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Asset and liability modelling for participating policies with guarantees

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Abstract

We study the problem of asset and liability management of participating insurance policies with guarantees. We develop a scenario optimization model for integrative asset and liability management, analyze the tradeoffs in structuring such policies, and study alternative choices in funding them. The nonlinearly constrained optimization model can be linearized through closed form solutions of the dynamic equations. Thus large-scale problems are solved with standard methods. We report on an empirical analysis of policies offered by Italian insurers. The optimized model results are in general agreement with current industry practices. However, some inefficiencies are identified and potential improvements are highlighted.

Earlier versions of this paper were presented at Frontiers in Finance, Paris, April 2001, Multinational Finance Society meeting, Garda, IT, June 2001, Forecasting Financial Markets 2001, London, June 2001, and seminars at Athens University of Economics and Business, Brunel University and University of Rome, and benefited from the comments of seminar participants. The comments of two anonymous referees led to additional empirical analysis and improvements in the interpretation of the results.

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1 Introduction

Insurance products become increasingly more innovative in order to face competitive pressures. Insurance policies today come with guarantees on the minimum rate of return, bonus provisions, and surrender options. These features make them attractive for investors who seek not only insurance but also investment vehicles. However, the new policies are much more complex to price than traditional insurance products, and we have witnessed an interest in applying financial pricing techniques to the valuation of insurance liabilities. The focus is shifting away from the traditional actuarial pricing, from static models to stochastic models; see Vanderhoof and Altman (1998), Babbel and Merill (1999), and Embrechts (2000). Among the most complex insurance products today we find participating policies with guarantees.

1.1 Features of policies with guarantees

Financial products with guarantees on the minimum rate of return come in two distinct flavors: *maturity guarantees* and *multi-period guarantees*. In the former case the guarantee applies only at maturity of the contract, and returns above the guarantee at some time before maturity offset shortfalls at other times. In the later case the time to maturity is divided into subperiods—quarterly or biannually—and the guarantee applies at the end of each period. Hence, excess returns in one sub-period can not be used to finance shortfalls in other sub-periods. Such guaranteed products appear in insurance policies, guaranteed investment contracts, and some pension plans, see, e.g., Hansen and Miltersen (2001).

With the historically low interest rates of the last decade the management of such policies is becoming more challenging. Reliance on fixed-income assets is unlikely to yield the guaranteed rate of return. For instance, Italian guaranteed rates after 1998 are at 3%. The difference between the guaranteed rate and the ten-year yield is only 1%, which is inadequate for covering the firm's costs. In Germany the guaranteed rates after 1998 are at 3.5% differing from the ten-year yield only by 0.5%. Danish products offered guarantees of 3% until 1999, which were reduced to 2% afterwards. In Japan Nissan Mutual Life failed on a \$2.56 billion liability arising from a 4.7% guaranteed investment, highlighting the difficulties faced by this industry.

In response to market pressures and regulatory conditions insurers offer currently very conservative guaranteed returns. Policyholders are compensated, however, by *participating* in the firm's profits, receiving a *bonus* whenever the return of the firm's portfolio exceeds the guarantee, creating a *surplus* for the firm. Bonuses may be distributed only at maturity, at multiple periods until maturity, or using a combination of distribution plans. The earlier *unit-linked* policies would pay a benefit—upon death or maturity—which was the greater of the guaranteed amount and the value of the reference portfolio. These simple were maturity guarantees with bonus paid at maturity as well. At the other extreme of complexity we have the modern UK insurance policies. These policies declare at each subperiod a fraction of the surplus as *reversionary* bonus which is then guaranteed. The remaining surplus is managed as an *investment reserve*, and is returned to customers as terminal bonus if it is positive at maturity or upon death. These policies are multi-period guarantees with bonuses paid in part at intermediate times and in part at maturity. Further discussion on the characteristics of products with guarantees is found in Kat (2001) and the papers cited below.

In this paper we consider multi-period guarantees with bonuses that are paid at each subperiod and are subsequently guaranteed. The bonus is contractually determined as a fraction of the portfolio excess return above the guaranteed rate during each subperiod. The guaranteed rate is also contractually specified. To understand the nature of this product we illustrate in Figure 1 the growth of a liability that participates by 85% in a given portfolio while it guarantees a return of at least 3% in each period. The liability is *lifted* every time a bonus is paid and the minimum guarantee applies to the increased liability: what is given can not be taken away.

1.2 Current models

The pricing of the option embedded in the early products with guarantees was addressed in the seminal papers of Brennan and Schwartz (1976) and Boyle and Schwartz (1977). They analyzed unit-linked maturity guarantee policies. Perhaps the most complete analysis of modern life insurance contracts—complete in the sense that it prices in a unified framework several components of the policy—is due to Grosen and Jørgensen (2000). They decompose the liability of modern participating policies with guarantees into a risk-free bond (the minimum guarantee), a bonus option, and a surrender option. The first two taken together are a European contract and all three together are an American contract, and the authors develop numerical techniques for pricing both. Hansen and Miltersen (2001) extend this model to the pricing of contracts with a smoothing surplus distribution mechanism of the form used by most Danish life-insurance companies and pension plans. They use the model to study different methods for funding these products, either by charging the customers directly or by keeping a share of the surplus. Similarly, Bacinello (1999) develops pricing models that permit her to study the interplay between the volatility of the underlying asset portfolio, the

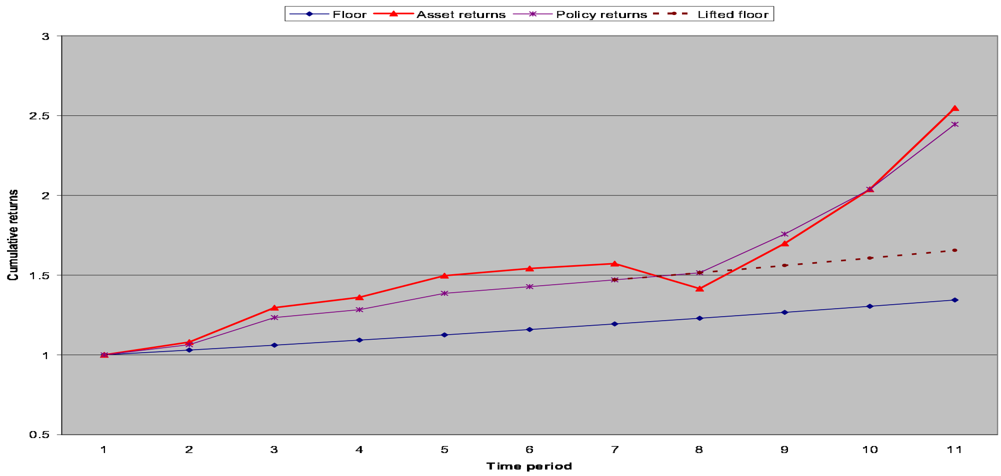


Figure 1: Typical returns of the asset portfolio and a participating policy with multi-period guaranteed return of 3% and participation rate 85%. The guarantee applies to a liability that is lifted every time a bonus is paid as illustrated at period seven. The asset portfolio experienced substantial losses at period seven while the liability grew at the 3% guaranteed rate. Subsequent superior returns of the assets allowed the firm to recover its losses by the tenth period and achieve a positive net return at maturity.

participation level for determining bonuses, and the guaranteed rate. Boyle and Hardy (1997) take this line of inquiry in a different direction by analyzing alternative reserving methods for satisfying the guarantee. More practical aspects of the problem are studied by Giraldi et al. (2000).

It is worth noting that current literature assumes the asset side is given *a priori* as a well-diversified portfolio which evolves according to a given stochastic process. For instance Brennan-Schwartz, Grosen–Jørgensen and Bacinello assume a geometric Browning motion, while Miltersen and Persson (1999) rely on the Heath-Jarrow-Morton framework and price multi-period guaranteed contracts linked either to a stock investment or the short-term interest rate. There is nothing wrong with these approaches, of course, except that part of the problem of the insurance companies is precisely to determine the structure of the asset portfolio. Indeed all of the above references carry out simulations for different values of the volatility of the assets. Brennan and Schwartz (1979) devote a section to the analysis of "misspecification of the stochastic process". Bacinello goes on to suggest that the insurance company should structure several reference portfolios according to their volatility and offer its customers choices among different triplets of guaranteed rate, bonus provision, and asset portfolio volatility. To this suggestion of endogenizing the asset decision we subscribe. It is a prime example of *integrated financial product management* advocated by Holmer and Zenios (1995).

Independently from the literature that prices the option embedded in the liabilities we have seen an interest in the use of portfolio optimization models for asset and liability management for insurance companies. The most prominent example is for a Japanese insurance firm—not too surprising given what has transpired in the Japanese financial markets—the Yasuda Kasai model developed by the Frank Russel Company. This model received coverage not only in the academic literature but also in the press, see Cariño and Ziemba (1998). Other successful examples include the Towers Perrin model of Mulvey and Thorlacius (1998), the CALM model of Consigli and Dempster (1998) and the Gjensidige Liv model of Høyland (1998). These models have been successful in practical settings but their application does not cover participating policies with guarantees. One reason is that insurance firms pursued integrated asset and liability management strategies for those products they understood well. This has been the case for policies that encompass mostly actuarial risk such as the fire and property insurance of the Yasuda Kasai model. Another reason is that the technology of scenario optimization through large-scale stochastic programming has only recently been developed into computable models, see, e.g., Censor and Zenios (1997). Finally, the combination of a guarantee with a bonus provision introduces nonlinearities which complicate the model.

1.3 Contributions of this paper

This paper extends current asset and liability management literature to address problems that have been studied thus far only from a pricing perspective. We develop a scenario optimization asset and liability management model for multi-period participating policies with guarantees. Our model optimizes the choice of an asset portfolio to deliver the contractual obligations of the policy while maximizing shareholder value.

The specific contributions of this paper, and the findings from the empirical analysis, are as follows:

1. A model that endogenizes the asset structure which has been considered exogenous in the pricing literature on guaranteed products. Exogeneity of the asset returns is justified for unit-linked policies, but is not quite acceptable for modern participating policies whereby the issuer has control of the asset portfolio.

2. An analysis of the tradeoffs between shareholders and policyholders in offering guaranteed products. In this respect our paper addresses pre-

cisely the suggestion made by Bacinello by structuring optimal asset portfolios. Thus it allows the insurance company to present policyholders with choices that are efficient.

3. The identification of optimal asset portfolio volatility for each target guaranteed return. Too low volatility is associated with low expected returns and portfolios that are unlikely to meet the guarantee. High volatility portfolios are associated with higher expected returns, and are more likely to meet the guarantee. However, the embedded option is in this case expensive and may erode shareholder value. Siglienti (2000) argued that portfolios with more than 10 to 15% in equities are likely to destroy shareholder value. We find that, indeed, too high equity content destroys shareholder value, but for properly optimized portfolios the cutoff point is around 20 to 25% in equities.

4. Flexibility in financing the guarantee either through reserving or by issuing long- or short-term debt. The model explicitly recognizes that the reserves will depend on the asset structure—a fact also recognized in Boyle-Hardy—and optimizes this asset structure viz-a-viz the liability.

5. A benchmark of the policies offered by Italian insurers against optimized policies. We see that policies backed by optimized portfolios dominate in risk-return space policies backed by the typical portfolios of Italian insurers.

The paper is organized as follows: Section 2 defines the dynamics of assets and liabilities and develops the optimization model. Section 3 shows how the model analyzes risk and return tradeoffs for different policies, and addresses questions pertaining to the cost of the guarantee and ways for funding this cost. It also analyzes tradeoffs between policyholders and shareholders, and examines the ability of the firm to satisfy regulatory requirements. Section 4 benchmarks the Italian policies and looks at international diversification and investments in credit products. The solution of the dynamic equations is given in Appendix A. Appendix B presents extensive results with the use of the model and can be obtained from the authors.

2 The Scenario Optimization Nonlinear Programming Model

We develop now the model for asset and liability management for multi-period participating policies with guarantees. It is a mathematical program that models stochastic variables using discrete scenarios. All portfolio decisions are made at $t = 0$ in anticipation of an uncertain future. At the end of the planning horizon the impact of these portfolio decisions in different scenarios is evaluated and risk aversion is introduced through a utility function. Portfolio decisions optimize the expected utility over the specified horizon.

2.1 Features of the model

In the model we consider three accounts: *(i)* a liability account that grows according to the contractual guaranteed rate and bonus provision, *(ii)* an asset account that grows according to the portfolio returns, net any payments due to death or policy surrenders, and *(iii)* a shortfall account that monitors lags of the portfolio return against the guarantee. In the base model shortfall is funded by equity but later we introduce alternative reserving methods.

The multi-period dynamics of these accounts are conditioned on discrete scenarios of realized asset returns and the composition of the asset portfolio. Within this framework a regulatory constraint on leverage is imposed. At maturity the difference between the asset and the liability accounts is the surplus realized by the firm after it has fulfilled its contractual obligations. In the policies considered here this surplus remains with the shareholders. Of course this surplus is a random variable, and a utility function is introduced to incorporate risk aversion.

2.2 Notation

We let Ω denote the index set of scenarios $l = 1, 2, \ldots N$, $\mathcal{U}$ the universe of available asset instruments, and $t = 1, 2, \ldots, T$, discrete points in time from today ($t = 0$) until maturity T. The data of the problem are as follows:

r^l_{it}, rate of return of asset i during the period $t - 1$ to t in scenario l.

r^l_{ft}, risk free rate during the period $t - 1$ to t in scenario l.

g , minimum guaranteed rate of return.

α, participation rate indicating the percentage of portfolio return paid to policyholders.

ρ, regulatory equity to debt ratio.

Λ^l_t, probability of abandon of the policy due to lapse or death at period t in scenario l.

The variables of the model are defined as follows:

x_i, percentage of initial capital invested in the ith asset.

y^l_{At}, expenses due to lapse or death at time t in scenario l.

z^l_t, shortfall below the guaranteed rate at time t in scenario l.

A^l_t, asset value at time t in scenario l.

E^l_t, total equity at time t in scenario l.

L^l_t, liability value at time t in scenario l.

R^l_{Pt}, portfolio rate of return during the period $t-1$ to t in scenario l.

y^{+l}_t, excess return over g at time t in scenario l.

y^{-l}_t, shortfall return under g at time t in scenario l.

2.3 Variable dynamics and constrains

We invest the premium collected (L_0) and the equity required by the regulators ($E_0 = \rho L_0$) in the asset portfolio. Our initial endowment $A_0 = L_0(1+\rho)$ is allocated to assets in proportion x_i such that $\sum_{i \in \mathcal{U}} x_i = 1$, and the dynamics of the portfolio return are given by

$$R^l_{Pt} = \sum_{i \in \mathcal{U}} x_i r^l_{it}, \text{ for } t = 1, 2, \ldots T, \text{ and for all } l \in \Omega. \tag{1}$$

The investment variables are nonnegative so that short sales are not allowed.

We now turn to the modelling of the liability account. Liabilities will grow at a rate which is at least equal to the guarantee. Excess returns over g are returned to the policyholders according to the participation rate α. The dynamics of the liability account are given by

$$L^l_t = (1 - \Lambda^l_t) L^l_{t-1} (1 + \max\left[\alpha R^l_{Pt}, g\right]), \text{ for } t = 1, 2, \ldots T, \text{ and for all } l \in \Omega. \tag{2}$$

The max operator introduces a discontinuity in the model. To circumvent this difficulty we introduce variables y^{+l}_t and y^{-l}_t to measure the portfolio excess return over the guaranteed rate, and the shortfall below the guarantee, respectively. They satisfy

$$\alpha R^l_{Pt} - g = y^{+l}_t - y^{-l}_t, \text{ for } t = 1, 2, \ldots T, \text{ and for all } l \in \Omega, \tag{3}$$

$$y_t^{+l} \geq 0,\ y_t^{-l} \geq 0,\ y_t^{+l} y_t^{-l} = 0, \text{ for } t = 1, 2, \ldots T, \text{ and for all } l \in \Omega. \quad (4)$$

Only one of these variables can be nonzero at any given time and in a given scenario.

The dynamics for the value of the liability are rewritten as

$$L_t^l = (1 - \Lambda_t^l) L_{t-1}^l (1 + g + y_t^{+l}), \text{ for } t = 1, 2, \ldots T, \text{ and for all } l \in \Omega. \quad (5)$$

Liabilities grow at least at the rate of g. Any excess return is added to the liabilities and the guarantee applies to the lifted liabilities.

At each period the insurance company makes payments due to policyholders abandoning their policies because of death or lapse. Payments are equal the value of the liability times the probability of abandonment, i.e.,

$$y_{At}^l = \Lambda_t^l L_{t-1}^l (1 + g + y_t^{+l}), \text{ for } t = 1, 2, \ldots T, \text{ and for all } l \in \Omega. \quad (6)$$

Whenever the portfolio return is below the guaranteed rate we need to infuse cash into the asset portfolio in order to meet the final liabilities. The shortfall account is modelled by the dynamics

$$z_t^l = y_t^{-l} L_{t-1}^l, \text{ for } t = 1, 2, \ldots T, \text{ for all } l \in \Omega. \quad (7)$$

In the base model shortfalls are funded through equity. We assume that equity is reinvested at the risk-free rate and is returned to the shareholders at the end of the planning horizon. (This is not all the shareholders get; they also receive dividends.) The dynamics of the equity are given by

$$E_t^l = E_{t-1}^l (1 + r_{ft}^l) + z_t^l, \text{ for } t = 1, 2, \ldots T, \text{ and for all } l \in \Omega. \quad (8)$$

By assuming the risk free rate as the alternative rate at which the shareholders could invest their money we analyze the *excess* return offered to shareholders by the participating contract modelled here, over the benchmark risk free investment. In principle one could use the firm's internal rate of return as the alternative rate, and analyze the excess return offered by the policy modelled here over the firm's other lines of business. In this setting, however, the problem would not be to optimize the asset allocation to maximize shareholder value, since this would already be endogenous in the internal rate of return calculations. Instead we could determine the most attractive features for the policyholders—g and α—that will make the firm indifferent in offering the new policy or maintaining its current line of business. This approach deserves further investigation. For the purpose of optimizing alternative policies for the shareholders, while satisfying the contractual obligations to the policyholders, the estimation of excess return over the risk free rate is a reasonable benchmark. In sections 3.2 and 3.4 we consider other alternatives for funding the shortfalls through long-term debt or short-term borrowing.

We now have the components needed to model the asset dynamics, taking into account the cash infusion that funds shortfalls, z_t^l, and the outflows due to actuarial events y_{At}^l, i.e.,

$$A_t^l = A_{t-1}^l(1 + R_{Pt}^l) + z_t^l - y_{At}^l, \text{ for } t = 1, 2, \ldots T, \text{ and for all } l \in \Omega. \quad (9)$$

In order to satisfy the regulatory constraint the ratio between the equity value and liabilities must exceed ρ. That is,

$$\frac{V_{ET}^l}{L_T^l} \geq \rho, \text{ for all } l \in \Omega, \quad (10)$$

where V_{ET}^l is the value of equity at the end of the planning horizon T. If the company sells only a single policy the value of its equity will be equal to the final asset value—which includes the equity needed to fund shortfall—minus the final liability due to the policyholders, and we have

$$V_{ET}^l = A_T^l - L_T^l. \quad (11)$$

Having described the assets and liability accounts in a way that the key features of the policy—guaranteed rate and bonus provisions—are accounted for, we turn to the choice of an appropriate objective function. We model the goal of a for-profit institution to maximize the return on its equity, and, more precisely in this case, to maximize any excess return on equity after all liabilities are paid for. Since return on equity is scenario dependent we maximize the expected value of the utility of excess return. This expected value is converted into a certainty equivalent for easy reference. The objective function of the model is to compute the maximal Certainty Equivalent Excess Return on Equity (CEexROE) given by

$$\text{CEexROE} \doteq U^{-1}\left\{ \underset{x}{\text{Maximize}} \ \frac{1}{N}\sum_{l\in\Omega} U\left\{\frac{A_T^l - L_T^l}{E_T^l}\right\}\right\}, \quad (12)$$

where $U\{\cdot\}$ denotes the decision maker's utility function and $A_T^l - L_T^l$ is the shareholder's reward in scenario l. We assume a power utility function with constant relative risk aversion of the form $U(V) = \frac{1}{\gamma}V^\gamma$, where $V \geq 0$, and $\gamma < 1$. In the base model we assume $\gamma = 0$ in which case the utility function is the logarithm corresponding to growth-optimal policies for the firm. In section 3.5 we study the effect of changing the risk aversion parameter.

As a byproduct of our model we calculate the cost of funding the guaranteed product. Every time the portfolio return drops below the guaranteed rate, we counterbalance the erosion of our assets by infusing cash. This cost can be charged either to the policyholders, as soon as they enter the insurance contract, or covered through shareholder's equity or by issuing debt. These choices entail a tradeoff between the return to shareholders and return to policyholders. We study in the next section this tradeoff.

The cost of the guarantee is the expected present value of reserves required to fund shortfalls due to portfolio performances below the guarantee. The dynamic variable E_t^l models precisely the total funds required up to time t, valued at the risk–free rate. However, E_t^l also embeds the initial amount of equity required by the regulators. This is not a cost and it must be deducted from E_t^l. Thus, the cost of the guarantee is given as the expected present value of the final equity E_T^l adjusted by the regulatory equity, that is,

$$\bar{O}_G = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{E_T^l}{\prod_{t=1}^{T}(1+r_{ft}^l)} - \rho L_0\right). \tag{13}$$

$\bar{O}_G$ is the expected present value of the reserves required to fund this product. This can be interpreted as the cost to be paid by shareholders in order to benefit from the upside potential of the surplus. A more precise interpretation of $\bar{O}_G$ is as the *expected downside risk* of the policy. This is not the risk-neutral price of the participating policies with guarantees that would be obtained under an assumption of complete markets for trading the liabilities arising from such contracts. This is the question addressed through an options pricing approach in the literature cited above, Brennan-Schwartz, Boyle-Schwartz, Bacinello, Grosen-Jørgensen, Hansen-Miltersen, Miltersen-Persson.

2.4 Linearly constrained optimization model

The model defined in the previous section is a nonlinearly constrained optimization model and is computationally intractable for large scale applications. However, the nonlinear constraints (5)–(9) are definitional constraints which determine the value of the respective variables at the end of the horizon. We solve these dynamic equations analytically (Appendix A) to obtain end–of–horizon analytic expressions for A_T^l, L_T^l, and E_T^l. These expressions are substituted in the objective function to obtain the equivalent linearly constrained nonlinear program below. The regulatory constraint (10), however, can not be linearized. For solution purposes the regulatory constraint is relaxed and its validity is tested *ex post.* Empirical results later on demonstrate that the regulatory constraint is not binding for the policies considered here and the generated scenarios of asset returns. However, there is no assurance that this will always be the case, and we may need to resort to nonlinearly constrained optimization for solving this model.

$$
\begin{aligned}
\underset{x\geq 0}{\text{Maximize}}\quad & \frac{1}{N}\sum_{l\in\Omega} U\Bigg\{\Bigg[(1+\rho)\prod_{t=1}^{T}(1+R_{Pt}^{l})+ \\
& +\sum_{t=1}^{T}\Big(y_t^{-l}-\Lambda_t^l\,(1+g+y_t^{+l})\Big) \\
& \prod_{\tau=t+1}^{T}(1+R_{P\tau}^{l})\prod_{\tau=1}^{t-1}(1+g+y_\tau^{+l})(1-\Lambda_\tau^l)+ \\
& -\prod_{t=1}^{T}(1-\Lambda_t^l)(1+g+y_t^{+l})\Bigg] \\
& /\Bigg[\rho\prod_{t=1}^{T}(1+r_{ft}^{l})+\sum_{t=1}^{T}y_t^{-l}\phi(t,T)\prod_{\tau=1}^{t-1}(1-\Lambda_\tau^l)(1+g+y_\tau^{+l})\Bigg]\Bigg\}
\end{aligned}
\tag{14}
$$

$$
\text{s.t.}\qquad \sum_{i\in\mathcal{U}} x_i = 1, \tag{15}
$$

$$
\alpha R_{Pt}^{l} - g = y_t^{+l} - y_t^{-l}, \qquad \text{for } t=1,2,\ldots T, \text{ and for all } l\in\Omega, \tag{16}
$$

$$
R_{Pt}^{l} = \sum_{i\in\mathcal{U}} x_i r_{it}^{l}, \qquad \text{for } t=1,2,\ldots T, \text{ and for all } l\in\Omega. \tag{17}
$$

The inverse utility function U^{-1} of the optimal objective value of this problem is the CEexROE.

2.5 Model extensions

We point out possible extensions of this model. Periodic premia can readily be incorporated, as well as bonus policies based on averaging portfolio performance. Guaranteed rates and bonus rates that are functions of time, g_t and α_t, are easy to incorporate. Similarly we can incorporate liabilities due to lapse, although a lapse model must first be built and calibrated such as the one given by Asay, Bouyoucos and Marciano (1993) or Nielsen and Zenios (1996). Incorporating participation rates that are functions of the asset returns—as is the case with the UK insurance policies—complicates the model and requires additional work.

The base model developed here funds shortfalls through equity. Extensions to deal with the funding of shortfalls through long- or short-term debt are given in sections 3.2 and 3.4, respectively. Furthermore, unlimited access to equity for funding shortfalls is assumed in the base model. We could do away with this assumption by imposing additional constraints, but this would complicate the model rendering it computationally intractable. The probability of insolvency is analyzed through post-optimality analysis in section 3.3, and is used to guide the debt structure in funding shortfalls using a combination of equity and debt.

3 Model Testing and Validation

We now turn to the empirical testing of the model. First, we show that the model quantifies the tradeoffs between the different targets of the insurance firm: providing the best products for its policyholders, providing the highest excess return to its shareholders, satisfying the guarantee at the lowest possible cost and with high probability. Some interesting insights are obtained on the structure of the optimal portfolios as the tradeoffs vary across the spectrum. Second, we analyze different debt structures whereby the cost of the guarantee is funded through equity or through debt with either long or short maturities. Third, we will see from the empirical results that the Italian insurance industry operates at levels which are close to optimal but not quite so. There is room for improvement either by offering more competitive products or by generating higher excess returns for the benefit of the shareholders. How are the improvements possible? The answer is found in the comparison of the optimal portfolios generated by our model with benchmark portfolios. We will see that the benchmark portfolios generate tradeoffs in the space of cost of guarantee *vs* net excess return on equity that are inefficient. The optimized portfolios lead to policies with the same cost but higher excess return on equity.

The asset classes considered in our study are 23 stock indexes of the Milano Stock Exchange, and three Salomon Brother indexes of the Italian Government bonds (Appendix B). We employ a simple approach for generating scenarios using only the available data without any mathematical modelling, by bootstrapping a set of historical records. Each scenario is a sample of returns of the assets obtained by sampling returns that were observed in the past. Dates from the available historical records are selected randomly, and for each date in the sample we read the returns of all assets classes realized during the previous month. These samples are scenarios of *monthly* returns. To generate scenarios of returns for a long horizon—say 10 years—we sample

120 monthly returns from different points in time. The compounded return of the sampled series is one scenario of the 10-year return. The process is repeated to generate the desired number of scenarios for the 10-year period. With this approach the correlations among asset classes are preserved.

Additional scenarios could also be included, although methods for generating them should be specified. Model-based scenario generation methods for asset returns are popular in the insurance industry—e.g., the Wilkie (1995) model—and could be readily incorporated into the scenario optimization model. Alternatively, one could use expert opinion or "scenario proxies" as discussed in Dembo et al. (2000).

For the numerical experiments we bootstrap monthly records from the ten year period Jan. 1990 to Feb. 2000. The monthly returns are compounded to yearly returns. For each asset class we generate 500 scenarios of returns during a 10 year horizon ($T = 120$ months). We consider an initial liability $L_0 = 1$ for a contract with participation rate $\alpha = 85\%$ and equity to liability ratio $\rho = 4\%$. The model is tested for guarantees ranging from 1% to 15%.

The probability that a policyholder *abandons* the policy is $Prob(death) + Prob(lapse)$. In our experiments we set lapse probabilities to zero and use probabilities of death from the Italian mortality tables. For each model run we determine the net annualized after–tax CEexROE

$$(\sqrt[T]{\text{CEexROE}} - 1)(1 - \kappa), \tag{18}$$

where κ is the tax rate set at 51%.

3.1 Analysis with the base model

The tradeoffs between the guaranteed rate and the net CEexROE is shown in Figure 2. Figure 3 shows the optimal asset allocation among the broad classes of bonds and stocks for different target guaranteed returns.

At first glance the portfolio structures appear puzzling. One expects that as the guarantee increases the amount of stock holdings should grow. However, we observe that for low guarantees (less than 7%) the holdings in stock increases with lower guarantees. For low g the embedded option is far out of the money even when our assets are mostly in equity and very volatile. The asset allocation strategy maximizes CEexROE by taking higher risks in the equities market. A marginal increase of the shortfall cost allows higher CEexROE. This is further clarified in Figure 4, showing the tradeoff between cost of the guarantee and net annualized CEexROE. At values of g less than 7% the option embedded in the liability is out-of-the-money and any excess return is passed on to the shareholders thus improving CEexROE. As

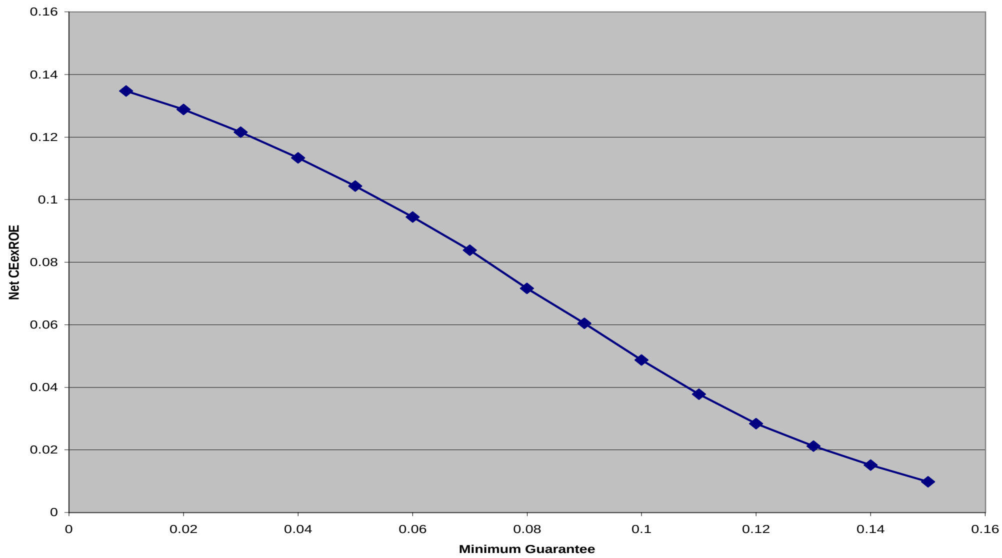


Figure 2: Net CEexROE (annualized) for different levels of the guarantee.

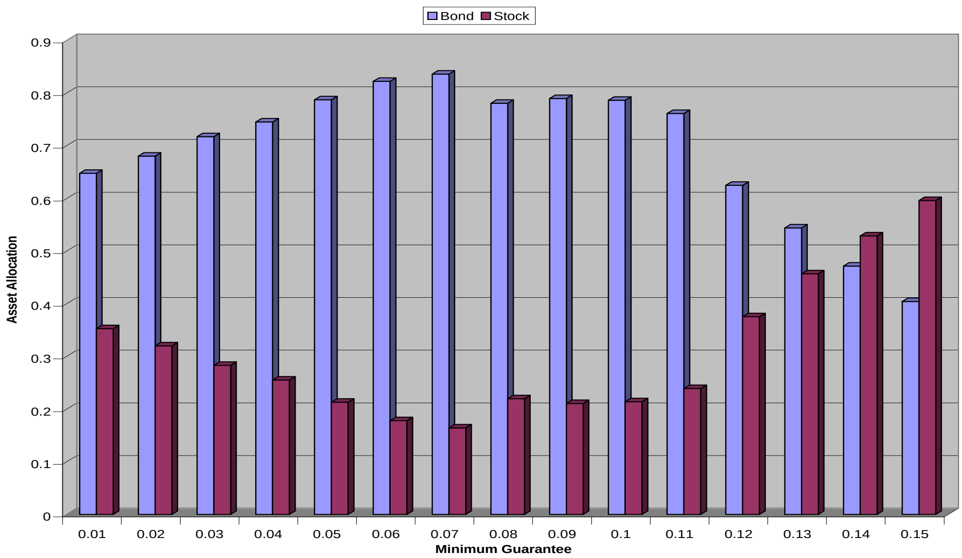


Figure 3: Broad asset allocation for different levels of the guarantee.

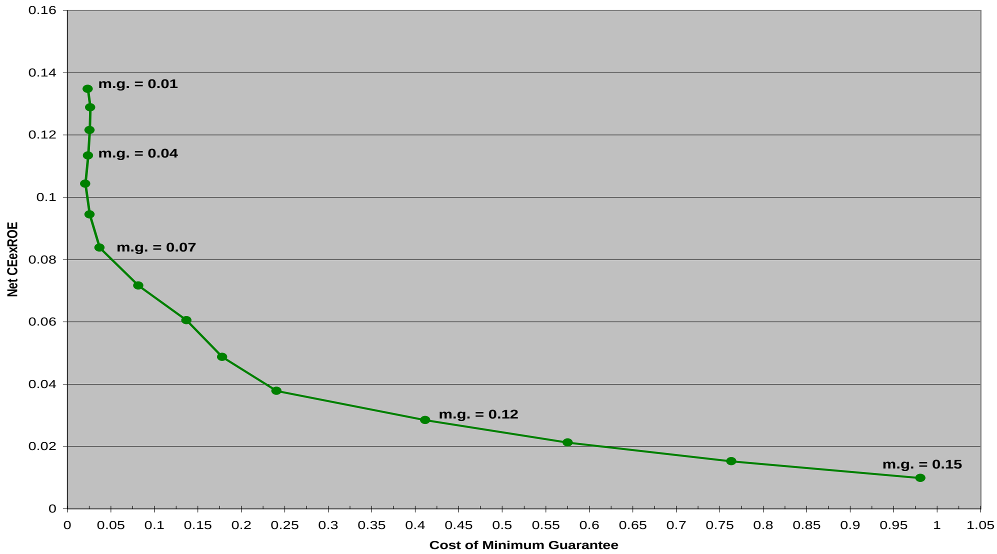


Figure 4: Cost of the guarantee *vs* net CEexROE.

the guarantee increases above 7% the option goes deeper into the money, the cost of the guarantee increases significantly and CEexROE erodes. Note from Figure 3 that higher values of the guarantee must be backed by aggressive portfolios with high equity content, but in this case the portfolio volatility is not translated into high CEexROE but into higher guaranteed returns for the policyholders. This is consistent with the conclusion of Siglienti (2000) that excessive investments in equity destroy shareholder value. However, for the guaranteed rates of 3 to 4% offered by Italian insurers it appears that the optimal portfolios consist of 20 to 25% in equities, as opposed to 15% that was obtained by Siglienti using simulations.

Finally, we show in Figure 5 the distribution of the equity to liability ratio (cf. eqn. 10) for a guarantee of 5%. Similar figures were obtained for guarantees ranging from 1% to 10%. This figure shows that for different values of the guarantee the minimum ratio of equity to liability is greater than the regulatory requirement. For the type of policies analyzed here, and for the scenarios sampled from the past ten bullish years, the regulatory constraint is satisfied without explicitly including it in the model.

3.2 Financing shortfalls through long term debt

So far we have assumed that the cost of the guarantee is covered by shareholders. It is possible, however, that such costs are charged to the policyholder

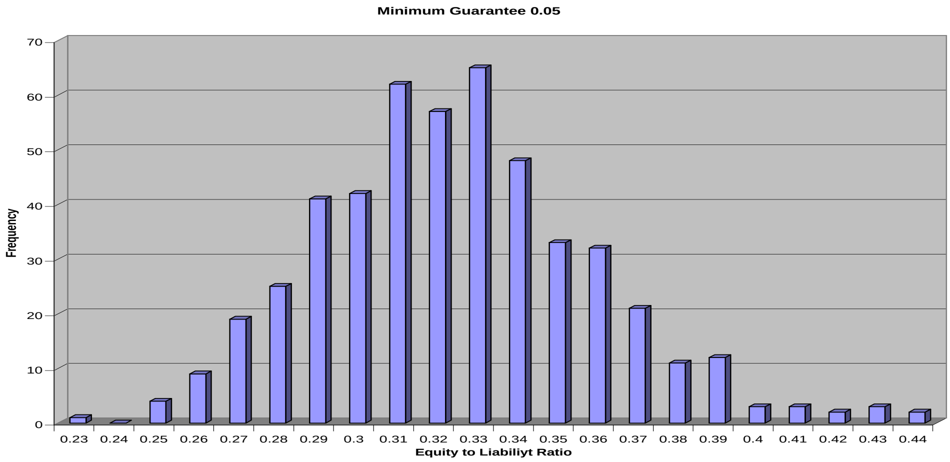


Figure 5: Distribution of equity-to-liability ratio at the end of the planning horizon for a guarantee of 5%.

or be funded by issuing debt. (Note that for mutual insurance firms the policyholders are the shareholders so the point of who pays for the cost is mute. However the issue of raising debt remains.) In either case there are advantages and disadvantages. In particular, if we let the policyholder assume the total cost, we run the risk of not being competitive, loose market share, and experience increased lapse. If we issue debt, we are liable for interest payments at the end of the planning horizon which could reduce our final return. Furthermore, companies face leverage restrictions. It may not be possible to cover all the cost of the guarantee by issuing debt because it will increase the leverage of the company beyond what is allowed by the regulators or accepted by the market.

Another important point in pursuing this question concerns the maturity of the issued debt. To issue long term debt we determine the amount of cash that we need to borrow in order to cover, with a certain probability, future expenditures due to shortfalls over all scenarios. If we indicate by β a confidence level we are searching for the β–percentile O_G^β such that the cost of the guarantee O_G^l in scenario l satisfies

$$P\left(O_G^l \geq O_G^\beta | l \in \Omega\right) = \beta. \tag{19}$$

The cost of the guarantee in scenario l is given by equation (13) as

$$O_G^l = \frac{E_T^l}{\prod_{t=1}^T (1 + r_{ft}^l)} - \rho L_0. \tag{20}$$

Note that O_G^β need not to be raised through the issue of debt only. It is just the reserves needed to fund shortfalls. Strategic considerations will subdivide O_G^β among policyholder charges, C_G, issue of debt or direct borrowing from money markets, D_G, and/or equity supplement, E_S. Thus, we have

$$O_G^\beta = C_G + D_G + E_S. \tag{21}$$

Given the debt structure implied in (21) we determine the final income I_T^l, for each scenarios $l \in \Omega$, as

$$I_T^l = A_T^l - L_T^l - D_G(1 + r_b)^T + (C_G - J_S) \prod_{t=1}^{T} (1 + r_{ft}^l), \tag{22}$$

where J_S are the fixed costs (in percentage of the initial liability) and r_b is the borrowing interest rate. Debt structures for which at least one $I_T^l < 0$ should be discarded as leading the firm into insolvency, even if the probability of such events is very low.

The net Return–on–Equity (ROE) corresponding to a given debt structure in each scenario is given by

$$ROE^l = \frac{I_T^l(1 - \kappa)}{\rho L_0 + E_S}. \tag{23}$$

This is not the *ex ante* excess return on equity optimized with the base mode, but the *ex post* realized total return on equity achieved when the structure of debt has also been specified. This measure can be used to analyze the probability of insolvency when all the cost of the guarantee is funded by shareholders instead of being charged, at least in part, to policyholders.

In Appendix B we report results with the analysis described here. Tables are generated to study the tradeoffs between leverage, policyholder charges, and shareholder returns. Similarly, we can study the effects of different guaranteed returns to the policyholder charges and shareholder returns.

3.3 Insolvency risks

So far we analyzed alternative decisions based only on the net CEexROE and market constrains (policyholder charges, leverage, etc.). Our analysis is missing a measure of risk of the ROE. It is not yet clear how alternative guarantees and debt allocations according to eqn. (21) affect the risk of ROE in eqn. (23). One could argue that the risk aversion of the decision maker is embedded in the utility function of the optimization model. This is true, but the utility function was used only to guide decisions on the asset side, and estimating the net total CEROE from (23) does not incorporate risk aversion when choosing a debt structure. Furthermore, the utility function ensures the solvency of the fund by covering shortfalls with infusion of equity. However, under certain conditions no external sources of equity will be available. The analysis we carry out here compensates for these omissions. It considers the risk of insolvency when structuring the debt structure, thus incorporating risk aversion in structuring the debt in addition to structuring the asset portfolio.

Define $\bar{R}_I$ as the expected excess return over the risk free rate for this line of business and $\bar{r}_f$ as the expected risk free rate. The rate at which we must discount the final income I_G^l is given by $R_\mu = \bar{r}_f + \bar{R}_I$. For our shareholders I_G^l represents the value of the equity at the end of the planning period and they are willing to stay in this business if the discounted value of this equity is not less than the initial capital invested. The shareholders will keep their shares if the *Excess Value per Share* (*EVS*) is greater than zero with a high probability. Recalling that the initial amount of equity is $\rho L_0 + E_S$ (E_S could be equal to zero) the *EVS* in each scenario is given by

$$EVS^l = \frac{I_G^l(1-\kappa)}{(1+R_\mu)^T} - (\rho L_0 + E_S). \tag{24}$$

The risk related to a specific debt allocation is given by the probability that EVS is less than zero, i.e., $P_{EVS}^- = P(EVS^l < 0 | l \in \Omega)$. This is the probability of insolvency and can be determined by calculating the EVS^l for each $l \in \Omega$, order from the lowest to the highest and look for the rank of the first EVS^l that is negative, i.e.,

$$P_{EVS}^- = \frac{\text{rank}(EVS^l < 0)}{N}. \tag{25}$$

The EVS can be used to determine the amount of policyholder charges required to make P_{EVS}^- equal to a given confidence level. Recall that I_G^l, and consequently EVS^l, is a function of C_G, E_S, and D_G. If we fix E_G then I_G is a function of C_G (D_G is determined from eqn. 21). Through a linesearch we can determine C_G^* such that

$$P[EVS(C_G^*) < 0] = \beta. \tag{26}$$

In our experiments we set $\bar{R}_I = 6\%$ and the probability of insolvency $\beta = 1\%$. Figure 6 shows the results of the line search which solves equation (26) for different values of equity supplement E_S. We observe that for guarantees higher than 6% the CEROE increases. How is it possible that higher guarantees can yield higher returns? The puzzle is resolved if we note that the increase in returns is accompanied by a significant increase of policyholder charges. The increases in the policyholder charges fund the guarantee and preserve equity from falling below its present value.

Significant increase in charges would be unacceptable to policyholders and would lead to increased lapses. Our analysis can be used as a demarcation criterion between “good” and “bad” levels of the guarantee. For instance the Italian insurance industry offers products with g in the range 3% to 4%. Our analysis shows that they could consider increasing g up to 6% without significant increase of charges to policyholders or reduction of CEROE. (One may justify the difference from the operating guarantee of 4% to the peak optimized value of 6% as the cost of running the business. If so this cost is high.) For guarantees above 6% we note a substantial increase to policyholder charges at a marginal improvement in CEROE, and this is clearly unacceptable to both policyholders and shareholders.

3.4 Short term financing of shortfalls

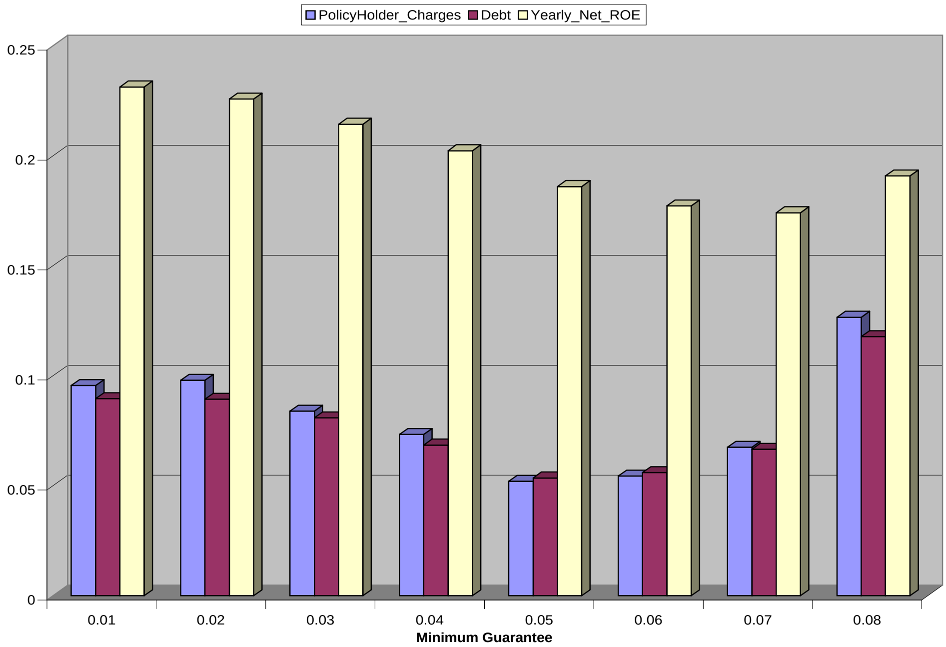
To this point our analysis has determined the cost of the shortfalls O_G^{β} and funded it through a combination of debt D_G, charges to policyholders C_G, and equity E_S. Now, let us fix C_G and E_S, and let D_G fluctuate according to the shortfall O_G^l realized in each scenario. This is equivalent to funding part of the shortfall through short term financing. Instead of issuing a bond for a notional equal to D_G and maturity T, we will borrow money when a shortfall occurs. The debt for each scenario is given by

$$D_G^l = O_G^l - C_G - E_S. \tag{27}$$

We assume that it is possible to borrow money at a spread δ over the risk-free rate. The definition of the final income becomes

$$I_T^l = A_T^l - L_T^l - D_G^l \prod_{t=1}^{T}(1 + r_{ft}^l + \delta) + (C_G - J_S) \prod_{t=1}^{T}(1 + r_{ft}^l). \tag{28}$$

We can apply the analysis of the previous section to determine policyholder charges C_G, and estimate the distribution of D_G^l. We solve equation (26) and display in Figure 7 the C_G^* for different levels of the guarantee, and for $\delta = 2\%$. Note that policyholder charges C_G^* are substantially lower than those obtained by solving (26) in the previous section as reported in Figure 6. This is expected as short-term financing of the cost is a dynamic strategy, as opposed to the fixed strategy of issuing long-term debt. These findings are consistent with the comparison of the two reserving methods in Boyle-Hardy. Since D_G^l is scenario dependent, it compensates for those scenarios with high shortfalls, while it is low (or null) for those scenarios with low shortfalls.



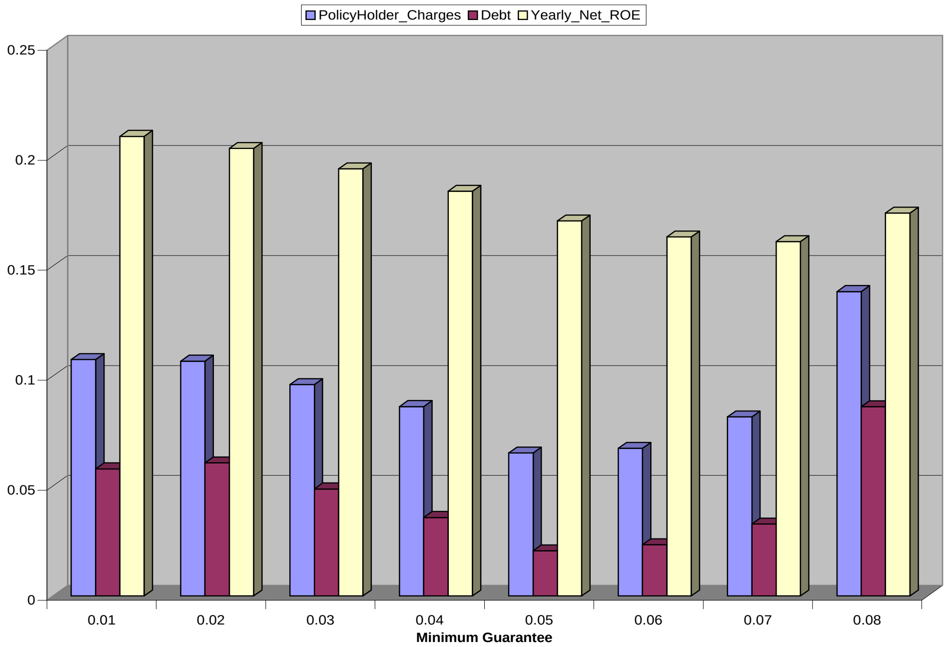


Figure 6: The levels of policyholder charge, debt and net CEROE such that the probability of insolvency is $P[EVS(C_G^*) < 0] = 1\%$. We show these levels for equity supplement $E_S = 0.0$ (top) and $E_S = 0.02$ (bottom).

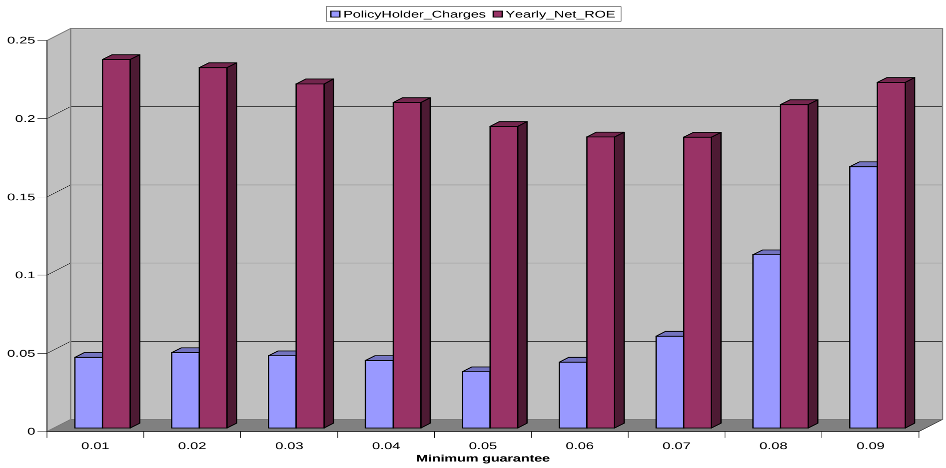


Figure 7: The levels of policyholder charges, and net CEROE for different guarantee such that $P[EVS(C_G^*) < 0] = 1\%$.

3.5 Choice of utility function

The decision maker's risk aversion specifies unique asset portfolio to back each guaranteed policy. Clearly increased risk aversion will lead to more conservative portfolios with higher contents of fixed income. The result will be a simultaneous reduction in both the CEexROE to shareholders and the cost of shortfalls required to fund the policy. Figure 8 illustrates the tradeoff as the risk aversion parameter γ varies from 0 (base case) to -2 (increased risk aversion) for five different target guarantees.

Note that for low target guarantees increased appetite for risk results in higher CEexROE for a marginal increase in cost of the guarantee. For higher target guarantees (e.g., 15%) we note a substantial increase in the cost of the guarantee as the embedded option goes deep in the money when we increase the risk tolerance and invest into volatile assets. These results confirm our expectations on model performance, and allow users to generate efficient tradeoffs that are consistent with the contractual obligations and the firm's risk tolerance.

4 Benchmarks of Italian Insurance Policies

In order to asses the effectiveness of our model we compare the optimal portfolios with industry benchmarks. We take as benchmark a set of portfolios with a fixed broad asset allocation between bonds and stocks, and random allocation among specific assets. In order to be consistent with the usual

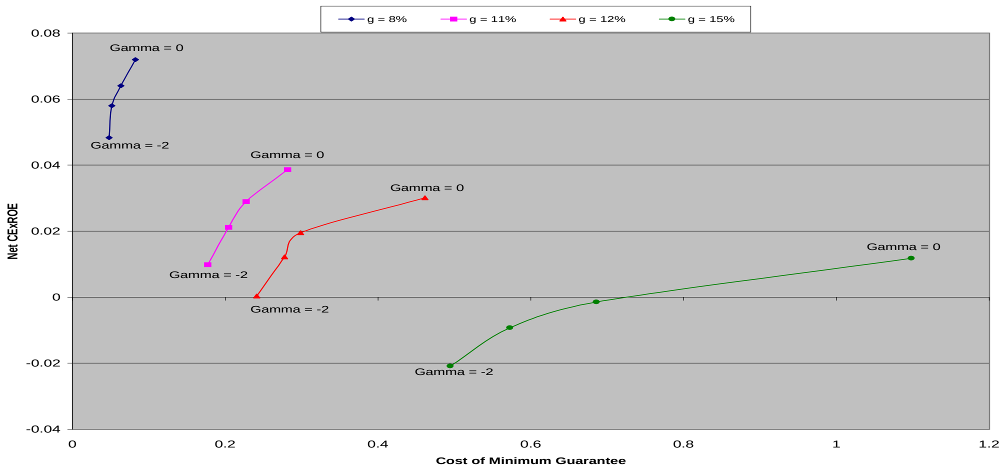


Figure 8: Tradeoff of CEexROE against cost of the guarantee with varying risk aversion for target guarantees 8% (left), 11%, 12% and 15% (right).

fixed-mix strategies and follow industry practices we set the broad asset allocation between bonds and stocks to 90/10, 80/20, and 70/30. The results of this experiment are reported in Figure 9. Note that the optimized portfolios always dominate the benchmark portfolios in the cost-of-guarantee *vs* CEexROE space. This figure justifies the integrative approach taken in this paper, whereby the insurance policy is analyzed *jointly* with the asset allocation decision instead of being analyzed for an *a priori* fixed asset portfolio.

The results of this section can be extended to incorporate other assets permitted by regulations, such as mortgages, corporate bonds and international sovereign debt. Italian insurers are allowed to invest up to 10% of the value of their portfolio in international assets. We run the base model for a guarantee of 4%, and allowing investments in the Morgan Stanley stock indices for USA, UK and Japan and the J.P. Morgan Government bond indices for the same countries. The internationally diversified portfolio achieves CEexROE of 0.14 at a cost of the guarantee of 0.02. By contrast, the results of Figure 4 show that domestic investments in the Italian markets fund the guarantee at the same cost but yield a CEexROE of only .11. Similarly, investments in the US Corporate bond market improves the CEexROE to .16, but this comes at an increase of the cost to 0.033.

The results of this section are in general agreement with the current practices of Italian insurers. However, the optimized results suggest that improved policies and associated asset strategies are still possible.

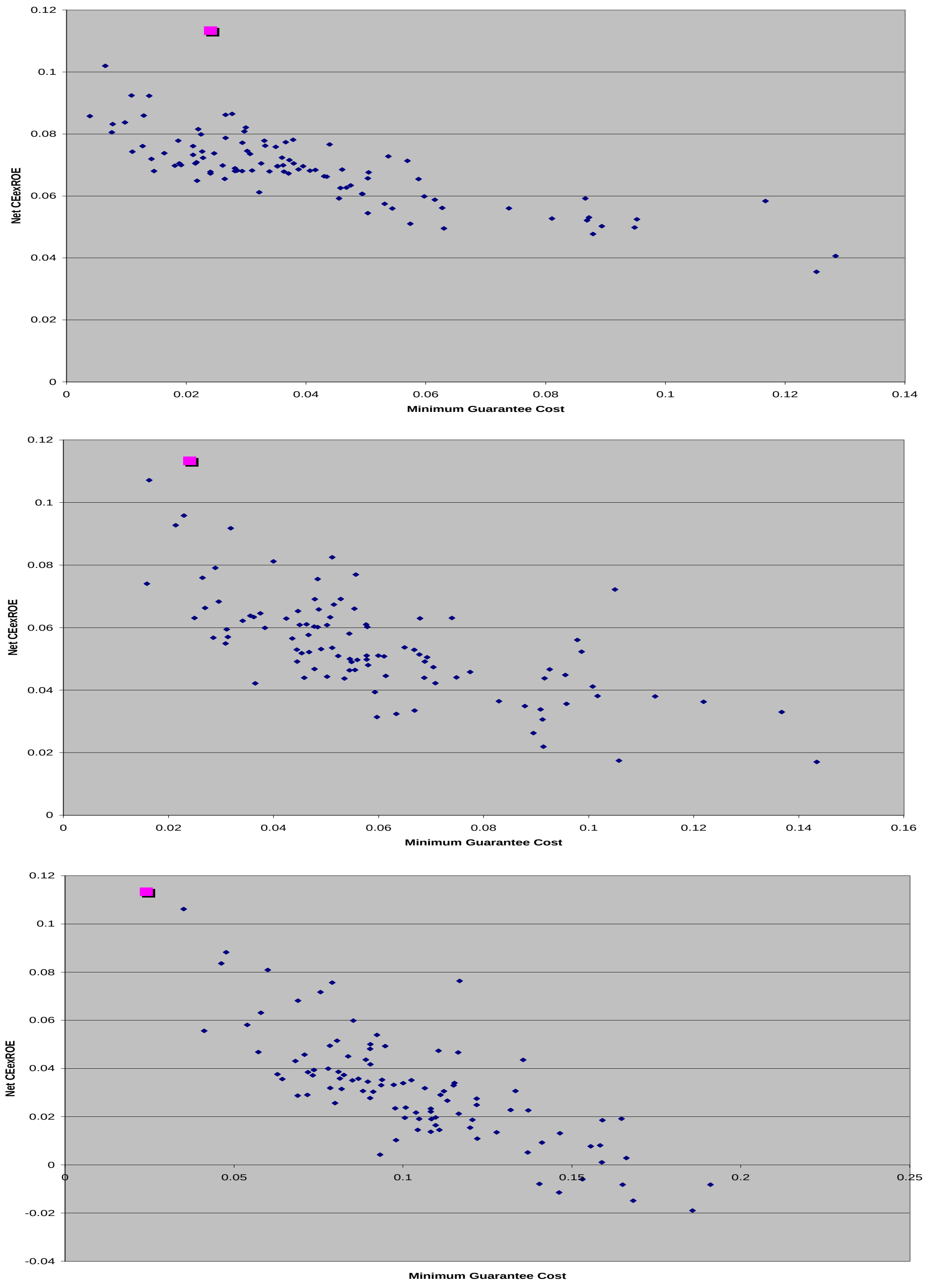


Figure 9: Performance of benchmark portfolios (diamonds) against the optimized portfolio (square) for $g = 4\%$. Asset allocation for the benchmark portfolios is set to 90/10 (bonds/stocks), 80/20, and 70/30, respectively, from top to bottom.

5 Conclusions

We have developed an integrative asset and liability management model for endowments with guarantees. It has been demonstrated that the integrative model generates asset structures for specific insurance policies that are efficient as opposed to asset strategies developed in a non-integrated model.

Several interesting conclusions can be drawn from the use of the model on data from the Italian insurance industry. First, we have quantified the tradeoffs between the different targets of the insurance firm: providing the best products for its policyholders, providing the highest excess return to its shareholders, satisfying the guarantee at the lowest possible cost and with high probability. Some interesting insights are obtained on the structure of the optimal portfolios. In particular we observe that too little equity in the portfolio and the insurer cannot meet the guarantee, while too much equity destroys shareholder value.

Second, we have analyzed different debt structures whereby the cost of the guarantee is funded through equity or through debt with either long or short maturities. The effects of these choices on the cost of the guarantee and on the probability of insolvency can be quantified, thus providing guidance to management for the selection of policies.

Third, we have seen from the empirical analysis that Italian insurers operate at levels which are close to optimal but not quite so. There is room for improvement either by offering more competitive products or by generating higher excess returns for the benefit of the shareholders and/or the policyholders.

A significant extension for the long time horizons of the products considered would be to a multi-stage model where decisions are revised at time instances after $t = 0$ until maturity. Such dynamic stochastic programs with recourse have been developed for asset and liability management by the references given in the introduction. However, for the highly nonlinear problem we are addressing here such models are difficult to develop. The linearization of the single-stage model developed in appendix A does not apply directly to multistage formulations. Specialized algorithms for geometric programming must be employed for the solution of multistage extensions of this model.

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A Solving the nonlinear dynamic equations

In this section we show how to solve the nonlinear equations (5)–(9) in order to obtain the objective function (12). At time $t = 0$, the liability is the pure premium L_0. At $t = 1$ (to simplify the notation we drop the scenario superscript) we have

$$L_1 = L_0(1 - \Lambda_1)(1 + g + y_1^+). \tag{29}$$

At $t = 2$ we use the value of L_1 from (29) to obtain

$$\begin{aligned} L_2 &= L_1(1 - \Lambda_2)(1 + g + y_2^+) \\ &= L_0(1 - \Lambda_2)(1 - \Lambda_1)(1 + g + y_1^+)(1 + g + y_2^+). \end{aligned} \tag{30}$$

Applying this process recursively for each t we obtain the final liability as

$$L_T = L_0 \prod_{t=1}^{T} (1 - \Lambda_t)(1 + g + y_t^+). \tag{31}$$

For the equity dynamics we have that $E_0 = \rho L_0$. At $t = 1$

$$E_1 = \rho L_0(1 + r_{f1}) + y_1^- L_0. \tag{32}$$

At $t = 2$ and substituting for E_1 and L_1 from (32) and (29) we obtain

$$\begin{aligned} E_2 &= E_1(1 + r_{f2}) + y_2^- L_1 \\ &= \rho L_0(1 + r_{f1})(1 + r_{f2}) + L_0 y_1^-(1 + r_{f2}) \\ &+ L_0 y_2^-(1 - \Lambda_1)(1 + g + y_1^+). \end{aligned} \tag{33}$$

At $t = 3$ we have

$$\begin{aligned} E_3 &= E_2(1 + r_{f3}) + y_3^- L_2 = \\ &= \rho L_0(1 + r_{f1})(1 + r_{f2})(1 + r_{f3}) + L_0 y_3^-(1 + r_{f2})(1 + r_{f3}) \\ &+ L_0\, y_3^-(1 + r_{f3})(1 - \Lambda_1)(1 + g + y_1^-) \\ &+ L_0\, y_3^-(1 - \Lambda_2)(1 - \Lambda_1)(1 + g + y_1^+)(1 + g + y_2^+). \end{aligned} \tag{34}$$

Applying this process recursively for each t we obtain after some simple algebra

$$E_T = L_0 \left[\rho \prod_{t=1}^{T} (1 + r_{ft}) + \sum_{t=1}^{T} \left(y_t^- \phi(t, T) \prod_{\tau=1}^{t-1} (1 - \Lambda_\tau)(1 + g + y_\tau^+) \right) \right], \tag{35}$$

where $\phi(t,T) = \prod_{\tau=t+1}^{T}(1+r_{f\tau})$ is the cumulative return of the short rate during from t to T.

With the same arguments it is possible to show that

$$y_{At} = L_0\,\Lambda_t\,(1+g+y_t^{+})\prod_{\tau=1}^{t-1}(1-\Lambda_\tau)(1+g+y_\tau^{+}). \tag{36}$$

For the asset dynamics we have that $A_0 = L_0(1+\rho)$. At $t=1$,

$$\begin{aligned} A_1 &= A_0(1+R_{P1}) + y_1^{-}L_0 - y_{A1} \\ &= L_0(1+\rho)(1+R_{P1}) + y_1^{-}L_0 - y_{A1}. \end{aligned} \tag{37}$$

At $t=2$ substituting L_1 from (29) we obtain

$$\begin{aligned} A_2 &= A_1(1+R_{P2}) + y_2^{-}L_1 - y_{A2} \\ &= L_0(1+\rho)(1+R_{P1})(1+R_{P2}) + y_1^{-}L_0(1+R_{P2}) \\ &\quad - y_{A1}(1+R_{P2}) + y_2^{-}L_1 - y_{A2}. \end{aligned} \tag{38}$$

The value of the assets at maturity is given by $A_T =$

$$\begin{aligned} &L_0(1+\rho)\prod_{t=1}^{T}(1+R_{Pt})L_0\sum_{t=1}^{T}y_t^{-}\prod_{\tau=t+1}^{T}(1+R_{P\tau})\prod_{\tau=1}^{t-1}(1+g+y_\tau^{+})\prod_{\tau=1}^{t-1}(1-\Lambda_\tau) \\ &-\sum_{t=1}^{T}y_{At}\prod_{\tau=t+1}^{T}(1+R_{P\tau}). \end{aligned} \tag{39}$$

By substituting y_{At} with the expression in (36) we obtain

$$\begin{aligned} A_T &= L_0(1+\rho)\prod_{t=1}^{T}(1+R_{Pt}) \\ &\quad + L_0\sum_{t=1}^{T}y_t^{-}\prod_{\tau=t+1}^{T}(1+R_{P\tau})\prod_{\tau=1}^{t-1}(1+g+y_\tau^{+})(1-\Lambda_\tau) \\ &\quad - L_0\sum_{t=1}^{T}\Lambda_t\,(1+g+y_t^{+})\prod_{\tau=t+1}^{T}(1+R_{P\tau})\prod_{\tau=1}^{t-1}(1+g+y_\tau^{+})(1-\Lambda_\tau). \end{aligned} \tag{40}$$

Collecting terms we obtain

$$\begin{aligned} A_T &= L_0(1+\rho)\prod_{t=1}^{T}(1+R_{Pt}) \qquad (41) \\ &\quad + L_0\sum_{t=1}^{T}\left(y_t^{-} - \Lambda_t\,(1+g+y_t^{+})\right)\prod_{\tau=t+1}^{T}(1+R_{P\tau})\prod_{\tau=1}^{t-1}(1+g+y_\tau^{+})(1-\Lambda_\tau). \end{aligned}$$

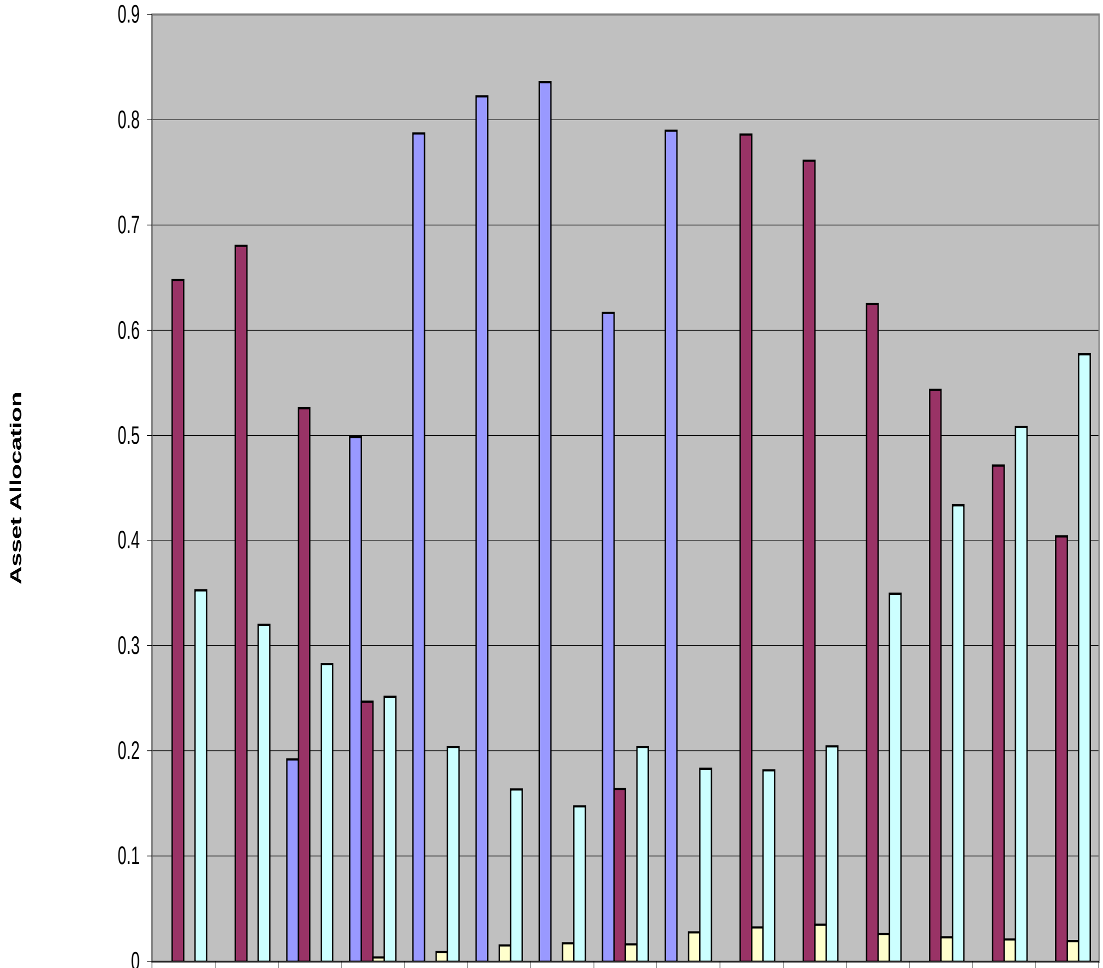
Code	Description
SBGVNIT.1-3	Salomon Brother Italian Government Bond 1-3 years
SBGVNIT.3-7	Salomon Brother Italian Government Bond 3-7 years
SBGVNIT.7-10	Salomon Brother Italian Government Bond 7-10 years
ITMSBNK	Milan Mib Historic Banks
ITMSAUT	Milan Mib Historic Cars
ITMSCEM	Milan Mib Historic Chemicals
ITMSCST	Milan Mib Historic Construction
ITMSDST	Milan Mib Historic Distribution
ITMSELT	Milan Mib Historic Electronics
ITMSFIN	Milan Mib Historic Finance
ITMSFPA	Milan Mib Historic Finance Holdings
ITMSFMS	Milan Mib Historic Finance Misc.
ITMSFNS	Milan Mib Historic Finance Services
ITMSFOD	Milan Mib Historic Food
ITMSIND	Milan Mib Historic Industrials
ITMSINM	Milan Mib Historic Industrials Misc
ITMSINS	Milan Mib Historic Insurance
ITMSPUB	Milan Mib Historic Media
ITMSMAM	Milan Mib Historic MineralsMetals
ITMSPAP	Milan Mib Historic Paper
ITMSMAC	Milan Mib Historic Plants & Machine
ITMSPSU	Milan Mib Historic Pub. Util. Serv
ITMSRES	Milan Mib Historic Real Estate
ITMSSER	Milan Mib Historic Services
ITMSTEX	Milan Mib Historic TextileClothing
ITMST&T	Milan Mib Historic Transportation & Tourism

Table 1: Asset classes used in testing the model.

B Asset classes and Further Empirical Results

The asset classes used in testing the model are given in Table 1. They consist of bond indices for short, medium and long-term debt of the Italian government, and stock indices of the major industrial sectors traded in the Milano stock exchange.

The broad asset allocation shown in Figure 3 is broken down among the different indices as shown in Figure 10. Table 2 displays net CEexROE and $\bar{O}_G$ for the range of guarantees reported in Figure 4. The difference of cost



	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1	0.11	0.12	0.13	0.14	0.15
SBGVNIT.1-3	0	0	0.19178	0.49818	0.78706	0.82202	0.83583	0.61661	0.78934	0	0	0	0	0	0
SBGVNIT.3-7	0.64738	0.67995	0.52544	0.24668	0	0	0	0.16398	0	0.78617	0.76099	0.62464	0.54353	0.47134	0.4041
ITMSFMS	0	0	0	0.00346	0.00905	0.01492	0.0169	0.01589	0.02759	0.03237	0.03491	0.02615	0.023	0.02076	0.01903
ITMSPSU	0.35262	0.32005	0.28278	0.25167	0.20389	0.16306	0.14727	0.20351	0.18307	0.18146	0.20411	0.34921	0.43346	0.5079	0.57687

Minimum Guarantee

Figure 10: Asset allocation for different levels of guarantee.

g	CEexROE	$\bar{O}_G$
0.01	0.1347	0.0235
0.02	0.1288	0.0266
0.03	0.1215	0.0256
0.04	0.1134	0.0240
0.05	0.1043	0.0208
0.06	0.0944	0.0255
0.07	0.0838	0.0370
0.08	0.0717	0.0819
0.09	0.0605	0.1372
0.1	0.0488	0.1780
0.11	0.0378	0.2407
0.12	0.0284	0.4117
0.13	0.0212	0.5755
0.14	0.0152	0.7635
0.15	0.0098	0.9809

Table 2: Net CEexROE and cost of the guarantee ($\bar{O}_G$) for different levels of guarantee (g).

between guarantee levels $g = 0.01$ and $g = 0.05$ is just 0.2%. Further results on the distribution of equity to liability, for different levels of guarantee, are shown in Figure11. These results are in agreement with Figure 5 presented in the main paper.

B.1 Leverage, policyholder charges and shareholder returns

In Table 3 we summarize data that assist the decision maker to take a position according to her strategic views and constrains. If no entries are displayed these choices cannot be implemented, either because some I_T^l are negative (this occurs when charges to policyholders are very low and high debt levels yield a negative final income), or because the amount of money necessary to cover shortfalls is absorbed by the policyholder charges, implying negative debt levels.

For example, by choosing a leverage level equal to 0.5, the highest yearly net CEROE is 0.183. Note that, if the firm wishes to achieve higher performance level, the leverage should also increase. Also, observe the inverse relation between leverage and policyholder charges. The greater the amount

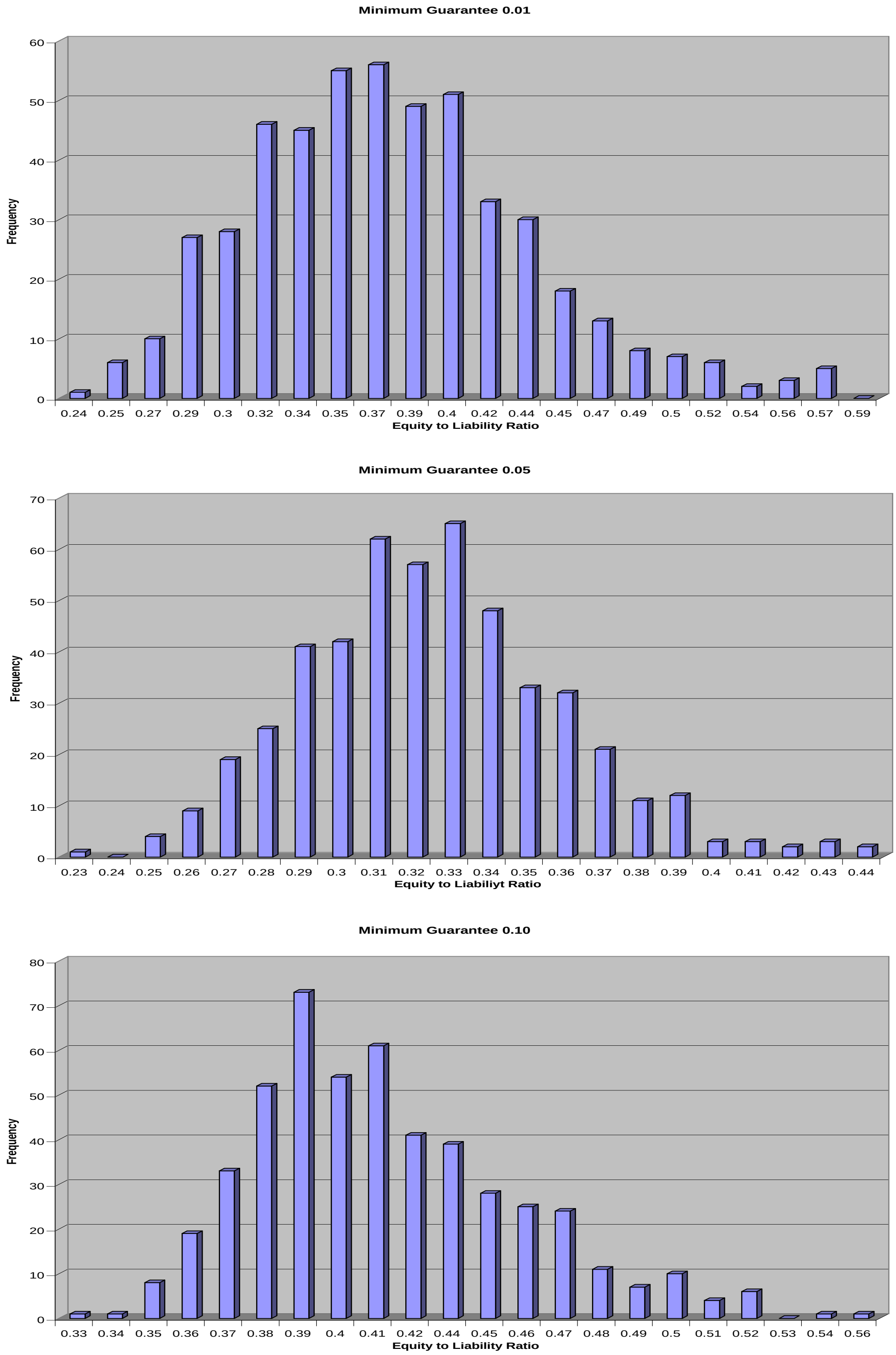


Figure 11: Equity-to-liability ratio at the end of the planning horizon for different levels of the guarantee.

we charge to the policyholder, the lower is the leverage level and the higher the annualized net CEROE.

Our model can generate similar tables to study the many interactions of endowment with guarantee. For example, we could be interested in investigating the effect of different guarantee levels to the policyholder charges and yearly returns. We first estimate, at a given confidence level β, the cost of the guarantee O_G^β, and then apportion this cost to policyholders (C_G in eqn. 21) and fund the rest through debt or equity surcharge. Depending on C_G we observe a change in the CEROE to shareholders.

Table 4 shows this relationship. We observe the same behavior we had seen between $\bar{O}_G$ and net CEexROE. The model chooses more aggressive strategies for low g because it is then possible to achieve higher levels of CEexROE at little cost. Recall that we are working with percentiles and the impact of aggressive strategies is much more evident on the tails. When the guarantee is low at $g = 0.01$ we need higher policyholder charges to reach the highest return, while for $g = 0.05$ lower charges are required.

The results in Table 3 should be examined taking into account the measure of risk P_{EVS}^- associated with the CEROE of every combination of policyholder charges and leverage level. The probabilities corresponding to Table 3 are shown in Table 5. Observe that the upper–left entry has a P_{EVS}^- equal to 0.58. This means that in 58% of the cases the present value of the final equity is less than the amount invested today by the shareholder, even though, net CEROE is acceptable (12%). This position is risky. The reason why this position is quite risky is due to the fact that we are asking our shareholders to fund the total β–percentile cost of the guarantee. No charges are passed on to policyholders.

Leverage levels	Policyholder Charges 0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.121125	0.124595	0.128295	0.132256	0.136515	0.141118	0.146123	0.151602	0.15765	0.164391
0.125	0.123946	0.127684	0.131656	0.135891	0.14043	0.145317	0.150612	0.156387	0.16274	0.169795
0.25	0.126654	0.13064	0.13486	0.139346	0.144137	0.14928	0.154834	0.160873	0.167495	0.174827
0.375	0.12926	0.133474	0.137923	0.142638	0.147659	0.153033	0.158821	0.165097	0.17196	0.179538
0.5	0.13177	0.136197	0.140857	0.145783	0.151014	0.156599	0.162599	0.169089	0.176169	0.183968
0.625	0.134193	0.138817	0.143673	0.148794	0.154219	0.159997	0.16619	0.172875	0.180151	0.188151
0.75	0.136533	0.141343	0.146381	0.151682	0.157285	0.163242	0.169612	0.176475	0.183932	0.192114
0.875	0.138798	0.143781	0.148989	0.154458	0.160227	0.166348	0.172882	0.179909	0.18753	0.195879
1	0.140991	0.146137	0.151505	0.15713	0.163053	0.169327	0.176013	0.183191	0.190964	0.199468
1.125	0.143118	0.148417	0.153935	0.159706	0.165774	0.172189	0.179016	0.186335	0.19425	0.202896
1.25	0.145182	0.150626	0.156285	0.162194	0.168396	0.174944	0.181903	0.189353	0.197399	0.206177
1.375	0.147188	0.152769	0.15856	0.164599	0.170928	0.177601	0.184682	0.192255	0.200423	
1.5	0.149138	0.154849	0.160766	0.166927	0.173375	0.180165	0.187362	0.19505	0.203333	
1.625	0.151037	0.156871	0.162907	0.169183	0.175744	0.182644	0.18995	0.197745		
1.75	0.152886	0.158837	0.164986	0.171371	0.178039	0.185044	0.192452	0.200349		
1.875	0.154688	0.160751	0.167007	0.173497	0.180265	0.187369	0.194875			
2	0.156446	0.162616	0.168974	0.175562	0.182427	0.189624	0.197222			
2.125		0.164433	0.17089	0.177572	0.184528	0.191814				
2.25		0.166207	0.172757	0.179529	0.186571	0.193942				
2.375		0.167938	0.174577	0.181435	0.188561					
2.5		0.16963	0.176354	0.183294	0.190499					
2.625			0.178089	0.185108						
2.75			0.179785	0.186879						
2.875			0.181443							
3			0.183064							
3.125										
3.25										
3.375										

Table 3: Net CEROE for different combinations of leverage and policyholder charges. The table is built for a guarantee $g = 4\%$ at a confidence level $\beta = 1\%$.

Policyholder Charges	Minimum Guarantee 0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
0	0.144564	0.139163	0.135832	0.13177	0.130433	0.120909	0.110348	0.099402
0.01	0.148057	0.142648	0.139726	0.136197	0.136011	0.126397	0.115193	0.102442
0.02	0.151703	0.146281	0.143803	0.140857	0.141965	0.132226	0.120278	0.105562
0.03	0.155517	0.150077	0.148086	0.145783	0.148361	0.138457	0.125641	0.10877
0.04	0.15952	0.154056	0.152599	0.151014	0.155289	0.145166	0.131326	0.112075
0.05	0.163732	0.158239	0.157375	0.156599	0.162863	0.152453	0.13739	0.115487
0.06	0.168182	0.162651	0.162452	0.162599	0.171238	0.16045	0.143903	0.119017
0.07	0.1729	0.167323	0.167876	0.169089	0.180626	0.169337	0.150957	0.122676
0.08	0.177925	0.172291	0.173703	0.176169	0.191338	0.17937	0.15867	0.126479
0.09	0.183304	0.177599	0.180007	0.183968		0.190924	0.167202	0.130443
0.1	0.189093	0.183304	0.18688	0.192664			0.176778	0.134586

Table 4: The relation between net CEROE, policyholder charges and guarantee. The table is built with confidence level $\beta = 1\%$ and liability (debt-to-equity ratio) equal to 0.5.

Leverage levels	Policyholder Charges 0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.58	0.522	0.462	0.4	0.344	0.278	0.208	0.148	0.096	0.042
0.125	0.534	0.478	0.416	0.366	0.302	0.242	0.172	0.112	0.072	0.02
0.25	0.508	0.444	0.394	0.338	0.274	0.212	0.15	0.1	0.06	0.012
0.375	0.476	0.416	0.368	0.306	0.252	0.188	0.134	0.092	0.042	0.012
0.5	0.444	0.396	0.346	0.284	0.226	0.162	0.118	0.076	0.032	0.006
0.625	0.418	0.374	0.322	0.266	0.212	0.152	0.106	0.068	0.022	0.004
0.75	0.404	0.366	0.304	0.258	0.198	0.144	0.098	0.056	0.016	0.002
0.875	0.4	0.354	0.286	0.234	0.184	0.136	0.092	0.05	0.012	0.002
1	0.378	0.33	0.28	0.224	0.162	0.124	0.088	0.04	0.012	0.002
1.125	0.37	0.318	0.266	0.216	0.156	0.114	0.078	0.036	0.008	0.002
1.25	0.364	0.31	0.264	0.208	0.146	0.108	0.074	0.032	0.008	0.002
1.375	0.356	0.296	0.254	0.2	0.146	0.104	0.07	0.026	0.004	
1.5	0.35	0.286	0.24	0.196	0.142	0.098	0.062	0.026	0.004	
1.625	0.332	0.282	0.234	0.188	0.136	0.096	0.06	0.02		
1.75	0.322	0.276	0.224	0.178	0.132	0.094	0.054	0.016		
1.875	0.316	0.266	0.22	0.162	0.126	0.092	0.052			
2	0.314	0.266	0.214	0.156	0.122	0.086	0.05			
2.125		0.264	0.214	0.15	0.118	0.084				
2.25		0.264	0.208	0.148	0.116	0.08				
2.375		0.26	0.202	0.146	0.112					
2.5		0.244	0.202	0.146	0.11					
2.625			0.198	0.146						
2.75			0.196	0.144						
2.875			0.196							
3			0.19							
3.125										
3.25										
3.375										

Table 5: Relationship between P^{-}_{EVS}—the probability that excess value per share will fall below zero— leverage and policyholder charges. The table is built for a guarantee $g = 4\%$ and confidence level $\beta = 1\%$.