RENÉ VAN DEN BRINK, PETER BORM, RUUD HENDRICKX
and GUILLERMO OWEN

CHARACTERIZATIONS OF THE β- AND THE DEGREE NETWORK POWER MEASURE

ABSTRACT. A symmetric network consists of a set of positions and a set of bilateral links between these positions. For every symmetric network we define a cooperative transferable utility game that measures the "power" of each coalition of positions in the network. Applying the Shapley value to this game yields a network power measure, the β-measure, which reflects the power of the individual positions in the network. Applying this power distribution method iteratively yields a limit distribution, which turns out to be proportional to the well-known degree measure. We compare the β-measure and degree measure by providing characterizations, which differ only in the normalization that is used.

KEY WORDS: symmetric networks, network power, β-measure, degree measure, stationary power distribution

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1. INTRODUCTION

An undirected graph, which consists of a set of positions and a set of links between pairs of positions, can represent various types of networks. Examples of such *symmetric networks*, in which the roles of the two positions on each link are symmetric, are exchange networks, communication networks, disease transmission networks and control networks. On the other hand, in *asymmetric networks* the positions on a link have different roles. One can think of buyer–seller networks or firm structures. The purpose of this article is to measure "power" or "control" of positions in symmetric networks.

For every symmetric network we define a cooperative transferable utility game that measures the worth or power

of coalitions of positions. Applying the Shapley value (Shapley, 1953) to this network power game yields the *β-measure*, which is discussed in van den Brink and Gilles (2000) and van den Brink and Borm (2002) for asymmetric networks. The idea behind the β-measure is that each position in a network has an initial weight equal to 1, and measuring power is seen as fairly redistributing this weight to all its neighbours. This measure fits well with *power dependence theory* as developed by Emerson (1962) since the power value of a position decreases when its neighbours have more other neighbours. We provide an axiomatic characterization of the β-measure using four logically independent properties that are based on graph-manipulation.

Instead of taking initial weights equal to 1, it seems natural to take weights that already reflect some power of the positions. In this way one obtains weighted β-measures. Similar as done in Borm et al. (2002) for asymmetric networks, we consider a sequence of weighted β-measures. Starting with the (unweighted) β-measure, we compute in each step a new weighted β-measure, taking the outcome of the previous step as input weights. We show that this sequence has a limit, which is proportional to the well-known *degree measure* for symmetric networks. This degree measure assigns to every position just its number of direct neighbours. We provide an axiomatic characterization of the degree measure that is similar to that of the β-measure, where the only difference is the normalization that is used.

The article is organized as follows. In Section 2 we briefly discuss some graph and game theoretic preliminaries. In Section 3 we introduce network power games and introduce the β-measure for symmetric networks. In Section 4 we provide an axiomatic characterization of the β-measure. In Section 5 we discuss the sequence of weighted β-measures, show that its limit is proportional to the degree measure, and provide an axiomatic characterization of the degree measure. Finally, in Section 6 we make some concluding remarks.

2. PRELIMINARIES

In this section we discuss some graph and game theoretic preliminaries. A *symmetric network* or undirected graph is a pair (N, G) where N is a finite set of *positions* or nodes, and $G \subseteq \{\{i, j\} | i, j \in N, i \neq j\}$ is a set of symmetric *edges* or links between these positions. So, we assume the networks to be *irreflexive*, i.e., $\{i, i\} \notin G$ for all $i \in N$. The collection of all (irreflexive) networks is denoted by $\mathcal{G}$. We often refer to these just as graphs.

If $\{i, j\} \in G$, then positions i and j are called *neighbours* and are *incident* to the edge $\{i, j\}$. We denote the set of all neighbours of position $i \in N$ in network (N, G) by

$$R_{(N,G)}(i) = \{j \in N \mid \{i, j\} \in G\}.$$

For a set of positions $S \subseteq N$ we denote $R_{(N,G)}(S) = \bigcup_{i \in S} R_{(N,G)}(i)$. If $R_{(N,G)}(i) = \emptyset$, then position i is called an *isolated* position. Position $i \in N$ is called a *pending* position if $|R_{(N,G)}(i)| = 1$. We denote the set of isolated positions in network (N, G) by $I(N, G)$ and the set of pending positions by $P(N, G)$.

For every graph $(N, G) \in \mathcal{G}$ and set of positions $T \subseteq N$, the induced subgraph $(T, G(T))$ is given by $G(T) = \{\{i, j\} \in G \mid \{i, j\} \subseteq T\}$. A network (N, G) is *connected* if for every pair of positions $i, j \in N$ there exists a sequence of positions $h_1, \ldots, h_p$ such that $h_1 = i, h_p = j$, and $\{h_k, h_{k+1}\} \in G$ for all $k \in \{1, \ldots, p - 1\}$. A set of positions $T \subseteq N$ is a *component* in $(N, G) \in \mathcal{G}$ if it is a maximally connected subset of N in (N, G), i.e., if the graph $(T, G(T))$ is connected and for every $i \in N \setminus T$ the graph $(T \cup \{i\}, G(T \cup \{i\}))$ is not connected. The set of components in graph (N, G) that consist of at least two nodes is denoted by $\mathcal{B}(N, G)$. (Note that $\mathcal{B}(N, G)$ is a partition of $N \setminus I(N, G)$.) The component of (N, G) containing node $i \in N$ is denoted by $B_i(N, G)$.

A *network power measure* for symmetric networks is a mapping p that assigns to every network $(N, G) \in \mathcal{G}$ a vector $p(N, G) \in \mathbb{R}^N$. We refer to this vector as a network power distribution for (N, G). A well-known network power measure is the *degree measure*, which assigns to every position in a

network its number of neighbours. The *degree measure* thus is the power measure d given by

$$d_i(N, G) = |R_{(N,G)}(i)| \text{ for all } i \in N.$$

A (finite) *cooperative game with transferable utility* (or simply TU-game) is a pair (N, v) with finite set N of *players* and *characteristic function* $v: 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. A (single valued) solution for TU-games is a function f that assigns to every TU-game (N, v) a vector $f(N, v) \in \mathbb{R}^N$, representing a distribution of payoffs to the players. A well-known solution is the *Shapley value* (Shapley, 1953), which equally distributes the *dividends* $\Delta_v(S)$ (see Harsanyi, 1959) over all players in coalition $S \subseteq N, S \neq \emptyset$:

$$Sh_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N,$$

where $\Delta_v(S) = v(S)$ if $|S| = 1$, and recursively $\Delta_v(S) = v(S) - \sum_{\substack{T \subsetneq S \\ T \neq \emptyset}} \Delta_v(T)$ for all $S \subseteq N, |S| \geq 2$. For every $T \subseteq N, T \neq \emptyset$, the *unanimity game* u_T is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. It is well-known that every characteristic function $v: 2^N \to \mathbb{R}$ can be written as a linear combination of unanimity games in a unique way by $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$.

3. NETWORK POWER GAMES AND THE β-MEASURE

In order to measure power or control in networks, we assign to every network $(N, G) \in \mathcal{G}$ a cooperative game with transferable utility (N, v), whose set of players N corresponds to the set of positions in the network. In cooperative game theoretic tradition we take a conservative approach to measuring the worth of a coalition by assigning to every coalition of positions $S \subseteq N$ the number of neighbours of S that have no neighbours outside S. The *network power game* (N, v_G) corresponding to $(N, G) \in \mathcal{G}$ thus is given by

$$v_G(S) = |\{j \in R_{(N,G)}(S)|R_{(N,G)}(j) \subseteq S\}| \quad \text{for all } S \subseteq N.$$

Note that $v_G(N) = |N \setminus I(N,G)|$ for all $(N,G) \in \mathcal{G}$. The dividends of v_G are given by

$$\Delta_{v_G}(S) = |\{j \in N \mid R_{(N,G)}(j) = S\}| \quad \text{for all } S \subseteq N, S \neq \emptyset, \quad (1)$$

which is easily seen since $\sum_{\substack{T \subseteq S \\ T \neq \emptyset}} \Delta_{v_G}(T) = \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} |\{j \in N \mid R_{(N,G)}(j) = T\}| = |\{j \in N | \emptyset \neq R_{(N,G)}(j) \subseteq S\}| = |\{j \in R_{(N,G)}(S)|R_{(N,G)}(j) \subseteq S\}| = v_G(S)$. Hence, this game can be decomposed as $v_G = \sum_{i \in R_{(N,G)}(N)} u_{R_{(N,G)}(i)}$. So, every network power game is *totally positive* meaning that it can be expressed as a nonnegative sum of unanimity games (see, e.g. Vasil'ev, 1975; Hammer et al., 1977). As a corollary, a network power game is *convex* meaning that $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

The Shapley value of a network power game can be seen as a network power distribution of the underlying network. The corresponding power measure is called the β-measure:

$$\beta(N,G) = Sh(N, v_G) \text{ for all } (N,G) \in \mathcal{G}.$$

PROPOSITION 3.1. *For every* $(N,G) \in \mathcal{G}$ *we have*

$$\beta_i(N,G) = \sum_{j \in R_{(N,G)}(i)} \frac{1}{|R_{(N,G)}(j)|} \text{ for all } i \in N.$$

Proof. Using (1), we obtain

$$\beta_i(N,G) = Sh_i(N, v_G) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\Delta_{v_G}(S)}{|S|}$$

$$= \sum_{\substack{S \subseteq N \\ i \in S}} \frac{|\{j \in N | R_{(N,G)}(j) = S\}|}{|S|}$$

$$= \sum_{\substack{S \subseteq N \\ i \in S}} \sum_{\substack{j \in N \\ R_{(N,G)}(j) = S}} \frac{1}{|R_{(N,G)}(j)|} = \sum_{j \in R_{(N,G)}(i)} \frac{1}{|R_{(N,G)}(j)|}.$$
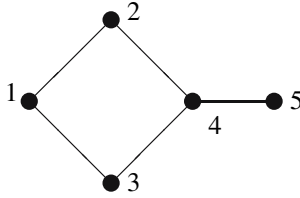
$\square$

Figure 1. Network (N, G) from Example 3.2.

So, it turns out that the β-measure distributes the weight over each position equally amongst its neighbours. Suppose that the positions in the network (N, G) are occupied by agents who will visit exactly one of their neighbouring positions. Then $\beta_i(N, G)$ can be interpreted as the expected number of neighbours that visit position i assuming that all neighbours of i visit all their neighbours with equal probability. The worth of a coalition of positions in the network power game then is the number of positions in the coalition that with probability one will visit a neighbour from inside the coalition.

EXAMPLE 3.2. Consider the network (N, G) with $N = \{1, 2, 3, 4, 5\}$ and $G = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$, see Figure 1. For this network we have $\beta(N, G) = (1, \frac{5}{6}, \frac{5}{6}, 2, \frac{1}{3})$, e.g. position 2 is visited by the agent in position 1 with probability $\frac{1}{2}$, and by the agent in position 4 with probability $\frac{1}{3}$, yielding an expected number of visits equal to $\frac{5}{6}$.

An important class of social networks consists of *exchange networks*. These networks consist of a set of positions that each own certain resources and a set of pairwise (symmetric) relations between these positions. The positions are occupied by agents. The pairwise relations between the positions describe the possibilities of the corresponding agents to engage in binary exchange processes. The possibilities of the agents to obtain favourable resource bundles depend on their positions in the exchange network. In this context the network power of a position refers to the possibilities of the

corresponding agent to obtain favourable exchange outcomes because of its position in the exchange network.

In measuring power in such networks it is usual that the power of a position increases if it has more neighbours. However, indirect relations might have a positive or a negative effect on the power of a position. The positive effect arises from the fact that the position can become intermediary between more pairs of other positions. The negative effect arises from the fact that the position can become less important as an intermediary for its direct neighbours whose exchange possibilities increase (see, e.g. Cook et al., 1983; Markovsky et al., 1998). This idea is based on *power dependence theory* as discussed by Emerson (1962) which states that if a direct relative of a position has more exchange possibilities then this relative is less dependent on that position. For a further discussion we refer to van den Brink (2002). Since, according to the β-measure the power of a position increases when it gets more neighbours, but decreases when its neighbours get more neighbours, the β-measure fits within power dependence theory.

4. A CHARACTERIZATION

In this section, we provide an axiomatic characterization of the β-measure using four properties. Let p be a network power measure. The first property is a normalization determining the total value of power to be distributed in each component. Since, we want to measure power resulting from the possibilities of positions to communicate with each other, we require that in each component with at least two positions the sum of the power values equals the number of positions in that component.

Component efficiency For every $(N, G) \in \mathcal{G}$ and $B \in \mathcal{B}(N, G)$ it holds that $\sum_{i \in B} p_i(N, G) = |B|$.

The second property is a boundary condition, which states that the power value of a position never exceeds the number of its neighbours, and is at least equal to the number of

its pending neighbours. The lower bound is motivated since pending positions are fully dependent on their neighbours. The upper bound reflects that a position cannot have more than full power over its direct neighbours, and thus also indirect power (i.e. power over neighbours of neighbours and so on) cannot be more than direct power over direct neighbours.

Reasonability For every $(N, G) \in \mathcal{G}$ it holds that $|R_{(N,G)}(i) \cap P(N, G)| \leq p_i(N, G) \leq |R_{(N,G)}(i)|$ for all $i \in N$.

Note that reasonability implies that all isolated positions have power value equal to zero. The third property states that cutting an edge into two pieces and putting two new positions at the two endings does not change the power of the positions that are not incident to the edge that is cut. This is a kind of invariance property implying that the power of a position is determined by a local structure.

Edge cutting independence Let (N, G) be such that $\{h, i\} \in G$. Then for $N' = N \cup \{j, k\}$ with $j, k \notin N$, and $G' = (G \setminus \{h, i\}) \cup \{\{h, j\}, \{i, k\}\}$ it holds that $p_\ell(N, G) = p_\ell(N', G')$ for all $\ell \in N \setminus \{h, i\}$.

The fourth property states that adding a new position to an existing position changes the power value of each neighbour of the existing position by the same amount. It is a kind of equal treatment property implying that positions are not discriminated with respect to the addition of a pending position to a common neighbour.

Pending node addition Let (N, G) and (N', G') be such that $N' = N \cup \{i\}$ with $i \notin N$, and $G' = G \cup \{h, i\}$ for some $h \in N$. Then $p_j(N, G) - p_j(N', G') = p_k(N, G) - p_k(N', G')$ for all $j, k \in R_{(N,G)}(h)$.

It is readily verified that the β-measure satisfies the four properties introduced above. To prove uniqueness, we start by showing that the first three properties uniquely determine the β-measure for star components. We call a component $B \in \mathcal{B}(N, G)$ a *star component* in $(N, G) \in \mathcal{G}$ if there exists an $h \in B$ with $h \in \{i, j\}$ for all $\{i, j\} \in G(B)$. We call h a *central* position in this star component.

LEMMA 4.1. *If a power measure p on $\mathcal{G}$ satisfies component efficiency, reasonability and edge cutting independence, and $B \in \mathcal{B}(N, G)$ is a star component in $(N, G) \in \mathcal{G}$, then $p_i(N, G) = \beta_i(N, G)$ for all $i \in B$.*

Proof. Let $B \in \mathcal{B}(N, G)$ be a star component in $(N, G) \in \mathcal{G}$ with central position h. Reasonability implies that $p_h(N, G) = |R_{(N,G)}(h)|$. If $|B| = 2$, then the result immediately follows. Otherwise, let $i, j \in R_{(N,G)}(h)$. Cut the edge $\{h, i\}$ and add the new positions x and i' to h and i, respectively. This results in a graph $(\overline{N}, \overline{G})$ with $\overline{N} = N \cup \{i', x\}$ and $\overline{G} = (G \setminus \{\{h, i\}\}) \cup \{\{h, x\}, \{i, i'\}\}$. By reasonability, $p_h(\overline{N}, \overline{G}) = |R_{(\overline{N}, \overline{G})}(h)| = |R_{(N,G)}(h)| = p_h(N, G)$. By edge cutting independence, $p_k(\overline{N}, \overline{G}) = p_k(N, G)$ for all $k \in B \setminus \{h, i\}$. So, by component efficiency, $p_x(\overline{N}, \overline{G}) = p_i(N, G)$. Next, cut the edge $\{h, j\}$ and add two new positions y and j' to h and j, respectively. This results in a graph (N', G') with $N' = \overline{N} \cup \{j', y\} = N \cup \{i', j', x, y\}$ and $G' = (\overline{G} \setminus \{\{h, j\}\}) \cup \{\{h, y\}, \{j, j'\}\} = (G \setminus \{\{h, i\}, \{h, j\}\}) \cup \{\{h, x\}, \{h, y\}, \{i, i'\}, \{j, j'\}\}$. By the same argument as before, $p_x(N', G') = p_i(N, G)$ and $p_y(N', G') = p_j(N, G)$.

Note that we can also construct the graph (N', G') from (N, G) by first cutting $\{h, i\}$ and adding y and i' and then cutting $\{h, j\}$ and adding x and j'. By the same reasoning as in the first construction, $p_y(N', G') = p_i(N, G)$ and $p_x(N', G') = p_j(N, G)$ and thus $p_i(N, G) = p_x(N', G') = p_j(N, G)$. Since, this holds for all $i, j \in R_{(N,G)}(h)$, together with $p_h(N, G) = |R_{(N,G)}(h)|$ and component efficiency, this implies $p_i(N, G) = \beta_i(N, G)$ for all $i \in B$. $\square$

Next we show that these properties uniquely determine the values for pending positions in any network.

LEMMA 4.2. *If a power measure p on $\mathcal{G}$ satisfies component efficiency, reasonability and edge cutting independence, then $p_i(N, G) = \beta_i(N, G)$ for all $(N, G) \in \mathcal{G}$ and all $i \in P(N, G)$.*

Proof. Let $(N, G) \in \mathcal{G}, i \in P(N, G), h \in R_{(N,G)}(i)$ and let $m = |R_{(N,G)}(h)| - 1$. Further, let the neighbours of h be labelled

by $R_{(N,G)}(h) = \{h_1, \ldots, h_m\}$. Consider the network $(N_i, G_i) \in \mathcal{G}$ given by $N_i = \{1, \ldots, n + 2m\}$ and $G_i = (G \setminus C) \cup D \cup E$ with $C = \{g \in G \mid h \in g, \ i \notin g\}$, $D = \bigcup_{k=1}^{m} \{h, n+k\}$ and $E = \bigcup_{k=1}^{m} \{h_k, n+m+k\}$. Since $B_i(N_i, G_i)$ is a star component of (N_i, G_i) we have that $p_i(N_i, G_i) = \beta_i(N_i, G_i)$ by Lemma 4.1. Since, (N_i, G_i) is obtained from (N, G) by cutting edges to which position i is not incident, edge cutting independence implies that $p_i(N, G) = p_i(N_i, G_i) = \beta_i(N_i, G_i) = \beta_i(N, G)$. □

Next we show uniqueness for components with two central positions. We call a component $B \in \mathcal{B}(N, G)$ a *double-centred star* component in network $(N, G) \in \mathcal{G}$ if there exists an edge $\{h, i\} \in G(B)$ such that $\{h, i\} \cap \{g, j\} \neq \emptyset$ for all $\{g, j\} \in G(B)$ and $R_{(N,G)}(h) \cap R_{(N,G)}(i) = \emptyset$. We call h and i the central positions in this double-centred star component. Note that a star component is a double-centred star component in which (at least) one of the central positions is pending.

LEMMA 4.3. *If a power measure p on $\mathcal{G}$ satisfies component efficiency, reasonability, edge cutting independence and pending node addition, and $B \in \mathcal{B}(N, G)$ is a double-centred star component in $(N, G) \in \mathcal{G}$, then $p_i(N, G) = \beta_i(N, G)$ for all $i \in B$.*

Proof. Let $B \in \mathcal{B}(N, G)$ be a double-centred star component in $(N, G) \in \mathcal{G}$ with central positions h and i. If B is a star component, then the result follows from Lemma 4.1. Otherwise, let $j \in R_{(N,G)}(i) \setminus \{h\}$. By Lemma 4.2, $p_j(N, G) = \beta_j(N, G)$.

Starting with $(N^0, G^0) = (N, G)$, construct a sequence of graphs $(N^t, G^t)_{t \in \mathbb{N}}$, where $N^t = N^{t-1} \cup \{n^t\}$, $n^t \notin N^{t-1}$, and $G^t = G^{t-1} \cup \{\{i, n^t\}\}$ for all $t \in \mathbb{N}$. Then by Lemma 4.2, $\lim_{t \to \infty} p_j(N^t, G^t) = \lim_{t \to \infty} \beta_j(N^t, G^t) = 0$. By pending node addition we then have $\lim_{t \to \infty} p_h(N^t, G^t) = p_h(N, G) - \beta_j(N, G)$. By reasonability, $p_h(N^t, G^t) \geq |R_{(N^t, G^t)}(h) \cap P(N^t, G^t)| = |R_{(N,G)}(h) \cap P(N, G)|$ for all $t \in \mathbb{N}$. Hence, $p_h(N, G) \geq |R_{(N,G)}(h) \cap P(N, G)| + \beta_j(N, G) = \beta_h(N, G)$. Similarly, one can show that $p_i(N, G) \geq \beta_i(N, G)$. Together with $p_j(N, G) = \beta_j(N, G)$ for all $j \in B \setminus \{h, i\}$ (which follows from Lemma 4.2) and component efficiency, the statement follows. □

Next, we show that in an *acyclic* component[1] where all positions are at distance at most two from a central position, the power values are determined by the four properties.

LEMMA 4.4. *Let $B \in \mathcal{B}(N, G)$ be such that there exists an $h \in B$ such that $i \in R_{(N,G)}(h) \cup R_{(N,G)}(R_{(N,G)}(h))$ for all $i \in B \backslash \{h\}$, and $(\{i\} \cup R_{(N,G)}(i)) \cap (\{j\} \cup R_{(N,G)}(j)) = \{h\}$ for all $i, j \in R_{(N,G)}(h)$. If power measure p on $\mathcal{G}$ satisfies component efficiency, reasonability, edge cutting independence and pending node addition then $p_i(N, G) = \beta_i(N, G)$ for all $i \in B$.*

Proof. By Lemma 4.2, $p_i(N, G) = \beta_i(N, G)$ for all $i \in B \cap P(N, G)$. Let $m = |R_{(N,G)}(h)| - 1$ and let the neighbours of h be labelled by $R_{(N,G)}(h) = \{h_1, \ldots, h_m\}$. If B is a star component, then the result follows from Lemma 4.1. Otherwise, let $i \in B \backslash (\{h\} \cup P(N, G))$, and let (N_i, G_i) be given by $N_i = \{1, \ldots, n + 2m\}$ and $G_i = (G \backslash C) \cup D \cup E$ with $C = \{g \in G \mid h \in g, \, i \notin g\}$, $D = \bigcup_{k=1}^{m} \{h, n + k\}$ and $E = \bigcup_{k=1}^{m} \{h_k, n + m + k\}$. Since $B_i(N_i, G_i)$ is a double-centered star component, $p_i(N_i, G_i) = \beta_i(N_i, G_i) = \beta_i(N, G)$ by Lemma 4.3. Since i is not incident to any edge that is cut going from (N, G) to (N_i, G_i), by edge cutting independence it then follows that $p_i(N, G) = p_i(N_i, G_i) = \beta_i(N_i, G_i) = \beta_i(N, G)$. Finally, since the power values of all positions in B except the central position h are determined as their β-outcomes, by component efficiency also $p_h(N, G) = \beta_h(N, G)$. $\square$

Finally, we can state our characterization of the β-measure.

THEOREM 4.5. *A power measure p on $\mathcal{G}$ is equal to the β-measure if and only if it satisfies component efficiency, reasonability, edge cutting independence and pending node addition.*

Proof. It is readily verified that the β-measure satisfies the four properties. For the converse, assume that the power measure p on $\mathcal{G}$ satisfies the properties, and let $(N, G) \in \mathcal{G}$. Let $h \in N$. If $h \in I(N, G)$, then $p_h(N, G) = \beta_h(N, G)$ by reasonability.

Otherwise, construct the graph (N', G') by cutting every edge not incident to h and adding a new node to each of the two positions involved. Then $B_h(N', G')$ is as described in Lemma 4.4 and $p_h(N', G') = \beta_h(N', G')$. As h is not involved in any edge that is cut, it follows from edge cutting independence that $p_h(N, G) = \beta_h(N, G)$. Hence, $p(N, G) = \beta(N, G)$. $\square$

Note that the four properties in Theorem 4.5 imply *anonymity* meaning that when permuting the labels of the positions in a graph the power distribution is obtained by permuting the payoffs correspondingly.

We end this section by showing logical independence of the four axioms of Theorem 4.5 using the following four alternative power measures.

1. The degree measure satisfies reasonability, edge cutting independence and pending node addition. It does not satisfy component efficiency.
2. Consider the power measure f given by $f_i(N, G) = 1$ if $i \in N \backslash I(N, G)$, and $f_i(N, G) = 0$ if $i \in I(N, G)$. In this power measure every non-isolated position keeps the power over itself. This power measure satisfies component efficiency, edge cutting independence and pending node addition. It does not satisfy reasonability.
3. Consider the power measure f given for all $i \in N$ by $f_i(N, G) = \beta_i(N, G)$ if $B_i(N, G)$ is a star component, and else by $f_i(N, G) = 0$ if $i \in P(N, G) \cup I(N, G)$ and $f_i(N, G) = |R_{(N,G)}(i) \cap P(N, G)| + 1$ otherwise. The difference with the β-measure is that in each component with at least two non-pending positions, these positions retain the full power over themselves. This power measure satisfies component efficiency, reasonability and pending node addition. It does not satisfy edge cutting independence.
4. Consider the power measure f given by $f_i(N, G) = \beta_i(N, G)$ if $i \in P(N, G) \cup I(N, G)$, and $f_i(N, G) = |R_{(N,G)}(i) \cap P(N, G)| + \frac{|R_{(N,G)}(i) \backslash P(N,G)|}{|R_{(N,G)}(i)|}$ if $i \in N \setminus (P(N, G) \cup I(N, G))$. The difference with the β-measure is that each non-pending (and non-isolated) position retains that part of the power over itself that

according to the β-measure would go to its non-pending neighbours. This power measure satisfies component efficiency, reasonability and edge cutting independence. It does not satisfy pending node addition.

The following example illustrates that the above given four power measures do not satisfy the corresponding axioms.

EXAMPLE 4.6. Consider the network (N, G) with $N = \{1, 2, 3, 4, 5\}$ and $G = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$. The β-measure of this network equals $(\frac{1}{2}, 1\frac{1}{3}, 2\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$. The degree measure of this network equals $(1, 2, 3, 1, 1)$, so its components do not add up to five, showing that this power measure does not satisfy component efficiency. Clearly, the second power measure given above yields power distribution $(1, 1, 1, 1, 1)$ and thus position 3 gets less than 2 being the number of its pending neighbours, showing that this power measure does not satisfy reasonability.

The third power measure yields $(0, 2, 3, 0, 0)$. Considering the network (N', G') given by $N' = N \cup \{6, 7\}$ and $G' = (G \backslash \{\{2, 3\}\}) \cup \{\{2, 6\}, \{3, 7\}\}$, the power measure for this network equals $(\frac{1}{2}, 1, 3, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3})$. So, the power value of positions 4 and 5 changed although an edge on which they are not incident is cut, showing that this power measure does not satisfy edge cutting independence.

Finally, the fourth power measure yields the power distribution $(\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for (N, G). Considering the network (N', G') given by $N' = N \cup \{6\}$ and $G' = G \cup \{\{3, 6\}\}$, the power distribution for this network equals $(\frac{1}{2}, 1\frac{1}{2}, 3\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. So, the power value of position 2 did not change while the power value of positions 4 and 5 decreased by adding position 6 as a pending position to position 3, showing that this power measure does not satisfy pending node addition.

5. AN ITERATIVE PROCEDURE AND THE DEGREE MEASURE

The idea behind the β-measure is that every position in a network has an initial weight equal to 1, and each of its

neighbours receives an equal share of this weight. Instead of taking initial weights equal to 1, it seems natural to take weights that already reflect the power of the positions. If we take the β-measure as initial weights, we obtain the second order measure β^2. Of course, this second order measure can be used as new input weights, and so on, yielding higher order measures. Starting with

$$\beta_i^0(N, G) = 1 \quad \text{for all } i \in N,$$

we recursively define the measures

$$\beta_i^t(N, G) = \sum_{j \in R_{(N,G)}(i)} \frac{\beta_j^{t-1}(N, G)}{|R_{(N,G)}(j)|} \quad \text{for all } i \in N, t \in \{1, 2, \ldots\}. \tag{2}$$

In particular, $\beta^1(N, G) = \beta(N, G)$. This sequence of measures has a limit, which is a stationary power distribution. A power distribution $p \in \mathbb{R}^N$ is a *stationary* power distribution of (2) if redistributing these weights according to (2) yields the same weights:

$$p_i(N, G) = \sum_{j \in R_{(N,G)}(i)} \frac{p_j(N, G)}{|R_{(N,G)}(j)|} \quad \text{for all } i \in N. \tag{3}$$

Borm et al. (2002) define a sequence similar to (2) for *directed* or *asymmetric* networks (N, D) with $D \subseteq N \times N$, and show that it has a limit, which is also a stationary distribution. Defining for every undirected network $(N, G) \in \mathcal{G}$ the corresponding directed network $(N, D(G))$ with $D(G) = \{(i, j) \in N \times N \mid \{i, j\} \in G\}$, the existence of a stationary distribution of (2) can be shown in a similar way as their result. Moreover, since for every $(N, G) \in \mathcal{G}$ and $i \in N$ we have that $\sum_{j \in R_{(N,G)}(i)} \frac{d_j(N,G)}{|R_{(N,G)}(j)|} = \sum_{j \in R_{(N,G)}(i)} \frac{|R_{(N,G)}(j)|}{|R_{(N,G)}(j)|} = |R_{(N,G)}(i)| = d_i(N, G)$, the degree measure yields a stationary power distribution of (2). In case the network (N, G) is connected, the corresponding directed network $(N, D(G))$ is strongly connected,[2] and it follows from standard results on such networks (see, e.g. Berger, 1993) that (2) has a unique component efficient stationary power distribution. If the network is not connected

then it can be shown that in every stationary power distribution the power in every component is distributed proportional to the degrees of the nodes in that component. By component efficiency of every measure $\beta^t, t \in \{1, 2, \dots\}$, this yields uniqueness also for unconnected networks. Summarizing we have the following proposition.

PROPOSITION 5.1. *For every $(N, G) \in \mathcal{G}$ the sequence defined by (2) has a limit, which is proportional to the degree measure of (N, G). This limit is the unique component efficient stationary power distribution of (N, G).*

In Section 3 we argued that the β-measure assigns to every position the expected number of visits if all agents visit each of their neighbours with equal probability. Now, suppose that given the new positions of the agents (i.e. there may be more than one agent in a position now) this process is repeated once more. Every agent starts from one of its neighbouring positions, and again everybody will move to each one of the (new) neighbouring positions with equal probability. The second order β-measure then gives the expected number of visits after this second turn. By iteratively applying this procedure the expected number of visits in the long run is proportional to the degree measure of the network.

The degree measure satisfies all properties stated in Theorem 4.5 except component efficiency. Instead, it satisfies an alternative normalization, which distributes twice the number of edges in every component with at least two nodes.

Degree efficiency For every $(N, G) \in \mathcal{G}$ and $B \in \mathcal{B}(N, G)$ it holds that $\sum_{i \in B} p_i(N, G) = 2|G(B)|$.

Replacing component efficiency by degree efficiency in Theorem 4.5 yields a characterization of the degree measure.

THEOREM 5.2. *A power measure p on $\mathcal{G}$ is equal to the degree measure if and only if it satisfies degree efficiency, reasonability, edge cutting independence and pending node addition.*

Note that the specific normalization is not essential for proving uniqueness in the proofs of Theorem 4.5 and the

preceding lemmas. Therefore the proof of Theorem 5.2 is a straightforward adaptation of the proof of Theorem 4.5 and is omitted.

6. CONCLUDING REMARKS

We provided axiomatic characterizations of the β- and degree measures differing only in the normalization that is used. We also showed that the limit of a recursive procedure which starts with the β-measure, and in each step gives as output a new weighted β-measure, taking as input weights the weighted β-measure obtained in the previous step, is proportional to the degree measure. Moreover, the degree measure is a stationary power distribution in the sense that it satisfies (3), and as such can be interpreted within the context of power dependence theory (see Emerson, 1962). On the other hand, Hendrickx et al. (2005) take an alternative approach. Instead of using weighted β-measures, they consider weighted Shapley values of the corresponding network power game. For various types of networks, they compute and interpret a Proper Shapley value as introduced by Vorob'ev (1998), which assigns to every network power game a particular weighted Shapley value such that these values are equal to the chosen weights. This yields a power measure π satisfying

$$\pi_i(N, G) = \sum_{j \in R_{(N,G)}(i)} \frac{\pi_j(N, G)}{\sum_{h \in R_{(N,G)}(j)} \pi_h(N, G)} \quad \text{for all } i \in N.$$

In defining the conservative network power game we followed the game theoretic tradition to assign to every coalition the minimal worth they can guarantee themselves. By definition, the dual game (N, v^*) of a TU-game (N, v) is given by $v^*(S) = v(N) - v(N \backslash S)$. Since $v_G^*(S) = v_G(N) - v_G(N \backslash S) = |R_{(N,G)}(N)| - |\{j \in R_{(N,G)}(N \backslash S) \mid R_{(N,G)}(j) \subseteq N \backslash S\}| = |R_{(N,G)}(N)| - |\{j \in R_{(N,G)}(N) \mid R_{(N,G)}(j) \cap S = \emptyset\}| = |\{j \in R_{(N,G)}(N) \mid R_{(N,G)}(j) \cap S \neq \emptyset\}| = |R_{(N,G)}(S)|$, it follows that the dual game of the conservative network power game corresponding to $(N, G) \in \mathcal{G}$ assigns to every coalition of positions $S \subseteq N$ the total

number of neighbours of S. This can be seen as an optimistic approach to network power measurement. Since, the Shapley value of a TU-game coincides with the Shapley value of its dual game, the β-measure also is equal to the Shapley value of this dual (optimistic) network power game. Moreover, linearity of the Shapley value implies that the β-measure equals the Shapley value of every convex combination of the conservative network power game and the optimistic network power game.

NOTES

1. A set of positions $\{i_1, \ldots, i_m\}$ with $i_1 = i_m, i_k \neq i_l$ for all $k, l \in \{1, \ldots, m-1\}$, and $\{i_k, i_{k+1}\} \in G$ for all $k \in \{1, \ldots, m-1\}$, is called a *cycle* in (N, G). A component that contains no cycles is called an *acyclic* component.
2. A directed graph (N, D) is *strongly connected* if for each pair of positions $i, j \in N, i \neq j$ there is a sequence of nodes $i_1, \ldots, i_p$ such that $i_1 = i, i_p = j$ and $(i_k, i_{k+1}) \in D$ for all $k \in \{1, \ldots, p-1\}$.

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Address for correspondence: R. Van den Brink, Department of Econometrics and Tinbergen Institute, Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands. E-mail: jrbrink@feweb.vu.nl

P. Borm and R. Hendrickx, CentER and Department of Econometrics and Operations Research, Tilburg University, Tilburg, The Netherlands.

G. Owen, Department of Mathematics, Naval Postgraduate School, Monterey, CA USA.