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AN EMPLOYMENT GAME BETWEEN GOVERNMENT AND FIRMS

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SUMMARY

Dynamic taxation of profits is a differential game between government and firms. The state equation is the accumulation of capital stock due to the investment of firms. The objective of the firms is the total stream of dividends. The objective of the government is total employment. With high tax revenues the government can stimulate public employment, but economic growth slows down, which is bad for private employment.

The time-inconsistent open-loop Stackelberg solution for this differential game is compared with the time-consistent feedback Stackelberg solution. The efficiency of the solutions and the sensitivity with respect to capital/labour intensiveness are investigated.

KEY WORDS Differential game theory Time inconsistency Optimal dynamic taxation Employment policy

1. INTRODUCTION

In the area of dynamics of the firm several models have been constructed for the dynamic behaviour of the firm (for a survey see Reference 1). In some of these models the effects of government measures such as the corporate tax rate and investment grants are investigated (see e.g. References 2–4). However, a drawback of these models is that the policy of the government is taken exogenously whereas it seems reasonable to consider not only that the firm will react to changes in government policy but also that the government will react to changes in the firm's policy. We can deal with this critique by modelling the problem as a dynamic game between government and a representative firm.

In this paper we focus on the employment problem of the government. In times of high unemployment the government is forced to make a choice, among other things, of whether to rely on the private sector to create employment or to create employment in the public sector. The second possibility is subject to more direct control but has to be financed by corporate taxation. High taxation implies less investment possibilities for firms, which in turn might imply less employment in the private sector and less future tax revenues. The question is whether or not the creation of extra public employment with high taxation is a good employment policy. The instrument of the government in this model is the corporate tax rate.

The firm wants to maximize the total stream of dividends. It is assumed that real wages are fixed and that the firm can sell what it wants and can attract the profit-maximizing amount of labour at each point in time. It follows that the crucial decision the firm has to make concerns the division of after-tax profits between investment and dividend.[2,3,5] Investment leads to a growth in capital stock with more profits in the future but leaves less dividend.

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The mathematical structure of the model proves to be similar to the mathematical structure of the Lancaster[6] model of capitalism. Because it can reasonably be assumed that the government has to decide on the tax policy before the firm makes decisions, the relevant solution concept for the game is the Stackelberg solution concept. The open-loop Stackelberg solution for the Lancaster game has been presented before in the literature,[7] but this result is not fully correct. The error is due to the fact that there is some misunderstanding about delta functions (see e.g. Reference 8). The correct solution will be derived in this paper and will be used to find the optimal employment policy. As usual, the open-loop Stackelberg solution displays time inconsistency.[9] The requirement of strong time consistency leads to the feedback Stackelberg solution, which can be obtained by Bellman's principle of optimality. Finally, it will be shown that only in some special cases is the open loop Stackelberg solution for this game efficient.

In general the solution of the employment game between government and firm leads to an initial period with a low corporate tax rate and high investments and a subsequent period with a high corporate tax rate and low investments. Both players are willing to be modest for a while in order to accumulate capital stock, which is beneficial for both of them. Under the requirement of strong time consistency the policy switch occurs earlier with lower total employment and lower total dividends. This is the correct model, when the government cannot commit itself to an announced policy and when the firm expects rational behaviour of the government at all times. The strongly time-consistent outcome is never efficient and the time-inconsistent outcome is only efficient when the production technology is labour-intensive or when the tax rate cannot become too low. Furthermore, it is shown that the policy switch occurs later for either a very labour-intensive or a very capital-intensive production technology.

The paper is organized as follows. Section 2 formulates the differential game between government and firm. Section 3 derives the relevant solutions for this differential game. In Section 4 the different outcomes are compared and interpreted. In Section 5 the effect of variations in the indicator of capital/labour intensiveness is analysed. Section 6 concludes the paper.

2. THE MODEL

Suppose that the representative firm is operating under a constant-returns-to-scale production technology of the Cobb–Douglas type

$$Q = K^{\alpha} L^{1-\alpha}, \quad 0 < \alpha < 1 \tag{1}$$

where Q denotes production, K denotes capital stock and L denotes labour in the private sector. It is assumed that the firm is not constrained on both output and labour markets. This implies that unemployment corresponds to classical unemployment in the sense of Malinvaud.[10] Furthermore, suppose that the real wage w/p, $0 < w/p < 1$, is fixed and take for simplicity $p = 1$. The assumption of a fixed real wage can be sustained by the theory of implicit contracts or efficiency wages[11] or by trade union behaviour.[12] The maximization of profit

$$\Pi = Q - wL \tag{2}$$

leads to the well known condition that the marginal productivity of labour equals the real wage, which implies that labour is a linear function of capital stock:

$$L = \left((1-\alpha)\frac{1}{w}\right)^{1/\alpha} K \tag{3}$$

Substitution of (1) and then (3) into (2) gives the profit as a linear function of capital stock:

$$\Pi = \beta K \tag{4}$$

where the rentability of capital stock β is given by

$$\beta = \alpha\left((1-\alpha)\frac{1}{w}\right)^{1/\alpha - 1} \tag{5}$$

The firm has to decide on the division of after-tax profits $(1-\tau)\Pi$, where τ denotes the corporate tax rate, between dividend D and investment I:

$$(1-\tau)\Pi = D + I \tag{6}$$

Investment can only be financed by retained earnings, so that

$$0 \leqslant I \leqslant (1-\tau)\beta K \tag{7}$$

Investment accounts for the growth of capital stock

$$\dot{K}(t) = I(t) \tag{8}$$

The firm's objective is to maximize the total stream of dividends over a planning period $[0, T]$:

$$\int_0^T D(t)\,\mathrm{d}t \tag{9}$$

Since the labour input L is a static variable in the optimization problem, it is correct to first maximize profits with respect to L and then maximize (9) (see e.g. Reference 1).

It is assumed that the government can use tax income $\tau\Pi$ to create public employment for the same wage w as in the private sector. Under the assumptions that the government has to pay back its debt and that the interest rate equals the discount rate, it does not make any difference whether the government can issue bonds or not. The government's objective is to maximize total employment over the planning period $[0, T]$:

$$\int_0^T \left(L(t) + \frac{\tau(t)\Pi(t)}{w}\right)\mathrm{d}t \tag{10}$$

The government's instrument is the corporate tax rate τ. It is assumed that

$$0 < \tau_1 \leqslant \tau \leqslant \tau_2 < 1, \qquad \tau_1 \neq \tau_2 \tag{11}$$

where τ_1 and τ_2 are the minimal and maximal tax rates respectively. It seems reasonable to state that there are always some taxes and that profits are never taxed away completely.

In essence the results of the paper are not affected by introducing depreciation of the capital stock or discounting of the objective functions. The crucial assumptions are that investments are irreversible, that the planning horizon is finite and that neither the firm nor the government can borrow. In order to relax the last assumption, a much more difficult model with financial markets would be required. The assumption of irreversible investments is not too severe, since the qualitative results of the paper will not change as long as investments are only partly reversible. An infinite planning horizon will certainly change the results. Such a model should be combined with adjustment costs of capital accumulation. However, the importance of the future after T can also be described with a salvage value $qK(T)$ and for $q < 1$ the qualitative results of the paper still hold (see in a different context also Reference 5).

Substitution of (6) and then (4) into (9) leads to the following behavioural model for the firm:

$$\underset{I(\cdot)}{\text{maximize}} \int_0^T \{[1-\tau(t)]\beta K(t) - I(t)\}\ \mathrm{d}t \tag{12}$$

subject to (7)

Substitution of (5) into (3) leads to $L = [(1-\alpha)\beta/\alpha w]K$ and substitution of this result and then (4) into (10) leads to the following behavioural model for the government:

$$\underset{\tau(\cdot)}{\text{maximise}} \int_0^T \left(\frac{1-\alpha}{\alpha} + \tau(t)\right) \frac{\beta}{w} K(t)\ \mathrm{d}t \tag{13}$$

subject to (11)

Government and firm maximize their objectives subject to the dynamic constraint (8) with initial condition $K(0) = K_0$.

The strategic dynamic interaction between government and firm is described by the differential game (12), (13). This differential game is in structure similar to the Lancaster[6] game of capitalism, which was further investigated by Hoel,[13] Pohjola[7] and Başar *et al.*[14] The government plays the role of the workers and the firm plays the role of the capitalists. In the next section several relevant equilibria for the differential game (12), (13) will be derived.

3. STRATEGIC EQUILIBRIA

In this section game equilibria for the differential game (12), (13) are derived. It is essential to establish first whether the mood of play is co-operative or non-co-operative, whether the players act simultaneously or sequentially, whether the players can commit themselves or not and what information is available to the players.[15] It is reasonable to assume here that the game is non-co-operative and that the government chooses the corporate tax rate policy before the firm chooses the investment policy. Therefore the Stackelberg solution concept seems appropriate. The 'open-loop' Stackelberg solution requires that the players commit themselves from the beginning to a strategy for the whole period and assumes that the players have no information on the 'state' of the system, which in this game is the level of capital stock. This solution displays time inconsistency,[9] which means that the government will have an incentive to deviate from the announced policy at some later point in time. This result is well known in the literature on capital taxation.[16] The government may announce that it will not tax capital in order to encourage accumulation, but once the capital is in place, the government may be tempted to renege on its promise because taxation of existing capital is non-distortionary. The time inconsistency of the open-loop Stackelberg solution was detected earlier in the control literature but under a different heading; namely, it can also be stated that the principle of optimality does not generalize to the open-loop Stackelberg solution.[17]

The assumption that each player in principle reconsiders its strategy at each point in time requires that the players are not committed to an announced strategy and that they have information on the state of the system (this is sometimes referred to as the requirement of 'strong time consistency'[18]). This requirement leads to the 'feedback' Stackelberg solution.[17] In order to be able to check the efficiency of the two solutions, the Pareto optimal strategy sets will also be derived. In this section it will be assumed that the maximal corporate tax rate τ_2 is at least $\frac{1}{2}$.

Proposition 1

(i) If $\tau_1 \geqslant 1 - 1/2\alpha$, the open-loop Stackelberg solution for the differential game (12), (13) with the government as leader and the firm as follower is given by

$$\begin{aligned} \tau(t) = \tau_1 \quad \text{and} \quad I(t) = \beta(1-\tau_1)\exp[(1-\tau_1)\beta t]K_0 \quad \text{for } t \in [0, t_0^{\mathrm{OS}}) \\ \tau(t) = \tau_1 \quad \text{and} \quad I(t) = 0 \quad \text{for } t \in (t_0^{\mathrm{OS}}, T] \end{aligned} \tag{14}$$

where $t_0^{\mathrm{OS}} = T - 1/\beta(1-\tau_1)$.

(ii) If $\tau_1 < 1 - 1/2\alpha$, the open-loop Stackelberg solution for the differential game (12), (13) with the government as leader and the firm as follower is given by

$$\begin{aligned} \tau(t) = \tau_1 \quad \text{and} \quad I(t) = \beta(1-\tau_1)\exp[(1-\tau_1)\beta t]K_0 \quad \text{for } t \in [0, t_1^{\mathrm{OS}}) \\ \int \tau(t)\,\mathrm{d}t = (2\alpha - 1)/\beta \text{ over } (t_1^{\mathrm{OS}}, T] \quad \text{and} \quad I(t) = 0 \quad \text{for } t \in (t_1^{\mathrm{OS}}, T] \end{aligned} \tag{15}$$

where $t_1^{\mathrm{OS}} = T - 2\alpha/\beta$.

Proof. The proof resembles Pohjola's[7] derivation of the open-loop Stackelberg solution for the Lancaster game. However, Pohjola is not correct in his conclusion that the costate q_G (see below), which he denotes by z, changes after the follower's policy switch. The open-loop Stackelberg solution results from the sequential application of Pontryagin's maximum principle and the Stackelberg equilibrium concept. It fits in the framework of Wishart and Olsder's[19] paper on discontinuous Stackelberg solutions.

The firm is follower and maximizes (12) subject to (7) and (8). This is a control problem with a mixed constraint. The way to handle these problems is described in Reference 20 (pp. 269–312). The Hamiltonian function for this maximization problem is given by

$$H_F(K, I, p_F, t) = [1 - \tau(t)]\beta K - I + p_F I \tag{16}$$

and the Lagrangian function is given by

$$L_F(K, I, p_F, \rho_1, \rho_2, t) = [1 - \tau(t)]\beta K - I + p_F I + \rho_1 I + \rho_2\{[1 - \tau(t)]\beta K - I\} \tag{17}$$

where p_F is the costate and ρ_1 and ρ_2 are the Lagrange multipliers. The necessary and sufficient conditions are:

$$I(t) \text{ maximizes } H_F(K(t), I, p_F(t), t) \text{ subject to (7)} \tag{18}$$

$$-1 + p_F(t) + \rho_1(t) - \rho_2(t) = 0 \tag{19}$$

$$\dot{p}_F(t) = -\beta[1 - \tau(t)] - \rho_2(t)\beta[1 - \tau(t)], \qquad p_F(T) = 0 \tag{20}$$

$$\rho_1(t) \geqslant 0 \quad (= 0 \text{ iff } I(t) > 0) \tag{21}$$

$$\rho_2(t) \geqslant 0 \quad (= 0 \text{ iff } I(t) < [1 - \tau(t)]\beta K(t)) \tag{22}$$

The solution is given in Table I.

Table I. The open-loop Stackelberg solution

	$t \in [0, t_1^{\mathrm{OS}})$	$t \in [t_1^{\mathrm{OS}}, T]$
$\rho_1(t)$	0	$1 - p_F(t) > 0$
$\rho_2(t)$	$p_F(t) - 1 > 0$	0
$I(t)$	$[1 - \tau(t)]\beta K(t)$	0
$\dot{p}_F(t)$	$-p_F(t)\beta[1 - \tau(t)]$	$-\beta[1 - \tau(t)]$

When the firm's investment rate is defined as

$$i(t)=\begin{cases}0 & \text{if } I(t)=0\\ 1 & \text{if } I(t)=[1-\tau(t)]\beta K(t)\end{cases} \tag{23}$$

then the solution can be rewritten as

$$i(t)=\begin{cases}0 & \text{if } p_{\mathrm{F}}(t)<1\\ 1 & \text{if } p_{\mathrm{F}}(t)>1\end{cases} \tag{24}$$

$$\dot{p}_{\mathrm{F}}(t)=-\beta\{[1-\tau(t)]\,[1-i(t)]+p_{\mathrm{F}}(t)[1-\tau(t)]\,i(t)\},\qquad p_{\mathrm{F}}(T)=0 \tag{25}$$

The Hamiltonian function for the maximization problem of the government is now given by

$$H_{\mathrm{G}}(K,p_{\mathrm{F}},\tau,p_{\mathrm{G}},q_{\mathrm{G}},t)=\left[\left(\frac{1-\alpha}{\alpha}+\tau\right)\frac{1}{w}+p_{\mathrm{G}}(1-\tau)i(p_{\mathrm{F}})\right]\beta K - \{(1-\tau)[1-i(p_{\mathrm{F}})]+p_{\mathrm{F}}(1-\tau)i(p_{\mathrm{F}})\}\beta q_{\mathrm{G}} \tag{26}$$

It follows that the government's optimal strategy is given by

$$\tau(t)=\begin{cases}\tau_1 & \text{if } g(t)<0\\ \tau_2 & \text{if } g(t)>0\end{cases} \tag{27}$$

where

$$g(t)=\left(\frac{1}{w}-p_{\mathrm{G}}(t)i(t)\right)K(t)+\{[1-i(t)]+p_{\mathrm{F}}(t)i(t)\}q_{\mathrm{G}}(t) \tag{28}$$

that the costate p_{G} is given by the adjoint system

$$\dot{p}_{\mathrm{G}}(t)=-\beta\left[\left(\frac{1-\alpha}{\alpha}+\tau(t)\right)\frac{1}{w}+p_{\mathrm{G}}(t)[1-\tau(t)]\,i(p_{\mathrm{F}}(t))\right],\qquad p_{\mathrm{G}}(T)=0 \tag{29}$$

and that the costate p_{G} is given by the adjoint system

$$\dot{q}_{\mathrm{G}}(t)=-\beta[1-\tau(t)]\left(\{p_{\mathrm{G}}(t)K(t)+[1-p_{\mathrm{F}}(t)]\,q_{\mathrm{G}}(t)\}\frac{\mathrm{d}i}{\mathrm{d}p_{\mathrm{F}}}(t)-i(t)q_{\mathrm{G}}(t)\right),\qquad q_{\mathrm{G}}(0)=0 \tag{30}$$

The costates p_{F} and p_{G} are monotonically decreasing. Suppose that $\hat{t}$ is the point in time where the firm switches from investment to dividend ($p_{\mathrm{F}}(\hat{t})=1$). The investment pattern becomes

$$i(t)=\begin{cases}0 & \text{if } t>\hat{t}\\ 1 & \text{if } t<\hat{t}\end{cases} \tag{31}$$

In the space of generalized functions (see Reference 8, Sections 1.3 and 2.2) this function has the derivative $-\delta(t-\hat{t})$, where δ denotes the so-called delta function. Furthermore, $\dot{p}_{\mathrm{F}}(\hat{t})=-\beta[1-\tau(\hat{t})]$. It follows that in the space of generalized functions $-\beta[1-\tau(t)]\,\mathrm{d}i(t)/\mathrm{d}p_{\mathrm{F}}$ behaves like $-\delta(t-\hat{t})$. The function iq_{G} vanishes because the function i is zero after $\hat{t}$ and the function q_{G} is zero before $\hat{t}$. Furthermore, the costate q_{G} is constant but non-zero after $\hat{t}$ and its value can be calculated as follows:

$$q_{\mathrm{G}}(\hat{t}+\varepsilon)=\int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon}\dot{q}_{\mathrm{G}}(t)\,\mathrm{d}t=\int_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon}-\delta(t-\hat{t})\{p_{\mathrm{G}}(t)K(t)+[1-p_{\mathrm{F}}(t)]\,q_{\mathrm{G}}(t)\}\,\mathrm{d}t \tag{32}$$

with $\varepsilon > 0$. Partial integration yields

$$q_G(\hat{t}+\varepsilon) = i(t)\{p_G(t)K(t) + [1 - p_F(t)]\,q_G(t)\}\Big|_{\hat{t}-\varepsilon}^{\hat{t}+\varepsilon} - \{p_G(t)K(t) + [1 - p_F(t)]\,q_G(t)\}\Big|_{\hat{t}-\varepsilon}^{\hat{t}}$$
$$= -p_G(\hat{t})K(\hat{t}) \tag{33}$$

Since $i(t) = 0$ and thus $K(t) = K(\hat{t})$ for $t > \hat{t}$, it follows that

$$g(t) = \left(\frac{1}{w} - p_G(\hat{t})\right)K(\hat{t}) \quad \text{for } t > \hat{t} \tag{34}$$

The adjoint systems for the costates p_F and p_G after the switch point $\hat{t}$ become

$$\dot{p}_F(t) = -\beta[1 - \tau(t)], \qquad p_F(\hat{t}) = 1, \qquad p_F(T) = 0 \tag{35}$$

$$\dot{p}_G(t) = -\beta\left(\frac{1-\alpha}{\alpha} + \tau(t)\right)\frac{1}{w}, \qquad p_G(T) = 0 \tag{36}$$

Firstly, $g(t) > 0$, so that $p_G(\hat{t}) < 1/w$ and $\tau(t) = \tau_2$ for $t > \hat{t}$, leads to a contradiction with (35) and (36), because it was already assumed that $\tau_2 \geqslant \frac{1}{2}$.

Secondly, if $\tau_1 < 1 - 1/2\alpha$, then $g(t) < 0$, so that $p_G(\hat{t}) > 1/w$ and $\tau(t) = \tau_1$ for $t > \hat{t}$, yields a contradiction with (35) and (36).

Thirdly, if $\tau_1 > 1 - 1/2\alpha$, then $g(t) = 0$, so that $p_G(\hat{t}) = 1/w$, contradicts with (35) and (36). The following conclusions can be drawn.

If $\tau_1 > 1 - 1/2\alpha$, then $g(t) < 0$, so that $\tau(t) = \tau_1$ for $t > \hat{t}$. The value of the switch point $\hat{t}$ can easily be found from the adjoint system (35). Furthermore, if $\tau_1 < 1 - 1/2\alpha$, then $g(t) = 0$, so that $p_G(\hat{t}) = 1/w$. The value of the switch point $\hat{t}$ as well as

$$\int \tau(t)\,\mathrm{d}t = (2\alpha - 1)/\beta \text{ over } (\hat{t}, T] \tag{37}$$

can be found from the adjoint systems (35) and (36) with $p_G(\hat{t}) = 1/w$.

Finally, the path of investment actions before the switch point can be found by integrating (8) with $I(t) = (1 - \tau_1)\beta K(t)$. Q.E.D.

In Proposition 1(ii) the government's strategy after t_1^{OS} is not unique. Owing to the error in his derivation, this non-uniqueness of the equilibrium was not found by Pohjola.[7] The government can choose, for example, an average tax rate $1 - 1/2\alpha$ or can choose to continue for a while with the minimal tax rate τ_1 and then switch to the maximal tax rate τ_2 at

$$t_2^{OS} = T - \frac{2\alpha(1-\tau_1) - 1}{\beta(\tau_2 - \tau_1)} \tag{38}$$

The values of the outcome of the game, however, are the same for all these possible strategies.

Proposition 2

The feedback Stackelberg equilibrium for the differential game (12), (13) with the government as leader and the firm as follower leads to the following path of actions:

$$\begin{aligned} \tau(t) = \tau_1 \quad &\text{and} \quad I(t) = \beta(1-\tau_1)\exp[(1-\tau_1)\beta t]\cdot K_0 \quad \text{for } t \in [0, t^{FS}) \\ \tau(t) = \tau_2 \quad &\text{and} \quad I(t) = 0 \quad \text{for} \quad t \in (t^{FS}, T] \end{aligned} \tag{39}$$

where $t^{FS} = T - 1/\beta(1 - \tau_2)$.

Proof. The feedback Stackelberg equilibrium results from the application of dynamic programming and the Stackelberg equilibrium concept (see e.g. Reference 14). The Hamilton–Jacobi–Bellman equations are given by

$$V_{Ft} + \max_{I \in [0,(1-\tau)\beta K]} [(1-\tau)\beta K - I + V_{FK} I] = 0 \tag{40}$$

$$V_{Gt} + \max_{\tau \in [\tau_1, \tau_2]} \left[\left(\frac{1-\alpha}{\alpha} + \tau \right) \frac{\beta}{w} K + V_{GK} I(\tau) \right] = 0 \tag{41}$$

where V_G and V_F are the value functions for the government and firm respectively and $I(\tau)$ denotes the rational investment decision of the firm at (t, K) given the tax rate chosen by the government at (t, K). This rational reaction results from the maximization in equation (40).

The rational reaction of the firm at (t, K) is

$$I = \begin{cases} 0 & \text{if } V_{FK}(t,K) < 1 \\ (1-\tau)\beta K & \text{if } V_{FK}(t,K) > 1 \end{cases} \tag{42}$$

The optimal action of the government at (t, K), given the rational reaction of the firm at (t, K) is

$$\tau = \begin{cases} \tau_1 & \text{if } I = (1-\tau)\beta K \text{ and } V_{GK}(t,K) > 1/w \\ \tau_2 & \text{if } I = 0 \text{ or } I = (1-\tau)\beta K \text{ and } V_{GK}(t,K) < 1/w \end{cases} \tag{43}$$

When the firm's investment rate is again defined as in (23), the equilibrium at (t, K) can then be written as

$$i = \begin{cases} 0 & \text{if } V_{FK}(t,K) < 1 \\ 1 & \text{if } V_{FK}(t,K) > 1 \end{cases} \tag{44}$$

$$\tau = \begin{cases} \tau_1 & \text{if } iV_{GK}(t,K) > 1/w \\ \tau_2 & \text{if } iV_{GK}(t,K) < 1/w \end{cases} \tag{45}$$

Because the problem is state-separable (see e.g. Reference 21), it is easy to check that $V_G(t,K) = p_G(t)K$ and $V_F(t,K) = p_F(t)K$ with

$$\dot{p}_G(t) = -\beta \left[\left(\frac{1-\alpha}{\alpha} + \tau(t) \right) \frac{1}{w} + p_G(t)[1-\tau(t)]i(t) \right], \qquad p_G(T) = 0 \tag{46}$$

$$\dot{p}_F(t) = -\beta\{[1-\tau(t)][1-i(t)] + p_F(t)[1-\tau(t)]i(t)\}, \qquad p_F(T) = 0 \tag{47}$$

satisfy the Hamilton–Jacobi–Bellman equations.

In the feedback Stackelberg equilibrium there is a period $(t^{FS}, T]$ where $i(t) = 0$ and thus $\tau(t) = \tau_2$. The point in time t^{FS} can be found from the adjoint system for the costate p_F. Because it is assumed that $\tau_2 \geqslant \frac{1}{2}$, so that $\tau_2 > 1 - 1/2\alpha$, the value of the costate p_G at t^{FS} is larger than $1/w$. Furthermore, both costates p_F and p_G are monotonically decreasing. It follows that before the point in time t^{FS}, $i(t) = 1$ and $\tau(t) = \tau_1$. The path of investment actions before the switch point can again be found by integration of (8). Q.E.D.

Proposition 3

The Pareto optimal or efficient solutions for the differential game (12), (13), with λ and $1-\lambda$, $0 \leqslant \lambda \leqslant 1$, denoting the relative weights of the objective functions of the government and firm respectively, in terms of investment rates are given by

(i) if $\lambda > (1-\lambda)w$,

$$\begin{aligned} \tau(t) &= \tau_1 \quad \text{and} \quad i(t)=1 \quad \text{for } t \in [0, t_1^{\mathrm{P}}) \\ \tau(t) &= \tau_2 \quad \text{and} \quad i(t)=1 \quad \text{for } t \in (t_1^{\mathrm{P}}, t_2^{\mathrm{P}}) \\ \tau(t) &= \tau_2 \quad \text{and} \quad i(t)=0 \quad \text{for } t \in (t_2^{\mathrm{P}}, T], \end{aligned} \tag{48}$$

where

$$t_2^{\mathrm{P}} = T - \frac{1-\lambda}{\beta\left[\lambda\left(\frac{1-\alpha}{\alpha}+\tau_2\right)\frac{1}{w}+(1-\lambda)(1-\tau_2)\right]}$$

$$t_1^{\mathrm{P}} = t_2^{\mathrm{P}} - \frac{1}{\beta(1-\tau_2)}\ln\left(\frac{\lambda\left(\frac{1-\alpha}{\alpha}+1\right)\frac{1}{w}}{\lambda\left(\frac{1-\alpha}{\alpha}+\tau_2\right)\frac{1}{w}+(1-\lambda)(1-\tau_2)}\right)$$

(ii) if $\lambda < (1-\lambda)w$,

$$\begin{aligned} \tau(t) &= \tau_1 \quad \text{and} \quad i(t)=1 \quad \text{for } t \in [0, t_3^{\mathrm{P}}) \\ \tau(t) &= \tau_1 \quad \text{and} \quad i(t)=0 \quad \text{for } t \in (t_3^{\mathrm{P}}, T] \end{aligned} \tag{49}$$

where

$$t_3^{\mathrm{P}} = T - \frac{1-\lambda}{\beta\left[\lambda\left(\frac{1-\alpha}{\alpha}+\tau_1\right)\frac{1}{w}+(1-\lambda)(1-\tau_1)\right]}$$

(iii) if $\lambda = (1-\lambda)w$,

$$\begin{aligned} \tau(t) &= \tau_1 \quad \text{and} \quad i(t)=1 \quad \text{for } t \in [0, t_4^{\mathrm{P}}) \\ \tau(t) &\in [\tau_1, \tau_2] \quad \text{and} \quad i(t)=0 \quad \text{for } t \in (t_4^{\mathrm{P}}, T] \end{aligned} \tag{50}$$

where $t_4^{\mathrm{P}} = T - \alpha/\beta$.

Proof. The efficient or Pareto optimal solutions result again from control theory with a mixed constraint applied to the weighted sum of the two objective functionals. The optimal co-operative strategy is given by

$$i(t) = \begin{cases} 0 & \text{if } p(t)-1+\lambda<0 \\ 1 & \text{if } p(t)-1+\lambda>0 \end{cases}$$

$$\tau(t) = \begin{cases} \tau_1 & \text{if } \lambda/w < \max(p(t), 1-\lambda) \\ \tau_2 & \text{if } \lambda/w > \max(p(t), 1-\lambda) \end{cases} \tag{51}$$

where the costate p is given by the adjoint system

$$\dot{p}(t) = -\beta\left[\lambda\left(\frac{1-\alpha}{\alpha}+\tau(t)\right)\frac{1}{w}+(1-\lambda)[1-\tau(t)][1-i(t)]+p(t)[1-\tau(t)]i(t)\right], \quad p(T)=0 \tag{52}$$

Again there is a point in time where the firm switches from investment to dividend. The value of the corporate tax rate τ after that point in time depends only on the value of λ, since $p(t) < 1 - \lambda$. The switch points for the different values of λ can then be found from the adjoint system for the costate p. If $\lambda/w \leqslant 1 - \lambda$, then the value of the corporate tax rate τ before that switch point is τ_1. If $\lambda/w > 1 - \lambda$, then the government switches from the minimal tax rate τ_1 to the maximal tax rate τ_2 at the point in time where the costate p is equal to λ/w. Q.E.D.

In the preceding analysis it is tacitly assumed that the values of the model parameters are such that the switches in policy in the planning period actually occur. It should be noted, however, that this is not always the case. For example, if $\beta(1 - \tau_2)T < 1$, then the firm will never invest in the feedback Stackelberg equilibrium and the government will always choose the maximal tax rate τ_2.

In the next sections these results will be used for an analysis of the model presented in Section 2.

4. THE 'BEST' EMPLOYMENT POLICY

Most strategic equilibria show a switch from a period with a low corporate tax rate and a high investment rate to a period with a high corporate tax rate and a low investment rate. The government is willing to postpone a high tax rate if the firm is willing to postpone the pay-out of dividends and vice versa, in order to create higher future profits.

The best employment result is achieved when government and firm co-operate with the total employment over the planning period as common objective. As can be seen from Proposition 3(i) with $\lambda = 1$, this implies that the firm only invests and does not pay out dividend, which is to be expected. However, in the situation of decentralized decision making the best employment result is achieved in the open-loop Stackelberg behavioural equilibrium. Under the assumption of Proposition 1(i), which means a labour-intensive technology or a high minimal corporate tax rate, the open-loop Stackelberg solution is efficient but with full weight on the objective functional of the firm ($\lambda = 0$ in Proposition 3(ii)). It is to be expected that in this case only the minimal corporate tax rate is levied. Under the assumption of Proposition 1(ii), which means a capital-intensive technology with a low minimal corporate tax rate, the open-loop Stackelberg solution is not efficient. The structure of the solution resembles the structure of the efficient solution of Proposition 3(iii), but the switch point differs. However, the open-loop Stackelberg solution dominates the feedback Stackelberg equilibrium in the sense that both players are better off. It is easy to show that in both behavioural models the equilibrium value of the firm's objective functional is equal to the level of capital stock at the switch point (which is equal to the final level of capital stock). In addition, the firm invests longer in the open-loop Stackelberg equilibrium than in the feedback Stackelberg equilibrium: $t_0^{\mathrm{OS}} > t^{\mathrm{FS}}$, $t_1^{\mathrm{OS}} > t^{\mathrm{FS}}$. Furthermore, it is not difficult to show that the feedback Stackelberg equilibrium is in fact equal to the open-loop Nash equilibrium of this model with a mixed constraint. As a result the government as leader is worse off than in the open-loop Stackelberg equilibrium. Figure 1 illustrates what happens. As open-loop Stackelberg equilibrium, the one from Proposition 1(ii) with investment switch t_1^{OS} and tax switch t_2^{OS}, according to equation (32), is chosen.

Two numerical examples might clarify the results. In Example 1 the efficient open-loop Stackelberg equilibrium from Proposition 1(i) appears. In Example 2 the open-loop Stackelberg equilibrium with investment switch t_1^{OS} and tax switch t_2^{OS} is chosen.

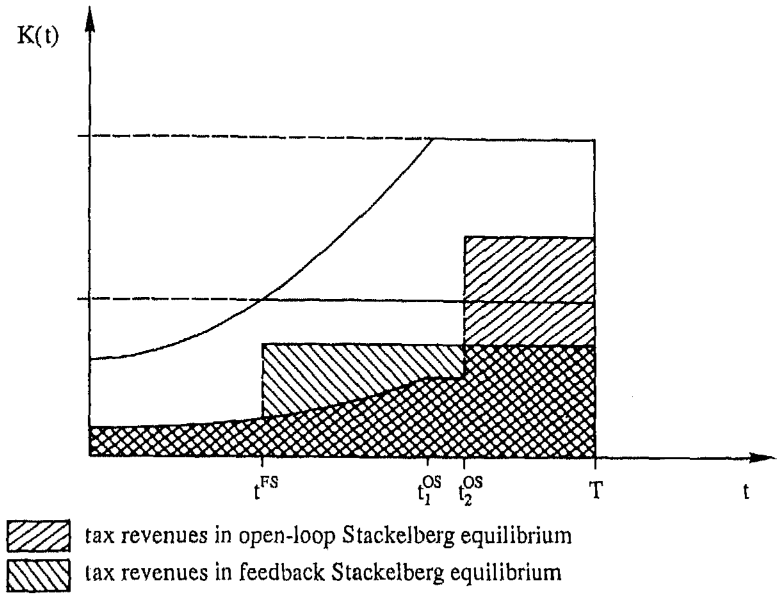

Figure 1. Capital accumulation in different strategic equilibria

Example 1

Suppose $w = \frac{1}{2}$, $K(0) = 1$, $T = 8$, $\tau_1 = 0{\cdot}25$, $\tau_2 = 0{\cdot}75$ and $\alpha = 0{\cdot}25$ (Table II).

Table II. Example 1 ($\alpha = 0{\cdot}25$)

	FBS	OLS/Pareto ($\lambda = 0$)
Investment switch	3·26	6·42
Tax switch	3·26	8
Total employment	295·64	998·75
Private	243·84 (82%)	918·85 (92%)
Public	51·80 (18%)	79·91 (8%)
Total dividend	7·87	58·12
Final capital	7·87	58·12

FBS, feedback Stackelberg solution; OLS, open-loop Stackelberg solution.

Example 2

Suppose $w = \frac{1}{2}$, $K(0) = 1$, $T = 8$, $\tau_1 = 0{\cdot}25$, $\tau_2 = 0{\cdot}75$ and $\alpha = 0{\cdot}75$ (Table III).

It is interesting to note that private employment as a percentage of total employment increases for the more efficient outcomes. The reason is that the more efficient outcomes have a longer period of investment.

The well known drawback of the open-loop Stackelberg equilibrium is that the leader's strategy is time-inconsistent. It will be immediately clear that the government's optimal strategy after the firm has stopped investing, at that point in time, is to levy the maximal

Table III. Example 2 ($\alpha = 0\cdot75$)

	FBS	OLS	Pareto ($\lambda = \frac{1}{3}$)
Investment switch	1·28	5·49	6·12
Tax switch	1·28	7·58	8
Total employment	6·31	39·44	42·35
Private	2·26 (36%)	20·89 (47%)	24·20 (57%)
Public	4·05 (64%)	18·55 (53%)	18·15 (43%)
Total dividend	1·77	11·53	12·88
Final capital	1·77	11·53	15·35

corporate tax rate, which is not prescribed by the open-loop Stackelberg equilibrium. It follows that the essential questions are whether the government will deviate from the announced tax policy or not and whether the firm will believe the government's announcement or not. If the government can commit itself to an announced policy or if the government has a strong reputation, the open-loop Stackelberg behavioural model is appropriate. If the government cannot commit itself or has a bad reputation, the feedback Stackelberg equilibrium results, which leads to worse outcomes for both employment and the value of the firm. In this way benchmarks are set for the analysis of the trade-off between commitments or reputation on the one hand and the effectiveness of an employment policy on the other hand.

5. SENSITIVITY FOR CAPITAL/LABOUR INTENSIVENESS

The real wage w and the bounds τ_1 and τ_2 on the corporate tax rate τ are supposed to be fixed. Because the real wage w is fixed, the elasticity parameter α of the Cobb–Douglas production function (1) represents the capital/labour intensiveness of the production technology in the model. For α close to zero the production technology is very labour-intensive and for α close to one the production technology is very capital-intensive. For the feedback Stackelberg strategic equilibrium the point in time where the switch occurs to a higher tax rate and the pay-out of dividends occurs is given by t^{FS} in Proposition 2. For the open-loop Stackelberg strategic equilibrium this switch point is given by t_0^{OS} in Proposition 1(i) for $\alpha \leqslant 1/2(1-\tau_1)$ and by t_1^{OS} in Proposition 1(ii) for $\alpha > 1/2(1-\tau_1)$. With equation (5) these switch points are a function of α, the indicator of capital/labour intensiveness.

It is easy to show that the rentability of the capital stock β, given by equation (5), is minimal for $\alpha = 1 - w$ with value $1 - w$ and that $\lim_{\alpha\downarrow 0}\beta = \infty$ and $\lim_{\alpha\uparrow 1}\beta = 1$. It is also easy to show that α/β is maximal for α satisfying

$$\alpha + \ln\left((1-\alpha)\frac{1}{w}\right) = 0 \tag{53}$$

which implies $\alpha > 1 - w$ and that $\lim_{\alpha\downarrow 0}(\alpha/\beta) = 0$ and $\lim_{\alpha\uparrow 1}(\alpha/\beta) = 1$.

Figure 2 shows the switch points as a function of α, where w, τ_1 and τ_2 take the same values as in Examples 1 and 2. Since the graphs of t_0^{OS} and t_1^{OS} intersect for $\alpha = 1/2(1-\tau_1)(=\frac{2}{5})$, it follows that the minimum of t_0^{OS} and t_1^{OS} represents the switch point for the open-loop Stackelberg equilibrium as a function of α. As was stated before, the switch point t^{FS} for the feedback Stackelberg equilibrium lies uniformly under the switch point $\min(t_0^{OS}, t_1^{OS})$. Typical for both equilibria is that for α close to zero or a very labour-intensive production technology the switch occurs close to the end of the planning horizon. In this case the firm waits with the pay-out of dividends until the very end of the planning period. The reasons are that the long

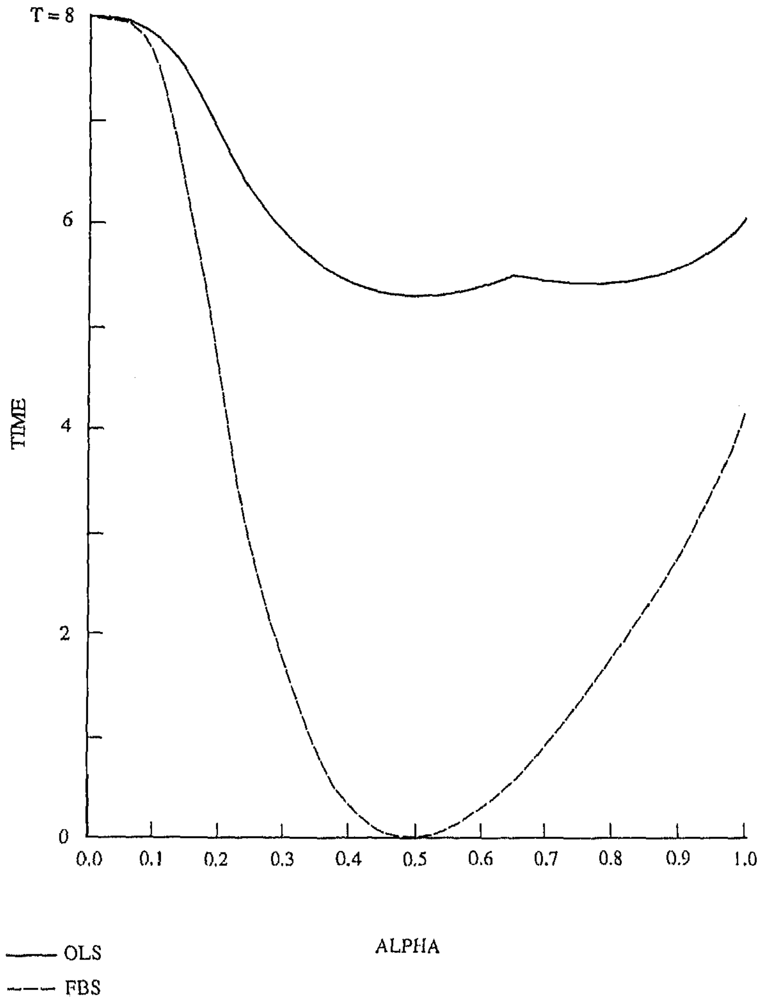

Figure 2. The switch point from investment to dividend

period of investment leads to a large capital stock and that $\lim_{\alpha \downarrow 0} \beta = \infty$, so that according to equation (4) the profits are very high. The government is also satisfied, because the long period of investment and the very labour-intensive production technology lead to a lot of private employment, and high profits at the end lead to high tax revenues at the end. For α close to unity or a very capital-intensive production technology the switch occurs not so late, but later than for a mixed production technology. In the feedback Stackelberg equilibrium the switch occurs earliest for $\alpha = 1 - w$ $(=\frac{1}{2})$. For the chosen values of w and τ_1 the same applies for the open-loop Stackelberg equilibrium.

6. CONCLUSIONS

A differential game is played between the government and a representative firm. The firm wants to maximize the total stream of dividends and can determine investments. The government wants to maximize total employment, which is the sum of public and private employment, and can determine the corporate tax rate. If the government can commit itself or has a strong reputation, the open-loop Stackelberg model is the correct behavioural model. Otherwise the feedback Stackelberg model should be used. The game is in structure similar to Lancaster's game of capitalism. The main difference is that in the model of this paper a mixed constraint appears in the optimization problem. More importantly, however, is that the paper corrects the open-loop Stackelberg solutions for this type of model.

The feedback Stackelberg equilibrium leads to less accumulation of capital stock and gives worse results for both the government and the firm. In the absence of commitments the effectiveness of an employment policy depends on the government's reputation. Typically a switch occurs from investment with low taxes to the pay-out of dividends with high taxes. A very labour-intensive production technology leads to a very short period of dividend payments. For a very capital-intensive production technology the period of dividend payments is also relatively short, but not as short as for a very labour-intensive production technology.

A first suggestion for further research is to extend the basic model in order to investigate the precise effects of other production technologies. Afterwards the model should be embedded in a more general macroeconomic context.

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