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Intertemporal Insurance

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Intertemporal Insurance [1]

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Abstract : This paper develops a discrete-time general equilibrium model of insurance using standard techniques of intertemporal finance. The underlying source of uncertainty is modeled as a marked point process. The paper begins by characterizing Walrasian equilibrium on the event tree generated by the accident process. The corresponding Arrow-Debreu-Radner contingent-commodity prices allow the pricing of insurance contracts. A transformation of the underlying probability measure gives an alternative characterization of insurance contract prices plus accumulated payouts as martingales. A direct application of the usual dynamic spanning argument demonstrates that one insurance contract for each type of accident suffices, at least generically, to achieve market completeness. The theory is illustrated by a simple example in which consumers have Cobb-Douglas preferences and experience accidents at a rate which varies across individuals but remains constant over time, the traditional setting for much of insurance theory.

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1 Introduction

Insurance and finance share common intellectual roots in the Arrow-Debreu theory of contingent commodities. It seems curious, therefore, that in addressing the question "How do markets deal with risk?" the general equilibrium theories of insurance and finance have gone such separate ways.

From the pioneering efforts of Malinvaud [1972, 1973] through the recent work of Cass, Chichilnisky, and Wu [1996], the general equilibrium theory of insurance has remained an essentially static theory, preoccupied with understanding how the law of large numbers might serve to alleviate problems of market incompleteness. On the surface, this concern with market incompleteness does not seem misplaced. Accidents are intrinsically personal affairs, and insurance contracts reflect the personalized nature of the underlying events: when an accident happens to someone, the insurance contract compensates the individual for the injury to his person or property. As Malinvaud pointed out, because accidents happen to individuals, the size of the state space, and hence the required number of contingent contracts, grows exponentially with the number of agents, straining our willingness to believe in the efficacy of Arrow-Debreu contingent contracts. Appeal to the law of large numbers offers a way out. When pricing identically and independently distributed risks in a large population, simple insurance contracts serve as at least a partial substitute for a much larger number of Arrow-Debreu contingent contracts, and in the limit insurance will be provided at actuarially fair rates. Cass, Chichilnisky, and Wu [1996] generalize this argument substantially.

This paper abandons the traditional approach to insurance, adopting instead the perspective of modern finance. Specifically, insurance contracts are treated as nothing more than a particular type of financial security, personalized but otherwise no different from any other. In contrast to traditional insurance theory, our approach is intrinsically intertemporal, regarding an insurance contract as a payment today for the promise of compensation if and when an accident happens sometime in the future. We use this intertemporal characterization of accident processes to justify a claim of market completeness through dynamic spanning in the fashion of Kreps [1982]. Insurance pricing in our view might be nearly actuarially fair, but it need not be. Instead, insurance contracts are priced just as in Harrison and Kreps' [1979] synthesis of financial asset pricing theory with asset prices plus accumulated dividends evolving as a martingale and asset returns satisfying a modified version of CAPM.

Although our theory of intertemporal insurance generalizes easily to continuous time, in this paper time is discrete. Section 2 applies "standard finance," as represented by Dothan [1990], Duffie [1988], or Huang and Litzenberger [1988], to insurance markets. We begin with a formal description of a discrete time marked point process, a class of stochastic process particularly well suited to accidents. Using a marked point process as the basic source of uncertainty, we define an intertemporal exchange economy on the event tree generated by the accident process and characterize the Walrasian equilibrium involving trade in Arrow-Debreu-Radner (ADR) time-event contingent commodities on this tree. The corresponding ADR time-event contingent prices allow us to price insurance contracts, which are typically **not** standard ADR contracts, as redundant securities. A transformation of the underlying probability measure gives an alternative characterization of insurance contract prices plus accumulated payouts as martingales in the fashion of Harrison-Kreps [1979]. With this standard Walrasian equilibrium in place, we then "remove" the ADR contingent commodities, leaving only the insurance contracts to deal with risks. A direct application of the usual dynamic spanning argument (as presented, for example, in Huang and Litzenberger [1988]) demonstrates that one insurance contract for each type of accident suffices, at least generically, to achieve market completeness. Section 3 illustrates the theory in a simple setting in which consumers have Cobb-Douglas preferences and experience accidents at a rate which varies across individuals but remains constant over time, the traditional setting for much of insurance theory. Section 4 offers some conclusions and directions for future research.

2 The model

The key to dynamic spanning in the finance literature, as formulated by Kreps [1982], is that uncertainty resolves "nicely." Uncertainty resolves nicely for a Weiner process because of path continuity. Uncertainty also resolves nicely for accident processes, but for a different reason: measured on a sufficiently fine time scale, accidents are relatively rare events and, as a consequence, economic agents have time to react to the news of their occurrence.

In our framework, the length of a time interval should be viewed as very short, perhaps a nanosecond. Consequently, at most dates nothing happens. Occasionally, there is an accident, but never more than one. The representation we choose for such an accident process is a discrete time,

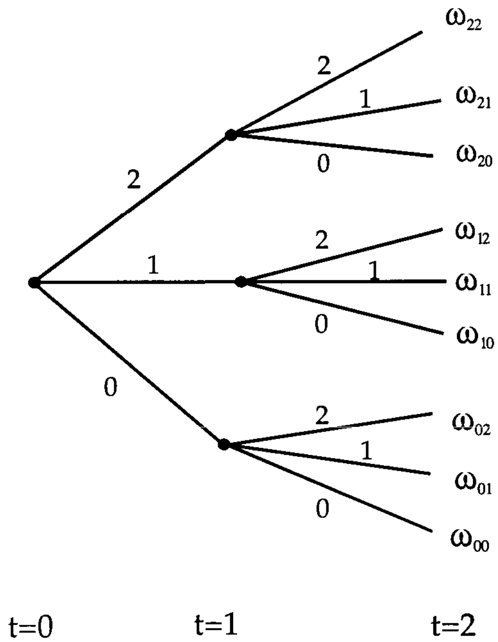

Figure 1: The sample space.

marked point process. To simplify the discussion, we confine attention to a model of pure exchange wit h a single commodity available for consumption at each date.

2.1 Characterizing the accident process

Let $\mathcal{T} := \{0, 1, \ldots, T\}$ represent the time set.[1] An accident can happen at any date $t \geq 1$, but never more than one accident at any given date. If an accident happens at date t, it is assumed to occur and be known to all agents in the economy prior to trade or consumption at date t.

We assume that any accident occurring at date t can be classified into one of a finite number of accident types indexed by $\mathcal{K} := \{0, \ldots, K\}$ with $k = 0$ signifying "no accident." All uncertainty in the economy is captured by the probability space $(\Omega, \mathcal{F}, P)$ with state space $\Omega := \mathcal{K}^T$, σ-algebra $\mathcal{F} = 2^{\Omega}$, and probability measure P. Throughout this paper, we assume that $P(\omega) > 0$ for all $\omega \in \Omega$.

Figure 1 illustrates for the case $T = K = 2$ which contains nine sample points. The point ω_{20}, for example, corresponds to a realization in which

[1] We assume that $T < \infty$ in this section, but the analysis generalizes quite directly to the infinite horizon case.

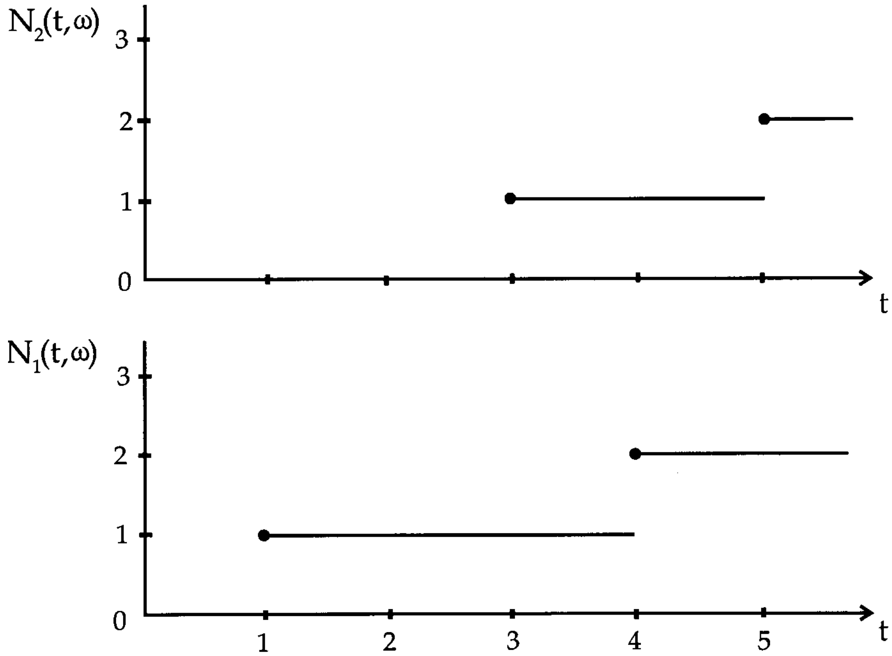

Figure 2: A realization of the accident process.

there is an accident of type 2 at date 1 and no accident at date 2. For $k > 0$, let $N_k\colon T \times \Omega \to \mathbf{Z}_+$ represent the stochastic process which counts the number of accidents of type k: i.e., $N_k(t,\omega)$ is the number of accidents of type k which have occurred up to date t. N_k is nondecreasing, and $N_k(0,\omega) = 0$ for all $\omega \in \Omega$. For each $t \geq 1$, assume that $\Delta N_k(t) := N_k(t) - N_k(t-1) \in \{0,1\}$ for all $k \in \{1,\ldots,K\}$ and $\Delta N_k(t) = 1$ for at most one $k \in \{1,\ldots,K\}$. The $\mathbf{Z}_+^K$-valued stochastic process defined by

$$N(t,\omega) := (N_1(t,\omega),\ldots,N_K(t,\omega))$$

is called a discrete time, K-variate counting process or, equivalently, a marked point process.[2] Figure 2, which imitates a similar figure for a continuous time counting process in Brémaud [1981, p. 20], illustrates a typical realization for an economy with two accident types: (i) accidents of type 1 occur at dates 1 and 4; (ii) accidents of type 2 occur at dates 3 and 5; and (iii) there is no accident at date 2. Note that there is never more than one accident of any type at any given date.

In the usual way, the stochastic process N generates a filtration, a non-

[2] The standard context for these processes is continuous time as in Brémaud [1981] or Last and Brandt [1995]. Here we have simply transcribed the concepts to discrete time.

decreasing sequence of a-algebras

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$$

with the property that $\mathcal{F}_t$ is the coarsest σ-algebra with respect to which the random variable $N(t): \Omega \to \mathbf{Z}_+$ is measurable. Let f_t denote the partition of Ω which generates the σ-algebra $\mathcal{F}_t$. By definition of the accident process, $f_0 = \{\Omega\}$ and $f_T = \{\{\omega\} \mid \omega \in \Omega\}$, corresponding to the absence of information at date $t = 0$ and complete information at date T. In Figure 1, the partition f_1 at date 1 consists of the three sets

$$a_{10} := \{\omega_{00}, \omega_{01}, \omega_{02}\}, \quad a_{11} := \{\omega_{10}, \omega_{11}, \omega_{12}\}, \quad a_{12} := \{\omega_{20}, \omega_{21}, \omega_{22}\}$$

representing the events (i) no accident at date 1; (ii) accident of type 1 at date 1; and (iii) accident of type 2 at date 2 respectively.

2.2 Describing the economy

Consumers in this economy are indexed by the finite set $I = \{1, \ldots, n\}$. For each consumer $i \in I$, a consumption process is a function $x_i: \mathcal{T} \times \Omega \to \mathbf{R}$ and an endowment process a function $w_i: \mathcal{T} \times \Omega \to \mathbf{R}$, each adapted to the filtration generated by the accident process N. Let L denote this vector space, the set of all functions $x: \mathcal{T} \times \Omega \to \mathbf{R}$ such that $x^{-1}(G) \in \mathcal{F}_t$ for every Borel subset G of $\mathbf{R}$ and for every $t \in \mathcal{T}$. Equivalently, processes adapted to the accident filtration are constant on the sets $\{t\} \times a_t$.

Contingent commodities provide a convenient way to represent these consumption or endowment processes. The (t, a_t)-contingent commodity, representing one unit of consumption in event $a_t \in f_t$ at date t, is represented by the indicator function $1(t, a_t): \mathcal{T} \times \Omega \to \mathbf{R}$ defined by[3]

$$1(t, a_t)(t', \omega') = \begin{cases} 1 & \text{if } t' = t \text{ and } \omega' \in a_t; \\ 0 & \text{otherwise.} \end{cases}$$

Using these contingent commodities as a basis, a consumption process $x_i \in L$ for consumer $i \in I$ has the representation

$$x_i = \sum_{t=0}^{T} \sum_{a_t \in f_t} x_i(t, a_t) 1(t, a_t)$$

[3] We write $1(t, a_t)$ rather than the more usual $1_{(t, a_t)}$ for typographical convenience.

and an endowment process the representation

$$w_i = \sum_{t=0}^{T} \sum_{a_t \in f_t} w_i(t, a_t) 1(t, a_t).$$

Similarly, consumption at date t can be written

$$x_i(t) = \sum_{a_t \in f_t} x_i(t, a_t) 1(t, a_t)$$

and endowment at date t as

$$w_i(t) = \sum_{a_t \in f_t} w_i(t, a_t) 1(t, a_t).$$

Letting

$$L_+ := \{ x \in L \mid x(t, a_t) \geq 0 \ \forall \ t \in \mathcal{T} \ \& \ a_t \in f_t \}$$

represent the nonnegative orthant of L, assume that each consumer $i \in I$ has consumption set $X_i = L_+$, endowment $w_i \in L_+$, and preference relation $\succeq_i$ on $X_i \times X_i$ which is a complete preordering and strongly monotonic. Walrasian prices are given by a linear functional $p: L \to \mathbf{R}$ with representation

$$p(x) = \pi \cdot x = \sum_{t \in \mathcal{T}} \sum_{a_t \in f_t} \pi(t, a_t) x(t, a_t)$$

where $\pi(t, a_t)$ is the price of a (t, a_t)-contingent commodity. Just as for consumption processes x, we can also view $\pi \in L_+$ as a stochastic process $\pi: \mathcal{T} \times \Omega \to \mathbf{R}$ adapted to the accident filtration. With respect to the Walrasian price system π, consumer i has budget set[4]

$$\beta_i(\pi) := \{ x_i \in X_i \mid \pi \cdot x_i \leq \pi \cdot w_i \}$$

and demand set

$$\phi_i(\pi) := \{ x_i \in X_i \mid \beta_i(\pi) \cap P_i(x_i) = \emptyset \}$$

where

$$P_i(x_i) := \{ x_i' \in X_i \mid x_i' \succ_i x_i \}$$

is the strict preference set of consumer i. An allocation $x: I \to L_+$ is **feasible** if

$$\sum_{i \in I} x_i = \sum_{i \in I} w_i.$$

A Walrasian equilibrium for this exchange economy consists of a feasible allocation x and a price system π such that $x_i \in \phi_i(\pi)$ for all $i \in I$.

[4]The notation adopted here is that of Ellickson [1993]

2.3 The pricing of insurance

The Walrasian equilibrium described above requires a large number of contingent commodities, one such commodity for each date-event (t, a_t). They are, of course, purely theoretical constructs with little resemblance to actual insurance contracts. However, since markets are complete when these ADR contingent contracts are present, any additional contracts which we choose to introduce are redundant assets and, consequently, they can be priced. Suppose, therefore, we introduce a collection of redundant assets which resemble actual insurance contracts, indexed by $j \in J$. Associated with each security $j \neq 0$ is a dividend process $d_j \in L_+$ where $d_j(t, a_t)$ represents the payout of insurance policy j at date-event (t, a_t), measured in units of the consumption good at (t, a_t). We assume that $d_j(0, \Omega) = 0$ for all $j \neq 0$. As for ADR prices and consumption processes, each dividend process d_j can be viewed as a stochastic process adapted to the accident filtration, reflecting the fact that payouts on an insurance contract must be based only on accidents which have already happened and not on those which are yet to come. In addition to these insurance policies, we also assume there exists a riskfree asset available at each date-event (t, a_t) which costs one unit of the consumption good at (t, a_t) and returns $d_0(t, a_{t+1}) = 1 + r(t, a_t)$ of the consumption good at each successor event $a_{t+1} \subset a_t$, $a_{t+1} \in f_{t+1}$. We will refer to $r(t, a_t)$ as the riskfree rate.

From now on, we adopt the price normalization[5] $\pi(0) = 1$ so that the Walrasian price functional has the representation

$$\pi \cdot x = x(0) + \sum_{t=1}^{T} \sum_{a_t \in f_t} \pi(t, a_t) x(t, a_t).$$

Given the ADR prices π, define for each event $a_t \in f_t$ and $a_{t+1} \in f_{t+1}$, where $a_{t+1} \subset a_t$, the **martingale conditional probability**

$$Q(a_{t+1} \mid a_t) := \frac{\pi(t+1, a_{t+1})}{\sum_{a'_{t+1} \subset a_t} \pi(t+1, a'_{t+1})}. \tag{1}$$

[5]Because the initial and terminal information partitions have a special structure,

$$f_0 = \{\Omega\} \quad \text{and} \quad f_T = \{\{\omega\} \mid \omega \in \Omega\},$$

it is convenient to abuse notation slightly, writing $\pi(0)$ and $x(0)$ in place of $\pi(0, \Omega)$ or $x(0, \Omega)$ and $\pi(T, \omega)$ or $x(T, \omega)$ in place of $\pi(T, \{\omega\})$ or $x(T, \{\omega\})$

Assuming that prices of all insurance assets are ex dividend, we define for each $t < T$ the price process S_j for asset j according to the relation

$$\pi(t, a_t)S_j(t, a_t) = \sum_{s=t+1}^{T} \sum_{\substack{a_s \in f_s \\ a_s \subset a_t}} \pi(s, a_s) d_j(s, a_s),$$

the Arrow-Debreu valuation of the dividend stream following the date-event (t, a_t). At terminal nodes, $S_j(T, \omega) = 0$. Note that $S_j \in L_+$ so that security prices can also be viewed as nonnegative stochastic processes adapted to the accident filtration. As with payouts, the security price $S_j(t, a_t)$ is measured in units of the (t, a_t) - consumption good.

As a simple consequence of the tree structure of the filtration,

$$\pi(t, a_t)S_j(t, a_t) = \sum_{a_{t+1} \subset a_t} \pi(t+1, a_{t+1}) \left[S_j(t+1, a_{t+1}) + d_j(t+1, a_{t+1}) \right]. \tag{2}$$

where the sum is over events a_{t+1} belonging to the partition f_{t+1} and contained in a_t. In the special case of the riskfree asset, which costs one unit of the consumption good at (t, a_t) and pays $1 + r(t, a_t)$ at each of the immediate successor nodes, the above condition specializes to

$$\pi(t, a_t) = \sum_{a_{t+1} \subset a_t} \pi(t+1, a_{t+1})(1 + r(t, a_t))$$

or, equivalently,

$$1 + r(t, a_t) = \frac{\pi(t, a_t)}{\sum_{a_{t+1} \subset a_t} \pi(t+1, a_{t+1})} \tag{3}$$

for all $t < T$. Using the definitions of the martingale conditional probability and the riskfree rate, equation (2) can now be written

$$S_j(t, a_t) = \frac{\sum_{a_{t+1} \subset a_t} \left[S_j(t+1, a_{t+1}) + d_j(t+1, a_{t+1}) \right] Q(a_{t+1} \mid a_t)}{1 + r(t, a_t)}.$$

Letting

$$r(t) := \sum_{a_t \in f_t} r(t, a_t) 1(t, a_t)$$

denote the interest rate at date t and

$$S_j(t) := \sum_{a_t \in f_t} S_j(t, a_t) 1(t, a_t),$$

the price of the j^{th} asset at date t, we have

$$S_j(t) = \frac{E_Q[S_j(t+1) + d_j(t+1) \mid \mathcal{F}_t]}{1 + r(t)} \tag{4}$$

where $E_Q[\ \cdot \mid \mathcal{F}_t]$ denotes conditional expectation relative to the sigma-algebra $\mathcal{F}_t$ under the martingale measure Q.

Using the riskfree rate to discount insurance asset prices and their payouts, define

$$S_j^*(t) := \frac{S_j(t)}{\prod_{s=0}^{t-1}(1 + r(s))}$$

and

$$d_j^*(t) := \frac{d_j(t)}{\prod_{s=0}^{t-1}(1 + r(s))}.$$

Define the cumulative discounted dividend process for security j as

$$D_j^*(t) := \sum_{s=0}^{t} d_j^*(t).$$

THEOREM 1: For each insurance contract $j \in J$, $S_j^* + D_j^*$ is a martingale with respect to the measure Q.

PROOF: From equation (4) it follows that

$$\begin{aligned} S_j^*(t) + D_j^*(t) &= \frac{E_Q[S_j(t+1) + d_j(t+1) \mid \mathcal{F}_t]}{\prod_{s=0}^{t}(1 + r(s))} + D_j^*(t) \\ &= E_Q[S_j^*(t+1) + D_j^*(t) + d_j^*(t+1) \mid \mathcal{F}_t] \\ &= E_Q[S_j^*(t+1) + D_j^*(t+1) \mid \mathcal{F}_t]. \end{aligned} \tag{5}$$

■

The rate of return of the j^{th} insurance contract at date t + 1 can be decomposed into a predictable and an innovation component,

$$\frac{S_j(t+1) + d_j(t+1)}{S_j(t)} - 1 = \mu_j(t+1) + \nu_j(t+1) - \nu_j(t)$$

where

$$\mu_j(t+1) := E_P\left[\frac{S_j(t+1) + d_j(t+1)}{S_j(t)} \mid \mathcal{F}_t\right] - 1$$

is the predictable component,

$$\nu_j(t+1) := \nu_j(t) + \frac{S_j(t+1)+d_j(t+1)}{S_j(t)} - E_P\left[\frac{S_j(t+1)+d_j(t+1)}{S_j(t)} \mid \mathcal{F}_t\right]$$

is the innovation component and $\nu_j(0)$ is an arbitrary constant. Define the likelihood ratio process z by

$$z(t) := E_P\left[\frac{Q}{P} \mid \mathcal{F}_t\right]$$

for all $t \geq 1$. The following theorem establishes z as an aggregate risk factor for the economy, expressing the excess return of the j^{th} insurance contract in terms of the conditional covariances between the innovation component of the return and the aggregate risk factor.[6]

THEOREM 2: For each insurance contract $j \in J$ and each $t \geq 1$,

$$\mu_j(t+1) - r(t) = -\frac{1}{z(t)}\text{covar}_\text{P}[\nu_j(t+1), z(t+1)] \tag{6}$$

PROOF: From equation (4),

$$\begin{aligned}
S_j(t) &= E_Q\left[\frac{S_j(t+1)+d_j(t+1)}{1+r(t)} \mid \mathcal{F}_t\right] \\
&= \frac{1}{z(t)}E_P\left[\left(\frac{S_j(t+1)+d_j(t+1)}{1+r(t)}\right)z(t+1) \mid \mathcal{F}_t\right] \\
&= \frac{S_j(t)}{z(t)(1+r(t))}E_P\left[\left(\frac{S_j(t+1)+d_j(t+1)}{S_j(t)}\right)z(t+1) \mid \mathcal{F}_t\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
1+r(t) &= \frac{1}{z(t)}E_P\left[\left(\frac{S_j(t+1)+d_j(t+1)}{S_j(t)}\right)z(t+1) \mid \mathcal{F}_t\right] \\
&= \frac{1}{z(t)}E_P[(1+\mu_j(t+1)+\nu_j(t+1)-\nu_j(t))z(t+1) \mid \mathcal{F}_t] \\
&= 1+\frac{1}{z(t)}E_P[(\mu_j(t+1)+\nu_j(t+1)-\nu_j(t))z(t+1) \mid \mathcal{F}_t]
\end{aligned}$$

[6]We follow closely the discussion leading up to Theorem 6.4 of Dothan [1990].

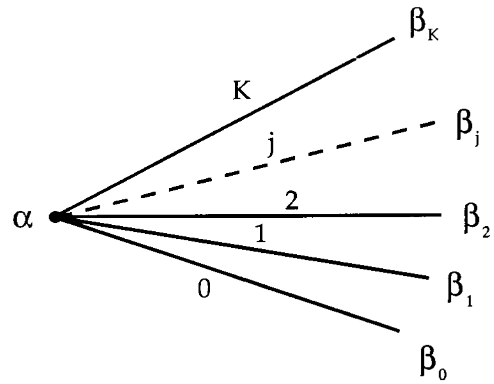

Figure 3: Event α and its successors

where in the last step we use the fact that z is a P-martingale. Rearranging and using the fact that both z and ν_j are P-martingales,

$$\begin{aligned} \mu_j(t+1) - r(t) &= -\frac{1}{z(t)} E_P[(\nu_j(t+1) - \nu_j(t))(z(t+1) - z(t)) \mid \mathcal{F}_t] \\ &= -\frac{1}{z(t)} \mathrm{covar_P}[\nu_j(t+1), z(t+1)] \end{aligned}$$

■

2.4 Dynamic spanning

In the preceding section, insurance contracts were introduced as assets which are redundant in the presence of the ADR contingent contracts. We claim that, as a consequence of dynamic spanning in the sense of Kreps [1982], the ADR contingent contracts are dispensable: all that is required, at least generically, are the insurance contracts we have introduced.

Although the machinery is rather elaborate, the basic idea behind dynamic spanning is quite simple. In an event tree context, the key to dynamic spanning is the index of the filtration, the maximum number of branches leaving any node of the event tree: the number of accident types plus one in our model. As shown by Kreps [1982], the number of securities required for dynamic spanning is no greater than the index of the filtration. What this means for us is that, in addition to the riskfree asset, all that is required for dynamic spanning is one insurance contract covering each type of accident.

To illustrate, assume that $\mathcal{J} = \mathcal{K}$ where security $j = k \geq 1$ pays one unit of the consumption good at date t if and only if an accident of type k

occurs at that date.[7] Let α represent an arbitrary event $a_{t-1} \in f_{t-1}$ and

$$B_t(\alpha) = \{ a_t \subset a_{t-1} \mid a_t \in f_t \} = \{ \beta_0, \beta_1, \ldots, \beta_K \}$$

the collection of its immediate successors (see Figure 3). Let

$$\theta_i(t, \alpha) = (\theta_i^0(t, \alpha), \theta_i^1(t, \alpha), \ldots, \theta_i^K(t, \alpha))^T$$

represent the port folio of securities purchased by consumer i at date-event $(t-1, \alpha)$ and

$$\theta_i(t+1, \beta) = (\theta_i^0(t+1, \beta), \theta_i^1(t+1, \beta), \ldots, \theta_i^K(t+1, \beta))^T$$

the portfolio acquired at date-event (t, β), $\beta \in B_t(\alpha)$ where T denotes transpose. Note that, by definition, the trading process θ_i is not only adapted to the filtration, but also predictable: i.e., for each t, $\theta_i(t)$ is measurable with respect to $\mathcal{F}_{t-1}$. Requiring predictability captures the economically natural restriction that a consumer must buy insurance prior to acquiring knowledge whether the insured event will occur. Finally, let

$$\Delta_w x_i(t, \beta) := x_i(t, \beta) - w_i(t, \beta)$$

represent the net trade of consumer i at date-event (t, β). For each $\beta \in B_t(\alpha)$, budget balance in the spot market at date-event (t, β) requires

$$\Delta_w x_i(t, \beta) = \sum_{j \in J} \theta_i^j(t, \alpha)[S_j(t, \beta) + d_j(t, \beta)] - \sum_{j \in J} \theta_i^j(t+1, \beta) S_j(t, \beta).$$

which, letting

$$c_i(t, \alpha, \beta) := \Delta_w x_i(t, \beta) + \sum_{j \in J} \theta_i^j(t+1, \beta) S_j(t, \beta),$$

can be written

$$\sum_{j \in J} \theta_i^j(t, \alpha)[S_j(t, \beta) + d_j(t, \beta)] = c_i(t, \alpha, \beta) \tag{7}$$

For each event α, there are $K+1$ such equations, one for each of the successor events $\beta \in B_t(\alpha)$, and $K+1$ unknowns, the components of the portfolio

$$\theta_i(t, \alpha) = (\theta_i^0(t, \alpha), \theta_i^1(t, \alpha), \ldots, \theta_i^K(t, \alpha)).$$

[7] In the terminology of Kreps [1982], these insurance contracts are long-lived securities.

For each date-event (t, α), define the $K+1$ by $K+1$ matrices

$$S(t,\alpha) = \begin{pmatrix} S_0(t,\beta_0) & S_1(t,\beta_0) & \dots & S_K(t,\beta_0) \\ S_0(t,\beta_1) & S_1(t,\beta_1) & \dots & S_K(t,\beta_1) \\ \vdots & \vdots & \ddots & \vdots \\ S_0(t,\beta_K) & S_1(t,\beta_K) & \dots & S_K(t,\beta_K) \end{pmatrix}$$

and

$$D(t,\alpha) = \begin{pmatrix} d_0(t,\beta_0) & d_1(t,\beta_0) & \dots & d_K(t,\beta_0) \\ d_0(t,\beta_1) & d_1(t,\beta_1) & \dots & d_K(t,\beta_1) \\ \vdots & \vdots & \ddots & \vdots \\ d_0(t,\beta_K) & d_1(t,\beta_K) & \dots & d_K(t,\beta_K) \end{pmatrix}.$$

The system of equations of type (7) at date-event (t, α) then takes the form

$$[S(t,\alpha) + D(t,\alpha)]\theta_i(t,\alpha) = c_i(t,\alpha) \tag{8}$$

where

$$c_i(t,\alpha) = (c_i(t,\alpha,\beta_0), c_i(t,\alpha,\beta_1), \dots, c_i(t,\alpha,\beta_K))^T.$$

An ADR equilibrium allocation is said to be dynamically spanned by the set of securities J provided there is a portfolio $\theta_i(t, \alpha)$ which solves equation (8) for every date-event (t, α) and every consumer $i \in I$. If T is finite and the matrix $S(t,\alpha) + D(t,\alpha)$ is invertible for every date-event (t, α), then finding a set of dynamically spanning portfolio trades is straightforward. Using equation (8), we solve first for the portfolios purchased at date $T-1$ and then work back recursively to the initial holdings at date 0. Assuming that securities are priced ex dividend, for each security j we have $S_j(T, \omega) = 0$ for all $\omega \in \Omega$ Consequently, for $t = T$ and $\alpha \in f_{T-1}$, equation (8) reduces to

$$D(T,\alpha)\theta_i(T,\alpha) = c_i(T,\alpha).$$

Since consumers will carry no portfolio holdings beyond date T, $c_i(T, \alpha, \beta)$ is simply the net trade $\Delta_w x_i(T, \beta)$ at the successor node β. Therefore, provided the dividend matrix $D(T, \alpha)$ is invertible, the equation can be solved for a unique portfolio $\theta_i(T, \alpha)$ for each $\alpha \in f_{T-1}$ and each consumer $i \in I$.

Moving back a date, consider an event $\alpha \in f_{T-2}$. From the computations at date T, the required portfolios $\theta_i(T, \beta)$ are known for each successor event $\beta \in B_t(\alpha)$ and consequently $c_i(T-1, \alpha)$ can be computed. Therefore, provided that the matrix $S(T-1, \alpha) + D(T-1, \alpha)$ is invertible, equation (8) can be solved for a unique portfolio $\theta_i(T-1, \alpha)$ for each $\alpha \in f_{T-2}$.

Continuing recursively in this fashion, portfolios $\theta_i(t, \alpha)$ can be computed for each consumer $i \in I$ and date-event (t, α) until finally we reach $t = 0$. Since the initial partition is trivial, $f_0 = \{\Omega\}$, we can write $\theta_i(1, \Omega) = \theta_i(1)$. Assuming that consumers have no initial endowments of securities, $\theta_i(0) = 0$. Equation (8) then reduces to the requirement $c_i(0, \alpha) = 0$ or

$$\Delta_w x_i(0) + \sum_{j \in J} \theta_i^j(1) S_j(0) = 0 \tag{9}$$

which requires the initial purchase of assets at date 0 to offset the net trade at date 0.

Of course, for this procedure to work the matrix $S(t, \alpha) + D(t, \alpha)$ must be invertible at each step which, according to Kreps [1982], will hold generically in models of this sort. Since we have nothing to add to that discussion, we instead use an explicit example to illustrate how the argument applies within the context of insurance markets.

3 An example

This section illustrates our model of intertemporal insurance with a simple example close in spirit to traditional models of insurance markets: consumers face hazard rates which are independent of the past history of the accident process and constant over time. We begin by deriving the Arrow-Debreu-Radner prices and the equilibrium net trades in contingent commodities. Within this setting two insurance regimes are then considered, one providing short-term insurance on next period's events and the other providing long-term contracts paying a unit of the consumption good every time an accident occurs in the future.

In our model an accident can be any event — a flood, a car wreck, or an earthquake for example — which affects either the endowment or the preferences of a consumer. However, as Malinvaud [1972, 1973] and others have emphasized, what seems to set insurance markets apart from other markets for risk is that insurance typically deals with individual risk. Using the terminology of general equilibrium theory, what this means is that contingent commodities or securities which deal with such risks must be personalized. Fortunately, this is easily handled within our model.

To demonstrate the marketing of insurance contracts tailored to individual risk, consider an example in which the identity of the insured is the only characteristic distinquishing one accident from another: $\mathcal{K} = I \cup \{0\}$ where

$k \geq 1$ represents an accident happening to individual k and $k = 0$ an accident happening to no one.[8] Assume that consumer $i \in I$ has von-Neumann Morgenstern utility

$$u_i(x_i) = \ln x_i(0) + \sum_{t=1}^{T} \delta^t \sum_{a_t \in f_t} P(a_t) \ln x_i(t, a_t) \tag{10}$$

where $\delta \in [0, 1)$ and $P(a_t)$ is the probability of event $a_t \in f_t$. Let

$$w(t, a_t) := \sum_{i \in I} w_i(t, a_t)$$

denote the aggregate endowment of the (t, a_t) - contingent commodity and

$$w(0) := \sum_{i \in I} w_i(0)$$

the aggregate endowment at date 0. Using the normalization $\pi(0) = 1$, ADR market clearing prices are given by

$$\pi(t, a_t) = \frac{\delta^t w(0) P(a_t)}{w(t, a_t)}. \tag{11}$$

For any $a_t \in f_t$ and $a_{t+1} \in f_{t+1}$ where $a_{t+1} \subset a_t$, assume that the conditional probability

$$P(a_{t+1} \mid a_t) = \begin{cases} \lambda_i & \text{if } \Delta N_i(t+1, a_{t+1}) = 1 \text{ for } i \in I \text{, and} \\ \lambda_0 & \text{if } \Delta N_i(t+1, a_{t+1}) = 0 \text{ for all } i \in I. \end{cases}$$

To simplify computations, assume that

$$w_i(t, a_t) = \begin{cases} 0 & \text{if } \Delta N_i(t, a_t) = 1 \\ Y & \text{otherwise} \end{cases}$$

where $Y > 0$ is the same for all consumers. Aggregate endowment is then

$$w(t, a_t) = \begin{cases} (n-1)Y & \text{if } \Delta N_i(t, a_t) = 1 \text{ for some } i \in I; \\ nY & \text{otherwise.} \end{cases}$$

[8] At each date t an accident can happen to anyone, but — as a consequence of our point process assumptions — an accident never happens to more than one person at any given date. Our framework allows for accidents which affect many consumers at the same time, as with a flood or an earthquake, but only when explicitly coded into the set of accident types $\mathcal{K}$.

Thus, the hazard rate for an accident happening to consumer i remains constant over time and, when an accident happens to i, it destroys her entire endowment at the date on which the accident occurs.

Under these assumptions, equation (1) yields the following expressions for the martingale conditional probabilities:

(i) (No accident.) If $\Delta N_i(t+1, a_{t+1}) = 0$ for all $i \in I$, then

$$Q(a_{t+1} \mid a_t) = \frac{\lambda_0}{\lambda_0 + \frac{n}{n-1}(1-\lambda_0)}.$$

(ii) (Accident to consumer i.) If $\Delta N_i(t+1, a_{t+1}) = 1$, then

$$Q(a_{t+1} \mid a_t) = \frac{\lambda_i}{\left(\frac{n}{n-1}\right)\lambda_0 + 1 - \lambda_0}.$$

Thus, provided that the number of consumers is greater than one, risk adjustment lowers the "risk neutral" probability of no accident and increases the "risk neutral" probability of each accident.

The riskfree rate $r(t)$ varies depending on whether an accident has occurred at date t or not. Evaluating equation (3):

(i) (No accident at t.) If $\Delta N_i(t, a_t) = 0$ for all $i \in I$, then

$$1 + r(t, a_t) = \frac{1}{\delta\left[\lambda_0 + \frac{n}{n-1}(1-\lambda_0)\right]}.$$

(ii) (Accident at t.) If $\Delta N_i(t, a_t) = 1$ for some $i \in I$, then

$$1 + r(t, a_t) = \frac{1}{\delta\left[\left(\frac{n}{n-1}\right)\lambda_0 + 1 - \lambda_0\right]}$$

From equation (11), ADR prices are

$$\pi(t, a_t) = \begin{cases} \delta^t P(a_t) & \text{if } \Delta N_i(t, a_t) = 0 \text{ for all } i \in I; \\ \left(\frac{n}{n-1}\right)\delta^t P(a_t) & \text{if } \Delta N_i(t, a_t) = 1 \text{ for some } i \in I. \end{cases}$$

Consequently, equilibrium wealth for consumer $i \in I$ is

$$
\begin{aligned}
\pi \cdot w_i &= w_i(0) + \sum_{t=1}^{T} \sum_{a_t \in f_t} \pi(t, a_t) w_i(t, a_t) \\
&= Y + nY \sum_{t=1}^{T} \delta^t \sum_{a_{t-1} \in f_{t-1}} P(a_{t-1}) \sum_{a_t \subset a_{t-1}} P(a_t \mid a_{t-1}) \frac{w_i(t, a_t)}{w(t, a_t)} \\
&= Y + Y \left(\frac{\delta(1-\delta^T)}{1-\delta} \right) \left[\lambda_0 + \frac{n}{n-1}(1 - \lambda_0 - \lambda_i) \right].
\end{aligned}
$$

From now on, suppose the hazard rate of no accident is $\lambda_0 = 1/4$, the number of consumers is even, and the population is split into two equal size groups: $I = I_1 \cup I_2$ with

- $\lambda_i = 1/n$ for the **high risk** consumers $i \in I_1$; and
- $\lambda_i = 1/2n$ for the **low risk** consumers $i \in I_2$.

If $\Delta N_i(t+1, a_{t+1}) = 0$ for all $i \in I$ (the no accident case), then

$$Q(a_{t+1} \mid a_t) = \frac{n-1}{4n-1}.$$

If $\Delta N_i(t, a_t) = 1$ for some $i \in I_1$ (an accident happens to a high risk consumer), then

$$Q(a_{t+1} \mid a_t) = \frac{4}{4n-1}.$$

And, finally, if $\Delta N_i(t, a_t) = 1$ for some $i \in I_2$ (an accident happens to a low risk consumer), then

$$Q(a_{t+1} \mid a_t) = \frac{2}{4n-1}.$$

As n, the number of consumers, approaches infinity, the martingale probability of no accident approaches one-fourth and the martingale probabilities of an accident to either a high or a low risk consumer approach zero. In each of these cases,

$$\lim_{n \to \infty} \frac{Q(a_{t+1} \mid a_t)}{P(a_{t+1} \mid a_t)} = 1$$

so that, consistent with the treatment in Malinvaud [1972, 1973], with a large number of consumers there is little difference between actuarial and martingale insurance pricing in this setting.

Table 1: Equilibrium trades and net trades

$k(t)$	$x_1(t,a_t)$	$x_2(t,a_t)$	$\Delta_w x_1(t,a_t)$	$\Delta_w x_2(t,a_t)$
0	380	400	-10	10
i	342	360	342	360
i'	342	360	-48	-30

We now specialize even further by assuming that $n = 10$, $\delta = 12/13$, Y = 390, and $T = cm$. The riskfree rate becomes

$$r(t,a_t) = \begin{cases} 0 & \text{if there is no accident at date } t\text{; and} \\ 1/9 & \text{if there has been an accident at date } t. \end{cases}$$

The martingale conditional probabilities are

$$Q(a_{t+1} \mid a_t) = \begin{cases} 3/13 & \text{if there is no accident at date } t+1; \\ 4/39 & \text{if an accident happens to } i \in I_1 \text{ at } t+1; \\ 2/39 & \text{if an accident happens to } i \in I_2 \text{ at } t+1. \end{cases}$$

In equilibrium, high risk consumers $i \in I_1$ have wealth $p \cdot w_i = 4,940$ while low risk consumers $i \in I_2$, facing half the risk, have wealth $p \cdot w_i = 5,200$. Consumer $i \in I$ receives

$$x_i(t,a_t) = \frac{(1-\delta)\delta^t P(a_t)\, p \cdot w_i}{\pi(t,a_t)}$$

as her equilibrium allocation of the (t, a_t) - contingent commodity. Table 1 shows the equilibrium trades and net trades of the (t, a_t) - contingent commodity for a high-risk consumer (consumer 1) and a low-risk consumer (consumer 2) under three conditions: (a) there is no accident at date t (the accident type or "mark" $k(t) = 0$), (b) there is an accident to consumer i ($k(t) = i$), and (c) there is an accident to some consumer $i' \neq 1$ ($k(t) = i'$). Note that gross trades depend only on the "macro risk" in the economy, i.e., whether the total endowment is nY or $(n-1)Y$, while net trades also depend on who has the accident.

Although we allow at most one accident per "nanosecond," Malinvaud's concern with the impact of individualized risk on the required number of Arrow-Debreu contingent commodities manifests itself dramatically in this intertemporal context: each node of the event tree is followed by 11 branches so that there are 11 contingent commodities at date 1, 11^2 at date 2, 11^3 at date 3 and so forth. We now consider how these Arrow-Debreu equilibria are implemented under two insurance regimes, one offering short-term insurance to consumers and the other offering long-term contracts. Either regime cuts the number of securities needed for spanning dramatically: 10 short-term insurance contracts per date with the first regime or simply 10 insurance cotracts of infinite duration under the second regime, plus a riskfree asset.

3.1 Short-term insurance

We know that to achieve dynamic spanning, it is necessary to offer a separate insurance contract for each type of accident. In the first regime we consider, insurance contracts are short-term: one unit of insurance issued at date t on accidents of type k at date $t+1$ returns a payout at $t+1$ of

$$d_k(t+1, a_{t+1}) = \begin{cases} 1 & \text{if an accident of type } k \text{ occurs at date } t+1; \\ 0 & \text{otherwise.} \end{cases}$$

Because security prices are ex dividend and this contract is short-term, it is worthless at date $t+1$: $S_k(t+1) = 0$. According to the fundamental equation (4) of asset pricing, the price of this asset depends both on the riskfree rate and on who is being insured. For $a_f \in f_t$, let $\beta_k \in f_{t+1}$ be the immediate successor event in which an accident of type k occurs. For a high-risk consumer $i \in I_1$,

$$\begin{aligned} S_i(t, a_t) &= \frac{\sum_{k=0}^{10} \lambda_k d_i(t+1, \beta_k)}{1 + r(t, a_t)} = \frac{4/39}{1 + r(t, a_t)} \\ &= \begin{cases} 4/39 & \text{if there were no accidents at date } t; \\ 6/65 & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, for a low-risk consumer $i \in I_2$,

$$S_i(t, a_t) = \begin{cases} 2/39 & \text{if there were no accidents at date } t; \\ 3/65 & \text{otherwise} \end{cases}$$

As the proof of Theorem 2 indicates, the CAPM equation (6) can be written in the form

$$\mu_j(t+1) - r(t) = -E_P[(\nu_j(t+1) - \nu_j(t))(z(t+1) - z(t))/z(t) \mid \mathcal{F}_t].$$

Table 2 presents the relevant data for an insurance contract on a high-risk consumer, the first three columns under the assumption there was no

Table 2: Short-term insurance for a high-risk consumer

	No accident at t			Accident at t		
$k(t+1)$	0	i	i'	0	i	i'
$\mu_i(t+1)$	-1/40	-1/40	-1/40	1/12	1/12	1/12
$\mu_i(t+1) - r(t)$	-1/40	-1/40	-1/40	-1/36	-1/36	-1/36
$(z(t+1) - z(t))/z(t)$	12/13	40/39	40/39	12/13	40/39	40/39
$\nu_i(t+1) - \nu_i(t)$	-39/40	351/40	-39/40	-13/12	317/12	-13/12
$P(a_{t+1} \mid a_t)$	1/4	1/10	13/20	1/4	1/10	13/20

accident at date t and the last three assuming there was an accident at date t. Each of these two groups is in turn divided into three columns corresponding to what happens at date $t+1$: (a) no accident ($k(t+1) = 0$), (b) an accident to the insured ($k(t+1) = i$) and (c) an accident to someone other than the insured ($k(t+1) = i'$).[9] If there is no accident at date t, the predictable return on an insurance contract for a high-risk consumer exceeds the riskless rate by – 1/40. Using the information provided in the last three rows, the right hand side of equation (6) evaluates to

$$\frac{1}{4}\left(-\frac{39}{40}\right)\left(\frac{12}{13}\right) + \frac{1}{10}\left(\frac{351}{40}\right)\left(\frac{40}{39}\right) + \frac{13}{20}\left(-\frac{39}{40}\right)\left(\frac{40}{39}\right) = -\frac{1}{40}$$

as claimed. A similar computation using the final three columns shows that equation (6) also holds when there is an accident at date t.

Table 3 presents the corresponding data for a low-risk consumer. It is straightforward once again to verify that the CAPM relation (6) holds for these contracts as well. Note that the only difference between these tables is in the rows reporting the increment to the innovation component of the rate of return, $\nu_i(t+1) - \nu_i(t)$, and the conditional probabilities $P(a_{t+1} \mid a_t)$ of the events. In particular, the predictable rate of return is the same for both risk classes.

Turning now to the issue of dynamic spanning, equation (7) must be modified slightly to account for the fact that the insurance securities we

[9] There are eleven distinct successor events at date t + 1, but all fall into one of these three types.

Table 3: Short-term insurance for a low-risk consumer

	No accident at t			Accident at t		
$k(t+1)$	0	i	i'	0	i	i'
$\mu_i(t+1)$	-1/40	-1/40	-1/40	1/12	1/12	1/12
$\mu_i(t+1) - r(t)$	-1/40	-1/40	-1/40	-1/36	-1/36	-1/36
$(z(t+1) - z(t))/z(t)$	12/13	40/39	40/39	12/13	40/39	40/39
$\nu_i(t+1) - \nu_i(t)$	-39/40	741/40	-39/40	-13/12	247/12	-13/12
$P(a_{t+1} \mid a_t)$	1/4	1/20	7/10	1/4	1/20	7/10

are considering are short-term. At date t there are two securities of type $k = i$ in existence: the "old" contracts issued at $t-1$ and paying off at t and the "new" contracts issued at t and paying off at $t+1$. Since contracts issued yesterday are worthless today (i.e., their ex dividend price is zero), we reserve $S_k(t, \alpha)$ to represent the price of contracts issued at date t and history $\alpha \in f_t$ on an accident of type k. With this modification, equation (7) becomes

$$\sum_{j=0}^{10} \theta_i^j(t, \alpha) d_j(t, \beta_k) = \Delta_w x_i(t, \beta_k) + \sum_{j=0}^{10} \theta_i^j(t+1, \beta_k) S_j(t, \beta_k) \qquad (12)$$

for each successor event $\beta_k \in B_t(\alpha)$.

Because we have assumed an infinite horizon in this example, we cannot solve for the trading portfolios by backward recursion. However, we might suspect that the portfolios are stationary, and that turns out to be the case. In equilibrium, neither the high-risk nor the low-risk consumer holds any of the riskfree asset. Each consumer buys $\theta_i^i(t, a_t) = 351$ units of insurance on accidents to herself and sells $\theta_i^k(t, a_t) = -39$ units of insurance to each consumer $k \neq i$. It is easy to verify that equation (12) is satisfied with this portfolio for each $\alpha \in f_t$ and $\beta_k \in B(\alpha)$ for all $t \geq 0$.

3.2 Long-term insurance

The short-term insurance contracts of the preceding section are, of course, simply Radner's variation on Arrow-Debreu contingent commodities: i.e., contingent commodities traded at dates $t > 0$ rather than at the beginning of time. Although they do reduce dramatically the number of contingent contracts or securities required for spanning, in the present context they seem only slightly more realistic than their ADR counterparts: issuing a new insurance contract every nanosecond puts a heavy burden on our tacit assumption of no transactions costs! Taking a step closer to reality, we now consider long-term insurance contracts. Specifically, an insurance policy on an accident of type k is a long-term obligation which, at a price $S_k(t, a_t)$ at date-event (t, a_t), returns one unit of the consumption good at each subsequent date-event at which an accident of type k occurs.

For the sake of symmetry, we also replace the riskfree asset with a bond which pays one unit of the consumption good at every date-event (t, a_t) from time one forward. For $a_t \in f_t$ let $\beta_k \in f_{t+1}$ be the immediate successor event in which an accident of type k occurs. From equation (4), the price of the bond at (t, a_t) is

$$\begin{aligned} S_0(t, a_t) &= \frac{\sum_{k=0}^{10} \lambda_k [S_0(t+1, \beta_k) + d_0(t+1, \beta_k)]}{1 + r(t, a_t)} \\ &= \frac{1 + \sum_{k=0}^{10} \lambda_k S_0(t+1, \beta_k)}{1 + r(t, a_t)}. \end{aligned} \tag{13}$$

Of the eleven successor events which could occur at $t + 1$, only two are distinct macro states in this simple economy: either an accident occurs or is does not. Let α^*, α^0 denote events in f_t in which an accident does or does not occur respectively, and let $\beta^*, \beta^0 \in f_t$ denote the corresponding events in f_{t+1}. Applied to all $a_t \in f_t$, equation (13) reduces to two distinct equations,

$$S_0(t, \alpha^0) = \frac{1 + (3/13) S_0(t+1, \beta^0) + (10/13) S_0(t+1, \beta^*)}{1 + r(t, \alpha^0)}$$

and

$$S_0(t, \alpha^*) = \frac{1 + (3/13) S_0(t+1, \beta^0) + (10/13) S_0(t+1, \beta^*)}{1 + r(t, \alpha^*)}$$

Assuming bond prices are stationary, so that

$$S_0(t, \alpha^0) = S_0(t, \beta^0) \quad \text{and} \quad S_0(t, \alpha^*) = S_0(t, \beta^*),$$

we conclude that

$$S_0(t, a_t) = \begin{cases} 13 & \text{if there is no accident at date } t; \\ 117/10 & \text{if there is an accident at date } t. \end{cases}$$

Table 4: Long-term insurance for a high-risk consumer

	No accident at t			Accident at t		
$k(t+1)$	0	i	i'	0	i	i'
$\mu_i(t+1)$	0	0	0	1/9	1/9	1/9
$\mu_i(t+1)-r(t)$	0	0	0	0	0	0
$(z(t+1)-z(t))/z(t)$	12/13	40/39	40/39	12/13	40/39	40/39
$\nu_i(t+1)-\nu_i(t)$	0	13/12	-1/10	0	13/18	-1/9
$P(a_{t+1} \mid a_t)$	1/4	1/10	13/20	1/4	1/10	13/20

The prices of the insurance contracts are found in essentially the same way. Equation (4) implies that for insurance on a high-risk consumer $i \in I_1$,

$$\begin{aligned} S_i(t,a_t) &= \frac{\sum_{k=0}^{10} \lambda_k [S_i(t+1,\beta_k) + d_i(t+1,\beta_k)]}{1+r(t,a_t)} \\ &= \frac{\lambda_i + \sum_{k=0}^{10} \lambda_k S_i(t+1,\beta_k)}{1+r(t,a_t)}. \end{aligned}$$

Exploiting the fact there are only two distinct macro states at any date and the hypothesis that the insurance contract price should be stationary, we conclude that for insurance on a high-risk consumer $i \in I_1$,

$$S_i(t,a_t) = \begin{cases} 4/3 & \text{if there is no accident at date } t; \\ 6/5 & \text{if there is an accident at date } t \end{cases}.$$

In the same way, for a low-risk consumer $i \in I_2$ we obtain

$$S_i(t,a_t) = \begin{cases} 2/3 & \text{if there is no accident at date } t; \\ 3/5 & \text{if there is an accident at date } t. \end{cases}$$

The CAPM equation (6) applies as well to these long-term insurance contracts. Table 4 presents the relevant data for an insurance contract on a high-risk consumer, organized just as in Tables 2 or 3: the first three columns assume there was no accident at date t while the last three assume there was an accident at date t, and each of these two groups is in turn divided into three columns corresponding to what happens at date $t+1$.

Table 5 gives the corresponding data for a low-risk consumer. Imitating our earlier calculation for the short-term insurance contracts, it is easy to

Table 5: Long-term insurance for a low-risk consumer

	No accident at t			Accident at t		
$k(t+1)$	0	i	i'	0	i	i'
$\mu_i(t+1)$	0	0	0	1/9	1/9	1/9
$\mu_i(t+1) - r(t)$	0	0	0	0	0	0
$(z(t+1) - z(t))/z(t)$	12/13	40/39	40/39	12/13	40/39	40/39
$\nu_i(t+1) - \nu_i(t)$	0	7/5	-1/10	0	14/9	-1/9
$P(a_{t+1} \mid a_t)$	1/4	1/20	7/10	1/4	1/20	7/10

verify that the equation (6) holds for these long-term contracts as well. Once again the predictable rates of return are the same for both risk classes.

Turning finally to the issue of dynamic spanning, equation (7) becomes

$$\sum_{j=0}^{10} \theta_i^j(t,\alpha)[S_j(t,\beta_k) + d_j(t,\beta_k)] = \Delta_w x_i(t,\beta_k) + \sum_{j=0}^{10} \theta_i^j(t+1,\beta_k) S_j(t,\beta_k) \tag{15}$$

for each successor event $\beta_k \in B_t(\alpha)$. Once again the hypothesis that stationary portfolios can achieve dynamic spanning turns out to be correct, but now, in contrast to the short-term insurance regime, low- and high-risk consumers hold somewhat different portfolios. Each high-risk consumer $i \in I_1$ sells $\theta_i^0(t,\alpha_t) = -10$ units of the riskfree asset, buys $\theta_i^i(t,a_t) = 352$ units of insurance on accidents to herself and sells $\theta_i^j(t,a_t) = -38$ units of insurance to each consumer $j \neq i$. Each low-risk consumer $i \in I_2$ buys $\theta_i^0(t,\alpha_t) = 10$ units of the riskfree asset, buys $\theta_i^i(t,a_t) = 350$ units of insurance on accidents to herself and sells $\theta_i^j(t,a_t) = -40$ units of insurance to each consumer $j \neq i$. It is easy to verify that these portfolios satisfy equation (15) for each $\alpha \in f_t$ and $\beta_k \in B(\alpha)$ for all $t \geq 0$ and that the security markets clear.

Our example follows Malinvaud in assuming that an accident never affects more than one individual. What happens if, as in the case of an earthquake or a flood, many consumers are affected simultaneously? Malinvaud's framework leaves no scope for such a possibility, while the finance-based approach to insurance offered here covers such cases with no change in the theory. Suppose, for example, there are n consumers but only two types of accident so that $\mathcal{K} = \{0,1,2\}$ as before. For the sake of interpretation, imagine a world consisting of two regions, either of which can experience an earthquake at date t. To apply the theory developed in this paper, we must assume there is never more than one earthquake at a given date, and there may be none. The set of consumers is divided into two equal subsets: I_1 who reside in region 1 and I_2 who reside in region 2. For each $i \in I := I_1 \cup I_2$, assume preferences are once again represented by the utility function (10) but the endowment of consumer $i \in I$ is now given by

$$w_i(t, a_t) = \begin{cases} 0 & \text{if } \Delta N_k(t, a_t) = 1 \text{ and } i \in I_k \text{ at date } t; \\ Y & \text{otherwise.} \end{cases}$$

In other words, if an earthquake strikes at date t and consumer i resides in the region in which it strikes, the consumer loses her entire endowment at that date; otherwise, she is unaffected. It is easy to see that in this case our results remain essentially unchanged: in particular, ADR prices, equilibrium net trades, and security prices are those we would obtain by setting $n = 2$ in the insurance example treated earlier in this section. However, assuming $n/2$ consumers reside in each region, the results in the earthquake case remain invariant as n increases: insurance can be priced "competitively" using the methodology of finance despite the irrelevance of an appeal to the law of large numbers.

4 Conclusion

As will be apparent to those familiar with the finance literature, the research reported here only begins to tap the potential for applying the tools of intertemporal finance to insurance markets. Insurance contracts are clearly more complex than the simple instruments described in this paper, typically insuring a variety of types of accident over varying periods of time with options to renew and the like. All such contracts are "redundant assets" in this setting and, as such, can be priced using martingale measure. Insurance contracts also typically pay out in real rather than nominal terms, a distinction we have not addressed in our single-commodity version of the model but which clearly can be addressed in an extension of the model. Development of some version of a mutual fund theorem, in which consumers buy insurance on themselves and invest in a market portfolio of insurance contracts on others, is another obvious possibility.

As the theoretical discussion clearly shows, there is no reason to assume hazard rates are independent of the past history of the process and constant or that the effect of an accident is confined to the date at which it occurs. When one medical problem strikes, it may announce the increased chance of other problems arising. And an accident today may put a worker out of commission for months or years to come.

Finally, and undoubtedly most obvious to those schooled in finance theory, the results extend naturally to continuous time. Our key hypothesis, that at most one accident happens at any date, is only an approximation in discrete time. As our earthquake example illustrates, a suitable interpretation of accident type takes much of the sting out of this assumption: an earthquake can affect a large number of people, but earthquakes are isolated events. Nevertheless, the clearest justification for our hypothesis comes in continuous time. Much of the formalism of this paper is aimed at making the transition from discrete to continuous time as effortless as possible.

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