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Capital-Labor Substitution and Equilibrium Indeterminacy*

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Abstract

Empirical evidence indicates that the elasticity of capital-labor substitution for the aggregate U.S. economy is below unity. In contrast, the existing indeterminacy literature has mostly restricted attention to a Cobb-Douglas production function which imposes a substitution elasticity exactly equal to unity. This paper examines the quantitative relationship between capital-labor substitution and the conditions needed for equilibrium indeterminacy (and belief-driven fluctuations) in a one-sector growth model. With variable capital utilization, the substitution elasticity has little quantitative impact on the minimum degree of increasing returns needed for indeterminacy. However, when capital utilization is constant, a below-unity substitution elasticity sharply raises the minimum degree of increasing returns. In this version of the model, lower substitution elasticities impose a higher adjustment cost on labor hours that cannot be mitigated by shifts in the capital utilization rate. Overall, our results show that empirically-plausible departures from the Cobb-Douglas specification can make indeterminacy more difficult to achieve.

Keywords: *Capital-Labor Substitution*, *Equilibrium Indeterminacy*, *Capital Utilization, Real Business Cycles, Sunspots.*

JEL Classification: E30, E32.

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1 Introduction

It is well-known that the elasticity of substitution between capital and labor in production can have an important influence on transition dynamics in the standard neoclassical growth model.[1] One might therefore expect this elasticity to influence the characteristics of fluctuations near the model's steady state. Rational belief-driven fluctuations (i.e., stationary sunspot equilibria) can arise in this class of models when the steady state is locally indeterminate.[2]

The existing indeterminacy literature has mostly restricted attention to a Cobb-Douglas production function which exhibits a unitary elasticity of substitution between capital and labor inputs. One piece of evidence against the Cobb-Douglas specification is the cyclical behavior of factor income shares in postwar U.S. data. A Cobb-Douglas specification implies that factor income shares are constant over the business cycle. In contrast, labor's share of income in U.S. data is countercyclical, while capital's share is procyclical. Over the period 1949.Q1 to 2004.Q4, the correlation coefficient between the cyclical component of labor's share and the cyclical component of real GDP is -0.26.[3]

Direct empirical estimates provide further evidence against the Cobb-Douglas specification. Chirinko (2008) reviews the many studies that have attempted to estimate the elasticity of capital-labor substitution using various econometric methods. He concludes that "the weight of the evidence suggests a value of σ [the elasticity parameter] in the range of $0.40 - 0.60$." Two recent studies that support this conclusion are Klump, McAdam, and Willman (2007) and Chirinko, Fazzari, and Meyer (2007).

In light of the evidence against the Cobb-Douglas specification, we seek to examine the quantitative relationship between the elasticity of capital-labor substitution and the minimum degree of increasing returns needed for local indeterminacy. Recent work by Pintus (2006, p. 643) has explored this issue in a calibrated one-sector growth model where the elasticity parameter lies in the range of $2.16 - 13.37$. Pintus' numerical analysis employs a utility function that is close to risk-neutral in consumption—an assumption that is not consistent with balanced growth within the model. Given the idiosyncratic nature of his numerical results, the aim of this paper is to provide a more complete picture of the quantitative link between capital-labor substitution and equilibrium indeterminacy in a plausibly calibrated one-sector growth model.

The framework for our analysis is an extended version of the model of Guo and Lansing (2007). The model allows for a variable rate of capital utilization that affects capital depreci-

[1] See, for example, Barro and Sala-i-Martin (1995, p. 45), Klump and de La Grandville (2000), Klump and Preissler (2000), Turnovsky (2002), and Smetters (2003), among others.

[2] See Benhabib and Farmer (1999) for a survey of this literature.

[3] The cyclical components are obtained by detrending each series with the Hodrick-Prescott filter, using a smoothing parameter of 1600. Data on labor's share of U.S. national income is from http://www.bls.gov/data, using series ID PRS85006173.

ation, along the lines of Wen (1998).[4] Capital depreciation can be mitigated by endogenous maintenance expenditures, along the lines of McGrattan and Schmitz (1999). When these features are shut down, the model collapses to one with constant utilization and depreciation rates, as in Benhabib and Farmer (1994) and Farmer and Guo (1994).

Following Klump and de La Grandville (2000) and Klump and Preissler (2000), we employ a "normalized" version of the standard constant-elasticity-of-substitution (CES) production function so that all steady-state allocations and factor income shares are held constant as the input substitution elasticity is changed. The normalization procedure identifies a family of CES production functions that are distinguished only by the elasticity parameter, and not by the steady-state allocations which are used to approximate the model's local dynamics. In practical terms, the normalization procedure amounts to recalibrating the model to "match the facts" each time the elasticity parameter is varied. Klump and Saam (2008) emphasize that normalization is necessary to avoid "arbitrary and inconsistent results."

Movements of labor's share of income in the model are linked directly to movements in the ratio of hours worked to utilized capital. For reasonable calibrations of the model, labor hours are more volatile than utilized capital in response to exogenous shocks. We show that in order to match the cyclical behavior of labor's share in the data, the model requires the elasticity of capital-labor substitution to be below unity, in agreement with the empirical studies summarized by Chirinko (2008).

We use numerical methods to examine the quantitative impact of capital-labor substitution on the minimum degree of increasing returns needed for local indeterminacy. We consider two versions of the model: one version allows for variable rates of capital utilization and depreciation, while the other version restricts these rates to be constant.

For the model with variable capital utilization and depreciation, we find that higher elasticities of capital-labor substitution cause the minimum degree of increasing returns for indeterminacy to decline monotonically, albeit gradually. Intuitively, a higher elasticity makes indeterminacy easier to obtain because it allows equilibrium labor hours to respond more freely to belief shocks, rather than being tightly coupled to utilized capital which responds more sluggishly. When the elasticity is unity (the Cobb-Douglas case), the model requires increasing returns-to-scale of around 1.08 for indeterminacy. When the elasticity is raised to 5, which far exceeds any empirical estimate, the minimum degree of increasing returns declines to around 1.05. When the elasticity is lowered to 0.4, as suggested by the U.S. empirical evidence, the minimum degree of increasing returns rises by a small amount to 1.09. For this version of the model, the substitution elasticity has little quantitative impact on the conditions needed for local indeterminacy.

[4] Instead of affecting depreciation, an alternative setup is one where the variable rate capital utilization is linked to a cost term that appears in the household budget constraint.

For the model with constant capital utilization and depreciation, we again find that higher substitution elasticities produce a monotonic decline in the minimum degree of increasing returns, but the slope of the quantitative relationship is now much steeper. For the Cobb-Douglas case, the model requires an implausibly large degree of increasing returns-to-scale for indeterminacy—around 1.5, confirming the results of Benhabib and Farmer (1994) and Farmer and Guo (1994). When the elasticity is raised to 5, the threshold degree of increasing returns drops sharply to the more plausible value of 1.1. But this result is a double-edged sword; when the elasticity is lowered to 0.4, as suggested by the U.S. empirical evidence, the threshold jumps sharply to around 2.3. Hence, for this version of the model, a realistic value for the substitution elasticity would essentially eliminate the possibility of local indeterminacy.

The intuition for why the substitution elasticity has a larger quantitative impact in the second version of the model is straightforward. When the capital utilization margin is shut-down, lower elasticities cause labor hours to become more tightly coupled to the stock of physical capital itself, rather than utilized capital. In effect, lower substitution elasticities impose a higher adjustment cost on labor hours that cannot be mitigated by shifts in the capital utilization rate. Consequently, equilibrium labor hours can respond less freely to belief shocks.

Overall, our results show that empirically-plausible departures from the Cobb-Douglas specification can make indeterminacy more difficult to achieve. It should be noted, however, that other types of models may become more susceptible to local or global indeterminacy when the elasticity of capital-labor substitution is *below* unity, as suggested by the empirical evidence. Examples include the capitalist-worker model of Grandmont, Pintus, and de Vilder (1998) and the multisector growth model of Nishimura and Venditti (2004). Another recent example is the one-sector growth model of Wong and Yip (2007) where the substitution elasticity is not a parameter, but instead is assumed to be a decreasing linear function of the economy's aggregate capital-labor ratio.

2 The Model

We adopt the basic framework of Guo and Lansing (2007) which allows for variable capital utilization and endogenous maintenance expenditures. A special case of this model imposes constant capital utilization and depreciation rates, as in Benhabib and Farmer (1994) and Farmer and Guo (1994). We depart from the usual assumption of a Cobb-Douglas production function by introducing a "normalized" version of the standard CES production function.

2.1 Households

The economy is populated by a unit measure of identical, infinitely-lived households, each endowed with one unit of time. The representative household maximizes

$$\sum_{t=0}^{\infty} \beta^t \left[\log(c_t) - \frac{A n_t^{1+\gamma}}{1+\gamma} \right], \qquad A > 0, \tag{1}$$

subject to the budget constraint

$$c_t = w_t n_t + d_t, \tag{2}$$

where $\beta \in (0,1)$ is the subjective time discount factor, c_t is consumption, n_t is hours worked, $\gamma \geq 0$ is the inverse of the intertemporal elasticity of substitution in labor supply, w_t is the real wage, and d_t is dividends paid out by the firms which the household takes as given. The household's period utility function in (1) is consistent with balanced long-run growth, a feature that is commonly maintained in the real business cycle literature.

The first-order condition for the household's optimization problem is given by

$$A c_t n_t^{\gamma} = w_t. \tag{3}$$

2.2 Firms

There are a large number of identical competitive firms, each endowed with k_0 units of capital, that produce a single final good y_t using the following linearly homogeneous technology:

$$y_t = B \left[\alpha (u_t k_t)^{\psi} + (1-\alpha) n_t^{\psi} \right]^{\frac{1}{\psi}} X_t, \tag{4}$$

$$B > 0, \qquad \alpha \in (0,1), \qquad \psi \equiv \frac{\sigma - 1}{\sigma}, \qquad \sigma \in (0, \infty),$$

where u_t is the endogenous rate of capital utilization and k_t is the firm's stock of physical capital. The parameter ψ depends on the elasticity of substitution σ between utilized capital $u_t k_t$ and labor hours n_t. When $\sigma = 1$ (or $\psi = 0$), we recover the usual Cobb-Douglas production technology. When $\sigma \to 0$ (or $\psi \to -\infty$), the production technology takes a Leontief formulation such that utilized capital and labor become perfect compliments. When $\sigma \to \infty$ (or $\psi \to 1$), utilized capital and labor become perfect substitutes.

As described in the appendix, our normalization procedure recalibrates the parameters B and α in equation (4) each time the elasticity of substitution σ is varied so that all steady-state allocations and factor income shares are held constant. Other model parameters are also recalibrated each time that σ is varied. For expositional convenience, we omit the explicit notation $B(\sigma)$ and $\alpha(\sigma)$ where these and other parameters appear in the paper.

The symbol X_t represents a productive externality that takes the form

$$X_t = Y_t^{\frac{\eta}{1+\eta}}, \qquad \eta \geq 0, \tag{5}$$

where Y_t is the economy-wide average level of output per firm. In a symmetric equilibrium, all firms take the same actions such that $y_t = Y_t$, for all t. As a result, equation (5) can be substituted into (4) to obtain the following social technology that may display increasing returns-to-scale:

$$y_t = B^{1+\eta} \left[\alpha \left(u_t k_t\right)^{\psi} + (1-\alpha)\, n_t^{\psi}\right]^{\frac{1+\eta}{\psi}}, \tag{6}$$

where the degree of increasing returns is given by $1+\eta$. When $\eta = 0$, the model collapses to one with constant returns-to-scale at both the firm and social levels.

The law of motion for the capital stock is given by

$$k_{t+1} = (1-\delta_t)\, k_t + i_t, \qquad k_0 \text{ given}, \tag{7}$$

where $\delta_t \in (0,1)$ is the endogenous rate of capital depreciation and i_t is investment in new capital. We postulate that δ_t takes the form

$$\delta_t = \tau \frac{u_t^{\theta}}{(m_t/k_t)^{\phi}}, \qquad \tau > 0,\;\; \theta > 1, \text{ and } \phi \geq 0, \tag{8}$$

where m_t/k_t represents maintenance expenditures per unit of installed capital. When $\phi = 0$, we recover the depreciation technology of Wen (1998) which abstracts from maintenance activity. Our setup is motivated by the work of McGrattan and Schmitz (1999) who argue that maintenance and repair activity is "too big to ignore." When $\theta \to \infty$ and $\phi = 0$, the model collapses to one with constant utilization and depreciation rates, as in Benhabib and Farmer (1994) and Farmer and Guo (1994).

Under the assumption that the labor market is perfectly competitive, firms take w_t as given and choose sequences of n_t, u_t, m_t, and $k_{t+1,}$ to maximize the following discounted stream of expected dividends:

$$\sum_{j=0}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-1} \underbrace{\left[y_{t+j} - w_{t+j}\, n_{t+j} - i_{t+j} - m_{t+j}\right]}_{d_{t+j}}, \tag{9}$$

subject to the firm's production function (4), the law of motion for capital (7), and the depreciation technology (8). Firms act in the best interests of households such that dividends in period $t+j$ are discounted using the household's intertemporal discount factor given by $\beta^j \left(c_{t+j}/c_t\right)^{-1}$.

The firm's first-order conditions with respect to the indicated variables are

$$n_t \quad : \qquad \frac{(1-\alpha)\, y_t}{n_t} \left[\frac{n_t^{\psi}}{\alpha\, (u_t k_t)^{\psi} + (1-\alpha) n_t^{\psi}} \right] = w_t, \tag{10}$$

$$u_t \quad : \qquad \frac{\alpha y_t}{\theta k_t} \left[\frac{(u_t k_t)^{\psi}}{\alpha\, (u_t k_t)^{\psi} + (1-\alpha) n_t^{\psi}} \right] = \delta_t, \tag{11}$$

$$m_t \quad : \qquad \phi\, \delta_t k_t = m_t \tag{12}$$

$$k_{t+1} \quad : \qquad \frac{1}{c_t} = \frac{\beta}{c_{t+1}} \left\{ \frac{\alpha y_{t+1}}{k_{t+1}} \left[\frac{(u_{t+1} k_{t+1})^{\psi}}{\alpha\, (u_{t+1} k_{t+1})^{\psi} + (1-\alpha) n_{t+1}^{\psi}} \right] + 1 - (1+\phi)\, \delta_{t+1} \right\}, \tag{13}$$

together with the transversality condition $\lim_{t\to\infty} \beta^t\ (k_{t+1}/c_t) = 0$.

By combining the household's budget constraint (2), the law of motion for capital (7), and the firm's dividend (9), we obtain the following aggregate resource constraint

$$y_t = c_t + k_{t+1} - (1-\delta_t)\ k_t + m_t. \tag{14}$$

3 Analysis of Dynamics

The dimensionality of the dynamical system can be reduced as follows. First, equation (12) is used to eliminate m_t from the resource constraint (14). Second, equation (11) is used to eliminate δ_{t+1} and δ_t from the consumption Euler equation (13) and the resource constraint (14). This procedure yields the dynamical system

$$k_{t+1} = y_t \left\{ 1 - \frac{\alpha\,(1+\phi)}{\theta} \left[\frac{(u_t k_t)^{\psi}}{\alpha\, (u_t k_t)^{\psi} + (1-\alpha) n_t^{\psi}} \right] \right\} + k_t - c_t, \tag{15}$$

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} \left\{ \frac{\alpha y_{t+1}}{k_{t+1}} \left(\frac{\theta - 1 - \phi}{\theta} \right) \left[\frac{(u_{t+1} k_{t+1})^{\psi}}{\alpha\, (u_{t+1} k_{t+1})^{\psi} + (1-\alpha) n_{t+1}^{\psi}} \right] + 1 \right\}, \tag{16}$$

where y_t is governed by the social technology (6). The dimensionality of the system can be further reduced by eliminating u_t and n_t, followed by u_{t+1} and n_{t+1}.

Solving equation (8) for u_t and then eliminating m_t and δ_t as before yields

$$u_t = \left(\frac{\phi^{\phi}}{\tau} \right)^{\frac{1}{\theta}} \left(\frac{\alpha y_t}{\theta k_t} \right)^{\frac{1+\phi}{\theta}} \left[\frac{(u_t k_t)^{\psi}}{\alpha\, (u_t k_t)^{\psi} + (1-\alpha) n_t^{\psi}} \right]^{\frac{1+\phi}{\theta}}, \tag{17}$$

which collapses to $u_t = 1$ as $\theta \to \infty$. The additively-separable term $\alpha\,(u_t k_t)^{\psi} + (1-\alpha) n_t^{\psi}$ in the denominator can be eliminated using the social technology (6). Doing so and then multiplying by k_t leads to the following expression:

$$(u_t k_t)^{\psi} = \left[\frac{\phi^{\phi}}{\tau} \left(\frac{\alpha}{\theta} \right)^{1+\phi} B^{(1+\phi)\psi} \right]^{\frac{\psi}{\theta-(1+\phi)\psi}} y_t^{\frac{[(1+\phi)(1+\eta)-\theta]\psi}{(1+\eta)[\theta-(1+\phi)\psi]}} k_t^{\frac{[\theta-1-\phi]\psi}{\theta-\psi(1+\phi)}}. \tag{18}$$

Next, we combine (3) and (10), and then once again use (6) to eliminate the additively-separable term $\alpha\,(u_t k_t)^{\psi} + (1-\alpha) n_t^{\psi}$. The resulting expression for n_t is

$$n_t = \left[\frac{(1-\alpha)\, B^{\psi}}{A} \right]^{\frac{1}{1+\gamma-\psi}} y_t^{\frac{1-\psi/(1+\eta)}{1+\gamma-\psi}} c_t^{\frac{-1}{1+\gamma-\psi}}. \tag{19}$$

The next step is to substitute the above expressions for $u_t k_t$ and n_t (and $u_{t+1} k_{t+1}$ and n_{t+1}) into the dynamical system and the social technology (6). The latter substitution yields a nonlinear equation for y_t in terms of k_t and c_t only. We log-linearize this equation around the normalized steady state (described in the Appendix) and then express y_t as an approximate power function in k_t and c_t. Iterating this function ahead one period generates an analogous function for y_{t+1} in terms of k_{t+1} and c_{t+1}. The approximate power functions for y_t and y_{t+1} are then substituted back into the dynamical system so that the only remaining variables are k_t, c_t, k_{t+1}, and c_{t+1}.

It is also useful to derive an approximate version of the equilibrium social technology in terms k_t and n_t. To accomplish this, we substitute the expression for $u_t k_t$ from (18) into (6). We then log-linearize the resulting expression around the normalized steady state and solve for y_t. The approximate social technology is given by

$$\log\left(y_t/\overline{y}\,\right) \;\simeq\; \alpha_k \log\left(k_t/\overline{k}\,\right) \,+\, \alpha_n \log\left(n_t/\overline{n}\,\right), \tag{20}$$

where $\overline{y}$, $\overline{k}$, and $\overline{n}$ are the normalized steady-state quantities. The production-function elasticities are given by

$$\alpha_k \;=\; \frac{\alpha\,(1+\eta)\,(\theta-1-\phi)}{\alpha\,[\theta-(1+\eta)\,(1+\phi)] \,+\, (1-\alpha)\,[\theta-\psi\,(1+\phi)]\,\left[\,\overline{n}\,/\,(\overline{u}\,\overline{k})\,\right]^{\psi}}, \tag{21}$$

$$\alpha_n \;=\; \frac{(1-\alpha)\,(1+\eta)\,[\theta-\psi\,(1+\phi)]}{\alpha\,[\theta-(1+\eta)\,(1+\phi)]\,\left[\,\overline{n}\,/\,(\overline{u}\,\overline{k})\,\right]^{-\psi} \,+\, (1-\alpha)\,[\theta-\psi\,(1+\phi)]}. \tag{22}$$

When $\psi = 0$, the above expressions are identical to those derived by Guo and Lansing (2007) for the Cobb-Douglas case. As usual, we restrict our analysis to the case of $\alpha_k < 1$, which implies that the productive externality is not strong enough to generate sustained endogenous growth. When $\eta = 0$, it is straightforward to verify that $\alpha_k + \alpha_n = 1$ for any value of ψ. That is, when the productive externality vanishes, the model exhibits constant returns-to-scale in production. This condition ensures that the individual firm's decision problem is concave.

3.1 Local Indeterminacy: Does Capital-Labor Substitution Matter?

The nonlinear dynamical system consists of equations (15) and (16) expressed in terms of k_t, c_t, $k_{t+1,}$ and c_{t+1}. The system is log-linearized around the normalized steady state to obtain:

$$\begin{bmatrix} \log\left(k_{t+1}/\overline{k}\,\right) \\ \log\left(c_{t+1}/\overline{c}\,\right) \end{bmatrix} = \mathbf{J} \begin{bmatrix} \log\left(k_t/\overline{k}\,\right) \\ \log\left(c_t/\overline{c}\,\right) \end{bmatrix}, \qquad k_0 \text{ given}, \tag{23}$$

where $\mathbf{J}$ is the Jacobian matrix of partial derivatives. The local stability properties of the steady state are determined by comparing the number of eigenvalues of $\mathbf{J}$ located inside the unit circle with the number of initial conditions. There is one initial condition represented by k_0. Hence, if both eigenvalues of $\mathbf{J}$ lie inside the unit circle, then the steady state is indeterminate (a sink) and the economy is subject to belief-driven fluctuations. This will occur if and only if

$$-1 \;<\; \det(\mathbf{J}) \;<\; 1 \quad \text{and} \quad -\left[1+\det(\mathbf{J})\right] \;<\; \mathrm{tr}(\mathbf{J}) \;<\; 1+\det(\mathbf{J}). \tag{24}$$

For our calibration, the most-binding condition among the necessary and sufficient conditions for local indeterminacy in (24) turns out to be $\det(\mathbf{J}) + \mathrm{tr}(\mathbf{J}) > -1$.

Figure 1 summarizes the stability properties of the steady state for the model with variable capital utilization and depreciation. Figure 2 is the analogous plot for the model with constant capital utilization and depreciation. For each pair of values for σ and η, we recalibrate the model using the normalization procedures described in the Appendix. The downward-sloping curve, which separates the regions labeled "Saddle" and "Sink, " plots the minimum required value of η for local indeterminacy. In both versions of the model, higher values of σ allow equilibrium indeterminacy to occur with a smaller externality parameter.

The intuition for why higher values of σ can make indeterminacy easier to obtain is straightforward. When agents become optimistic about the future, they will invest more today, thus raising next period's capital stock. To validate agents' optimistic expectations as a self-fulfilling prophecy, we require the next period's return on capital, net of depreciation, to rise in equilibrium. Other things equal, a higher value of σ allows labor hours n_t to respond more strongly to belief shocks, rather than being tightly coupled to either utilized capital $u_t k_t$ (Figure 1) or the stock of physical capital itself k_t (Figure 2), both of which respond more sluggishly than hours. The positive response of labor hours provides a direct boost to the return on capital, allowing agents' optimistic beliefs to become validated at a lower threshold degree of increasing returns.

In Figure 1, when $\sigma \to 0$ (Leontief), the model requires $\eta > 0.099$ for local indeterminacy. When $\sigma = 1$ (Cobb-Douglas), the model requires $\eta > 0.083$ for local indeterminacy, coinciding with results of Guo and Lansing (2007). When $\sigma = 5$, which far exceeds any empirical estimate for the U.S. economy, the model requires $\eta > 0.0496 \simeq 0.050$ for local indeterminacy.

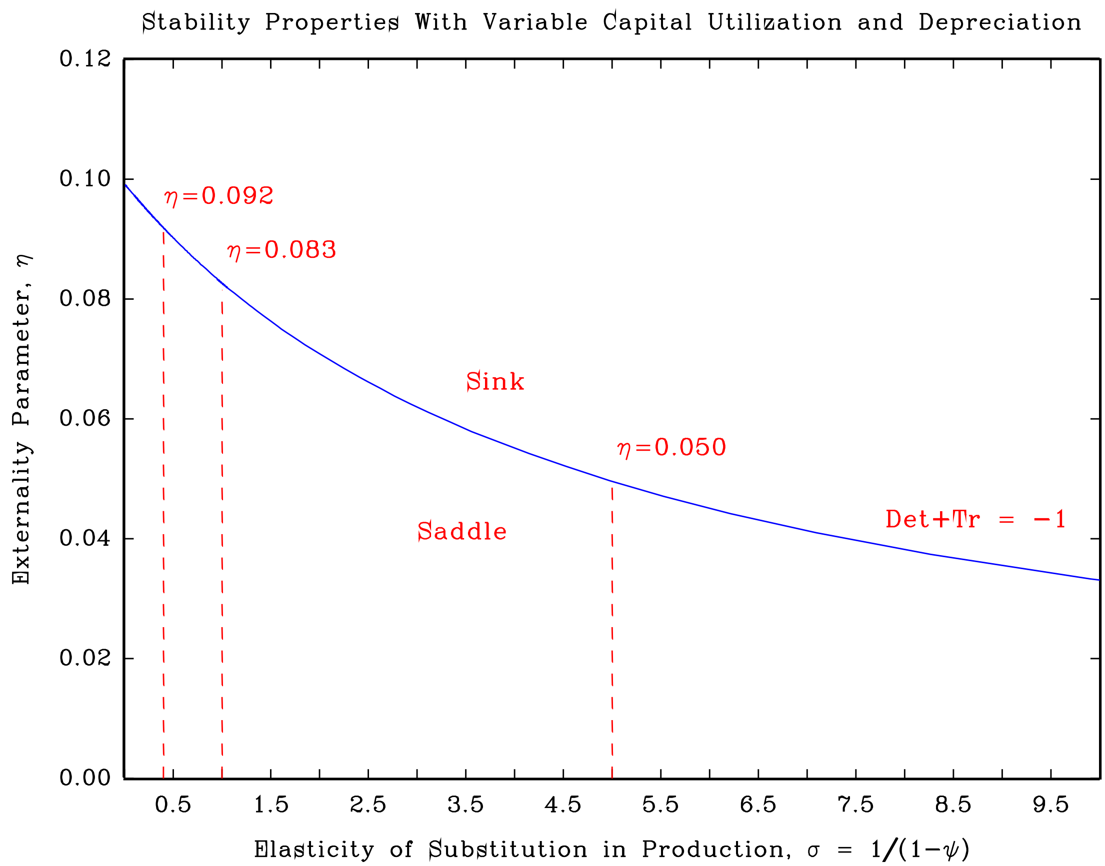

Figure 1: As σ increases, the externality threshold for indeterminacy declines gradually.

As described in the introduction, empirical studies indicate that the value of σ for the U.S. economy falls in the range of 0.4 to 0.6. In Figure 1, when $\sigma = 0.4$, the model requires $\eta > 0.092$ for local indeterminacy. This threshold corresponds to a relatively mild degree of increasing returns, one that remains within the realm of empirical plausibility. For example, Basu and Fernald (1997, Table 3, col. 1, p. 268) report a returns-to-scale estimate of 1.03 (standard error = 0.18) for the U.S. private business economy.

Although not shown, the stability curve in Figure 1 shifts upward by a small distance when maintenance expenditures are omitted from the model by setting $\phi = 0$. For example, when $\sigma = \{0.4,\ 1,\ 5\}$, the corresponding threshold values become $\eta > \{0.119,\ 0.104,\ 0.056\}$.

Relative to Figure 1, the level of the stability curve in Figure 2 is higher for any value of σ while the slope is steeper (taking into account the wider range for η plotted on the vertical axis). The intuition for the steeper curve in Figure 2 is straightforward. When the capital utilization margin is shutdown ($u_t = 1$), lower values of σ cause labor hours to become more

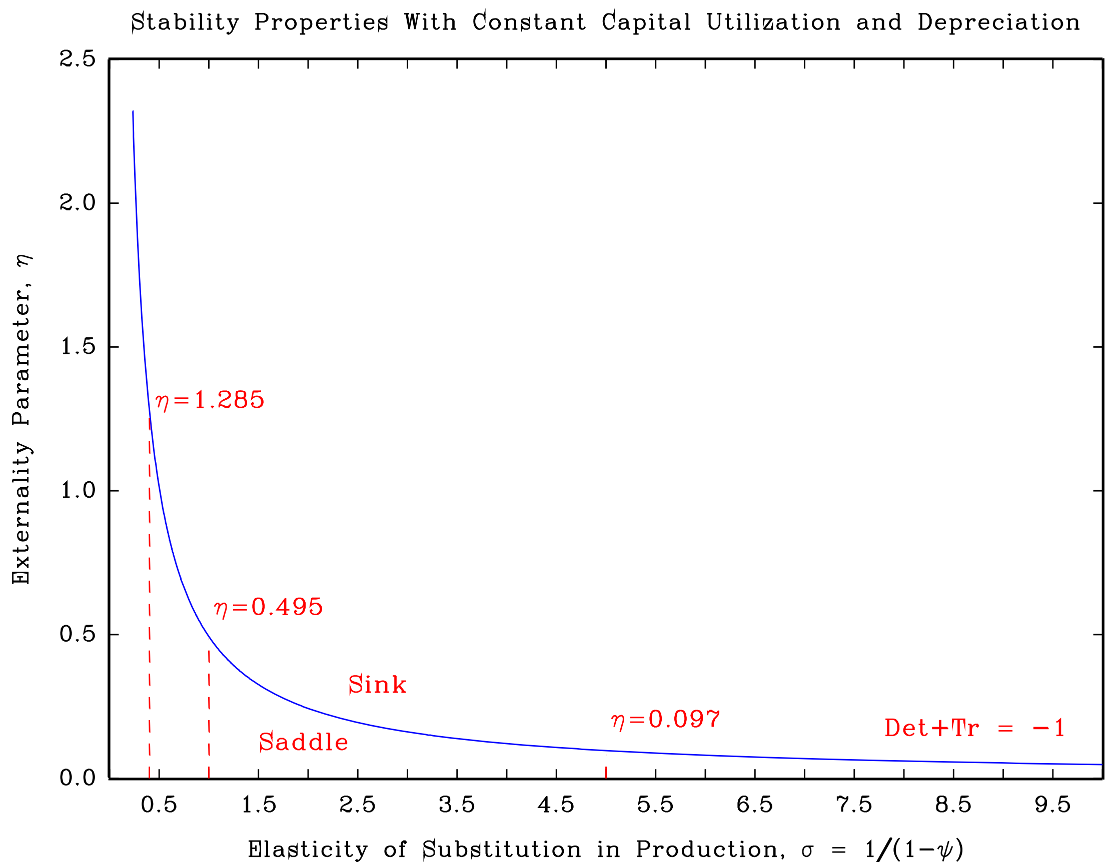

Figure 2: As σ increases, the externality threshold for indeterminacy declines sharply.

tightly coupled to k_t itself, rather than $u_t k_t$. In effect, lower values of σ impose a higher adjustment cost on labor hours that cannot be mitigated by shifts in u_t. The higher effective adjustment costs for labor hours make indeterminacy more difficult to achieve, thus requiring a higher threshold value for the externality parameter η.

In Figure 2, when $\sigma = 1$, the model requires $\eta > 0.495$ for indeterminacy—a value that is too large to be empirically plausible. In discrete time, it is well known that a necessary but not sufficient condition for local indeterminacy in the Cobb-Douglas case is

$$\eta \ > \ \frac{1+\gamma}{1-\alpha} - 1, \tag{25}$$

which implies that the equilibrium labor demand curve must be positively sloped and steeper than the labor supply curve.[5] Hintermaier (2003) derives a more-general necessary condition in the Cobb-Douglas case that extends to preferences which are non-separable in consumption

[5] In continuous time, Benhabib and Farmer (1994) show that condition (25) is both necessary and sufficient for local indeterminacy.

and leisure. For any concave utility function, he shows that a necessary but not sufficient condition for indeterminacy is $\eta > \alpha/(1-\alpha)$. For our calibration with $\gamma = 0$ and $\alpha = 0.3$, these two necessary conditions coincide and would require $\eta > 0.429$, a value that is somewhat below the true threshold value of 0.495 shown in Figure 1b.

Figure 2 also shows that when $\sigma = 5$, the threshold for indeterminacy drops sharply to $\eta > 0.097$, a value which is now within the range of empirical plausibility. However, the parameter setting $\sigma = 5$ is not empirically plausible. When $\sigma = 0.4$, as suggested by the U.S. empirical evidence, the threshold for indeterminacy jumps sharply to $\eta > 1.295$. Hence, for this version of the model, a realistic calibration for σ essentially eliminates any possibility of local indeterminacy. When $\sigma < 0.233$, indeterminacy is truly impossible in this version of the model because η must so large as to cause the equilibrium production-function elasticity of k_t to reach or exceed 1.0, thus implying non-existence of a steady state.

Pintus (2006) considers a one-sector growth model with constant capital utilization and depreciation rates. In his numerical examples, the coefficient of relative risk aversion (CRRA) for consumption in the agent's additively separable utility function is close to zero—a calibration that is not consistent with balanced growth within the model. He reports (p. 643) that when the CRRA is below 0.04 and the elasticity of capital-labor substitution exceeds 2.16, the model requires increasing returns-to-scale of around 1.03 for indeterminacy. In our framework, the CRRA for consumption is fixed at 1.0 (log utility). When $\sigma = 2.16$, the minimum degree of increasing returns for indeterminacy from Figure 2 is around 1.23. It follows that the lower threshold for indeterminacy in Pintus' numerical examples can be attributed to the assumption of very low curvature in the utility of consumption. A near-zero risk coefficient (or equivalently, a large elasticity of intertemporal substitution in consumption) implies a very small welfare loss from belief-driven cycles, making these cycles more likely to occur.[6] Pintus' model would require a higher degree of increasing returns for indeterminacy if the risk coefficient was increased to 1.0 (to achieve balanced growth in the model), or if the elasticity of capital-labor substitution was reduced to the empirically relevant range of 0.4 to 0.6.

4 Cyclical Behavior of Labor's Share of Income

Labor's share of income in the model is given by

$$\frac{w_t n_t}{y_t} = \frac{1-\alpha}{\alpha \left[n_t / (u_t k_t) \right]^{-\psi} + 1 - \alpha}, \tag{26}$$

which is obtained by rearranging the firm's first-order condition (10). The above expression shows that movements in labor's share over the business cycle are linked directly to movements

[6] The importance of very low curvature of utility in helping to achieve local indeterminacy is confirmed by Lloyd-Braga, Nourry, and Venditti (2006).

in the ratio $n_t/(u_t k_t)$.

For our calibration with indivisible labor $(\gamma = 0)$, labor hours n_t are more volatile than utilized capital $u_t k_t$ in response to exogenous shocks. A positive belief shock will therefore raise the ratio $n_t/(u_t k_t)$ while output y_t increases. Similarly, in the model with constant capital utilization, a positive belief shock will raise the ratio n_t/k_t while y_t increases.

When $\sigma > 1$, as in Pintus' (2006) numerical examples, we have $\psi > 0$ such that labor's share moves in the same direction as the pro-cyclical ratio $n_t/(u_t k_t)$. Hence, labor's share itself is pro-cyclical, which is not consistent with the postwar U.S. data. To achieve a countercyclical labor share, the term $w_t n_t / y_t$ must move in the opposite direction as the pro-cyclical ratio $n_t/(u_t k_t)$. This requirement is satisfied when $\sigma < 1$ such that $\psi < 0$. Intuitively, an elasticity of capital-labor substitution below unity ties labor hours more closely to utilized capital (or to the stock of physical capital itself), thus hindering the freedom of hours to respond to positive shocks so as to generate more labor income.

Gomme and Greenwood (1995) and Boldrin and Horvath (1995) also document the countercyclical behavior of labor's share of income in postwar U.S. data. Both papers develop models where labor contracts between workers and firms can break the direct link between the real wage and the marginal product of labor. The labor contracts can generate a countercyclical labor share even when the elasticity of capital-labor substitution is unity (the Cobb-Douglas case).

5 Conclusion

The weight of empirical evidence indicates that the elasticity of capital-labor substitution for the aggregate U.S. economy is below unity. In contrast, the indeterminacy literature has mostly restricted attention to a Cobb-Douglas production function which imposes a substitution elasticity exactly equal to unity. In a model with variable capital utilization, we showed that the Cobb-Douglas assumption, although counterfactual from the standpoint of the data, turns out to be fairly innocuous; the assumption does not significantly impact the minimum degree of increasing returns needed for local indeterminacy. However, in a model with constant capital utilization, the Cobb-Douglas assumption is by no means innocuous; it significantly understates the difficulty of achieving indeterminacy relative to a plausibly-calibrated model where the substitution elasticity is below unity. Overall, our results show that, depending on the model, findings of local indeterminacy may not be robust to empirically-plausible departures from the Cobb-Douglas specification.

A Appendix: Normalization Procedure

A.1 Model with Variable Capital Utilization and Depreciation

The normalized steady-state quantities are denoted by $\overline{n}$, $\overline{\delta}$, $\overline{k}$, $\overline{y}$, $\overline{c}$, $\overline{u}$, and $\overline{m}$. As the elasticity of capital-labor substitution σ is varied, the normalized quantities are held constant by the appropriate choice of parameters. The reference point that defines the normalized quantities is the Cobb-Douglas case with $\sigma = 1$ (or $\psi = 0$) and $B = 1$. Following Guo and Lansing (2007), straightforward computations for the Cobb-Douglas case yield.

$$\overline{n} = \left[\frac{1-\overline{\alpha}}{A - A\,\overline{\alpha}\,(1+\phi)\,/\theta}\right]^{\frac{1}{1+\gamma}}, \tag{A.1}$$

$$\overline{\delta} = \frac{\rho}{\theta - 1 - \phi}, \tag{A.2}$$

$$\overline{k} = \left[b\,\overline{\alpha}\,\overline{n}^{\mu_n} \,/\, \left(\theta\,\overline{\delta}\right)\right]^{\frac{1}{1-\mu_k}}, \tag{A.3}$$

$$\overline{y} = b\,\overline{k}^{\mu_k}\,\overline{n}^{\mu_n}, \tag{A.4}$$

$$\overline{c} = \left[1 - \overline{\alpha}\,(1+\phi)\,/\theta\right]\,\overline{y}, \tag{A.5}$$

$$\overline{u} = \left[\frac{\phi^{\phi}\ \overline{\delta}^{\,1+\phi}}{\tau}\right]^{\frac{1}{\theta}}, \tag{A.6}$$

$$\overline{m} = \left[\frac{\phi\,\overline{\alpha}}{\theta}\right]\overline{y}, \tag{A.7}$$

where b, μ_k, and μ_n represent combinations of parameters, $\rho \equiv 1/\beta - 1$ is the rate of time preference, and $\overline{\alpha} = 0.3$ such that labor's share of income for the Cobb-Douglas case is given by $1 - \bar{\alpha} = 0.70$. The definitions for b, μ_k, and μ_n are:

$$b \equiv \left[\frac{\phi^{\phi}}{\tau}\left(\frac{\overline{\alpha}}{\theta}\right)^{1+\phi}\right]^{\frac{\overline{\alpha}(1+\eta)}{\theta - \overline{\alpha}(1+\eta)(1+\phi)}}, \tag{A.8}$$

$$\mu_k \equiv \frac{\overline{\alpha}\,(1+\eta)\,(\theta - 1 - \phi)}{\theta - \overline{\alpha}\,(1+\eta)\,(1+\phi)}, \tag{A.9}$$

$$\mu_n \equiv \frac{(1-\overline{\alpha})\,(1+\eta)\ \theta}{\theta - \overline{\alpha}\,(1+\eta)\,(1+\phi)}. \tag{A.10}$$

Given a value for the externality parameter η, the elasticity of substitution σ is varied over a wide range of values. For all computations, we set $\beta = 0.99$ to obtain a quarterly real interest rate of 1 percent and $\gamma = 0$ to reflect "indivisible labor". The constant τ affects no result, so we set $\tau = 1$.

As σ takes on different values, the parameters ϕ, θ, and A are set to maintain the following calibration targets used by Guo and Lansing (2007): $\overline{m}/\overline{y} = 0.061$, $\overline{\delta} = 0.025$, and $\overline{n} = 0.3$. The remaining parameters for the general CES specification are α and B. As σ is varied, the parameter α is set to maintain the steady-state labor's income share at 0.7, while the parameter B is set to maintain the steady-state output level equal to the Cobb-Douglas value $\overline{y}$. In this way, all steady state quantities are maintained at the corresponding Cobb-Douglas values.

The normalization procedure is defined by the following calibration formulas

$$\phi = \overline{m}/\left(\overline{\delta}\,\overline{k}\right), \tag{A.11}$$

$$\theta = 1 + \phi + \rho/\overline{\delta}, \tag{A.12}$$

$$\alpha = \frac{\overline{\alpha}}{\overline{\alpha} + (1-\overline{\alpha})\left(\overline{u}\overline{k}/\overline{n}\right)^{\psi}} \tag{A.13}$$

$$B = \frac{\overline{y}^{\,\frac{1}{1+\eta}}}{\left[\alpha\left(\overline{u}\overline{k}\right)^{\psi} + (1-\alpha)\,\overline{n}^{\psi}\right]^{\frac{1}{\psi}}} \tag{A.14}$$

$$A = \frac{(1-\alpha)\,B^{\psi}\,\overline{y}^{\,\frac{-\psi}{1+\eta}}\left[1 - \overline{m}/\overline{y} - \overline{\delta}\,\overline{k}/\overline{y}\right]^{-1}}{\overline{n}^{\,1+\gamma-\psi}}, \tag{A.15}$$

where $\psi = (\sigma - 1)/\sigma$, $\overline{\alpha} = 0.3$, and $\overline{m}$, $\overline{\delta}$, $\overline{k}$, $\overline{n}$, and $\overline{y}$, are the steady-state quantities from the Cobb-Douglas case.

A.2 Model with Constant Capital Utilization and Depreciation

The steady-state quantities at the Cobb-Douglas reference point ($\psi = 0$ and $B = 1$) are now given by

$$\overline{n} = \left\{\frac{1-\overline{\alpha}}{A\left[1 - \delta\,\overline{\alpha}/(\rho+\delta)\right]}\right\}^{\frac{1}{1+\gamma}}, \tag{A.16}$$

$$\overline{k} = \left[\frac{\overline{\alpha}\,\overline{n}^{\,(1-\overline{\alpha})(1+\eta)}}{\rho+\delta}\right]^{\frac{1}{1-\overline{\alpha}(1+\eta)}}, \tag{A.17}$$

$$\overline{y} = \left[\overline{k}^{\,\overline{\alpha}}\,\overline{n}^{\,1-\overline{\alpha}}\right]^{1+\eta}, \tag{A.18}$$

$$\overline{c} = \left[1 - \delta\,\overline{\alpha}/(\rho+\delta)\right]\overline{y}, \tag{A.19}$$

where $\delta = 0.025$ is a constant parameter.

The normalization procedure is defined by the following calibration formulas

$$\alpha = \frac{\overline{\alpha}}{\overline{\alpha} + (1-\overline{\alpha})\left(\overline{k}/\overline{n}\right)^{\psi}} \tag{A.20}$$

$$B = \frac{\overline{y}^{\frac{1}{1+\eta}}}{\left[\alpha\overline{k}^{\psi} + (1-\alpha)\,\overline{n}^{\psi}\right]^{\frac{1}{\psi}}} \tag{A.21}$$

$$A = \frac{(1-\alpha)\,B^{\psi}\,\overline{y}^{\frac{-\psi}{1+\eta}}\left[1-\delta\overline{k}/\overline{y}\right]^{-1}}{\overline{n}^{1+\gamma-\psi}}, \tag{A.22}$$

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