequent investigation.

Acknowledgements

I would like to thank Bernt Arne Ødegaard for informing me about the insurance futures market, and Per

Erik Manne, Jørgen Aase Nielsen and Paul Embrechts for interesting suggestions.Versions of the paper have so far been presented at Isaac Newton Institute in Cambridge, Humboldt University, University of Oslo, The FIBE-conference in Bergen, Silivri, Istanbul, at the "22nd Seminar of the European Group of Risk and Insurance Economists" in Geneva and at the Chicago Board of Trade (CBOT) 1995 European Futures Research Symposium in Barcelona, Spain

References

Aase, K. K (1992). Dynamic equilibrium and the structure of premiums in a reinsurance market. *Geneva Papers on Risk and Insurance Theory* 17:2,93-136.

(1993a). A jump/diffusion consumption-based capital asset pricing model and the equity premium puzzle. *Mathematical Finance* 3, No. 2,65-84.

(1993b). Continuous trading in an exchange economy under discontinuous dynamics: A resolution of the Equity premium puzzle. *Scand. J. Mgmt 9,* Suppl., 3-28.

(1993c). Premiums in a dynamic model of a reinsurance market. *Scand. Actuarial J.* 2, 134-160.

(1993d). Equilibrium in a reinsurance syndicate; existence, uniqueness and characterization. *ASTIN BULLETIN* 23; 2, 185-211.

Abramowitz, M. and I. A. Stegun (1972). *Handbook of mathematical functions.* Dover Publications, Inc., New York.

Albrecht, P. (1994). *Katastrophenversicherungs-termingeschäfte.* Mannheimer Manuskripte zu Versicherungsbetribslehre Nr. 72, Mannheim.

Araujo, A. and P. Monteiro (1989). Existence without uniform conditions. *Journal of Economic Theory* 48,416-427.

Back K. (1991). Asset pricing for general processes. *Journal of Mathematical Economics* 20, 371-395.

Boel, R., P. Variya, and E. Wong (1975). Martingales on jump processes. I: Representation Results; II: Applications. *SIAM J. Control* 13,999-1061.

Borch, K. H. (1962). Equilibrium in a reinsurance market. *Econometrica* 30,424-444.

(1990). *Economics of Insurance.* (Eds. K.K. Aase and A. Sandmo). North-Holland, Amsterdam, New York, Oxford Tokyo.

Chicago Board of Trade Catastrophe Insurance Futures and Options Background Reports (April 1994), and CAT Catastrophe Insurance Futures and Options Reference Guide.

Chichilnisky, G. (1993). Markets with endogenous uncertainty - Theory and policy. *Theory and Decisions* (to appear).

Cummins, J. D. and H. Geman (1995). Pricing catastrophe insurance futures and call spreads: An arbitrage approach. Forthcoming in *The Journal of Fixed income* (March)

Delbaen, F. and J. Haezendonck (1989). A martingale approach to premium caluclation principles in an arbitrage free market. *Insurance: Mathematics and Economics* 8,269-277.

Doléans-Dade, C. (1970). Quelques applications de la formule de changement de variables pour les semi-martingales. *Zeitschrift für Wahrscheinlichkeitstheorie und verw. Geb.* 16, 181-194.

Duffie, D. (1992). *Dynamic Asset Pricing Theory.* Princeton, New Jersey.

(1989). *Futures markets.* Prentice-Hall International, Inc.

Duffie, D. and C. F. Huang (1985). Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities. *Econometrica* 53; 1337-56.

Duffie, D. and W. Zame (1989). The consumption-based capital asset pricing model. *Econometrica* 57, 1279-1297.

Föllmer, H and D. Sondermann (1986). Hedging of non-redundant contingent claims, in: W. Hildenbrand and A. Mas.Colell (eds.), *Contributions to Mathematical Economics,* 205-223.

Gihman, I. I. and A. V. Skorohod (1979). *The Theory of Stochastic Processes III.* Springer Verlag; Berlin, Heidelberg, New York.

Hull, J. (1989). *Options, futures and other derivative securities.* Prentice-Hall International, Inc.

Jacod, J. (1975). Multivariate point processes: Predictable projection, Radon-Nikodym derivative, representation of martingales. *Zeitschrift für Wahrscheinlichkeitstheorie und verw. Geb.* 31, 235-253.

Jacod, J. and A. N. Shiryayev (1987). *Limit Theorems for Stochastic Processes.* Springer Verlag; Berlin, Heidelberg New York.

Lucas, R. E., Jr. (1978). Asset prices in an exchange economy. *Econometrica* 46, 1429-45.

Meister, S. (1995). *Contributions to the mathematics of catastrophe insurance futures.* Unpublished Diplomarbeit, ETH Zürich.

Naik V., and M. Lee (1990). General equilibrium pricing of options on the market portfolio with discontinuous returns. *The Review of Financial Studies* 3,493-521.

Schweitzer, M. (1991). Options hedging for semimartingales. *Stochastic Processes and their Applications* 37,339-363.