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by
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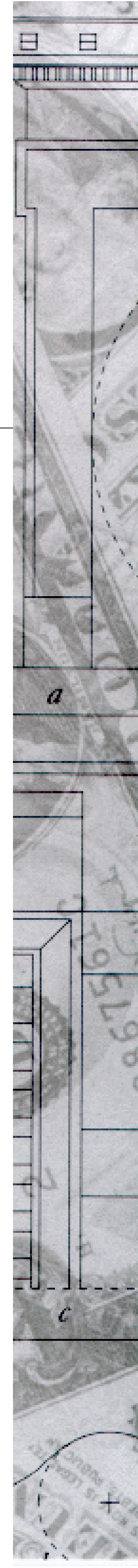

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Risk factor analysis and portfolio immunization in the corporate bond market

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Abstract

In this paper we develop a multi-factor model for the yields of corporate bonds. The model allows the analysis of factors which influence the changes in the term structure of corporate bonds. More than 98% of the variability in the corporate bond market is captured by the model, which is then used to develop credit risk immunization strategies. Empirical results are given for the U.S. market using data for the period 1992-1999.

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1 Introduction

The analysis of the risks in the corporate bond market is one of the most important topics in risk management and finance today. The attention is well justified given the size of the corporate bond market, its rapid rate of growth, and the large credit risk exposures of major dealers. The outstanding corporate debt in the United States in 2000 stands at an estimated value in excess of $250 billion, see Keenan [13]. The annual issuance of corporate bonds has been growing rapidly since the eighties. Following the dramatic decline in 1990 we have seen a resurgence of new issues—although we have witnessed a "flight to quality" as evidenced from the data in Figure 1.

In tandem with the growth of the market we have witnessed large credit risk exposures of major dealers. The credit risk assets of Mitsubushi Bank were $33 billion in fair market valuation, for Citicorp they stood at $32 billion and for Chase Manhattan at $26 billion. The top twenty dealers had credit risk assets in excess of $5 billion each (data for December 1995, reported in RISK Magazine in February 1997).

It is understandable, then, that the term structure of the yields of corporate bonds is receiving today the same attention that the term structure of interest rates received more than a decade ago. Figure 2 illustrates the dramatic changes in the shape and level of yield curves of various ratings viz-a-viz the treasury rates at different points in time. The volatility of the yields of different maturities, estimated over the period 1992 to 1999, is shown in Figure 3. Witness that the volatility of yields of different maturities for high quality bonds is slightly smaller than those of treasury securities. The situation changes substantially, at least in the short period, for lower rating bonds—both in magnitude and in shape. Furthermore, the volatility of these yields has been varying during this seven year period, as shown in Figure 4. While high quality bonds follow in volatility the treasury bonds, bonds of lower ratings behave differently.

It is the excess return implied by the corporate yields illustrated in Figure 2 that attract the attention of investors. The volatility trends of Figures 3 and 4 attract the attention of analysts and regulators. (The Basel Committee on Banking Supervision (1999) [1] is taking steps to amend its regulations and use credit risk models for regulatory capital.)

In this paper we develop a multi-factor model for the corporate bond market. The model is simple compared to the elaborate Monte Carlo simulation models promoted by systems like CreditMetrics, see J.P.Morgan [12], CreditRisk+, see Credit Suisse Financial Products [4], KMV's portfolio manager [14] or Algorithmic's portfolio credit risk model, see Iscoe et al. [10]. Why do we need a simpler model? While we subscribe to the significance of the Monte Carlo simulation approach to bond portfolio management, see Mulvey and Zenios [16], we will see that the multifactor model proposed here captures more than 98% of the variability in the corporate bond markets. Furthermore, the model is easy to calibrate using readily available data on bond yields and standard statistical

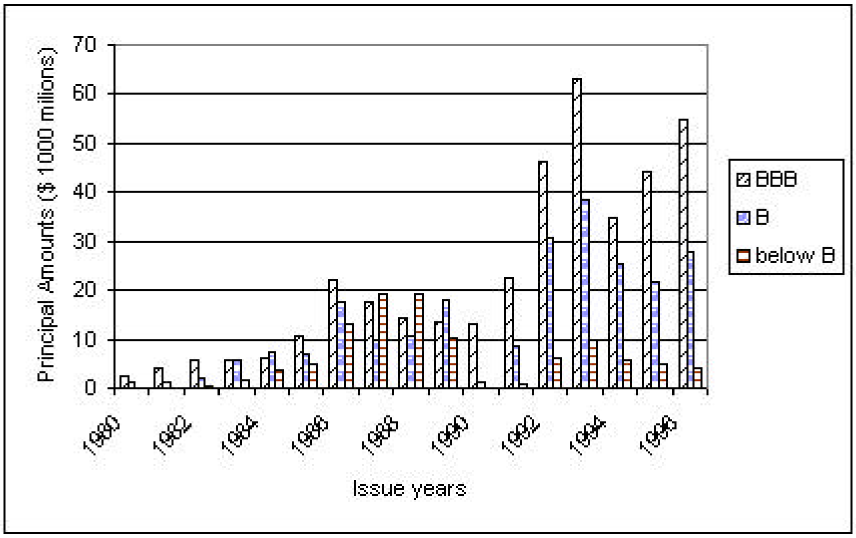
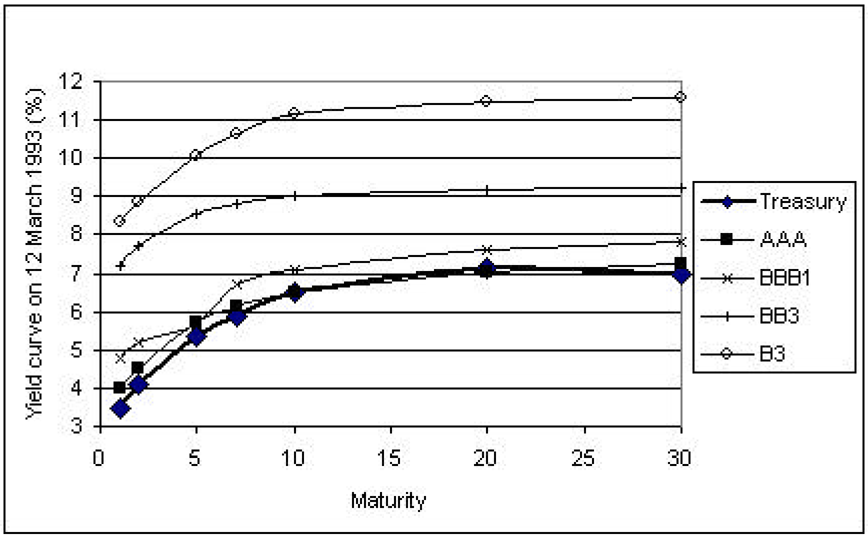


Figure 1: The annual issuance of non-convertible corporate bonds demonstrates a rapid growth of the corporate bond market in the 1990's with a flight to higher quality debt.

packages. Hence, the model strikes a balance between model complexity and accuracy. As such it provides a first approximation for modeling and managing credit risk which can be used by institutions that do not have the capability of elaborate computational models.

Other attempts have been made recently to identify the factors that explain changes of credit spreads, most notably Collin-Dufresne at al. [3] and Elton et al. [8]. Both approaches differ from ours in one significant respect. They try to identify financial or economic variables to explain the spreads. The former paper runs regressions which explain approximately 25% of the credit spread changes. The authors identify that a single factor explains most of the residuals, but they could not identify this factor. Elton et al. build models that explain 67 to 85% (depending on the industry sector) of the spread changes by three factors: compensation for expected defaults, compensation for state taxes, and compensation for systematic risk relative to government bond returns. We take a different technical approach in analyzing the yield and spread changes. As a result we can explain more than 98% of the variability, However, no financial or economic interpretation can be attached to the factors idenitifed from our analysis. In a nutshell the other approaches have modest explanatory power but their results have economic and financial meaning, while ours has much higher explanatory power but does not admit readily any interpretation. The



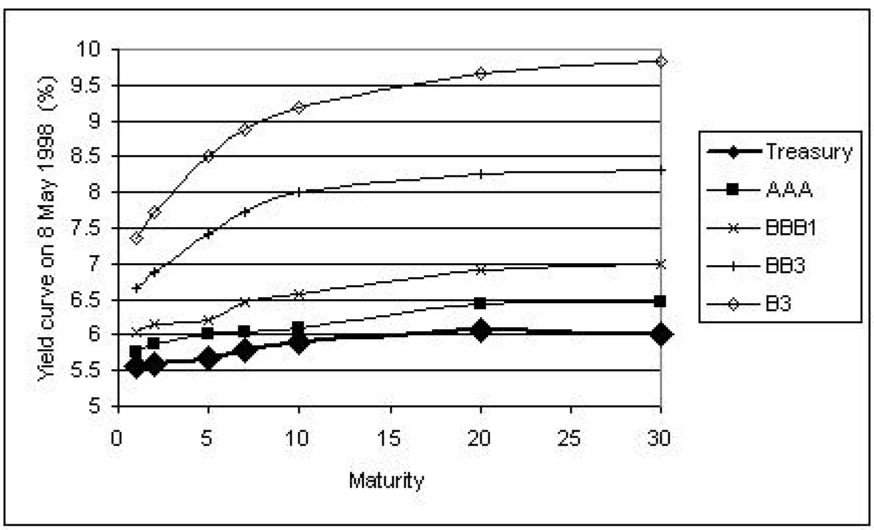


Figure 2: Yield curves of treasury securities and corporate bonds of different ratings change substantially with time.

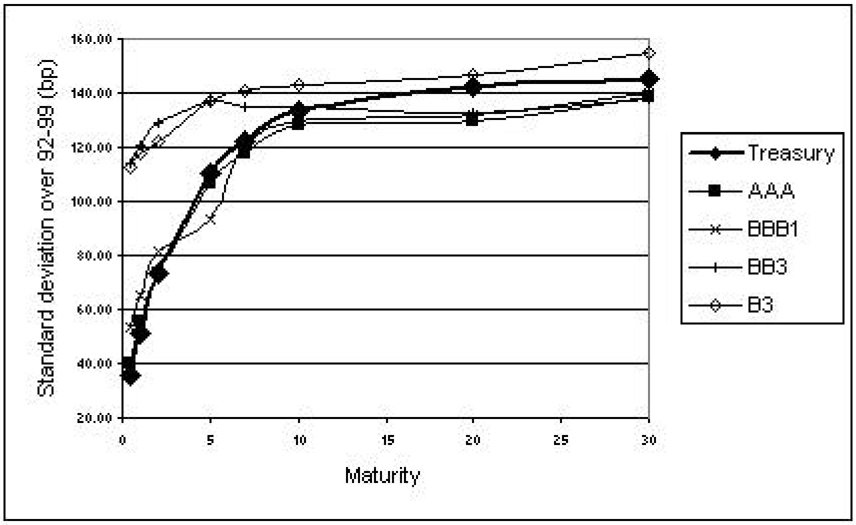


Figure 3: The volatility of the yields varies both with the maturity and with the quality rating of the bond; changes are observed in both magnitude and shape.

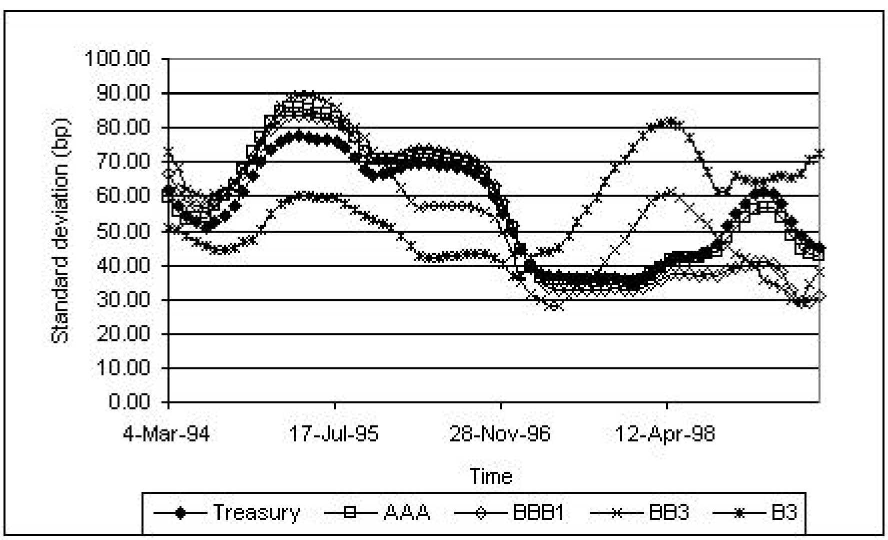


Figure 4: Changes in the volatility of the corporate bond market as evidenced by standard deviation of the 10-years yields of different ratings, computed on a 2-year moving window using weekly data.

AAA	6M	1Y	2Y	5Y	7Y	10Y	20Y	30Y
6M	1.000							
1Y	0.956	1.000						
2Y	0.935	0.965	1.000					
5Y	0.921	0.943	0.976	1.000				
7Y	0.940	0.934	0.966	0.990	1.000			
10Y	0.904	0.931	0.957	0.982	0.990	1.000		
20Y	0.899	0.918	0.938	0.965	0.976	0.985	1.000	
30Y	0.896	0.907	0.924	0.954	0.966	0.977	0.986	1.000

Table 1: Correlations of weekly yield changes for AAA industrial bonds from March 1992 to July 1999.

systematic factor analysis carried out here can provide some insights to those searching for the appropriate economic and financial variables. Furthermore, our factor analysis can be used to develop credit risk immunization strategies without attaching any meaning to the factors.

The paper is organized as follows: Section 2 analyzes the data for yield and spread changes, and develops the multifactor risk model, focusing on the analysis of the yields of a single rating class. Section 3 reports the results of estimating the factor model. Section 4 discusses immunization models using the results of the factor model. Section 5 provides a significant extension of the models to analyze simultaneously multiple credit rating classes.

2 Measuring the risk of yields on corporate bonds

Following current tradition, see Garbade [9], we analyze and model the *changes* of the yields of different maturities. The first step in modeling the changes of the yields of corporate bonds is the examination of the correlations between yield changes for different maturities. We estimate the correlations for bonds of different ratings and of different industries. The correlations of weekly yield changes are shown in Tables 1–4 for the *industrial* sector. All data in our study are obtained from Bloomberg Financial Services for the period 1992-1999.

We observe that changes in the yields of bonds of different maturities are imperfectly correlated. The correlations are the highest for close maturity dates and the lowest for the most distant maturities. The correlation coefficients for securities with the "next available" or the "previous available" maturity date are more than 0.95 for all quality ratings. As we move to bonds of distant maturity dates the correlations decline to a lowest of 0.68 (observed for the B3 bonds).

Furthermore, we note that the correlation coefficients are lower for the bond sectors of lower quality, as evidenced mainly from the correlations among bonds of distant maturities. Conversely, the volatilities of the yield changes of the

BBB1	6M	1Y	2Y	5Y	7Y	10Y	20Y	30Y
6M	1.000							
1Y	0.957	1.000						
2Y	0.929	0.963	1.000					
5Y	0.917	0.953	0.982	1.000				
7Y	0.893	0.928	0.962	0.971	1.000			
10Y	0.892	0.921	0.949	0.960	0.988	1.000		
20Y	0.885	0.905	0.931	0.940	0.973	0.980	1.000	
30Y	0.883	0.899	0.917	0.926	0.961	0.975	0.985	1.000

Table 2: Correlations of weekly yield changes for BBB1 industrial bonds from March 1992 to July 1999.

BB3	6M	1Y	2Y	5Y	7Y	10Y	20Y	30Y
6M	1.000							
1Y	0.965	1.000						
2Y	0.946	0.973	1.000					
5Y	0.908	0.940	0.972	1.000				
7Y	0.873	0.899	0.929	0.969	1.000			
10Y	0.881	0.908	0.931	0.969	0.971	1.000		
20Y	0.888	0.908	0.924	0.954	0.954	0.976	1.000	
30Y	0.880	0.896	0.910	0.942	0.941	0.964	0.993	1.000

Table 3: Correlations of weekly yield changes for BB3 industrial bonds from March 1992 to July 1999.

B3	6M	1Y	2Y	5Y	7Y	10Y	20Y	30Y
6M	1.000							
1Y	0.980	1.000						
2Y	0.929	0.957	1.000					
5Y	0.748	0.799	0.894	1.000				
7Y	0.654	0.706	0.810	0.964	1.000			
10Y	0.657	0.704	0.798	0.950	0.962	1.000		
20Y	0.637	0.674	0.768	0.930	0.947	0.975	1.000	
30Y	0.682	0.712	0.797	0.932	0.938	0.963	0.990	1.000

Table 4: Correlations of weekly yield changes for B3 industrial bonds from March 1992 to July 1999.

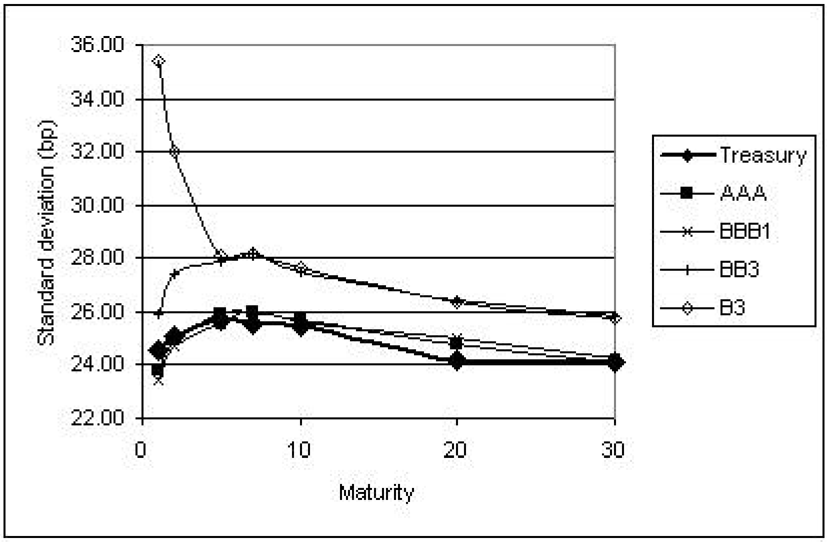


Figure 5: The volatility of the yield changes varies both with the maturity and with the quality rating of the bond; changes are observed in both magnitude and shape.

lower rating bonds are higher than those of the high quality bonds and of the treasury securities (see Figure 5). The same observations hold for other sectors we have analyzed (*financial* and *utilities*).

As a further step in measuring the risk of yields on corporate bonds we estimate the volatilities and correlations on the changes of the *credit spreads*. Figure 6 shows the volatility of the spread changes on corporate bonds of different maturities and different credit ratings. The volatilities of credit spreads vary with maturity and the variation is more significant for the low quality bonds. It is noteworthy to compare the volatilities of the spread changes of Figure 6 with the volatilities of yield changes of Figure 5: for high quality bonds the spread volatility is only 25% of the yield volatility, and for low rating bonds it is more than 50%. This is expected since yields of corporate bonds are driven to some extent by the yields of the treasury securities. The analysis indicates that the volatility of high quality bonds is mostly due to term structure changes, while for low quality bonds their volatility is affected to a lesser extent by term structure movements. Disentangling interest rate risk from credit spread risk is precisely the motivation of the factor models developed in this paper.

The correlations of the changes of the spreads of bonds of different maturities and of different ratings are shown in Tables 5–8. As in the case of the yield correlations, we observe that changes in the spreads of bonds of different matu-

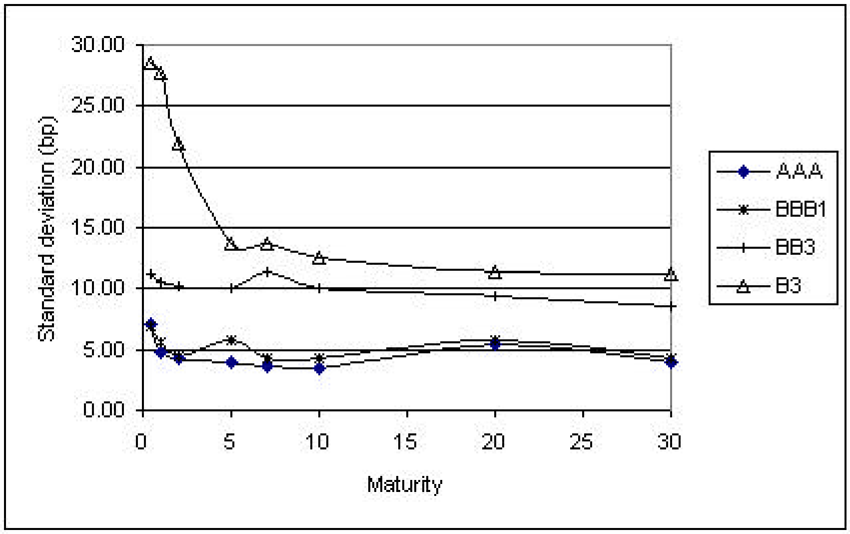


Figure 6: The volatility of the spread changes varies both with the maturity and with the quality rating of the bond; changes are observed in both magnitude and shape.

rities are imperfectly correlated. The correlations are higher for close maturity dates and lower for distant maturities. The correlation coefficients of the spread changes are significantly lower than the yields correlations.

Unlike the yield correlations though the spread correlations are higher for lower bond ratings! Note, from Figure 7, the correlations of yield and spread changes between the 6-month and 30-year bonds. This figure reinforces our earlier observation that the volatility of high quality bonds is mostly due to term structure changes, while for low quality bonds their volatility is affected more by spread changes.

2.1 Concluding observations

From the preceeding analysis we conclude that changes in yields and spreads are not perfectly correlated for bonds of different maturities. The correlation coefficients vary with credit rating and the volatilities and correlations of spreads are lower than those of the yields. Spread correlations for lower quality bonds are higher than those of high quality bonds, while the situation is reversed for yield correlations.

These conclusions give the impression of a chaotic corporate bond market! What should be done? It is not surprising the corporate bond managers resort to elaborate Monte Carlo simulations to capture the risk exposure of their port-

AAA	6M	1Y	2Y	5Y	7Y	10Y	20Y	30Y
6M	1.000							
1Y	0.541	1.000						
2Y	0.408	0.552	1.000					
5Y	0.177	0.233	0.333	1.000				
7Y	0.168	0.206	0.234	0.528	1.000			
10Y	0.147	0.118	0.274	0.287	0.404	1.000		
20Y	0.093	-0.002	0.141	0.168	0.286	0.291	1.000	
30Y	0.127	0.111	0.166	0.217	0.242	0.326	0.276	1.000

Table 5: Correlations of weekly spread changes for AAA industrial bonds from March 1992 to July 1999.

BBB1	6M	1Y	2Y	5Y	7Y	10Y	20Y	30Y
6M	1.000							
1Y	0.550	1.000						
2Y	0.455	0.538	1.000					
5Y	0.279	0.341	0.425	1.000				
7Y	0.218	0.244	0.312	0.292	1.000			
10Y	0.213	0.202	0.334	0.340	0.509	1.000		
20Y	0.110	0.015	0.259	0.192	0.401	0.362	1.000	
30Y	0.188	0.204	0.259	0.292	0.401	0.463	0.425	1.000

Table 6: Correlations of weekly spread changes for BBB1 industrial bonds from March 1992 to July 1999.

BB3	6M	1Y	2Y	5Y	7Y	10Y	20Y	30Y
6M	1.000							
1Y	0.854	1.000						
2Y	0.798	0.873	1.000					
5Y	0.631	0.702	0.830	1.000				
7Y	0.503	0.560	0.655	0.781	1.000			
10Y	0.521	0.583	0.684	0.798	0.772	1.000		
20Y	0.511	0.535	0.607	0.677	0.641	0.764	1.000	
30Y	0.487	0.513	0.581	0.671	0.617	0.752	0.872	1.000

Table 7: Correlations of weekly spread changes for BB3 industrial bonds from March 1992 to July 1999.

B3	6M	1Y	2Y	5Y	7Y	10Y	20Y	30Y
6M	1.000							
1Y	0.976	1.000						
2Y	0.934	0.948	1.000					
5Y	0.672	0.702	0.782	1.000				
7Y	0.429	0.460	0.541	0.839	1.000			
10Y	0.444	0.462	0.537	0.797	0.809	1.000		
20Y	0.381	0.385	0.439	0.713	0.738	0.838	1.000	
30Y	0.532	0.541	0.596	0.810	0.760	0.840	0.893	1.000

Table 8: Correlations of weekly spread changes for B3 industrial bonds from March 1992 to July 1999.

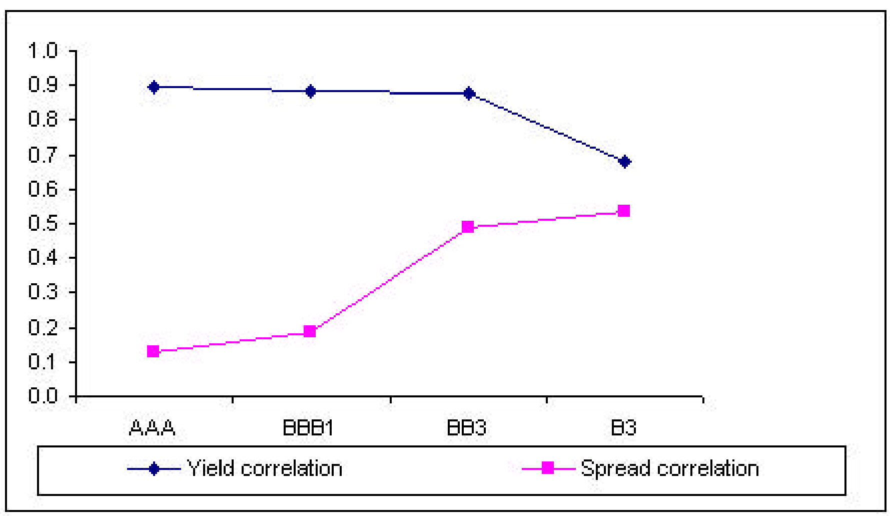


Figure 7: Yield and spread correlations between the 6-month and the 30-year bonds for different credit ratings.

folios. Systems like CreditMetrics, CreditRisk+ are perhaps invaluable, and the regulators' interest in credit risk modeling is justified. However, the analysis we present next shows that these markets are not as chaotic as they appear on first examination. A few factors can be used to capture most of the variability, and this observation has significant implications for credit risk portfolio immunization.

2.2 A multifactor model of risk

The manager of a portfolio of corporate bonds must identify those risk factors which influence the variations of yields or yield spreads along time, and to control the sensitivity of a portfolio return to movements in the risk factors. Appropriate factor immunization models have been developed by Garbade [9] for US Treasury bonds, by Dahl [5] for the Danish market and by D'Ecclesia and Zenios for the Italian market [6]. Here we develop a multifactor model for corporate bonds. Let:

P_i be the price, at time 0, of corporate bond i with maturity T,

C_{it} be the bond cashflows at time t,

y_t be the continously compounded yield of a zero-coupon bond which pays 1 dollar at t.

The bond price is given by

$$P_i = \sum_{t=1}^{T} C_{it} e^{-y_t t}. \tag{1}$$

We assume now that changes in yields can be expressed as a linear combination of K independent factors

$$dy_{t\tau} = \sum_{j=1}^{K} \beta_{jt} df_{j\tau} + \epsilon_{t\tau}, \tag{2}$$

where j denotes the independent factors, $j = 1, \ldots, K$, and τ denotes the calendar time. In this model $dy_{t\tau}$ denotes the change at time τ of the yield of a bond maturing at time t and $df_{j\tau}$ denotes the change in the level of factor j at time τ. β_{jt} denotes the sensitivity of yield changes to factor changes (*factor loadings*). This factor indicates the change of the t-maturity yield due to one unit change in factor j. $\epsilon_{t\tau}, t = 1, \ldots, T$, are error terms assumed to be independently normally distributed with zero mean and constant variance.

Notice that the factor loadings are assumed to be constant over calendar time, whereas the factor changes are time dependent. For simplicity the calendar time index τ will be omitted.

We use principal component analysis to determine the magnitude of changes in yield due to independent risk factors without making any assumption on the nature of the factors. Thus, we obtain a vector of factor loadings corresponding to each factor. We can then compute the factor modified duration for each bond. Differentiating equation (1) with respect to y_t we get

$$dP_i = -\sum_{t=1}^{T} C_{it} t e^{-y_t t} dy_t \tag{3}$$

When the yield curve is flat and only moves in parallel (i.e., $dy_t = dy$ for all $t = 1, \ldots, T$) the quantity $k_i = \frac{dP_i}{dy_t}$ is the well known dollar duration.

Substituting equation (2) into (3) we get the duration of a bond with respect to movements of the jth factor

$$k_{ij} = \frac{\partial P_i}{\partial f_j} = -\sum_{t=1}^{T} C_{it} t \beta_{jt} e^{-y_t t}. \tag{4}$$

k_{ij} is the factor modified duration of bond i with respect to the jth factor.

In order to immunize a bond portfolio from all factors that affect the shape of term structure of corporate bonds, we match the factor modified durations of assets and liabilities. A linear programming formulation for factor immunisation is given in the Appendix.

3 Risk factor analysis of corporate yields

We estimated the model of equation (1) using principal component analysis on corporate bonds of five different credit ratings (AAA to B3) in the industrial sector. The model is estimated on the excess returns over the risk free rate of the U.S. bond market. Data are obtained from the term structure curves (Bloomberg source) at preselected maturities (i.e., six months, one, two, five, seven, ten, twenty and thirty years). A standard package (MATLAB) is then used to analyze the vectors of excess returns of these yields and to estimate the principal components and the factor loadings.

In all the experiments we found that the first three factors explain more than 98% of the total variance of the yield changes. Indeed, the first factor explains more than 95% of the total variance while the second explains 2% of the total variance and the third explains 1%. These numbers speak favourably of the power of the model.

Figures 8–12 illustrate the factor loadings for bonds of different maturities and different ratings. Figures 8 and 9 confirm that factor loadings for Treasury and AAA classes are almost identical.

The first factor affects yield of different maturities by the same amount and it means that more than 95% of changes in yields can be explained by *parallel shifts*. The second factor, which explains more than 2% of changes, affects short and long term yields in opposite directions, at least in the medium period, and it indicates changes in the *steepness* of the yield curve. The third factor, which

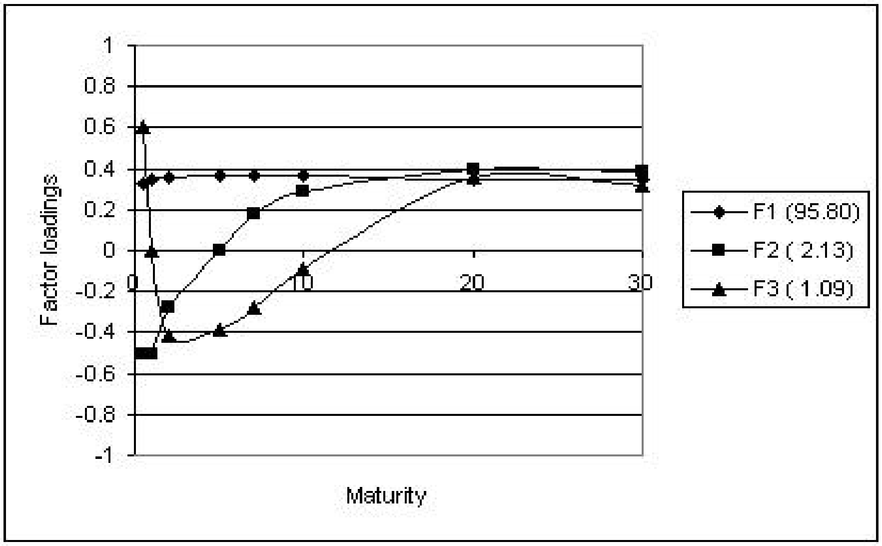


Figure 8: Factors loading for the Treasury yield changes.

explains more than 1% of changes, affects the *curvature* in the short period. The same qualitative factors (parallel shifts, steepness and curvature) affect the lower quality ratings. However the numerical values are substantially different.

In order to establish the stability of factor loadings we divide the period 1992-1999 in three subperiods and repeat the analysis on each subperiod. The factor loading are stable in magnitude and shape for all the rating classes.

Similar results were obtained by applying the principal component analysis to credit spreads. The results are not reported here for the sake of brevity and they can be obtained from the authors.

4 Implications for portfolio immunization

Once we have identified factors that affect the chanes of yields of different maturities and the corresponding factor loadings we can immunize our asset and liability portfolio from changes to these factors. Linear programming models for structuring immunized portfolios to hedge against factor changes are given in the Appendix. We built first immunized portfolios for a liability that has a specific rating and build a portfolio of bonds of different maturities that is immunised against changes of the factors affecting the rating class of interest.

We present the results obtained using model (27)–(29). All tests were done using bonds and liabilities in AAA class. Figures 13–15 show the results of the immunisation model when one, two and three factors, respectively, are intro-

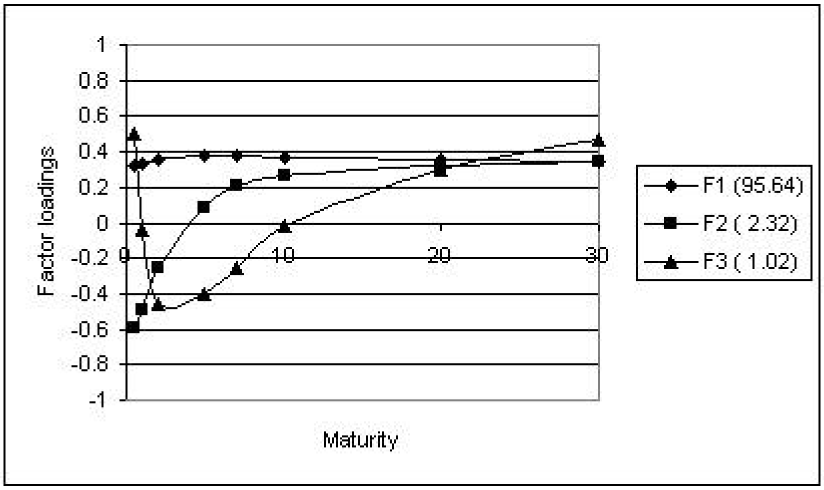


Figure 9: Factors loading for the AAA yield changes.

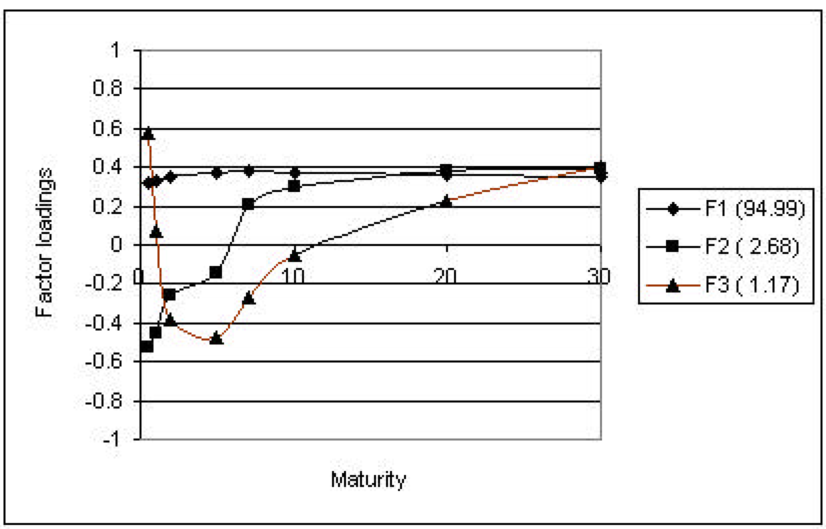


Figure 10: Factor loadings for the BBB1 yield changes.

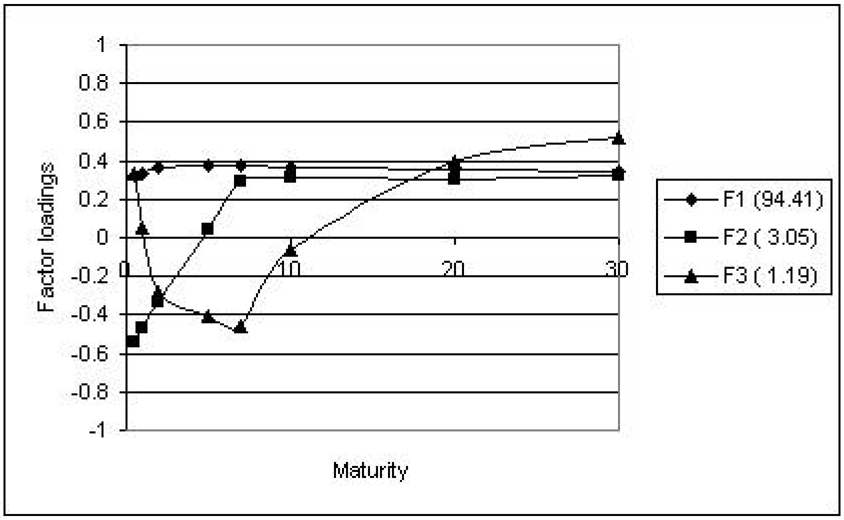


Figure 11: Factor loadings for the BB3 yield changes.

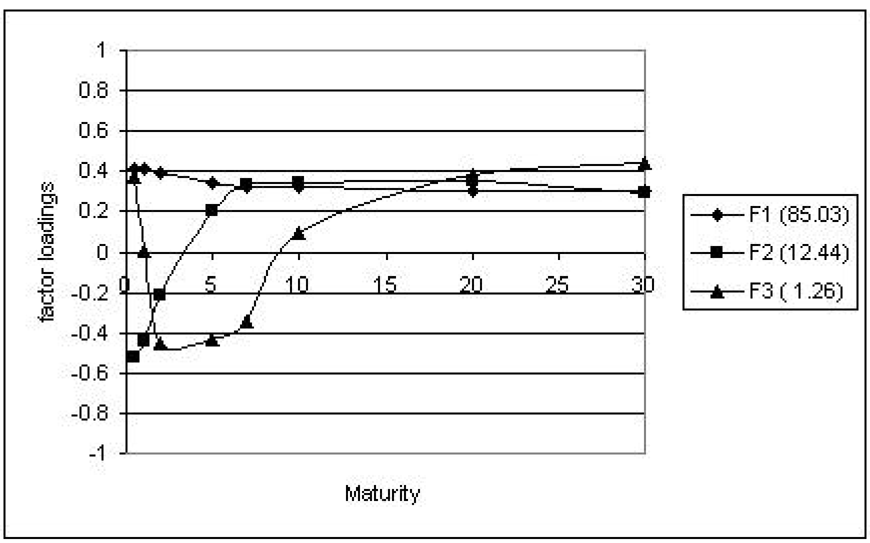


Figure 12: Factor loadings for the B3 yield changes.

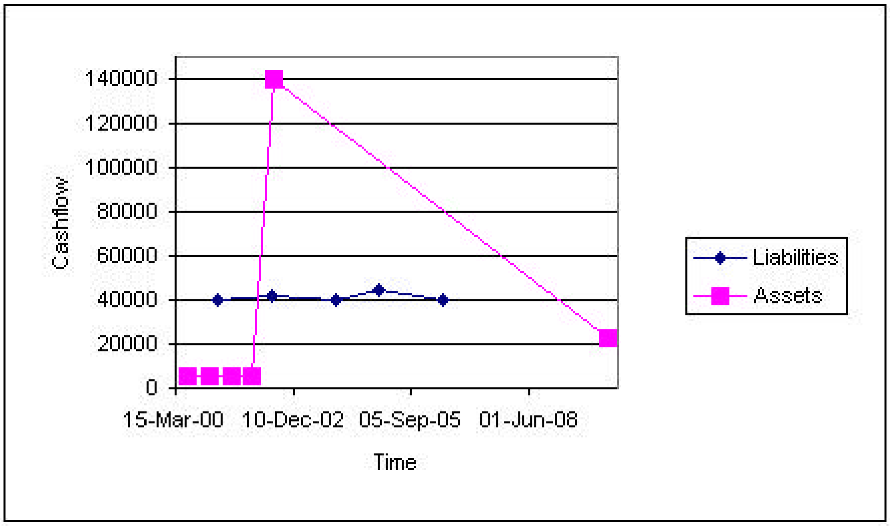


Figure 13: Cashflow pattern of a portfolio immunised against the first factor of the AAA corporate yield (2 bonds in the portfolio).

duced. We observe that the cashflow matching of assets and liabilities becomes tighter when increasing the number of factors. The advantage of the factor immunisation approach is exactly its flexibility. The portfolio manager can decide which factors to hedge against and which active risk positions to take.

5 Portfolio immunization with corporate bonds of multiple credit ratings

The result of the previous section, interesting as they may be on their own right, do not address an important question: How to deal with a portfolio of bonds with different credit ratings? The correlations across multiple credit rating classes should also be considered when building an immunized portfolio. The factor analysis of Section 3 permits a straightforward extension to incorporate multiple classes. We start with a formulation assuming the yield changes in different rating classes are independent and then formulate a model to deal with comovements of the yield curves.

Let L be the number of rating classes with l denoting the lth rating class. The fair value equation

$$P_i^l = \sum_{t=1}^{T} C_{it}^l e^{-y_t^l t} \tag{5}$$

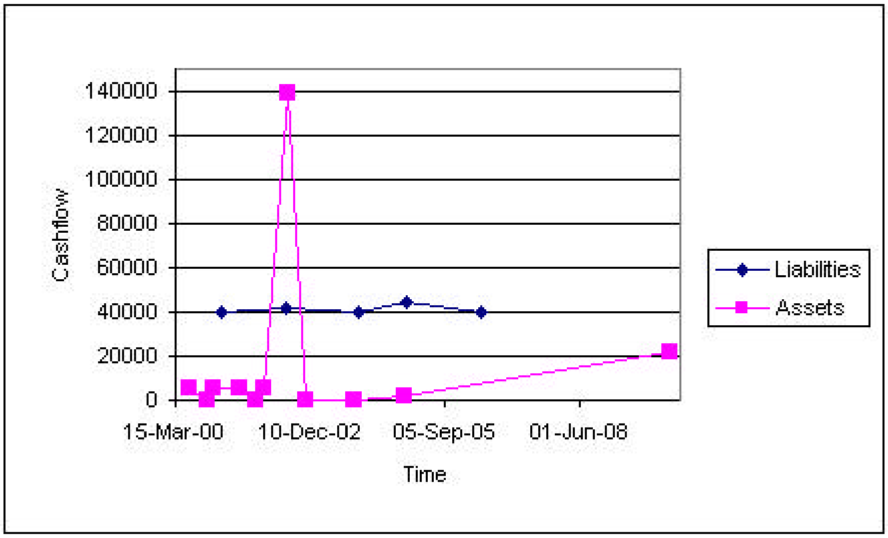


Figure 14: Cashflow pattern of a portfolio immunised against the first two factors parallel shift and steepening of the AAA corporate yield (3 bonds in the portfolio).

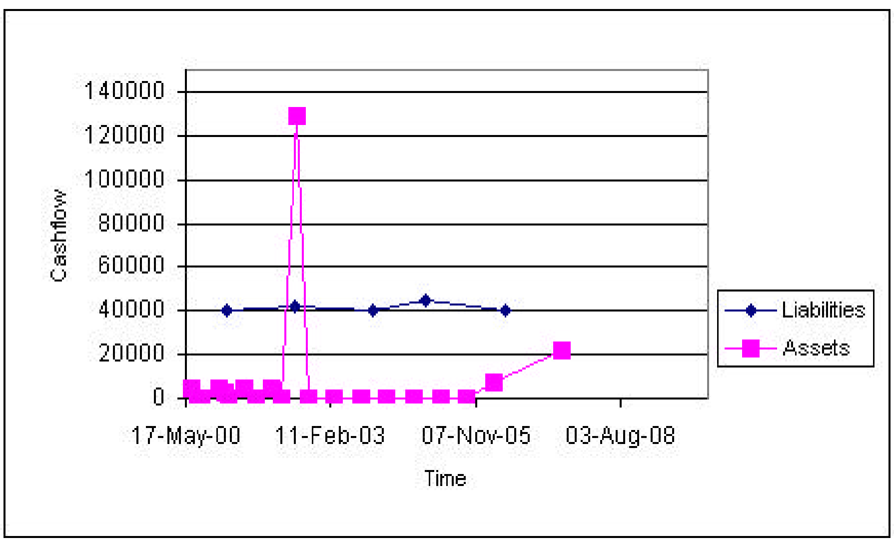


Figure 15: Cashflow pattern of a portfolio immunised against the first three factors of the AAA corporate yield (4 bonds in the portfolio).

is now indexed by the rating class.

A linear factor model for the yield changes of class l takes the form

$$dy_t^l = \sum_{j=1}^{K_l} \beta_{jt}^l df_j^l + \epsilon_t^l \tag{6}$$

where we assume that K_l independent factors explain the yield changes of the lth class. The change of bond prices due to factor changes is given by

$$dP_i^l = \sum_{j=1}^{K_l} \frac{\partial P_i^l}{\partial f_j{}^l} df_j^l = \sum_{j=1}^{K_l} k_{ij}^l df_j^l \tag{7}$$

Consider now a portfolio of assets from different rating classes. The portfolio fair value is

$$W = \sum_{i=1}^{m} \sum_{l=1}^{L} x_i^l P_i^l, \tag{8}$$

where x_i^l denotes the holdings of bond i, class l, in the portfolio. The change in the portfolio value due to changes in yield curves is

$$dW = \sum_{i=1}^{m} \sum_{l=1}^{L} \sum_{j=1}^{K_l} \frac{\partial P_i^l}{\partial f_j^l} x_i^l df_j^l = \sum_{i=1}^{m} \sum_{l=1}^{L} \sum_{j=1}^{K_l} k_{ij}^l x_i^l df_j^l \tag{9}$$

Assuming first that yield changes in different rating classes are independent, we can write an immunization model imposing the constraints that assets and liabilities have the same present value and the same sensitivity to changes in the risk factors. That is, the following are the immunization conditions

$$\sum_{l=1}^{L} \sum_{i=1}^{m} P_i^l x_i^l = P_L \tag{10}$$

where P_L is the present value of total liabilities and

$$\sum_{i=1}^{m} k_{ij}^l x_i^l = k_{jL}^l, \qquad for j = 1, \ldots, K \qquad l = 1, \ldots, L. \tag{11}$$

k_{jL}^l is the factor loading for thr liabilities with respect to factor j in rating class l.

If we assume that the liability is in a single class, say class $l = 0$, then the optimization model becomes

$$\underset{x}{\text{Minimize}} \quad \sum_{l=1}^{L}\sum_{i=1}^{m} P_{iq}^{l} x_i^l \tag{12}$$

$$\text{s.t.} \quad \sum_{l=1}^{L}\sum_{i=1}^{m} P_i^l x_i^l = P_L^0, \tag{13}$$

$$\sum_{i=1}^{m} k_{ij}^l x_i^l = \begin{cases} k_{jL}^0, & j = 1,\ldots,K_l, \quad l = 0, \\ 0, & j = 1,\ldots,K_l, \quad l = 1,\ldots,L. \end{cases} \tag{14}$$

Here P_{iq}^l is the quoted price of the bond i in class l and P_L^0 is the present value of total liabilities in class 0.

Note that constraints (14) force $x_i^l = 0$ for all i, when $l \neq 0$. Hence bond holdings in assets with rating different than the target rating $l = 0$ are excluded.

These constraints are relaxed as follows:

$$\sum_{i=1}^{m} k_{ij}^l x_i^l = k_{jL}^0, \text{for} l = 0 \tag{15}$$

$$\underline{k} \quad \sum_{i=1}^{M} k_{ij}^l x_i^l \quad \overline{k}, \text{for} l = 1,\ldots,L \tag{16}$$

where $\underline{k}$ and $\overline{k}$ are user specified constants. These constants ensure that the risk exposure to factors affecting credit classes other that the target class 0 is limited. Admitedly these are ad hoc contraints. The rigorous model would keep constraints (14). In which case the model reduces back to the single rating class model of section 4 unless we find a combination of bonds from different rating classes with modified durations that cancel out each other.

We compare the relaxed model introduced here with the single class model tested in section 4for immunizing identical streams of liabilities. We construct our asset portfolio choosing bonds from AAA and B3 classes. Figures 19–21 show that the results are similar to those obtained by using the single class model. However a small fraction, less than 1%, now consists of B3 bonds. The inclusions of lower rating bonds in the asset portfolio results in savings of the order of 0.03% to the cost of the portfolio but at a higher risk, due to the factors affecting the B3 bonds. However this risk has been restricted. The value of the relaxed model is precisely in limiting the credit risk exposure in some classes while immunizing against risk exposure in others.

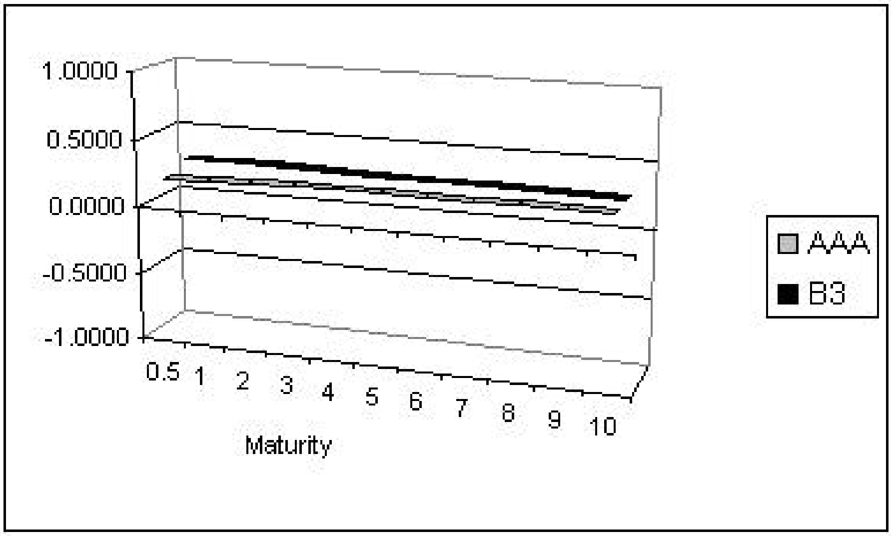


Figure 16: Factor loadings (1st factor) for yield change comovements of AAA and B3. The first factor explains 82% of the variability.

5.1 Capturing the correlation of the yields of different credit ratings

The limitation of the immunisation model (12)–(14) is the assumption of independence of the changes of the yield curves of different rating classes. This assumption is too strong and it forced us to impose the ad hoc constraints (15)–(16). We relax this assumption here, thus also obtaining a rigorous immunisation model.

In order to identify the factors affecting comovements of the yield curves of different rating classes we perform principal component analysis on an expanded correlation matrix. See for example the correlation matrix in Table 9 of the two classes AAA and B3. The upper left triangle submatrix coincides with the matrix in Table 1., and the bottom right triangle submatrix coincides with the matrix in Table 4. Moreover, now we have additional correlation information in the lower left submatrix.

The factor analysis of the expanded matrix identifies K factors that jointly affect yield changes of bonds of different maturities for both classes. It is instructive to visualize the factor loadings as illustrated in Figures 16– 18: for each class we plot the factor loadings of bonds of different maturities with respect to each one of the factors separately. For the two classes analyzed here we get three factors explaining more than 98% of the yield changes.

The immunisation model (12)–(14)can be easily extended to immunise against

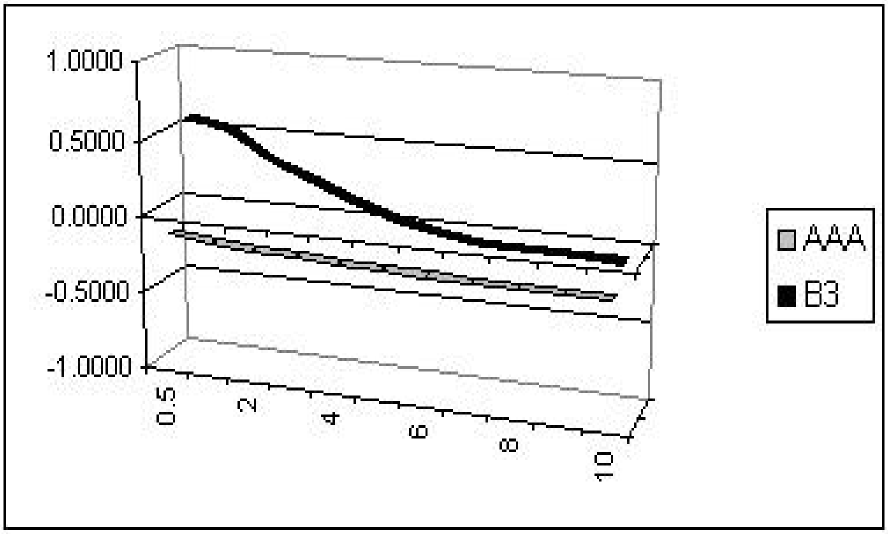


Figure 17: Factor loadings (2nd factor) for yield change comovements of AAA and B3.The second factor explains 12% of the variability.

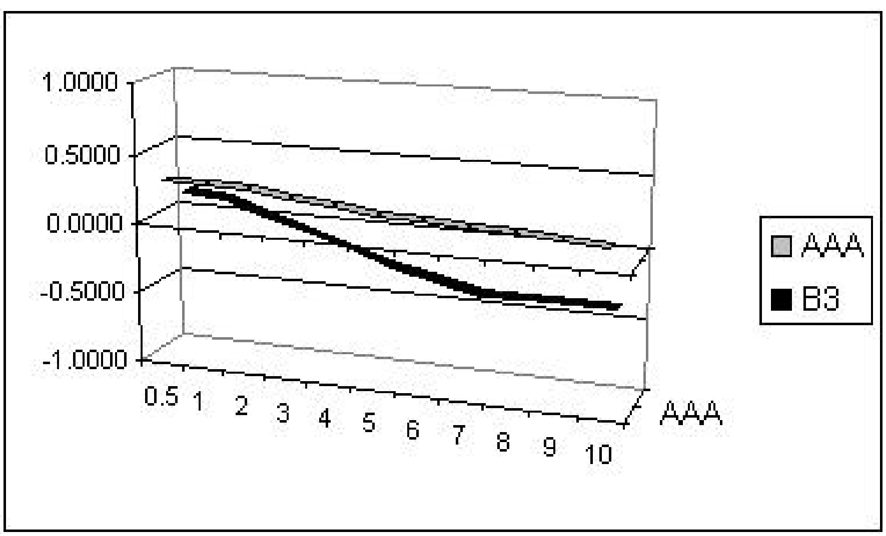


Figure 18: Factor loadings (3rd factor) for yield change comovements of AAA and B3. The third factor explains 4% of the variability.

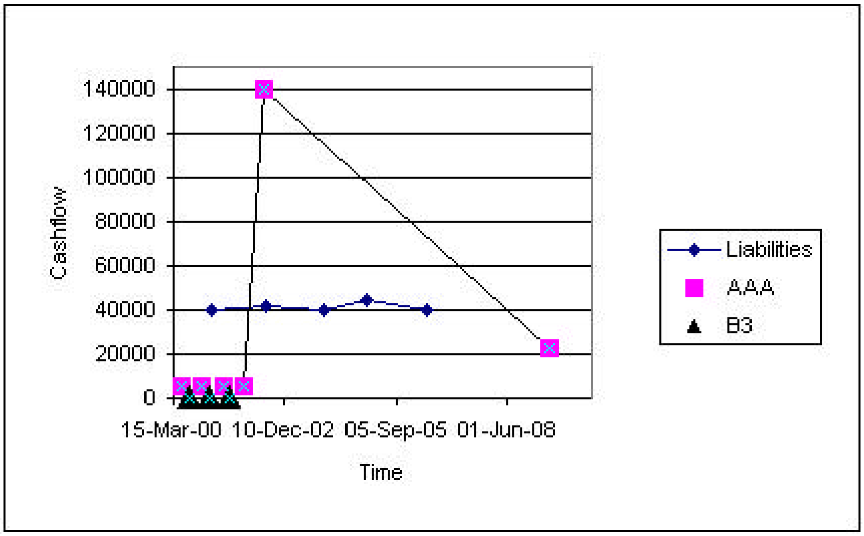


Figure 19: Cashflow pattern of a portfolio immunised against the first factor of the AAA and B3 corporate yields assuming independent changes across rating classes (3 bonds in the portfolio).

changes of the common factors modifying constraint (14) as follows

$$\sum_{l=1}^{L}\sum_{i=1}^{m}\tilde{k}_{ij}^{l}x_i^l = \sum_{l=1}^{L}\tilde{k}_{Lj}^{l}, \qquad j = 1, \ldots, K \tag{17}$$

where $\tilde{k}_{ij}^{l}$ is the factor modified duration of bond i in the lth class with factor j where factor loadings have been estimated on the expanded matrix and $\tilde{k}_{liabj}^{l}$ is the liabilities factor modified duration with respect to factor j in class l.

The results of this last model are shown in Figures 22– 24. The portfolios appear to be less diversified in terms of maturity rates than the portfolios obtained with the model of the previous section when we assumed that that yield changes of different rating were independent. Clearly if the model captures correlations among different ratings it might be possible to achieve a certain level of immunization with fewer bonds, carefully selected from the different credit ratings. We also note that as we add more factors to the immunization models (Figure 24) the portfolio composition tilts more towards the highly rated bonds, but at the expense of less diversification accross maturities. Credit risk is diversified at the expense of interest rate risk.

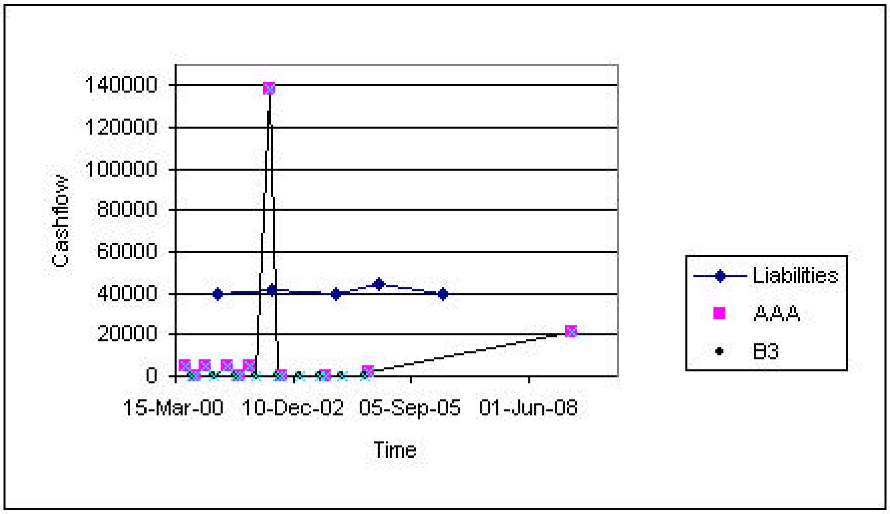


Figure 20: Cashflow pattern of a portfolio immunised against the first two factors of the AAA and B3 corporate yields assuming independent changes across rating classes (4 bonds in the portfolio).

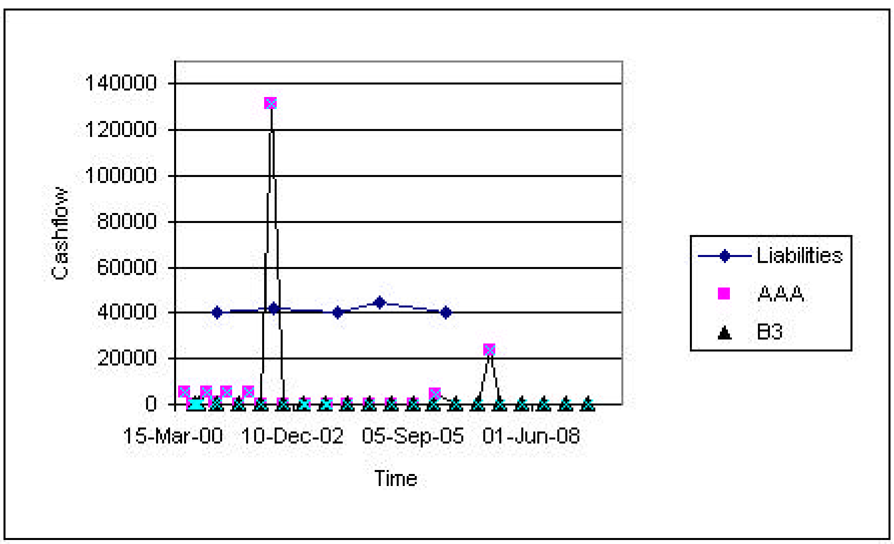


Figure 21: Cashflow pattern of a portfolio immunised against the first three factors of the AAA and B3 corporate yields assuming independent changes across rating classes (6 bonds in the portfolio).

	6M AAA	1Y AAA	2Y AAA	5Y AAA	7Y AAA	10Y AAA	6M B3	1Y B3	2Y B3	5Y B3	7Y B3	10Y B3
(AAA)												
6M	1.00											
1Y	0.96	1.00										
2Y	0.93	0.97	1.00									
5Y	0.92	0.94	0.98	1.00								
7Y	0.94	0.93	0.97	0.99	1.00							
10Y	0.90	0.93	0.96	0.98	0.99	1.00						
(B3)												
6M	0.62	0.58	0.57	0.55	0.54	0.55	1.00					
1Y	0.63	0.64	0.63	0.61	0.60	0.60	0.98	1.00				
2Y	0.72	0.73	0.74	0.73	0.72	0.71	0.93	0.96	1.00			
5Y	0.82	0.84	0.87	0.88	0.87	0.87	0.75	0.80	0.89	1.00		
7Y	0.81	0.83	0.85	0.88	0.88	0.88	0.65	0.71	0.81	0.96	1.00	
10Y	0.83	0.85	0.86	0.88	0.88	0.89	0.66	0.70	0.80	0.95	0.96	1.00

Table 9: Correlations of weekly yield changes for AAA and B3 bonds from March 1992 to July 1999.

	6M AAA	1Y AAA	2Y AAA	5Y AAA	7Y AAA	10Y AAA	6M B3	1Y B3	2Y B3	5Y B3	7Y B3	10Y B3
(AAA)												
6M	1.00											
1Y	0.54	1.00										
2Y	0.41	0.55	1.00									
5Y	0.18	0.23	0.33	1.00								
7Y	0.17	0.21	0.23	0.53	1.00							
10Y	0.15	0.12	0.27	0.29	0.40	1.00						
(B3)												
6M	0.15	0.17	0.16	0.08	-0.02	0.04	1.00					
1Y	0.05	0.20	0.16	0.09	-0.02	0.02	0.98	1.00				
2Y	0.07	0.17	0.21	0.09	-0.01	0.04	0.93	0.95	1.00			
5Y	0.07	0.14	0.17	0.23	0.06	0.06	0.67	0.70	0.78	1.00		
7Y	0.03	0.08	0.10	0.12	0.13	0.06	0.43	0.46	0.54	0.84	1.00	
10Y	0.07	0.08	0.11	0.10	0.07	0.15	0.44	0.46	0.54	0.80	0.81	1.00

Table 10: Correlations of weekly spread changes for AAA and B3 bonds from March 1992 to July 1999.

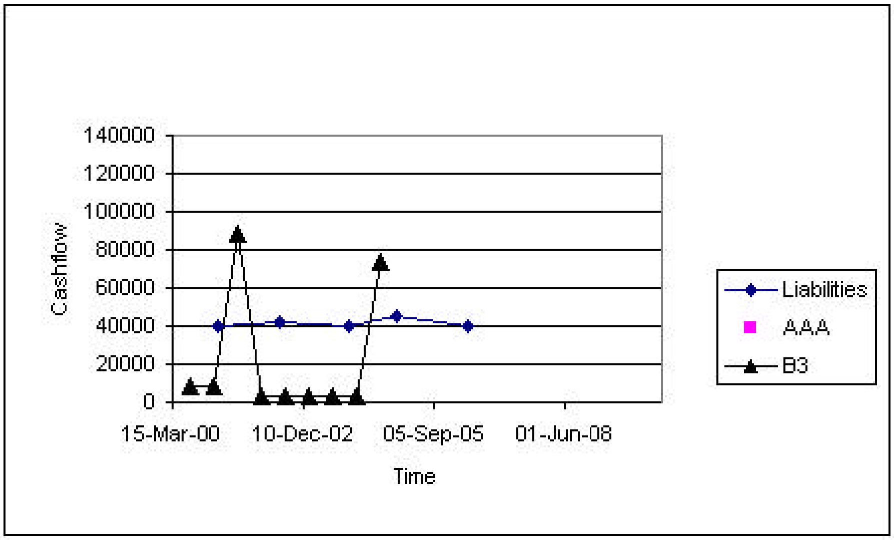


Figure 22: Cashflow pattern of a portfolio immunised against the first factor of the AAA and B3 corporate comovements of yield curves (2 bonds in the portfolio).

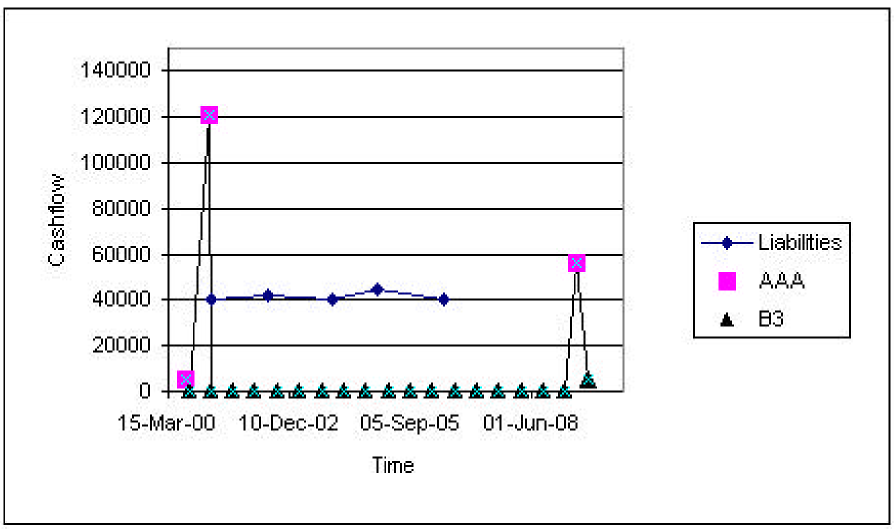


Figure 23: Cashflow pattern of a portfolio immunised against the first two factors of the AAA and B3 corporate comovements of yield curves (3 bonds in the portfolio).

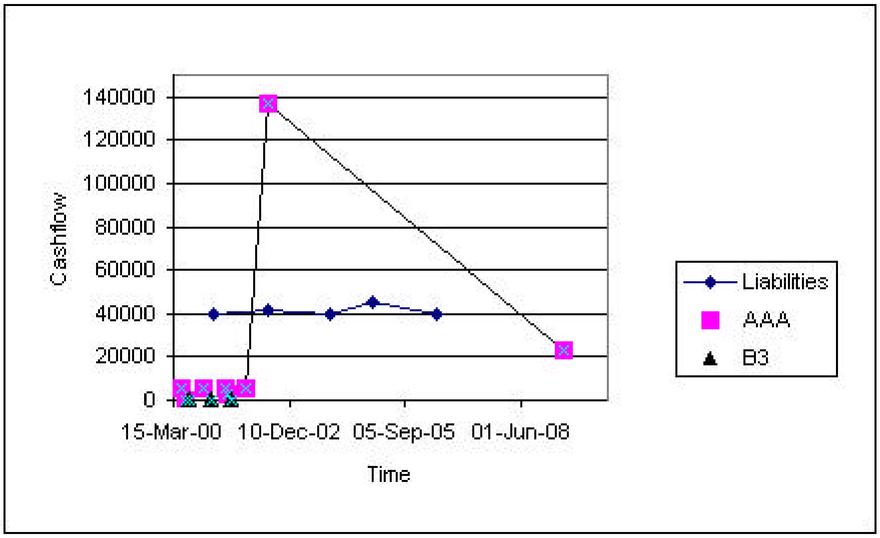


Figure 24: Cashflow pattern of portfolio immunisation against a parallel shift, steepening and curvature of the AAA and B3 corporate comovements of yield curves (4 bonds in the portfolio).

6 Conclusions

The multifactor models developed in this paper have been succesful in capturin gmore than 98% of the changes in yields and spreads in the corporate bonds. They do so when one credit rating class is analyzed at a time, thus ignoring comovements accross different credit classes, but also when multiple credit rating classes are analyzed simultaneously. The results of the factor analysis can be embedded in suitable immunization models. Again immunization strategies have been developed that hedge against the factors affecting a single rating class or for multiple rating classes, with our without an independence assumption. The results are encouraging although the shortcoming of the approach in attaching any financial or economic interpretation to the factors is recognized. We also point out that the analysis can be carried out simultaneously not only for multiple credit ratings but also for different corporate sectors (e.g., industrials and financials). Perhaps the factor analysis carried out here can also provide some insights in the search for the appropriate economic and financial variables.

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A The linear programming factor immunization models

In this appendix we formulate the linear programming models for structuring immunized portfolios. First we introduce some notation.

x_i is the quantity of ith bond in the portfolio, assumed throughout to be nonnegative so that short sales are excluded,

m is the number of available bonds,

K is the number of factors,

r_i is the yield to maturity of the ith bond,

r_P is the portfolio yield to maturity approximated by the following expression

$$r_P = \frac{\sum_{i=1}^{m} k_i x_i r_i}{\sum_{i=1}^{m} k_i x_i}; \tag{18}$$

P_i is the fair price of bond i,

P_{iq} is the quoted (market) price of bond i,

k_i is the dollar duration,

P_L is the present value of liabilities,

k_L is the dollar duration of the liabilities,

k_{jL} is the factor loading for factor j of the liabilities,

T_0 is the time horizon.

The standard linear programming model for portfolio immunization against parallel shifts, see Zenios [17], is given as:

$$\underset{x}{\text{maximize}} \quad r_P \tag{19}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} P_i x_i = P_L, \tag{20}$$

$$\sum_{i=1}^{m} k_i x_i = k_L. \tag{21}$$

To build a portfolio that is immunized from the changes of all factors we formulate a linear program with a constraint on matching the present value of assets with liabilities and additional constraints that match the factor loadings of assets with those of the liabilities. A direct extension of model (19)–(21) requires an oversimplification since the definition of a portfolio yield approximation needs assumptions of parallel shifts and a constant value in the denominator. Alternative immunization models, see Dahl [5], De Felice and Moriconi [7], can be formulated using an alternative objective function as follows

$$
\begin{aligned}
\underset{x}{\text{maximize}} \quad & \sum_{i=1}^{m} P_{iT_0} x_i && (22)\\
\text{s.t.} \quad & \sum_{i=1}^{m} P_i x_i = P_L, && (23)\\
& \sum_{i=1}^{m} k_{ij} x_i = k_{jL}, \qquad j = 1, \ldots, K. && (24)
\end{aligned}
$$

or

$$
\begin{aligned}
\underset{x}{\text{maximize}} \quad & \sum_{i=1}^{m} (P_i - P_{iq}) x_i && (25)\\
\text{s.t.} \quad & (23) \text{and } (24) \quad . && (26)
\end{aligned}
$$

Note that problem (25)–(26), due to constraint (23), is equivalent to

$$
\begin{aligned}
\underset{x}{\text{maximize}} \quad & \sum_{i=1}^{M} P_{iq} x_i && (27)\\
\text{s.t.} \quad & \sum_{i=1}^{m} P_i x_i = P_L, && (28)\\
& \sum_{i=1}^{m} k_{ij} x_i = k_{jL}, \qquad j = 1, \ldots, K. && (29)
\end{aligned}
$$

Model (22)–(24) requires estimation of future prices while model (27)–(29) do not require any additional computation as it uses quoted market prices. This is the model used in our empirical work.