Optimal bidding in a uniform price auction with multi-unit demand[1]

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Abstract

We partially characterize a class of symmetric, monotonic, Bayes-Nash equilibria to a uniform-price sealed-bid auction in an environment in which each demander wishes to purchase multiple units. © 1997 Elsevier Science S.A.

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1. Introduction

In many multi-unit auctions, the typical demander wishes to buy more than one unit from the set of objects being sold. The theoretical literature on auctions, however, has mostly studied environments in which each bidder can obtain at most one unit.[2] We model here a simple multi-unit uniform-price sealed-bid auction in an environment in which demanders have independent private valuations for obtaining up to two of the k identical units sold. The units are awarded to the k highest bidders, each of whom pays a per-unit price equal to the kth highest bid. We partially characterize the class of symmetric Bayes-Nash equilibria in which bids are strictly monotonically increasing in valuations.

2. Model and definitions

A seller offers k (≥ 2) identical units for sale. There are $n + 1 (> (k/2))$ demanders, indexed by i. Each demander i draws a pair of valuations v_1^i and v_2^i, where $v_1^i \geq v_2^i$, from a common probability

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[1]The views expressed in this article are those of the authors and do not necessarily reflect those of the OECD or of the governments of its member countries.

[2]Recent exceptions include Ausubel and Cramton (1996), Maskin and Riley (1992), Noussair (1995) and Engelbrecht-Wiggans and Kahn (1995), all of whom incorporate multi-unit demand in their models.

distribution $G(v_1,v_2)$ with density function $g(v_1,v_2)$. Assume that $G(0,0) = 0$ and $G(1,1) = 1$. G, n and k are common knowledge but v^i_j is known privately to demander i for $j = 1,2$ and for all i.

Each demander submits two bids simultaneously. b^i_1 and b^i_2 denote the two bids of bidder i and the bids are labelled so that $b^i_1 \geq b^i_2$. The bidders who make the k highest bids are each awarded a number of items equal to the number of bids they have among the top k and pay a per-unit price equal to the kth highest bid.[3]

A pure strategy is a mapping from valuations into bids $B(v_1,v_2) = (B_1(v_1,v_2), B_2(v_1,v_2))$; $[0,1]^2 \to R^{2+}$ where $B_1(v_1,v_2) \geq B_2(v_1,v_2)$, $\forall v_1,v_2$. A strategy $B(v_1,v_2)$ is said to be *separable* if $B(v_1,v_2) = (B_1(v_1), B_2(v_2))$, that is, if the demander's bid for his higher (lower) valued unit is independent of his lower (higher) valuation. A strategy $B(v_1,v_2)$ is *strictly monotonic* if $(\partial B_j(v_1,v_2)/ \partial v_j) > 0$ for $j = 1,2$ and for all v_j, and is *weakly monotonic* if $(\partial B_j(v_1,v_2)/\partial v_j) \geq 0$ for $j = 1,2$ and for all v_j. A *Bayes-Nash equilibrium* is a profile of strategies in which each player maximizes her expected payoff given the strategies of the other bidders. The equilibrium is *symmetric* if each player is using the same strategy. Since only symmetric equilibria are considered here, $B(v_1,v_2)$ designates both a strategy and a strategy profile. Let the probability distribution $F^B_\nu(x_\nu)$ be the distribution of the νth order statistic of the $2n$ bids made by n bidders using the strategy $B(v_1,v_2)$. x_ν denotes the νth lowest (or alternatively the $2n - v + 1st$ highest) of the $2n$ bids. Let $f^B_\nu(x_\nu)$ denote the corresponding density function.

3. Results

Lemma 1 shows that if an equilibrium strategy is strictly monotonic, then (i) it must be separable, and (ii) each bidder must always submit bids less than or equal to her valuations.

Lemma 1. If $B^*(v_1,v_2)$ is a strictly monotone symmetric equilibrium strategy, and $F^{B*}_\nu(x_\nu)$ is twice differentiable for $\nu = 2n - k + 1, 2n - k + 2, 2n - k + 3$, then (i) $B^*(v_1,v_2) = (B^*_1(v_1), B^*_2(v_2))$, and (ii) $B^*_j(v_j \leq v_j; \ j = 1,2; \ \forall v_j \in [0,1]$.

Proof. Let $B^*(v_1,v_2)$ be a symmetric Bayes-Nash equilibrium. In equilibrium, bidder i chooses b^i_1, $b^i_2 \geq 0$ to maximize expected profit, denoted by $E\pi^i(b^i_1, b^i_2)$, and given by:

$$\int_0^{b^i_2} (v^i_1 + v^i_2 - 2x_{2n-k+3})f^{B*}_{2n-k+3}(x_{2n-k+3})dx_{2n-k+3} + (v^i_1 + v^i_2 - 2b^i_2)(F^{B*}_{2n-k+2}(b^i_2) - F^{B*}_{2n-k+3}(b^i_2))$$

$$= \int_{b^i_2}^{b^i_1} (v^i_1 - x_{2n-k+2})f^{B*}_{2n-k+2}(x_{2n-k+2})dx_{2n-k+2} + (v^i_1 - b^i_1)(F^{B*}_{2n-k+1}(b^i_1) - F^{B*}_{2n-k+2}(b^i_1)). \qquad (1)$$

The first term in (1) corresponds to the case when the purchase price (the kth highest of all $2n + 2$ bids) is lower than bidder i's lower bid, in which case bidder i receives two units and pays a per-unit

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[3]Ties for the kth highest bid are broken by allocating the tied unit(s) to each of the tied bidders with equal probability. If there are d bids tied for the last c units, each of the tied bids is accepted with probability (c/d). Ties occur with probability zero in the type of equilibria considered in this paper.

price equal to the $k-2$nd highest bid submitted by the other n bidders. The second term corresponds to the case in which the per-unit purchase price equals bidder i's lower bid, in which case bidder i receives two units. The third term is the case when the purchase price is lower than bidder i's higher bid and higher than bidder i's lower bid, in which case bidder i receives one unit at a price equal to the $k-1$st highest of the other players' bids. The fourth term comprises the case when the purchase price equals bidder i's higher bid, in which case bidder i receives one unit. If the purchase price is higher than bidder i's higher bid, i receives zero units and the payoff to i is zero.

The first-order necessary conditions are:

$$\frac{\partial E\pi^i}{\partial b_j^i} = (v_j^i - b_j^i)f_{2n-k+j}^{B*}(b_j^i) - j(F_{2n-k+j}^{B*}(b_j^i))(F_{2n-k+j+1}^{B*}(b_j^i)) \quad \begin{aligned} &= 0 \text{ if } b_j^i > 0 \\ &\leq \text{ if } b_j^i = 0, \end{aligned} \tag{2}$$

for $j=1,2$. Setting b_j^i equal to $B_j^*(v_1^i,v_2^i)$ must be a solution to (2). The derivative of the first order condition with respect to v_j is:

$$(v_j^i - B_j^*(v_1,v_2))\frac{\mathrm{d}f_{2n-k+j}^{B*}(B_j^*(v_1,v_2))}{\mathrm{d}B_j^*}\frac{\partial B_j^*(v_1,v_2)}{\partial v_j} - jf_{2n-k+j}^{B*}(B_j^*(v_1,v_2))\frac{\partial B_j^*(v_1,v_2)}{\partial v_j}$$

$$+ jf_{2n-k+j+1}^{B*}(B_j^*(v_1,v_2))\frac{\partial B_j^*(v_1,v_2)}{\partial v_j} + \left(1 + \frac{\partial B_j^*(v_1,v_2)}{\partial v_j}\right)f_{2n-k+j}^{B*}(B_j^*(v_1,v_2)) = 0, \tag{3}$$

for $B_j^* > 0$. This can be rewritten as:

$$\frac{\partial B_j^*}{\partial v_j}$$

$$= \frac{-f_{2n-k+j}^{B*}(B_j^*(v_1,v_2))}{(v_j^i - B_j^*(v_1,v_2))\dfrac{\mathrm{d}f_{2n-k+j}^{B*}(B_j^*(v_1,v_2))}{\mathrm{d}B_j^*} - (1+j)f_{2n-k+j}^{B*}(B_j^*(v_1,v_2)) + jf_{2n-k+1+j}^{B*}(B_j^*(v_1,v_2))}. \tag{4}$$

Since the density function $f_{2n-k+j} \geq 0$ and B^* is strictly monotonic, it must be the case that $f_{2n-k+j} > 0$, implying that the numerator of (4) is strictly less than zero. The denominator is therefore strictly less than zero by the assumption of strict monotonicity.[4] Since $f_{2n-k+j} > 0$,[5] and the function f_{2n-k+j} is

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[4]Notice that the second derivatives of the profit function are given by:

$$\frac{\partial^2 E\pi_i}{\partial b_j^{i2}} = (v_j^i - b_j^i)\frac{\mathrm{d}f_{2n-k+j}^{B*}(b_j^i)}{\mathrm{d}b_j^i} - (1+j)f_{2n-k+j}^{B*}(b_j^i) + jf_{2n-k+j+1}^{B*}(b_j^i) \tag{5}$$

for $j=1,2$ and

$$\frac{\partial^2 E\pi^i}{\partial b_1^i \partial b_2^i} = \frac{\partial^2 E\pi^i}{\partial b_2^i \partial b_1^i} = 0 \tag{6}$$

Since the denominator of (4) is the same expression as in equation (5), (5) must be < 0, indicating that the second order necessary conditions for a maximum are satisfied by any solution to (2) if B_j^* is strictly monotonic.

[5]A weakly monotonic symmetric equilibrium strategy profile can have a flat portion at some $b_j^i > 0$ only if $f_{2n-k+j}^{B*}(b_j^i) = 0$. This is seen by noticing that the left side of (4) can equal 0 only if the numerator of the right side equals 0.

derived only from the strategies chosen by players other than i, equation (2) indicates that b_1^i must be independent of v_2^i and that b_2^i must be independent of v_1^i. From equation (2), it also follows that since $(F_{2n-k+j}(b_j^i) - F_{2n-k+j+1}(b_j^i)) \geq 0$ and $f_{2n-k+j}(b_j^i) > 0$ that $B_j^*(v_j) \leq v_j$. $\square$

Proposition 1 gives precise necessary conditions for a symmetric, strictly monotonic strategy profile to be a Bayes-Nash equilibrium.

Proposition 1. The strategy profile consisting of each player using the strategy $\bar{B}(v_1, v_2)$ is a strictly monotone symmetric Bayes-Nash equilibrium strategy only if $\bar{B}(v_1, v_2)$ equals $\beta(v_1, v_2) = (\beta_1(v_1), \beta_2(v_2))$, where $\beta(v_1, v_2)$ satisfies:

$$\beta_j(v_j) = v_j - j \frac{\sum_{l,m:2l+m=2n-k+j} H(V_1(\beta_j(v_j)), V_2(\beta_j(v_j)), \gamma, m, n, l)}{\sum_{q=2n-k+j}^{2n} \sum_{l,m:2l+m=q} \left(\frac{\partial H(\cdot)}{\partial V_1} V_1'(\beta_j(v_j)) + \frac{\partial H(\cdot)}{\partial V_2} V_2'(\beta_j(v_j)) \right)}; j = 1,2 \qquad (7)$$

with initial conditions $\beta_1(1) = 1$ and $\beta_2(0) = 0$, where $V_j(x) = \beta_j^{-1}(x)$, and where:

$$H(z_1, z_2, \gamma, m, n, l) =$$

$$\frac{n!}{l!m!(n-m-l)!} \left(\int_0^{z_1} \int_0^{z_2} g(v_1, v_2) dv_2 dv_1 \right)^l \left(\int_{z_1}^1 \int_0^{z_2} g(v_1, v_2) dv_2 dv_1 \right)^m \left(\int_{z_1}^1 \int_{z_2}^1 g(v_1, v_2) dv_2 dv_1 \right)^{n-m-l}.$$
$$(8)$$

Proof. Suppose all bidders are using a symmetric profile of strategies, $\bar{B}(v_1, v_2) = (\bar{B}_1(v_1), \bar{B}_2(v_2))$, which satisfies the necessary condition (2). Consider some bidder $y \neq i$. Since $\bar{B}_j(v_j)$ is strictly monotone increasing in v_j, it is invertible, and we can define the probability the bidder y submits two bids, one bid, and zero bids less than or equal to $\bar{b}_j^i$ as equalling respectively:

$$Prob[\bar{b}_2^y \leq \bar{b}_j^i, \bar{b}_1^y \leq \bar{b}_j^i] = \int_0^{\bar{B}_1^{-1}(\bar{b}_j^i)} \int_0^{\bar{B}_2^{-1}(\bar{b}_j^i)} g(v_1, v_2) dv_2 dv_1,$$

$$Prob[\bar{b}_2^y \leq \bar{b}_j^i, \bar{b}_1^y > \bar{b}_j^i] = \int_{\bar{B}_1^{-1}(\bar{b}_j^i)}^1 \int_0^{\bar{B}_2^{-1}(\bar{b}_j^i)} g(v_1, v_2) dv_2 dv_1,$$

and

$$Prob[\bar{b}_2^y > \bar{b}_j^i, \bar{b}_1^y > \bar{b}_j^i] = \int_{\bar{B}_1^{-1}(\bar{b}_j^i)}^1 \int_{\bar{B}_2^{-1}(\bar{b}_j^i)}^1 g(v_1, v_2) dv_2 dv_1,$$

$F_\nu^{\bar{B}}(x_\nu)$, is then the summation of $H(\cdot)$, where H is as defined in equation (8), over all combinations of l and m such that $\nu \leq 2l + m \leq 2n$, and such that $l, m \geq 0$. Then,

$$F_{\nu}^{\bar{B}}(\bar{b}_j^i) = \sum_{q=\nu}^{2n} \sum_{l,m:2l+m=q} H(\bar{B}_1^{-1}(\bar{b}_j^i), \bar{B}_2^{-1}(\bar{b}_j^i), \gamma, m, n, l) \tag{9}$$

It then follows easily that

$$f_{\nu}^{\bar{B}}(\bar{b}_j^i) = \sum_{q=\nu}^{2n} \sum_{l,m:2l+m=q} \left(\frac{\partial H}{\partial \bar{B}_1^{-1}} \frac{d\bar{B}_1^{-1}(\bar{b}_j^i)}{d\bar{b}_j^i} + \frac{\partial H}{\partial \bar{B}_2^{-1}} \frac{d\bar{B}_2^{-1}(\bar{b}_j^i)}{d\bar{b}_j^i} \right)$$

and that

$$F_{\nu}^{\bar{B}}(\bar{b}_j^i) - F_{\nu+1}^{\bar{B}}(\bar{b}_j^i) = \sum_{l,m:2l+m=\nu} H(\bar{B}_1^{-1}(\bar{b}_j^i), \bar{B}_2^{-1}(\bar{b}_j^i), \gamma, m, n, l).$$

Substituting into (2) and rearranging terms results in (7).

We now show that $\bar{B}_1(1) = 1$.[6] Consider the necessary condition (2) evaluated at $\bar{B}_1(1)$. $f_{2n-k+1}^{\bar{B}}(\bar{B}_1(1)) > 0$ by strict monotonicity. We need to show that $F_{2n-k+1}^{\bar{B}}(\bar{B}_1(1)) - F_{2n-k+2}^{\bar{B}}(\bar{B}_1(1)) = 0$. $F_{2n-k+1}^{\bar{B}}(\bar{B}_1(1)) - F_{2n-k+2}^{\bar{B}}(\bar{B}_1(1))$

$$= \sum_{l,m:2l+m=2n-k+1} \frac{n!}{l!m!(n-m-l)!} \left(\int_0^1 \int_0^{\bar{B}_2^{-1}(\bar{B}_1(1))} g(v_1,v_2)dv_2dv_1 \right)^l$$

$$\times \left(\int_1^1 \int_0^{\bar{B}_2^{-1}(\bar{B}_1(1))} g(v_1,v_2)dv_2dv_1 \right)^m \left(\int_1^1 \int_{\bar{B}_2^{-1}(\bar{B}_1(1))}^1 g(v_1,v_2)dv_2dv_1 \right)^{n-m-l}. \tag{10}$$

This expression equals zero unless $m=0$, $n-m-l=0$ and $2l+m=2n-k+1$. These last three statements can simultaneously be true only if $k=1$. $((m=0, n-m-l=0, 2l+m=2n-k+1) \Rightarrow (n=l, 2l=2n-k+1) \Rightarrow (2n=2n-k+1) \Rightarrow (k=1))$. Since we assume that $k>1$, the expression equals 0 for all relevant parameters.

A similar argument can be used to show that $\beta_2(0) = 0$. Consider (2) evaluated at $\bar{B}_2(0)$. Since $f_{2n-k+2}^{\bar{B}}(\bar{B}_2(0)) > 0$, it suffices to show that $\sum_{l,m:2l+m=2n-k+2} H(\bar{B}_1^{-1}(\bar{B}_2(0)), 0, \gamma, m, n, l) = 0$. The expression equals 0 unless $m=0$, $l=0$ and $2l+m=2n-k+2$. These last three events can only occur simultaneously if $2n+2=k$. Since $2n+2>k$ (there are more valuations than units) the expression equals 0 for all relevant parameters. $\square$

4. Concluding remarks

We have derived precise necessary conditions for a symmetric profile of strategies to be a strictly monotonic Bayes-Nash equilibrium to a simple auction game. Strict monotonicity comprises an interesting class of equilibria to consider, since it has the reasonable property that demanders with higher valuations submit higher bids. In equilibrium, each demander must submit $b_1^i \le v_1^i$ and $b_2^i \le v_2^i$.

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[6]It also must be the case that $\bar{B}_1(0) = 0$ by the fact that $\bar{B}_j(v_j) \le v_j$ and the non-negativity constraint on bids.

The underrevelation $(b_j^i < v_j^i)$ occurs for purely non-cooperative reasons, and does not require any collusion on the part of bidders. The equilibrium strategy must be separable because the trade-offs determining the choice of b_j^i are independent of the valuations of bidder i other than v_j^i (though they do depend on the strategies of players other than i). Increasing b_j^i increases the probability of acceptance of b_j^i, but not of i's other bids. Lowering b_j^i reduces the price paid for all of the units i receives in the event that b_j^i is the kth highest bid overall, but the benefit from a lower price depends only on j, the rank of the bid (and therefore how many of i's bids are accepted in the event that b_j^i is the marginal bid), and not on the actual amount of demander i's valuations.

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