Simple priorities and core stability in hedonic games*

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Abstract

In this paper we study hedonic games where each player views every other player either as a friend or as an enemy. Two simple priority criteria for comparison of coalitions are suggested, and the corresponding preference restrictions based on appreciation of friends and aversion to enemies are considered. We characterize internally

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stable coalitions on the proposed domains and show how these characterizations can be used for generating a strict core element in the first case and a core element in the second case. Moreover, we prove that an element of the strict core under friends appreciation can be found in polynomial time, while finding an element of the core under enemies aversion is NP-hard.

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1 Introduction

The study of the hedonic aspect of coalition formation goes back to Drèze and Greenberg (1980) who stress the dependence of a player's utility on the composition of members of her coalition. The formal model of a hedonic game was introduced by Banerjee, Konishi and Sönmez (2001) and Bogomolnaia and Jackson (2002). In their work, the focus on the identity of the members of a coalition determines the structure of the game: the latter consists of a preference ranking, for each player, over the coalitions that player may belong to.

Despite the simplicity of the model, it turned out that the question of the existence of a *core stable partition*, that is, a partition of the set of all players for which there is no group of individuals who can all be better off by forming a new deviating coalition, does not have an easy answer. In this paper we restrict ourselves to hedonic games with *separable preferences*, i.e. games where the effect of a given player on another player's preferences is the same, regardless of which coalition the latter player is a member of. In such games every player partitions the society into desirable and undesirable coalitional partners (friends and enemies, respectively), and the division between friends and enemies guides the ordering of coalitions in the sense that adding a friend leads to a more preferable coalition, while adding an enemy leads to a less preferable coalition.

As shown by Banerjee, Konishi and Sönmez (2001), non-emptiness of the core is not guaranteed even if one restricts separability to *additive separability* (players' preferences are representable by an additive separable utility function) and imposes in addition *symmetry* (i.e. the players have the same reciprocal values for each other). For an excellent study of the role of symmetric additive separable preferences for non-emptiness of the core of a hedonic game the reader is referred to Burani and Zwicker (2003).

In this paper we restrict the domain of additive separable preferences by assuming that each player uses a simple priority criterion when comparing coalitions she may belong to. As a result, the class of additive separable preferences based on *appreciation of friends* and the class of additive separable preferences based on *aversion to enemies* are considered. The first preference domain corresponds to a situation in which every player in the game has very strong friends and very weak enemies: when comparing two coalitions she may belong to, a player who appreciates her friends pays attention first to the friends in either coalition. The coalition that contains more friends is declared by the player as better than the other, and if the two coalitions have the same number of friends, then the coalition with less enemies wins the comparison. The second preference domain displays a situation in which every player has very strong enemies and very weak friends, i.e. a player who is averse to her enemies looks first at the enemies in either coalition. The coalition that contains less enemies is declared by the player as better than the other, and if the two coalitions have the same number of enemies, then the number of friends is decisive for the comparison. Notice that both restrictions allow for indifferences in the corresponding rankings over coalitions. We characterize internally stable coalitions on the proposed domains and show how these characterizations can be used for generating a strict core element in the first case and a core element in the second case. Moreover, we prove that an element of the strict core under friends appreciation can be found in polynomial time, while finding an element of the core under enemies aversion is NP-hard.

The outline of the paper is as follows. Section 2 introduces the model of a hedonic game and presents the formal definitions of our domain restrictions. We introduce the notions of an internally stable coalition and of a deviation stable collection of coalitions as our basic tools in Section 3, and relate them to core stability. In Section 4 we present results with respect to the structure of an internally stable coalition provided that players' preferences are separable. However, in order to provide full characterization results we have to restrict the preferences either to friends appreciation or to enemies aversion. These results are collected in Section 5 and Section 6, respectively, where we present our core existence proofs as well. Section 7 is devoted to the computational complexity for finding a core stable element for hedonic games in our domains.

2 Preliminaries

Consider a finite set of players $N = \{1, 2, \dots, n\}$. A *coalition* is a non-empty subset of N. For each player $i \in N$, we denote by $\mathcal{N}_i = \{X \subseteq N \mid i \in X\}$ the collection of all coalitions containing i. A collection $\mathcal{C}$ of coalitions is called a *coalition structure* if $\mathcal{C}$ is a partition of N, i.e. the coalitions in $\mathcal{C}$ are pairwise disjoint and $\bigcup_{C \in \mathcal{C}} C = N$. By $\mathbf{C}^N$ we denote the set of all coalition structures of N. For each coalition structure $\mathcal{C} \in \mathbf{C}^N$ and each player $i \in N$, by $\mathcal{C}(i)$ we denote the coalition in $\mathcal{C}$ which contains i, i.e. $\{\mathcal{C}(i)\} = \mathcal{C} \cap \mathcal{N}_i$.

We assume that each player $i \in N$ is endowed with a preference $\succeq_i$ over $\mathcal{N}_i$, i.e. a binary relation over $\mathcal{N}_i$ which is reflexive, complete, and transitive. We denote by $P = (\succeq_1, \succeq_2, \ldots, \succeq_n)$ a profile of preferences $\succeq_i$ for all $i \in N$, and by $\mathcal{P}$ the set of all preference profiles. Moreover, we assume that the preference of each player $i \in N$ over coalition structures is *purely hedonic*, i.e. it is completely characterized by $\succeq_i$ in such a way that, for each $\mathcal{C}, \mathcal{C}' \in \mathbf{C}^N$, player i weakly prefers $\mathcal{C}$ to $\mathcal{C}'$ if and only if $\mathcal{C}(i) \succeq_i \mathcal{C}'(i)$.

A *hedonic game* is a pair (N, P) consisting of a finite set N of players and a preference profile $P \in \mathcal{P}$. This paper is devoted to the question whether there exists a coalition structure $\mathcal{C} \in \mathbf{C}^N$ which is stable in some sense. The corresponding stability notions are given in Section 3.

We now specify the preference domains that will be considered. For each $i \in N$, we let $G_i := G(\succeq_i) = \{j \in N : \{i,j\} \succeq_i \{i\}\}$ be the set of friends of player i, and its complement $B_i = N \setminus G_i$ the set of enemies of player i. Notice that, from $\{i\} \succeq_i \{i\}$, we have $i \in G_i$ for each $i \in N$. The next definition suggests two ways of how each player i ranks the sets in $\mathcal{N}_i$ depending on the numbers of her friends and enemies.

Definition 1 *Let $P = (\succeq_1, \succeq_2, \ldots, \succeq_n) \in \mathcal{P}$ be a profile of players' preferences.*

- *We say that P is based on **appreciation of friends** if, for all $i \in N$ and all $X, Y \in \mathcal{N}_i$, $X \succeq_i Y$ if and only if* (1) $|X \cap G_i| > |Y \cap G_i|$ *or* (2) $|X \cap G_i| = |Y \cap G_i|$ *and* $|X \cap B_i| \leq |Y \cap B_i|$*;*

- *We say that P is based on **aversion to enemies** if, for all $i \in N$ and all $X, Y \in \mathcal{N}_i$, $X \succeq_i Y$ if and only if* (1) $|X \cap B_i| < |Y \cap B_i|$ *or* (2)

$|X \cap B_i| = |Y \cap B_i|$ *and* $|X \cap G_i| \geq |Y \cap G_i|$.

Thus, if the preference profile is based on appreciation of friends, we have a priority for friends when comparing two coalitions. If the preference profile is based on aversion to enemies, each player looks first at her enemies when comparing two coalitions. In the following, the set of all preference profiles based on appreciation of friends is denoted by $\mathcal{P}^f$, and the set of all preference profiles based on aversion to enemies is denoted by $\mathcal{P}^e$.

It is not difficult to see that if players' preferences are induced by either way suggested by Definition 1, then each player $i \in N$ will be equipped with a preference relation over $\mathcal{N}_i$ with G_i being its top and $B_i \cup \{i\}$ being its bottom. The next example illustrates this point.

Example 1 *Let* $N = \{1, 2, 3\}$ *and* $G_1 = \{1, 2\}$, $G_2 = \{2\}$, $G_3 = \{1, 2, 3\}$. *Let* $P = (\succeq_1, \succeq_2, \succeq_3) \in \mathcal{P}$.

- *If* $P \in \mathcal{P}^f$, *then*
  - *the ranking over* $\mathcal{N}_1$ *is* $\{1, 2\} \succ_1 \{1, 2, 3\} \succ_1 \{1\} \succ_1 \{1, 3\}$,
  - *the ranking over* $\mathcal{N}_2$ *is* $\{2\} \succ_2 \{1, 2\} \sim_2 \{2, 3\} \succ_2 \{1, 2, 3\}$,
  - *the ranking over* $\mathcal{N}_3$ *is* $\{1, 2, 3\} \succ_3 \{1, 3\} \sim_3 \{2, 3\} \succ_3 \{3\}$.
- *If* $P \in \mathcal{P}^e$, *then*
  - *the ranking over* $\mathcal{N}_1$ *is* $\{1, 2\} \succ_1 \{1\} \succ_1 \{1, 2, 3\} \succ_1 \{1, 3\}$,
  - *the ranking over* $\mathcal{N}_2$ *is* $\{2\} \succ_2 \{1, 2\} \sim_2 \{2, 3\} \succ_2 \{1, 2, 3\}$,
  - *the ranking over* $\mathcal{N}_3$ *is* $\{1, 2, 3\} \succ_3 \{1, 3\} \sim_3 \{2, 3\} \succ_3 \{3\}$.

In fact, the preference profiles based on appreciation of friends and the preference profiles based on aversion to enemies belong to a more general class of preference profiles, namely the class of additive separable preferences. A profile $P \in \mathcal{P}$ of players' preferences is *additive separable* if, for all $i \in N$, there exists a function $v_i : N \to \mathbb{R}$ such that for all $X, Y \in N_i$, $X \succeq_i Y$ if and only if $\sum_{j \in X} v_i(j) \geq \sum_{j \in Y} v_i(j)$. We denote the set of all additive separable preferences by $\mathcal{P}^{as}$. For the preference profile $P \in \mathcal{P}^f$ in Example 1, one can take $v_1(1) = v_1(2) = v_2(2) = v_3(1) = v_3(2) = v_3(3) = 3$ and $v_1(3) = v_2(1) = v_2(3) = -1$. For the preference profile $P \in \mathcal{P}^e$ in the same example the choice can be $v_1(1) = v_1(2) = v_2(2) = v_3(1) = v_3(2) = v_3(3) = 1$ and $v_1(3) = v_2(1) = v_2(3) = -3$. More generally, when $P \in \mathcal{P}^f$, one can take, for each $i \in N$, $v_i(j) = n$ if $j \in G_i$, and $v_i(j) = -1$ otherwise; when $P \in \mathcal{P}^e$, one can take, for each $i \in N$, $v_i(j) = 1$ if $j \in G_i$, and $v_i(j) = -n$ otherwise. Therefore, we have $(\mathcal{P}^f \cup \mathcal{P}^e) \subset \mathcal{P}^{as}$.

Notice finally that the preferences in $\mathcal{P}^f \cup \mathcal{P}^e$ are also *separable* because for every player $i \in N$ we have that for every $j \in N$ and $X \in \mathcal{N}_i$ with $j \notin X$, we have $[X \cup \{j\} \succ_i X \Leftrightarrow j \in G_i]$ and $[X \cup \{j\} \prec_i X \Leftrightarrow j \in B_i]$. We denote the set of all separable preferences by $\mathcal{P}^s$. Hence, the relation among $\mathcal{P}^f$, $\mathcal{P}^e$, $\mathcal{P}^{as}$, and $\mathcal{P}^s$ is as follows: $(\mathcal{P}^f \cup \mathcal{P}^e) \subset (\mathcal{P}^{as} \cap \mathcal{P}^s) \subset \mathcal{P}$.

3 Core stability, deviation stability and internal stability

Let $N = \{1, 2, \ldots, n\}$ be a finite set of players and let $\mathbf{D}^N$ denote the set of all collections of disjoint non-empty coalitions. For each $\mathcal{D} \in \mathbf{D}^N$ and for each $i \in \bigcup_{D \in \mathcal{D}} D$, we denote by $\mathcal{D}(i)$ the coalition in $\mathcal{D}$ containing i, i.e. $\{\mathcal{D}(i)\} = \mathcal{D} \cap \mathcal{N}_i$. Notice that the empty collection of coalitions belongs to $\mathbf{D}^N$, i.e. $\emptyset \in \mathbf{D}^N$. Observe further that each coalition structure is also a collection of non-empty disjoint coalitions, and thus, $\mathbf{C}^N \subseteq \mathbf{D}^N$.

Definition 3 *Let* $P \in \mathcal{P}$, $X \subseteq N$ *and* $\mathcal{D} \in \mathbf{D}^N$.

- *We say that X is a* ***deviation*** *from $\mathcal{D}$ if $\emptyset \neq X \subseteq \bigcup_{D\in\mathcal{D}} D$, and $X \succ_i \mathcal{D}(i)$ for each $i \in X$.*

- *We say that X is a* ***weak deviation*** *from $\mathcal{D}$ if $\emptyset \neq X \subseteq \bigcup_{D\in\mathcal{D}} D$, $X \succeq_i \mathcal{D}(i)$ for each $i \in X$, and $X \succ_j \mathcal{D}(j)$ for at least one $j \in X$.*

By using these notions, we define now core stability and strict core stability.

Definition 4 *Let $P \in \mathcal{P}$ and $\mathcal{C} \in \mathbf{C}^N$.*

- *We say that $\mathcal{C}$ is* ***core stable*** *if a deviation from $\mathcal{C}$ does not exist.*

- *We say that $\mathcal{C}$ is* ***strictly core stable*** *if a weak deviation from $\mathcal{C}$ does not exist.*

Notice that $\{X\} \in \mathbf{D}^N$ for every non-empty coalition $X \subseteq N$. Similar to the notion of core stability for coalition structures, one can define the notion of internal stability for coalitions.

Definition 5 *Let $P \in \mathcal{P}$ and $X \subseteq N$ with $X \neq \emptyset$.*

- *We say that X is* ***internally stable*** *if there is no $Y \subseteq X$ which is a deviation from $\{X\}$.*

- *We say that X is* ***strictly internally stable*** *if there is no $Y \subseteq X$ which is a weak deviation from $\{X\}$.*

We denote by $W(N,P)$ the collection of all internally stable coalitions, and by $S(N,P)$ the collection of all strictly internally stable coalitions.

Remark 1 *Note that for each hedonic game (N,P) the sets $W(N,P)$ and $S(N,P)$ are non-empty since singletons are always (strictly) internally stable.*

Remark 2 *Observe that, for every $P \in \mathcal{P}$ and every $\mathcal{C} \in \mathbf{C}^N$, we have $\mathcal{C} \subseteq W(N,P)$ if $\mathcal{C}$ is core stable, and $\mathcal{C} \subseteq S(N,P)$ if $\mathcal{C}$ is strictly core stable.*

Theorem 1 stated below without proof is very useful because, when (strict) core stability is under consideration, it allows us to concentrate only on (weak) deviations which are (strictly) internally stable. Note that this theorem is basically a variation of Ray's (1989) well-known result on the credibility of blocking coalitions.

Theorem 1 *Let $P \in \mathcal{P}$. For every $\mathcal{C} \in \mathbf{C}^N$,*

- *$\mathcal{C}$ is core stable if and only if there does not exist any deviation from $\mathcal{C}$ which is internally stable,*
- *$\mathcal{C}$ is strictly core stable if and only if there does not exist any weak deviation from $\mathcal{C}$ which is strictly internally stable.*

When presenting our results the following concepts will allow us to provide constructive existence proofs.

Definition 6 *Let $P \in \mathcal{P}$ and $\mathcal{D} \in \mathbf{D}^N$.*

- *We say that $\mathcal{D}$ is **deviation stable** if, for each $\mathcal{C} \in \mathbf{C}^N$ with $\mathcal{D} \subseteq \mathcal{C}$, there does not exist any deviation X from $\mathcal{C}$ such that $X \in W(N,P)$ and $X \cap \left(\bigcup_{D \in \mathcal{D}} D\right) \neq \emptyset$.*
- *We say that $\mathcal{D}$ is **strictly deviation stable** if, for each $\mathcal{C} \in \mathbf{C}^N$ with $\mathcal{D} \subseteq \mathcal{C}$, there does not exist any weak deviation X from $\mathcal{C}$ such that $X \in S(N,P)$ and $X \cap \left(\bigcup_{D \in \mathcal{D}} D\right) \neq \emptyset$.*

Observe that, by definition, for each $\mathcal{D} \in \mathbf{D}^N$, $\mathcal{D} \subseteq W(N,P)$ if $\mathcal{D}$ is deviation stable, and $\mathcal{D} \subseteq S(N,P)$ if $\mathcal{D}$ is strictly deviation stable. Moreover, from Theorem 1, the following corollary can be obtained immediately.

Corollary 1 *Let $P \in \mathcal{P}$. For every $\mathcal{C} \in \mathbf{C}^N$,*

- *$\mathcal{C}$ is core stable if and only if $\mathcal{C}$ is deviation stable, and*
- *$\mathcal{C}$ is strictly core stable if and only if $\mathcal{C}$ is strictly deviation stable.*

A sketch of our constructions of core stable coalition structures in the following sections looks as follows:

- start with the empty collection of coalitions, which is (strictly) deviation stable;
- construct a new (strictly) deviation stable collection of disjoint nonempty coalitions by including a (strictly) internally stable coalition, and repeat this until a coalition structure (a partition of N) is obtained.

So that, from Corollary 1, we will finally obtain a (strictly) core stable coalition structure.

4 Separability and internal stability

It should be clear from the previous section that the notion of an internally stable coalition plays an important role in the study of the existence of core stable partitions in hedonic games. In what follows in this section we offer a first look at the structure of an internally stable coalition provided that the players in the corresponding hedonic game have separable preferences.

Let $N = \{1, 2, \ldots, n\}$ and $P \in \mathcal{P}^s$. Let $X \subseteq N$ with $X \neq \emptyset$, and $i \in X$. For each positive integer t, let $G_i^t(X) \subseteq X$ be a set defined as follows:

$$\begin{aligned} G_i^1(X) &= G_i \cap X, \\ G_i^{t+1}(X) &= \bigcup_{j \in G_i^t(X)} (G_j \cap X). \end{aligned}$$

Let $G_i^*(X) = G_i^t(X)$ for a positive integer t such that $G_i^t(X) = G_i^{t+1}(X)$. Notice that $G_i^*(X) \neq \emptyset$ for each $X \subseteq N$ with $X \neq \emptyset$ and $i \in X$; this fact follows simply from $i \in G_i$ for all $i \in N$. Moreover, we have $G_i^n(X) = G_i^{n+1}(X)$ for each $X \subseteq N$ and each $i \in X$, and $G_i^*(X) \subseteq X$ and $G_i^*(Y) \subseteq G_i^*(X)$ for each $Y \subseteq X$ and each $i \in Y$.

The interpretation of $G_i^*(X)$ for $X \subseteq N$ and $i \in X$ becomes more clear by noticing that, given separability, we can always construct a *directed graph* $H_{(N,P)} = (V, E)$ with set of vertices $V = N$ and set of directed edges $E = \{(i, j) \in N \times N \mid i \neq j,\ j \in G_i\}$. Observe then that, for every $X \subseteq N$ and $i, j \in X$, we have $j \in G_i^*(X)$ if and only if there exists a sequence $k_1, k_2, \ldots, k_m$ for some $m \geq 1$ such that $k_1, k_2, \ldots, k_m \in X$ and $k_{l+1} \in G_{k_l}$ for each $1 \leq l \leq m-1$. Thus, $j \in G_i^*(X)$ if and only if there exists a directed path from i to j via vertices belonging to X. If, for each $i, j \in X$ with $i \neq j$, there exist directed paths via vertices belonging to X from i to j and from j to i (i.e. $G_i^*(X) = X$ for each $i \in X$), then the induced subgraph H_X of $H_{(N,P)}$ by X is called *strongly connected*, where H_X is the directed graph such that $V_X = X$ and $E_X = E \cap (X \times X)$. The next lemma simply says then that, given separability, the induced subgraph by every strictly internally stable coalition is strongly connected.

Lemma 1 *Let $P \in \mathcal{P}^s$ and $X \subseteq N$ with $X \neq \emptyset$. Then, $X \in S(N, P)$ implies $G_i^*(X) = X$ for each $i \in X$.*

Proof. Suppose $X \in S(N,P)$ and there exists $i \in X$ such that $G_i^*(X) \neq X$. Since $G_i^*(X) \subseteq X$ for each $i \in X$, we have $X \setminus G_i^*(X) \neq \emptyset$. Because $X \cap G_j = G_i^*(X) \cap G_j$ for each $j \in G_i^*(X)$ and from $P \in \mathcal{P}^s$, we have $X \setminus G_i^*(X) \subseteq N \setminus G_j = B_j$ for each $j \in G_i^*(X)$. From $P \in \mathcal{P}^s$ and the non-emptiness of $X \setminus G_i^*(X)$, we have $G_i^*(X) \succ_j X$ for each $j \in G_i^*(X)$. That is, $G_i^*(X)$ is a deviation from $\{X\}$, and is also a weak deviation from $\{X\}$. This contradicts the assumption that $X \in S(N,P)$. ■

Example 2 shows that the reverse implication to the one in Lemma 1 may not hold when $P \in \mathcal{P}^s$. Every player in this example is indifferent among coalitions on the same row and, for each $i \in N$, the top row corresponds to G_i and the bottom row corresponds to $B_i \cup \{i\}$.

Example 2 *Let $N = \{1,2,3,4\}$ and players' preferences be as follows:*

1	**2**	**3**	**4**
12	123	34	134
1, 123, 124	12, 23, 1234	3, 134, 234	14, 34, 1234
1234	2, 124, 234	1234	4, 124, 234
13, 14	24	13, 23	24
134		123	

Notice that the directed graph corresponding to this game is strongly connected but for example $\{1,2\}$ is a weak deviation from N, i.e. N is not strictly internally stable.

Given $N = \{1,2,\ldots,n\}$ and $P \in \mathcal{P}^s$, our next result provides a sufficient condition for a strongly connected subgraph H_X of $H_{(N,P)}$ to be induced by a strictly internally stable coalition $X \subseteq N$. This sufficient condition requires X to be a *clique* in $H_{(N,P)}$, i.e. a subset of V such that $(i,j),(j,i) \in E$ for every $i,j \in X$. Notice that this requirement is equivalent to "$G_i^1(X) = X$

for each $i \in X$".

Lemma 2 *Let $P \in \mathcal{P}^s$ and $X \subseteq N$ with $X \neq \emptyset$. Then, $G_i^1(X) = X$ for each $i \in X$ implies $X \in S(N, P)$.*

Proof. Suppose $X \subseteq N$ with $X \neq \emptyset$ is such that $G_i^1(X) = X$ for each $i \in X$. If $|X| = 1$, then, in view of Remark 1, $X \in S(N, P)$. Suppose now that $|X| \geq 2$ and let $Y \subset X$ be a non-empty proper subcoalition of X. Then, from $Y \subset X$ and $G_i^1(X) = X$ for each $i \in X$, we have, for each $i \in Y$, $X = G_i \cap X \subseteq G_i$, and thus, $X \setminus Y$ is a non-empty subset of G_i for each $i \in Y$, i.e. $X \succ_i Y$ from $P \in \mathcal{P}^s$. Thus, there is no subcoalition of X that is a weak deviation from $\{X\}$, i.e. $X \in S(N, P)$. ∎

Notice again that the reverse implication to the one in Lemma 2 may not hold when $P \in \mathcal{P}^s$. This is shown in

Example 3 *Consider $N = \{1, 2, 3\}$ and let $\{1, 2, 3\} \succ_1 \{1, 2\} \sim_1 \{1, 3\} \succ_1 \{1\}$, $\{1, 2\} \succ_2 \{1, 2, 3\} \succ_2 \{2\} \succ_2 \{2, 3\}$, and $\{1, 2, 3\} \succ_3 \{1, 3\} \sim_3 \{2, 3\} \succ_3 \{3\}$. Notice that the grand coalition N is strictly internally stable and, hence, internally stable but it is not a clique in the directed graph corresponding to this game.*

As we shall see in the next two sections, the reverse implications in the above lemmas hold true when the preferences are based on appreciation of friends and aversion to enemies, respectively.

5 Appreciation of friends

We consider in this section hedonic games with preference profiles belonging to $\mathcal{P}^f$ and provide a characterization of strictly internally stable coalitions (Lemma 3). Then, we show via Lemma 4 how this characterization can be used for proving (in Theorem 2) the existence of a strictly core stable partition for all games in the friends appreciation domain.

Lemma 3 *Let $P \in \mathcal{P}^f$ and $X \subseteq N$ with $X \neq \emptyset$. Then, $X \in S(N,P)$ if and only if $G_i^*(X) = X$ for each $i \in X$.*

Proof. The fact that $X \in S(N,P)$ implies $G_i^*(X) = X$ for each $i \in X$ follows from Lemma 1. Suppose now that $X \subseteq N$ with $X \neq \emptyset$ and is such that $G_i^*(X) = X$ for each $i \in X$. If $|X| = 1$, then, in view of Remark 1, $X \in S(N,P)$. Suppose now that $|X| \geq 2$ and let $Y \subset X$ be a non-empty proper subcoalition of X. Then, we have $G_i^*(X) \setminus Y \neq \emptyset$ for each $i \in Y$, and thus, there exists a $j \in Y$ such that $G_j \cap (X \setminus Y) \neq \emptyset$. That is, $|X \cap G_j| > |Y \cap G_j|$, and from $P \in \mathcal{P}^f$, we have $X \succ_j Y$ for some $j \in Y$. It follows that there is no subcoalition of X which is a weak deviation from $\{X\}$. Therefore, we have $X \in S(N,P)$. ∎

Next, we show a useful property of the set of all strictly internally stable coalitions.

Lemma 4 *Let $P \in \mathcal{P}^f$. If $X, Y \in S(N,P)$ with $X \cap Y \neq \emptyset$, then $X \cup Y \in S(N,P)$.*

Proof. Suppose $X, Y \in S(N,P)$ with $X \cap Y \neq \emptyset$. From Lemma 3, it suffices to show that $G_i^*(X \cup Y) = X \cup Y$ for each $i \in X \cup Y$. Recall that we have, for each non-empty $Z \subseteq N$ and each $i \in Z$, $G_i^*(Z) \subseteq Z$, and $G_i^*(Z) \subseteq G_i^*(Z')$ if $Z \subseteq Z'$. Thus $G_i^*(X) \cup G_i^*(Y) \subseteq G_i^*(X \cup Y) \subseteq X \cup Y$ for each $i \in X \cup Y$. In the following, we show that $X \cup Y \subseteq G_i^*(X \cup Y)$ for each $i \in X \cup Y$.

Let $i \in X \cap Y$. By assumption, $G_i^*(X) = X$ and $G_i^*(Y) = Y$, and thus, we have $X \cup Y = G_i^*(X) \cup G_i^*(Y) \subseteq G_i^*(X \cup Y)$ for each $i \in X \cap Y$.

Let $i \in X \setminus Y$. By assumption, $G_i^*(X) = X$, and thus, $X \subseteq G_i^*(X \cup Y)$. Let $j \in X \cap Y$. Notice that such a j exists by assumption. Then $j \in$

$G_i^*(X \cup Y)$, and by definition $G_j^*(X \cup Y) \subseteq G_i^*(X \cup Y)$. Since $Y = G_j^*(Y) \subseteq G_j^*(X \cup Y)$, we have $Y \subseteq G_i^*(X \cup Y)$. Thus, $X \cup Y \subseteq G_i^*(X \cup Y)$ for each $i \in X \setminus Y$. By the same argument, one can show that $X \cup Y \subseteq G_i^*(X \cup Y)$ for each $i \in Y \setminus X$.

Now we can conclude that $X \cup Y \in S(N, P)$ if $X, Y \in S(N, P)$ with $X \cap Y \neq \emptyset$, and the proof is completed. ∎

Having described, for games with preference profiles belonging to $\mathcal{P}^f$, a characterization of strictly internally stable coalitions and the structure of the collection $S(N, P)$, we redirect our attention to the problem of core stability of such games.

For each $M \subseteq N$, we denote by $S^M(N, P)$ the collection of all strictly internally stable coalitions which are subsets of M, i.e. $S^M(N, P) = \{X \in S(N, P) \mid X \subseteq M\}$. Let $GS^M(N, P)$ be the collection of all largest coalitions among coalitions belonging to $S^M(N, P)$, i.e. $GS^M(N, P) = \{X \in S^M(N, P) \mid |X| \geq |Y| \text{ for all } Y \in S^M(N, P)\}$. Obviously, $S^N(N, P) = S(N, P)$. Notice that $\{i\} \in S^M(N, P)$ for each $i \in M$. Hence, $S^M(N, P)$ and $GS^M(N, P)$ are non-empty whenever M is non-empty.

The following proposition suggests a way for extending a strictly deviation stable collection of disjoint coalitions for $P \in \mathcal{P}^f$.

Proposition 1 *Let $P \in \mathcal{P}^f$, $\mathcal{D} \in \mathbf{D}^N \setminus \mathbf{C}^N$, and $M = N \setminus (\bigcup_{D \in \mathcal{D}} D)$. If $\mathcal{D}$ is strictly deviation stable, then $\mathcal{D} \cup \{D'\}$ is strictly deviation stable for each $D' \in GS^M(N, P)$.*

Proof. Suppose $\mathcal{D}$ is strictly deviation stable and let $D' \in GS^M(N, P)$. Observe that $\mathcal{D} \cup \{D'\} \in \mathbf{D}^N$ and $D' \in GS^M(N, P) \subseteq S^M(N, P) \subseteq S(N, P)$. In the following we show that $\mathcal{D} \cup \{D'\}$ is strictly deviation stable.

Since $\mathcal{D}$ is strictly deviation stable, every $X \in S(N, P)$ with $X \cap \left(\bigcup_{D \in \mathcal{D}} D\right) \neq$

$\emptyset$ cannot be a weak deviation from any $\mathcal{C} \in \mathbf{C}^N$ such that $(\mathcal{D} \cup \{D'\}) \subseteq \mathcal{C}$. Thus, a strictly internally stable coalition X with $X \cap \left(\bigcup_{D \in \mathcal{D}} D \cup D'\right) \neq \emptyset$ is a weak deviation from some $\mathcal{C} \in \mathbf{C}^N$ such that $(\mathcal{D} \cup \{D'\}) \subseteq \mathcal{C}$ only if $X \subseteq M$ and $X \cap D' \neq \emptyset$. Let $X \in S(N, P)$ with $X \subseteq M$ and $X \cap D' \neq \emptyset$. From $D' \in S(N, P)$ and Lemma 4, we have $X \cup D' \in S(N, P)$. Since $X, D' \subseteq M$, we have $X \cup D' \in S^M(N, P)$. Moreover, if $X \subseteq D'$, then, from $D' \in S(N, P)$, X cannot be a weak deviation from $\{D'\}$, and thus X cannot be a weak deviation from any $\mathcal{C} \in \mathbf{C}^N$ such that $(\mathcal{D} \cup \{D'\}) \subseteq \mathcal{C}$. Then, if $X \nsubseteq D'$ we have $|X \cup D'| > |D'|$, which contradicts $D' \in GS^M(N, P)$. Therefore, $\mathcal{D} \cup \{D'\}$ is strictly deviation stable. ■

Hence, we have the following theorem[1].

Theorem 2 *For each $P \in \mathcal{P}^f$, a strictly core stable coalition structure exists.*

Proof. Extend the empty collection of coalitions (which is strictly deviation stable) to a strictly deviation stable coalition structure of N in the way suggested by Proposition 1. Then, from Corollary 1, such a coalition structure is strictly core stable. ■

6 Aversion to enemies

The way of proving the existence of a strictly core stable partition for the case of appreciation of friends can be used also for the case of aversion to enemies.

First, we show a characterization of internally stable coalitions when $P \in$

[1] For a proof of the existence of a core stable element on a larger domain than $\mathcal{P}^f$ without using a characterization of internally stable coalitions the reader is referred to Alcalde and Revilla (2004).

$\mathcal{P}^e$.

Lemma 5 *Let $P \in \mathcal{P}^e$ and $X \subseteq N$ with $X \neq \emptyset$. Then, $X \in W(N,P)$ if and only if $G_i^1(X) = X$ for each $i \in X$.*

Proof. The fact that $G_i^1(X) = X$ for each $i \in X$ implies $X \in S(N,P) \subseteq W(N,P)$ follows from Lemma 2. To show the reverse implication, let $X \subseteq N$ with $X \cap B_i \neq \emptyset$ for some $i \in X$. Then, from $\{i\} \succ_i X$, $\{i\}$ is a deviation from $\{X\}$, i.e. $X \notin W(N,P)$. Thus, $X \subseteq G_i$ for each $i \in X$ if $X \in W(N,P)$. Obviously, $G_i^1(X) = X$ if $X \subseteq G_i$. ■

In other words, $W(N,P)$ is the collection of all non-empty coalitions of players in N such that, for each of these coalitions, the players like each other under preference profile $P \in \mathcal{P}^e$.

As analogy to $S^M(N,P)$ and $GS^M(N,P)$, we define $W^M(N,P) = \{X \in W(N,P) \mid X \subseteq M\}$ and $GW^M(N,P) = \{X \in W^M(N,P) \mid |X| \geq |Y| \text{ for all } Y \in W^M(N,P)\}$ for each $M \subseteq N$. Again, $W^M(N,P)$ and $GW^M(N,P)$ are non-empty whenever M is non-empty.

The following proposition suggests a way for extending a deviation stable collection of disjoint coalitions for $P \in \mathcal{P}^e$.

Proposition 2 *Let $P \in \mathcal{P}^e$, $\mathcal{D} \in \mathbf{D}^N \setminus \mathbf{C}^N$, and $M = N \setminus (\bigcup_{D \in \mathcal{D}} D)$. Then, if $\mathcal{D}$ is deviation stable, then $\mathcal{D} \cup \{D'\}$ is deviation stable for each $D' \in GW^M(N,P)$.*

Proof. The proof is similar to the proof of Proposition 1. Again, observe that $\mathcal{D} \cup \{D'\} \in \mathbf{D}^N$ and $D' \in GW^M(N,P) \subseteq W^M(N,P) \subseteq W(N,P)$. In the following we show that $\mathcal{D} \cup \{D'\}$ is deviation stable.

Since $\mathcal{D}$ is deviation stable, every $X \in W(N,P)$ with $X \cap \left(\bigcup_{D \in \mathcal{D}} D\right) \neq \emptyset$ cannot be a deviation from any $\mathcal{C} \in \mathbf{C}^N$ such that $(\mathcal{D} \cup \{D'\}) \subseteq \mathcal{C}$. Thus, an internally stable coalition X with $X \cap \left(\bigcup_{D \in \mathcal{D}} D \cup D'\right) \neq \emptyset$ is a deviation from some $\mathcal{C} \in \mathbf{C}^N$ such that $(\mathcal{D} \cup \{D'\}) \subseteq \mathcal{C}$ only if $X \subseteq M$ and $X \cap D' \neq \emptyset$. Let $X \in W(N, P)$ with $X \subseteq M$ and $X \cap D' \neq \emptyset$. From Lemma 5, we have $|X \cap B_i| = |D' \cap B_i| = 0$ for each $i \in X \cap D'$. Then, if $|X| \leq |D'|$ we have $D' \succeq_i X$ for each $i \in X \cap D'$, and thus, X cannot be a deviation from any $\mathcal{C} \in \mathbf{C}^N$ such that $(\mathcal{D} \cup \{D'\}) \subseteq \mathcal{C}$. Moreover, from $D' \in GW^M(N, P)$, we have $|X| \leq |D'|$ for every $X \in W^M(N, P)$. Therefore, $\mathcal{D} \cup \{D'\}$ is deviation stable. ■

With the help of this proposition, we are able to present our main result in this section.

Theorem 3 *For each $P \in \mathcal{P}^e$, a core stable coalition structure exists.*

Proof. Extend the empty collection of coalitions (which is deviation stable) to a deviation stable coalition structure of N in the way suggested by Proposition 2. Then, from Corollary 1, such a coalition structure is core stable. ■

The next example shows that the strict core may be empty when the players are averse to their enemies.

Example 4 *Consider $N = \{1, 2, 3\}$ under aversion to enemies with $G_1 = \{1, 2\}$, $G_2 = \{1, 2, 3\}$, and $G_3 = \{2, 3\}$. We have then the following preferences:*

1	**2**	**3**
12	123	23
1	12, 23	2
123	2	123
13		13

Clearly, the candidates for strictly core stable coalition structures should be core stable as well. As it can be easily seen from the graph that corresponds to *this game, there are two maximal cliques -* $\{1,2\}$ *and* $\{2,3\}$*. If one extends the empty collection of coalitions by* $\{1,2\}$*, then the resulting partition of* N *will be* $\{\{1,2\},\{3\}\}$*. Alternatively, if one starts by adding* $\{2,3\}$*, then the partition will be* $\{\{2,3\},\{1\}\}$*. The reader can easily check that these partitions are core stable. Notice however that* $\{2,3\}$ *and* $\{1,2\}$ *are weak deviations from* $\{\{1,2\},\{3\}\}$ *and* $\{\{2,3\},\{1\}\}$*, respectively. Therefore, no strictly core stable coalition structure exists.*

Remark 3 *For a comparison of the preference restriction based on aversion to enemies with other sufficient conditions for non-emptiness of the core of a hedonic game the reader is referred to Dimitrov et al.* (2004)*.*

7 Computational complexity

For general hedonic games the existence problem of a core stable coalition structure is shown to be NP-complete; moreover, if one imposes anonymity (the players pay attention only to the size of the corresponding coalitions) the problem remains NP-complete even when only strict preferences are allowed (see Ballester (2004)). Cechlárová and Hajduková (2002) study the computational complexity of the existence problem of a strict core element for the specific extension of the preferences over individuals to preferences over coalitions proposed by Cechlárová and Romero-Medina (2001) and show that when ties are included the existence problem is NP-complete. In contrast, according to Theorem 2 and Theorem 3, a strictly core stable coalition structure for the case of appreciation of friends and a core stable coalition structure for the case of aversion to enemies always exists. In other words, the two corresponding existence problems always have the answer "*Yes*", and of course, the answer can be obtained without computation.

In the following, the computational complexity for finding core stable coalition structures is considered. As it turns out, even if we know that a core element always exists, the problem for finding a core element for the case of aversion to enemies is NP-hard (Theorem 4). In contrast, a strict core element on the domain of appreciation of friends can be found in polynomial time as we show in Theorem 5. Here, we assume that each preference profile is given by a collection $(G_1, G_2, \ldots, G_n)$ of n subsets of N satisfying $i \in G_i$ for each $i \in N$. The collection $(G_1, G_2, \ldots, G_n)$ determines the set of friends for each player $i \in N$. Observe that the input size (the length for representing a collection $(G_1, G_2, \ldots, G_n)$) is $O(n^2)$.

We start with a lemma for the case of aversion to enemies.

Lemma 6 *For every $P \in \mathcal{P}^e$ and every $\mathcal{C} \in \mathbf{C}^N$, $GW(N, P) \cap \mathcal{C} \neq \emptyset$ if $\mathcal{C}$ is core stable.*

Proof. Let $\mathcal{C} \in \mathbf{C}^N$ be such that $GW(N, P) \cap \mathcal{C} = \emptyset$, and let $X \in GW(N, P)$. Then, we have $|X| > |\mathcal{C}(i)|$ for each $i \in X$. From Lemma 5, $X \subseteq G_i$ for each $i \in X$, i.e. $|X \cap B_i| = 0$ for each $i \in X$. Thus, $X \succ_i \mathcal{C}(i)$ for each $i \in X$, i.e. X is a deviation from $\mathcal{C}$. Therefore, $\mathcal{C}$ is not core stable. ■

This lemma allows us to formulate our result on computational complexity for finding *any* core stable partition in hedonic games with enemy averse preferences.

Theorem 4 *When $P \in \mathcal{P}^e$, the problem of finding a core stable coalition structure is NP-hard.*

Proof. The NP-hardness is shown by reduction from the Maximum Clique Problem, which is known to be NP-hard.

Maximum Clique Problem: For a given undirected graph $H = (V, E)$, find a clique with the maximum cardinality in H, where a clique K in H is a subset of V such that $\{i,j\} \in E$ for every $i,j \in K$ with $i \neq j$.

Let H be an undirected graph. Without loss of generality, we assume that $V = \{1,2,\ldots,n\}$. Consider the hedonic game (N,P) with $P \in \mathcal{P}^e$ such that $N = \{1,2,\ldots,n\}$ and $G_i = \{i\} \cup \{j \in N \mid \{i,j\} \in E\}$ for each $i \in N$. Notice that, from Lemma 5, for each $X \subseteq N$, $X \in W(N,P)$ if and only if X is a clique in H, and $GW(N,P)$ is the set of all cliques with the maximum cardinality. Then, from Lemma 6, to each core stable coalition structure belongs at least one clique X with the maximum cardinality. This implies that the problem of finding a core stable coalition structure when $P \in \mathcal{P}^e$ is at least as difficult as finding a clique with the maximum cardinality in an undirected graph. Therefore, the problem of finding a core stable coalition structure when $P \in \mathcal{P}^e$ is NP-hard. ■

Remark 4 *Notice that, from the proof of Theorem* 4*, the problem of finding a core stable coalition structure when* $P \in \mathcal{P}^e$ *remains NP-hard even if the mutuality condition "*$j \in G_i$ *if and only if* $i \in G_j$*" is imposed.*

In the following, we show that when $P \in \mathcal{P}^f$, a strictly core stable coalition structure can be found in polynomial time, by taking a graph theoretical approach.

Theorem 5 *When* $P \in \mathcal{P}^f$*, a strictly core stable coalition structure can be found in polynomial time.*

Proof. Let $N = \{1,2,\ldots,n\}$, $P \in \mathcal{P}^f$, and let $H_{(N,P)} = (V,E)$ be a directed graph such that $V = N$ and $E = \{(i,j) \in N \times N \mid i \neq j,\ j \in G_i\}$. It follows by Lemma 3 that $X \in S(N,P)$ if and only if the induced subgraph of $H_{(N,P)}$ by X is strongly connected. Moreover, $X \in GS(N,P)$ if and only if the induced subgraph of $H_{(N,P)}$ by X is a strongly connected component in

$H_{(N,P)}$ with the largest number of vertices. From Theorem 2, a strictly core stable coalition structure can be found as follows.

- Set $M := N$ and $\mathcal{C} := \emptyset$.
- Repeat the following until $M = \emptyset$:
  - Find a set $X \subseteq M$ such that the induced subgraph of $H_{(M,P)}$ by X is a strongly connected component in $H_{(M,P)}$ with the largest number of vertices.
  - Set $M := M \setminus X$ and $\mathcal{C} := \mathcal{C} \cup \{X\}$.
- Return $\mathcal{C}$.

Observe that the outcome is in fact the collection of all strongly connected components of $H_{(N,P)}$. An algorithm for finding all strongly connected components of a directed graph (i.e. a strong decomposition of a directed graph) is proposed by Tarjan (1972), which has running time $O(n^2)$. Therefore, a strictly core stable coalition structure can be found in polynomial time. ■

Remark 5 *Notice that the algorithm described in Theorem* 5 *delivers a unique decomposition of a directed graph into strongly connected subgraphs.*

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