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Linear Algebra and its Applications 423 (2007) 33–41
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Cospectral graphs and the generalized adjacency matrix

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Received 12 May 2006; accepted 31 July 2006
Available online 11 September 2006
Submitted by D.M. Cvetković

Abstract

Let J be the all-ones matrix, and let A denote the adjacency matrix of a graph. An old result of Johnson and Newman states that if two graphs are cospectral with respect to $yJ - A$ for two distinct values of y, then they are cospectral for all y. Here we will focus on graphs cospectral with respect to $yJ - A$ for exactly one value $\hat{y}$ of y. We call such graphs $\hat{y}$-cospectral. It follows that $\hat{y}$ is a rational number, and we prove existence of a pair of $\hat{y}$-cospectral graphs for every rational $\hat{y}$. In addition, we generate by computer all $\hat{y}$-cospectral pairs on at most nine vertices. Recently, Chesnokov and the second author constructed pairs of $\hat{y}$-cospectral graphs for all rational $\hat{y} \in (0, 1)$, where one graph is regular and the other one is not. This phenomenon is only possible for the mentioned values of $\hat{y}$, and by computer we find all such pairs of $\hat{y}$-cospectral graphs on at most eleven vertices.

AMS classification: 05C50; 05E99

Keywords: Cospectral graphs; Generalized spectrum; Generalized adjacency matrix

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[1] The research of E.R. van Dam has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.
[2] This work was done while J.H. Koolen was visiting Tilburg University and he appreciates the hospitality and support received from Tilburg University.

0024-3795/$ - see front matter
doi:10.1016/j.laa.2006.07.017

1. Introduction

For a graph Γ with adjacency matrix A, any matrix of the form $M = xI + yJ + zA$ with $x, y, z \in \mathbb{R}$, $z \neq 0$ is called a *generalized adjacency matrix* of Γ (as usual, J is the all-ones matrix and I the identity matrix). Since we are interested in the relation between Γ and the spectrum of M, we can restrict to generalized adjacency matrices of the form $yJ - A$ without loss of generality.

Let Γ and Γ' be graphs with adjacency matrices A and A', respectively. Johnson and Newman [7] proved that if $yJ - A$ and $yJ - A'$ are cospectral for two distinct values of y, then they are cospectral for all y, and hence they are cospectral with respect to all generalized adjacency matrices. If this is the case we will call Γ and Γ' $\mathbb{R}$-*cospectral*. So if $yJ - A$ and $yJ - A'$ are cospectral for some but not all values of y, they are cospectral for exactly one value $\hat{y}$ of y. Then we say that Γ and Γ' are $\hat{y}$-*cospectral*. Thus cospectral graphs (in the usual sense) are either 0-cospectral or $\mathbb{R}$-cospectral. For both types of cospectral graphs, many examples are known (see for example [5]). In Fig. 1 we give an example of both. This figure also gives examples of $\hat{y}$-cospectral graphs for $\hat{y} = \frac{1}{3}$ and $\hat{y} = -1$. Note that Γ and Γ' are $\hat{y}$-cospectral if and only if their complements are $(1 - \hat{y})$-cospectral. So we also have examples for $\hat{y} = 1$, $\frac{2}{3}$ and 2. If $\hat{y} = \frac{1}{2}$, one can construct a graph cospectral with a given graph Γ by multiplying some rows and the corresponding columns of $\frac{1}{2}J - A$ by -1. The corresponding operation in Γ is called *Seidel switching*. This shows that every graph with at least two vertices has a $\frac{1}{2}$-cospectral mate.

It is well known that, with respect to the adjacency matrix, a regular graph cannot be cospectral with a nonregular one (see [2, p. 94]). In [5] this result is extended to generalized adjacency matrices $yJ - A$ with $y \notin (0, 1)$. In [1] a regular-nonregular pair of $\hat{y}$-cospectral graphs is constructed for all rational $\hat{y} \in (0, 1)$. In the next section we shall see that $\hat{y}$ is rational for any pair of $\hat{y}$-cospectral graphs. Thus we have:

Theorem 1. *There exists a pair of $\hat{y}$-cospectral graphs, where one graph is regular and the other one is not, if and only if $\hat{y}$ is a rational number satisfying* $0 < \hat{y} < 1$.

In the final section we will generate all regular-nonregular $\hat{y}$-cospectral pairs on at most eleven vertices. The smallest such pair has only six vertices; it is the $\frac{1}{3}$-cospectral pair of Fig. 1. In Section 3 we shall construct $\hat{y}$-cospectral graphs for every rational value of $\hat{y}$. Therefore:

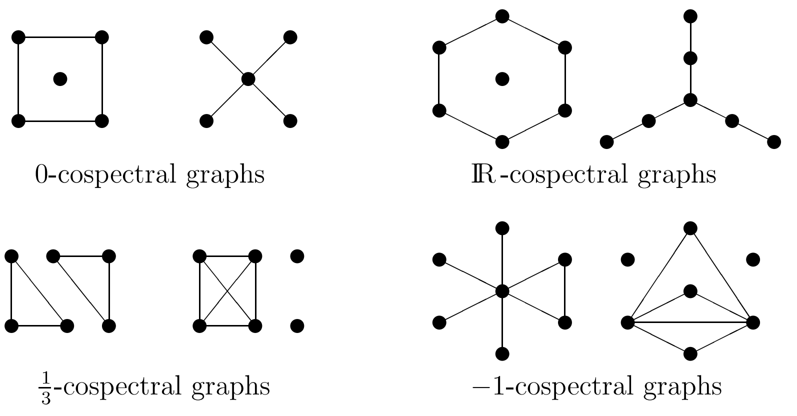

Fig. 1. Some examples of (generalized) cospectral graphs.

Theorem 2. *There exists a pair of* $\hat{y}$*-cospectral graphs if and only if* $\hat{y}$ *is a rational number.*

In the final section we also generate all pairs of $\hat{y}$-cospectral graphs on at most nine vertices.

2. The generalized characteristic polynomial

For a graph Γ with adjacency matrix A, the polynomial $p(x, y) = \det(xI + yJ - A)$ will be called the *generalized characteristic polynomial* of Γ. Thus $p(x, y)$ has integral coefficients, $p(x, y)$ can be interpreted as the characteristic polynomial of $A - yJ$, and $p(x, 0) = p(x)$ is the characteristic polynomial of A. The generalized characteristic polynomial is closely related to the so-called *idiosyncratic polynomial*, which was introduced by Tutte [9] as the characteristic polynomial of $A + y(J - I - A)$. We prefer the polynomial $p(x, y)$, because it has the important property that the degree in y is only 1. Indeed, for an arbitrary square matrix M it is known that $\det(M + yJ) = \det M + y\Sigma \operatorname{adj} M$, where $\Sigma \operatorname{adj} M$ denotes the sum of the entries of the adjugate (adjoint) of M. It is also easily derived from the fact that by Gaussian elimination in $xI + yJ - A$ one can eliminate all y-s, except for those in the first row. In this way we will obtain more useful expressions for $p(x, y)$ as follows. Partition A according to a vertex v, the neighbors of v and the remaining vertices ($\mathbf{1}$ denotes an all-ones vector, and $\mathbf{0}$ an all-zeros vector):

$$A = \begin{bmatrix} 0 & \mathbf{1}^\top & \mathbf{0}^\top \\ \mathbf{1} & A_1 & B \\ \mathbf{0} & B^\top & A_0 \end{bmatrix}.$$

Then

$$\begin{aligned} p(x, y) &= \det \begin{bmatrix} x+y & (y-1)\mathbf{1}^\top & y\mathbf{1}^\top \\ (y-1)\mathbf{1} & xI + yJ - A_1 & yJ - B \\ y\mathbf{1} & yJ - B^\top & xI + yJ - A_0 \end{bmatrix} \\ &= \det \begin{bmatrix} x+y & (y-1)\mathbf{1}^\top & y\mathbf{1}^\top \\ (-1-x)\mathbf{1} & xI + J - A_1 & -B \\ -x\mathbf{1} & J - B^\top & xI - A_0 \end{bmatrix} \\ &= p(x) + y \det \begin{bmatrix} 1 & \mathbf{1}^\top & \mathbf{1}^\top \\ (-1-x)\mathbf{1} & xI + J - A_1 & -B \\ -x\mathbf{1} & J - B^\top & xI - A_0 \end{bmatrix} \\ &= p(x) + y \det \begin{bmatrix} 1 & 2\mathbf{1}^\top & \mathbf{0}^\top \\ -\mathbf{1} & xI - A_1 & J - B \\ \mathbf{0} & J - B^\top & xI - A_0 \end{bmatrix} - xy \det \begin{bmatrix} 0 & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{1} & xI - A_1 & -B \\ \mathbf{1} & -B^\top & xI - A_0 \end{bmatrix} \end{aligned}$$

This expression provides the coefficients of the three highest powers of x in $p(x, y)$. A similar expression is used for the computations in Section 4.

Lemma 1. *Let* Γ *be a graph with* n *vertices,* e *edges, and generalized characteristic polynomial* $p(x, y) = \sum_{i=0}^{n}(a_i + b_i y)x^i$. *Then* $a_n = 1,\ b_n = 0,\ a_{n-1} = 0,\ b_{n-1} = n,\ a_{n-2} = -e$ *and* $b_{n-2} = 2e$.

Proof. By using the above expression for $p(x, y)$, and straightforward calculations. □

Thus the coefficient of x^{n-2} in $p(x, y)$ equals $e(2y-1)$. This implies the known fact that, for any $y \neq \frac{1}{2}$, the number of edges of a graph can be deduced from the spectrum of $yJ - A$. Note that a $\frac{1}{2}$-cospectral pair with distinct numbers of edges can easily be made by Seidel switching.

Now let Γ and Γ' be graphs with generalized characteristic polynomials $p(x, y)$ and $p'(x, y)$, respectively. It is clear that $p(x, y) \equiv p'(x, y)$ if and only if Γ and Γ' are $\mathbb{R}$-cospectral, and Γ and Γ' are $\hat{y}$-cospectral if and only if $p(x, \hat{y}) = p'(x, \hat{y})$ for all $x \in \mathbb{R}$, whilst $p(x, y) \not\equiv p'(x, y)$. If this is the case, then $a_i + \hat{y}b_i = a_i' + \hat{y}b_i'$ with $(a_i, b_i) \neq (a_i', b_i')$ for some i $(0 \leqslant i \leqslant n-3)$. This implies $\hat{y} = -\left(a_i - a_i'\right) / \left(b_i - b_i'\right)$. Thus we proved the mentioned result of Johnson and Newman, that there is only one possible value of $\hat{y}$. Moreover, we see:

Proposition 1. *Let Γ and Γ' be two $\hat{y}$-cospectral graphs then*

(i) *$\hat{y}$ is a rational number.*
(ii) *Let $|\hat{y}| = p/q$ with p and q relative primes. Then $|\hat{y}| \leqslant p \leqslant 4\left(1 + \frac{1}{2}\sqrt{n+1}\right)^{n+1}$.*

Proof. We have $\hat{y} = -\left(a_i - a_i'\right) / \left(b_i - b_i'\right)$ for some i with a_i, a_i', b_i and b_i' integral. Therefore $p \leqslant |a_i| + |a_i'|$. The Hadamard bound gives that the absolute value of the determinant of any $m \times m$ (0, 1)-matrix is at most $2^{-m}(m+1)^{(m+1)/2}$. Hence the coefficient a_i of the characteristic polynomial $\sum_{i=0}^{n} a_i x^i$ of any $n \times n$ (0, 1)-matrix satisfies

$$|a_i| \leqslant \binom{n}{i} 2^{i-n}(n-i+1)^{\frac{n-i+1}{2}} \leqslant \sum_{i=0}^{n} \binom{n}{i} 2^{i-n}(n+1)^{\frac{n-i+1}{2}} = 2^{-n}\left(2 + \sqrt{n+1}\right)^{n+1}.$$

Therefore $p \leqslant 4\left(1 + \frac{1}{2}\sqrt{n+1}\right)^{n+1}$. □

The generalized characteristic polynomial $p(x, y)$ of a graph Γ is related to the set of main angles $\{\beta_1, \ldots, \beta_\ell\}$ of Γ. Suppose the adjacency matrix A of Γ has ℓ distinct eigenvalues $\lambda_1 > \cdots > \lambda_\ell$ with multiplicities $m_1, \ldots, m_\ell$, respectively, then the main angle β_i is defined as the cosine of the angle between the all-ones vector $\mathbf{1}$ and the eigenspace of λ_i. For $i = 1, \ldots, \ell$, let V_i be an $n \times m_i$ matrix whose columns are an orthonormal basis for the eigenspace of λ_i. Then $\beta_i\sqrt{n} = \|V_i^\top \mathbf{1}\|$. Moreover, we can choose V_i such that $V_i^\top \mathbf{1} = \beta_i\sqrt{n}\mathbf{e}_1$ (where $\mathbf{e}_1$ is the unit vector in $\mathbb{R}^{m_i}$). Put $V = [V_1 \cdots V_\ell]$, then $V^\top AV = \Lambda$, where Λ is the diagonal matrix with the spectrum of A, and $V^\top \mathbf{1} = \sqrt{n}\left[\beta_1\mathbf{e}_1^\top \cdots \beta_\ell\mathbf{e}_1^\top\right]^\top$.

Assume that Γ and Γ' are cospectral graphs with the same angles. Then there exist matrices V and V' such that $V^\top AV = V'^\top A'V' = \Lambda$ and $V^\top \mathbf{1} = V'^\top \mathbf{1}$. Define $Q = VV'^\top$, then $Q^\top AQ = A'$ and $Q\mathbf{1} = Q^\top \mathbf{1} = \mathbf{1}$. This implies that $Q^\top(yJ - A)Q = yJ - A'$, so $yJ - A$ and $yJ - A'$ are cospectral for every $y \in \mathbb{R}$, hence Γ and Γ' have the same generalized characteristic polynomial.

Cvetković and Rowlinson [3] (see also [4, p. 100]) proved the following expression for $p(x, y)$ in terms of the spectrum and the main angles of Γ:

$$p(x, y) = p(x)\left(1 + yn\sum_{i=1}^{\ell} \beta_i^2/(x - \lambda_i)\right).$$

This formula also shows that the main angles can be obtained from $p(x, y)$, as can be seen as follows. Suppose $q(x) = \Pi_{i=1}^{\ell}(x - \lambda_i)$ is the minimal polynomial of A, put $r(x) = p(x)/q(x)$ and

$q(x, y) = p(x, y)/r(x)$. Then $q(x, y)$ is a polynomial satisfying $q(\lambda_j, 1) = n\beta_j^2 \Pi_{i \neq j}(\lambda_j - \lambda_i)$, which proves the claim. As a consequence, we also proved (a result due to Johnson and Newman [7]) that Γ and Γ' are $\mathbb{R}$-cospectral if and only if there exist an orthogonal matrix Q such that $Q^\top AQ = A'$ and $Q\mathbf{1} = \mathbf{1}$. The next theorem recapitulates the conditions we have seen for graphs being $\mathbb{R}$-cospectral.

Theorem 3. *Let Γ and Γ' be graphs with adjacency matrices A and A'. Then the following are equivalent*:

- *Γ and Γ' have identical generalized characteristic polynomials*,
- *Γ and Γ' are cospectral with respect to all generalized adjacency matrices*,
- *Γ and Γ' are cospectral*, *and so are their complements*,
- *Γ and Γ' are cospectral*, *and have the same main angles*,
- *$yJ - A$ and $yJ - A'$ are cospectral for two distinct values of y*,
- *$yJ - A$ and $yJ - A'$ are cospectral for any irrational value of y*,
- *$yJ - A$ and $yJ - A'$ are cospectral for any y with $|y| > 4\left(1 + \frac{1}{2}\sqrt{n+1}\right)^{n+1}$*,
- *there exist an orthogonal matrix Q*, *such that $Q^\top AQ = A'$ and $Q\mathbf{1} = \mathbf{1}$*.

3. A construction

We construct pairs of graphs Γ and Γ' on n vertices. For each pair the vertex set is partitioned into three parts with sizes a, b, and c for Γ, and a', b', and $c' = c$ for Γ'. Thus $a + b = a' + b' = n - c$. With these partitions Γ and Γ' are defined via their adjacency matrices A and A' as follows (O denotes the all-zeros matrix):

$$A = \begin{bmatrix} O & O & O \\ O & O & J \\ O & J & J - I \end{bmatrix}, \quad A' = \begin{bmatrix} O & J & O \\ J & O & J \\ O & J & J - I \end{bmatrix}.$$

So for the matrices $M = yJ - A$ and $M' = yJ - A'$ we get:

$$M = \begin{bmatrix} yJ & yJ & yJ \\ yJ & yJ & (y-1)J \\ yJ & (y-1)J & (y-1)J + I \end{bmatrix}, \quad M' = \begin{bmatrix} yJ & (y-1)J & yJ \\ (y-1)J & yJ & (y-1)J \\ yJ & (y-1)J & (y-1)J + I \end{bmatrix}.$$

Clearly rank $(M) \leqslant c + 2$, so the characteristic polynomial $p(x, y)$ of M has a factor x^{n-c-2}. Moreover rank $(M - I) \leqslant a + b + 1$, so $p(x, y)$ has a factor $(x - 1)^{c-1}$. In a similar way we find that the characteristic polynomial $p'(x, y)$ of M' also has a factor $x^{n-c-2}(x - 1)^{c-1}$. Define

$$r(x, y) = \frac{p(x, y)}{x^{n-c-2}(x-1)^{c-1}} \quad \text{and} \quad r'(x, y) = \frac{p'(x, y)}{x^{n-c-2}(x-1)^{c-1}}.$$

Then $r(x, y)$ and $r'(x, y)$ are polynomials of degree 3 in x and degree 1 in y. Clearly M and M' are cospectral if $r(x, y) = r'(x, y)$ for all $x \in \mathbb{R}$. Write

$$r(x, y) = t_0 + t_1 x + t_2 x^2 + t_3 x^3, \quad \text{and} \quad r'(x, y) = t'_0 + t'_1 x + t'_2 x^2 + t'_3 x^3,$$

where t_i and t'_i are linear functions in y. Then $t_3 = t'_3 = 1$, and $t_2 = t'_2 = -ny + c - 1$, because $-ny = -\text{trace}(M) = -\text{trace}(M')$, which equals the coefficient of x^{n-1} in $p(x, y)$ and $p'(x, y)$. We shall require that

$$b'(a' + c) = bc.$$

This means that Γ and Γ' have the same number of edges. In the previous section we saw that the number of edges determines the coefficient of x^{n-2} in the generalized characteristic polynomial. Therefore the above requirement gives $t_1 = t_1'$. Finally we shall use the fact that $r(x, y)$ and $r'(x, y)$ are the characteristic polynomials of the quotient matrices R and R' of M and M', respectively (the quotient matrices are the 3×3 matrices consisting of the row sums of the blocks). So if we choose $y = \hat{y}$ such that these quotient matrices have the same determinant we have $t_0 = t_0'$, and therefore M and M' have the same spectrum. We find

$$\det R = \det \begin{bmatrix} ya & yb & yc \\ ya & yb & yc - c \\ ya & yb - b & yc - c + 1 \end{bmatrix} = -yabc,$$

and

$$\det R' = \det \begin{bmatrix} ya' & yb' - b' & yc \\ ya' - a' & yb' & yc - c \\ ya' & yb' - b' & yc - c + 1 \end{bmatrix} = (1 - 2y)a'b'(c - 1).$$

Using $bc = b'(a' + c)$, this leads to $\hat{y} = a'(c - 1)/(2a'c - 2a' - ac - aa')$. So any choice of positive integers a, a', b, b', and c that satisfy $a + b = a' + b'$, $bc = b'(a' + c)$, and $2a'c - 2a' - ac - aa' \neq 0$ leads to a pair of $\hat{y}$-cospectral graphs with the above $\hat{y}$ (indeed, $\hat{y}$ is uniquely determined, hence Γ and Γ' are not $\mathbb{R}$-cospectral). For example $(a, a', b, b', c) = (2, 4, 3, 1, 2)$ leads to the two -1-cospectral graph of Fig. 1. Moreover, by a suitable choice of these numbers we can get every rational value of $\hat{y} > 1/2$. Indeed, write $\hat{y} = p/q$ with $2p - q \geqslant 2$, then

$$a = 2p - q - 1, \quad a' = a(p + 1), \quad b = p(a + 1), \quad b' = p, \quad \text{and} \quad c = p + 1$$

satisfy the required conditions and gives $\hat{y} = p/q$. As remarked in the introduction, $\frac{1}{2}$-cospectral graphs are easily made by use of Seidel switching, and we also saw that two graphs are $\hat{y}$-cospectral if and only if their complements are $(1 - \hat{y})$-cospectral. Thus we have:

Proposition 2. *A pair of $\hat{y}$-cospectral graphs exists for every rational $\hat{y}$.*

Variations on the above construction are possible. The $\hat{y}$-cospectral pairs, with $0 < \hat{y} < 1$, constructed in [1] (where one graph is regular and the other one not) are of a completely different nature.

4. Computer enumeration

By computer we enumerated all graphs with a $\hat{y}$-cospectral $\left(\hat{y} \neq \frac{1}{2}\right)$ mate on at most nine vertices. For fixed numbers of vertices (n) and edges (e) we generated all graphs with these numbers using *nauty* [8], and for each graph we computed $p(x, y)$ for $x = 0, \ldots, n$. Note that these $n + 1$ linear functions in y uniquely determine the polynomial $p(x, y)$. For each pair of graphs we compared the corresponding linear functions, giving a system of $n + 1$ linear equations in y. If the system had infinitely many solutions, then we concluded that the pair was $\mathbb{R}$-cospectral; and if it had a unique solution $\hat{y}$, then the pair was $\hat{y}$-cospectral. The results of these computations are given in Table 1. Note that we only considered the cases where $2e \leqslant \binom{n}{2}$ since, as mentioned before, the complement of a pair of $\hat{y}$-cospectral graphs is a pair of $(1 - \hat{y})$-cospectral graphs. In the table, the columns with e give the numbers of edges and the columns with # give the numbers of graphs

Table 1
Numbers of graphs with a $\hat{y}$-cospectral mate

Graphs on 5 vertices

e	#	$\mathbb{R}$	0
2	2	.	.
3	4	.	.
4	6	.	2
5	6	.	.

Graphs on 6 vertices

e	#	$\mathbb{R}$	0	$\frac{1}{3}$
2	2	.	.	.
3	5	.	.	.
4	9	.	4	.
5	15	.	2	.
6	21	.	2	2
7	24	.	2	.

Graphs on 7 vertices

e	#	$\mathbb{R}$	-1	0	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{3}$	$\frac{3}{4}$	1
2	2	.	.	.	.	.	.	.	.	.	.	.
3	5	.	.	.	.	.	.	.	.	.	.	.
4	10	.	.	4	.	.	.	.	.	.	.	.
5	21	.	.	3	.	.	4	.	.	.	.	2
6	41	2	.	12	.	.	6	.	.	.	.	.
7	65	2	2	6	2	.	8	4	.	.	.	.
8	97	4	.	14	.	.	18	2	.	.	.	.
9	131	6	.	6	.	2	11	2	2	.	2	6
10	148	6	.	10	.	.	4	2	2	2	.	9

Graphs on 8 vertices

e	#	$\mathbb{R}$	-1	0	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	1	2
2	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	5	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	11	.	.	4	.	.	.	.	.	.	.	.	.	.	.
5	24	.	.	5	.	.	.	.	.	.	.	.	.	.	.
6	56	2	.	21	.	2	2	.	.	.	.	.	.	.	.
7	115	4	.	22	.	6	2	.	.	.	.	.	.	.	.
8	221	10	.	58	.	15	8	.	2	.	.	.	.	.	.
9	402	30	.	44	.	17	6	.	.	.	.	.	.	.	.
10	663	55	.	117	.	44	20	.	2	2	.	.	.	11	.
11	980	87	.	67	.	55	31	2	.	.	.	2	.	4	.
12	1312	133	2	90	2	36	30	.	.	.	2	6	.	18	.
13	1557	173	.	58	.	21	22	.	.	.	.	7	2	37	.
14	1646	178	2	47	.	2	31	.	4	.	4	31	2	47	2

Graphs on 9 vertices

e	#	$\mathbb{R}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{3}{14}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{3}{10}$	$\frac{5}{16}$	$\frac{1}{3}$	$\frac{7}{20}$	$\frac{5}{14}$	$\frac{4}{11}$
2	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	11	.	.	.	.	.	.	4	.	.	.	.	.	.	.	.	.	.	.	.
5	25	.	.	.	.	.	.	5	.	2	.	.	2	.	.	.	.	.	.	.
6	63	2	.	.	.	.	.	24	.	.	.	.	10	.	.	.	.	.	.	.
7	148	6	.	.	.	.	2	47	.	.	.	.	28	.	.	.	2	.	.	.
8	345	18	.	.	.	2	.	130	.	4	.	.	82	.	.	.	8	.	.	.
9	771	61	.	.	.	.	.	165	2	21	4	.	222	.	2	2	20	.	2	.
10	1637	158	.	.	2	3	2	399	.	6	2	.	428	.	.	.	46	.	.	.
11	3252	383	2	.	2	4	2	403	.	8	2	.	800	.	2	.	79	.	.	.
12	5995	764	.	.	8	.	3	762	6	20	2	.	1335	6	7	.	221	.	.	.
13	10120	1469	.	.	2	4	.	724	2	16	6	.	1911	.	9	.	344	.	.	.
14	15615	2342	.	.	6	4	2	1133	.	14	11	.	2264	.	9	.	482	.	2	.
15	21933	3557	.	.	2	6	2	915	15	16	2	.	2004	2	19	.	505	.	.	2
16	27987	4624	.	2	5	15	.	1099	2	8	.	2	1489	3	4	.	576	2	.	.
17	32403	5619	.	.	4	9	.	858	2	6	5	.	690	2	.	3	320	.	.	.
18	34040	5805	.	.	6	10	.	736	2	10	.	.	760	.	6	.	270	.	.	.

e	#	$\frac{3}{8}$	$\frac{7}{18}$	$\frac{2}{5}$	$\frac{5}{12}$	$\frac{3}{7}$	$\frac{4}{9}$	$\frac{11}{20}$	$\frac{5}{9}$	$\frac{4}{7}$	$\frac{7}{12}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{9}{14}$	$\frac{2}{3}$	$\frac{11}{16}$	$\frac{7}{10}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{7}{8}$	1	$\frac{5}{4}$	$\frac{3}{2}$	2
2	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	11	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	25	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	63	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
7	148	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.
8	345	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
9	771	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.
10	1637	8	.	5	.	.	.	.	.	.	.	.	.	.	.	.	.	4	.	.	.	4	.	.	.
11	3252	14	.	4	2	.	.	.	.	.	.	.	.	.	.	.	.	5	.	.	.	12	2	.	.
12	5995	20	.	13	.	2	.	.	.	.	.	.	.	.	.	.	.	14	.	.	.	30	2	2	.
13	10120	70	.	16	4	2	.	.	.	.	.	.	.	.	.	.	.	14	.	.	.	45	.	6	.
14	15615	106	2	27	4	2	.	.	.	.	.	.	6	.	16	.	.	79	.	.	.	134	.	2	2
15	21933	100	2	30	12	5	.	.	.	2	2	4	12	2	28	.	.	61	.	4	.	168	2	7	2
16	27987	124	2	51	2	2	.	2	.	2	4	16	18	2	101	.	.	186	2	.	.	388	.	4	2
17	32403	92	.	39	2	4	2	.	2	.	10	13	53	.	112	2	.	216	.	.	.	434	.	8	.
18	34040	54	.	50	17	14	4	.	4	14	17	50	54	.	270	.	6	760	.	10	2	736	.	10	6

Table 2
Numbers of regular graphs $\hat{y}$-cospectral with nonregular graphs

n	k	#	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{4}{11}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{5}{12}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{4}{7}$	$\frac{7}{12}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$
6	2	2	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
8	2	3	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.
8	3	6	1	.	.	3	.	.	.	.	.	.	.	.	.	.	.
9	2	4	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.
9	4	16	.	1	.	1	.	.	2	1	2	.	2	1	2	1	1
10	2	5	2	1	.	1	.	.	.	.	.	.	.	.	.	.	.
10	3	21	1	2	1	5	.	.	1	.	1	.	.	.	.	.	.
10	4	60	.	4	.	3	.	.	4	.	.	1	1	.	1	1	.
11	4	266	.	45	.	22	1	2	5	.	.	.	.	.	.	1	.

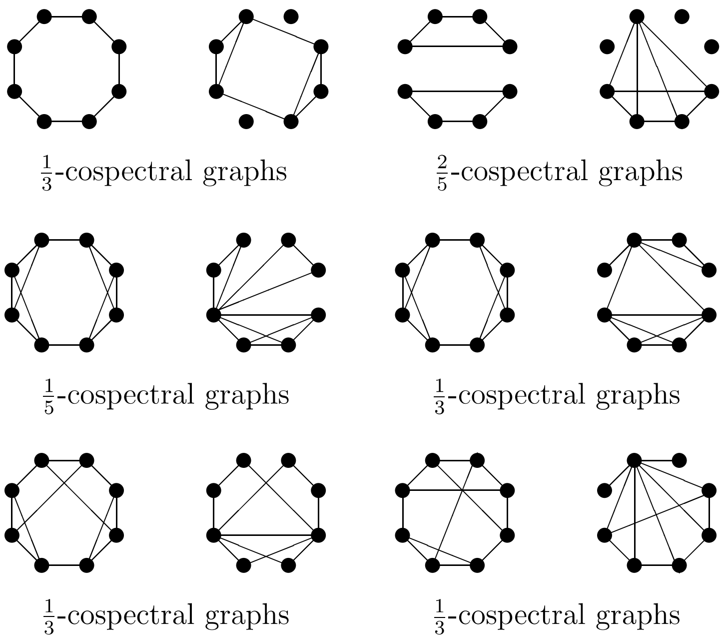

Fig. 2. Regular-nonregular $\hat{y}$-cospectral pairs on eight vertices.

with e edges. The columns with header $\mathbb{R}$ contain the number of graphs that have an $\mathbb{R}$-cospectral mate. The columns with a number $\hat{y}$ in the header contain the numbers of graphs that have a $\hat{y}$-cospectral mate. Note that this does not mean that a graph cannot be counted in more than one column; for example, of the triple of 0-cospectral graphs with seven vertices and five edges, one (the union of $K_{1,4}$ and an edge) also has a 1-cospectral mate, and another (the union of $K_{2,2}$, an isolated vertex, and an edge) also has a $\frac{1}{4}$-cospectral mate.

We may conclude that the 0-cospectral pair of Fig. 1 is the smallest pair of $\hat{y}$-cospectral graphs. The smallest pair of $\hat{y}$-cospectral graphs for $\hat{y} \neq 0$ is the pair of $\frac{1}{3}$-cospectral graphs in Fig. 1. This is also the smallest example where one graph is regular, and the other one not. The smallest $\mathbb{R}$-cospectral pair of graphs, and the smallest $\hat{y}$-cospectral pair of graphs for a negative $\hat{y}$ are also given in Fig. 1.

We remark that also in [6] all graphs with an $\mathbb{R}$-cospectral mate, as well as all graphs with a (usual) cospectral mate were enumerated (up to eleven vertices). The latter enumeration is different from our enumeration of graphs with a 0-cospectral mate since it also counts graphs with a $\mathbb{R}$-cospectral mate (whereas these are excluded in our enumeration).

We also enumerated all regular graphs with a nonregular $\hat{y}$-cospectral mate $\left(\hat{y} \neq \frac{1}{2}\right)$ on at most eleven vertices; see Table 2. The columns with a number $\hat{y}$ in the header contain the numbers of graphs that have a $\hat{y}$-cospectral mate. The column with n gives the number of vertices, the column with k gives the valency and the column with # gives the number of k-regular graphs with n vertices. The computations were restricted to $v \geqslant 2k + 1$ for a similar reason as before. We remark further that for missing pairs (v, k) in the considered range, such as $(11, 2)$, there are no regular graphs with a $\hat{y}$-cospectral mate $\left(\hat{y} \neq \frac{1}{2}\right)$. In Fig. 2 we give all regular-nonregular $\hat{y}$-cospectral pairs on eight vertices (up to complements).

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