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A Characterization of Association Schemes from Affine Spaces

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Abstract. We characterize the association schemes from affine spaces as the association schemes in which all relations are equivalence relations (when united with the identity relation). The schemes from affine spaces of dimension at least three are counterexamples of a conjecture of A. V. Ivanov [8, Problem 1.3] on amorphic schemes.

Keywords: Affine geometry, association schemes

1. Introduction

In this note we assume the reader has some basic knowledge of affine spaces and association schemes. For some background on affine spaces we refer the reader to [3], for some background on association schemes to [2]. Furthermore, we shall use the following terminology.

The irreflexive part of a relation $R \subset V \times V$ is defined as the relation $R \setminus I$, where I is the identity relation on V, i.e in the irreflexive part of a relation an element is not related to itself. An equivalence relation is called uniform if all its equivalence classes have the same size. When seen as a graph, the irreflexive part of such a relation is a disjoint union of cliques of the same size, hence it is a strongly regular graph (or a complete graph).

In a finite affine space each parallel class of lines naturally defines a (uniform) equivalence relation on the points of the affine space. Moreover, the irreflexive parts of the equivalence relations coming from all parallel classes in the affine space form the set of non-identity relations of an association scheme. We shall give a characterization of these association schemes by proving that they are the only association schemes in which every non-identity relation is the irreflexive part of an equivalence relation. This answers the question which association schemes can be "built" with the most elementary relations: equivalence relations.

For translation schemes the characterization follows from a result of Baer [1] (see also [9, Section 29]) on so-called geometric partitions of Abelian groups. Note that by the Veblen-Young theorem (cf. [3]) a finite affine space is a finite affine plane (not necessarily Desarguesian) or $AG(n, q)$, the n-dimensional affine geometry (vector space) over $GF(q)$, hence (of the affine spaces) only the affine planes which are not translation planes give rise to association schemes that are not translation schemes. Note however that the restriction to translation schemes is a rather strong one. Other related results have been obtained by Cameron [4, Thm. 2B.1], who showed that a $(v-1)$-class association scheme on v vertices must be the association scheme from the lines of $AG(n, 2)$, for some n. This result is implied by a result of Ferguson and Turull [6], who showed that a—not necessarily symmetric—association scheme with all valencies equal to 1 is the direct product of cyclic association schemes of prime power order (and they also give some other characterizations of these association schemes). Rao, Ray-Chaudhuri and Singhi [10] characterize all—not necessarily symmetric—association schemes for which the intersection parameters $p_{ij}^k \leq 1$, for all $i, j, k \neq 0$. These are precisely the association schemes with all valencies 1 and, if the number of vertices is odd, their symmetrized schemes (a symmetrized scheme is obtained by uniting each relation with its transpose).

Note that the association schemes that we shall characterize are precisely the association schemes for which the intersection parameters $p_{ii}^k = 0$ for all $i, k \neq 0$, $i \neq k$. We do not even have to restrict to symmetric association schemes, since a relation that is uniform and transitive is necessarily symmetric.

2. The Characterization

Suppose we have an association scheme with vertex set V and relations R_i, $i = 0, 1, \ldots, d$, with corresponding adjacency matrices A_i, $i = 0, 1, \ldots, d$, where R_0 is the identity relation. Assume that $R_i \cup R_0$ is an equivalence relation for all $i \neq 0$. Then on the point set V we define lines as the equivalence classes from all these equivalence relations. Note that this gives a linear space, i.e. through any two points there is precisely one line, and lines have size at least two.

LEMMA *Three noncollinear points lie in a subspace which is an affine plane.*

Proof. Consider three noncollinear points x, y and z. Without loss of generality x and y are on a line from relation R_1, and x and z are on a line from relation R_2. Now consider the set X of points p that are on a line from relation R_2 that intersects the line through x and y. Algebraically this definition is expressed by $(A_1 + I)(A_2 + I)_{xp} > 0$. Note that x, y and z are elements of X.

Since A_1 and A_2 commute, it follows that X and the lines from relations R_1 and R_2 through X form a grid. This implies that for all $u \in X$, $(A_1 + I)(A_2 + I)_{uw} > 0$ if and only if $w \in X$.

Now take two points u and w from X. We want to show that the entire line through u and w is contained in X. If u and w are on a line from relations R_1 or R_2, then this is clearly the case. Next, suppose that u and w are not on a line from relations R_1 and R_2 (u and w are noncollinear on the grid). Without loss of generality we may assume that u and w are on a line from relation R_3, hence $(A_3)_{uw} = 1$. It is also clear that $(A_1A_2)_{uw} = 1$, and so it follows from the equation $A_1A_2 = \sum_k p_{12}^k A_k$ that the intersection parameter p_{12}^3 equals 1. From the construction of X, and the equation $A_1A_2 = \sum_k p_{12}^k A_k$ it now follows that all vertices v such that $(A_3)_{uv} = 1$ are contained in X. Hence also in this case the line through u and w is contained in X, and we proved that X is a subspace.

Next, consider a line l in X, and a point p in X, not on l. To show that X is an affine plane, it suffices to show that there is a line in X containing p and parallel to l. If l is from relation R_1 or R_2, then this is obviously the case. If the line l is from relation R_i, $i \neq 1, 2$, then it follows that $p^i_{12} > 0$, and hence that the line through p from relation R_i is also contained in X. Thus in all cases there is a line in the subspace X through p, parallel to l, which proves that X is an affine plane. ■

THEOREM *Let $(V, \{R_i\}_{i=0,1,\ldots,d})$ be a d-class association scheme, $d > 1$, such that $R_i \cup R_0$ is an equivalence relation for all $i \neq 0$. Then the set of all equivalence classes of these equivalence relations is the set of lines in an affine space on point set V. In particular, all non-identity relations are isomorphic.*

Proof. As mentioned before, the set of lines on point set V is a linear space. Moreover, by the lemma we have that any three noncollinear points are contained in a subspace which is an affine plane. Since parallelism in these planes is an equivalence relation (this follows since if lines are parallel, then they are from the same relation; and each relation is by assumption (the irreflexive part of) an equivalence relation), and there exist three noncollinear points (since $d > 1$), it follows that the linear space is an affine space (see for example Thm. 2.7 of [3]). ■

The proof of the theorem uses the fact that we have an association scheme. It is not possible to only use the fact that we have a partition of the complete relation into the identity relation and the irreflexive parts of uniform equivalence relations, such that the adjacency matrices of any two relations commute (the union of any two relations is a disjoint union of grids), since there are such "schemes" that are not association schemes. For example, in $AG(n, q)$, $n \geq 3$, one might take one parallel class of planes for one relation, and the parallel classes of lines which are transversal to these planes for the other relations; here all relations commute, but this is not an association scheme (for example, by the theorem). In fact, what we found here are counterexamples of a conjecture on amorphic association schemes.

3. A False Conjecture on Amorphic Association Schemes

An association scheme is called amorphic if any way of fusing the non-identity relations again gives an association scheme. It is proven in [7] that in an amorphic d-class association scheme with $d \geq 3$ all non-identity relations are strongly regular graphs of Latin square type or strongly regular graphs of negative Latin square type. The association schemes from affine planes are amorphic of the first type. (For more on amorphic association schemes we refer to [5].)

A.V. Ivanov [8, Problem 1.3] conjectured that an association scheme in which all non-identity relations are strongly regular graphs must be amorphic. The association schemes from $AG(n, q)$, $n \geq 3$, are clearly counterexamples of this conjecture by the remarks just before this section (the parallel class of planes is a fusion of parallel classes of lines). Note also that the non-identity relations in these schemes are not of Latin square type or negative Latin square type. According to M. Klin [private communication], Ivanov probably intended to conjecture that a *primitive* association scheme in which all non-identity relations are strongly regular graphs must be amorphic. This weaker conjecture still stands.

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