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Density of the quotient of non-negative quadratic forms in normal variables with application to the F-statistic

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The density of the quotient of two non-negative quadratic forms in normal variables is considered. The covariance matrix of these variables is arbitrary. The result is useful in the study of the robustness of the F-test with respect to errors of the first and second kind. An explicit expression for this density is given in the form of a proper Riemann-integral on a finite interval, suitable for numerical calculation.

Keywords: Ratio of quotient of quadratic forms in normal variables, F-test, F-statistic, numerical evaluation of probability densities

1. Introduction

Let $Y \sim N_n(\mu, \boldsymbol{\Omega})$ be the n-variate normal distribution with expectation μ and covariance matrix $\boldsymbol{\Omega}$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two non-negative definite $n \times n$-matrices. Set

$$\begin{aligned} X &= (Y'\boldsymbol{B}Y)/(Y'\boldsymbol{A}Y) \\ F &= (\text{tr}(\boldsymbol{A})/\text{tr}(\boldsymbol{B}))X \end{aligned} \tag{1}$$

A relatively simple expression is given for the density g of X (or equivalently for the density of F). For numerical calculation, some eigenvalues and eigenvectors must be computed once and then a one-dimensional proper Riemann-integral on a finite interval must be evaluated for each point $x \in \mathbb{R}$ to get the value $g(x)$.

A special case with singular $\boldsymbol{A}$ and $\boldsymbol{B}$ arises with quotients of orthogonal projections. Let L and R be two orthogonal linear subspaces of $\mathbb{R}^n$ of dimensions l and r, respectively, ($l \geq 1$, $r \geq 1$, $l + r \leq n$). Set

$$\begin{aligned} X &= |Y_L|^2/|Y_R|^2 \\ F &= (r/l)X \end{aligned} \tag{2}$$

with $Y_L = \boldsymbol{P}_L Y$ and $\boldsymbol{P}_L$ the orthogonal projection matrix belonging to L; Y_R and R are similarly defined. Then Equation 1 leads to Equation 2 for $\boldsymbol{A} = \boldsymbol{P}_R$, $\boldsymbol{B} = \boldsymbol{P}_L$.

This result is useful in studying the robustness of the F-test in linear models. Let $\boldsymbol{Y} = \boldsymbol{Z}\beta + \varepsilon$ with $\boldsymbol{Z} \in \mathbb{R}^{n \times k}$ the (non-stochastic) matrix of explanatory variables and $\varepsilon \sim N_n(0, \boldsymbol{\Omega})$. Then $Y \sim N_n(\mu, \boldsymbol{\Omega})$ with $\mu = \boldsymbol{Z}\beta \in \mathbb{R}^n$. An (identifiable) hypothesis H_0 in terms of restrictions on β is equivalent to H_0: $\mu \in L_0$ with L_0 some linear subspace of $\mathscr{R}(\boldsymbol{Z})$. The usual F-statistic F for testing H_0: $\mu \in L_0$ against H_1: $\mu \in \mathscr{R}(\boldsymbol{Z}) - L_0$ is given by F in Equation 2, where L and R are determined by $L \perp L_0$, $L + L_0 = \mathscr{R}(\boldsymbol{Z})$ and $R \perp \mathscr{R}(\boldsymbol{Z})$, $R + \mathscr{R}(\boldsymbol{Z}) = \mathbb{R}^n$.

The usual assumption, $\boldsymbol{\Omega} = \sigma^2 I_n$, gives $F \sim F_r^l(\delta)$, the non-central F-distribution with degrees of freedom l and r and non-centrality parameter $\delta = |\mu_L|^2/\sigma^2$. Equivalently, X follows the distribution with density

$$\exp(-\tfrac{1}{2}\delta) \sum_{k=0}^{\infty} \frac{(\delta/2)^k}{k!} p(x; l/2 + k, r/2) \quad x > 0 \tag{3}$$

where $p(x; \rho_1, \rho_2)$ stands for the density of the beta-distribution of the second kind given by

$$p(x; \rho_1, \rho_2) = x^{\rho_1 - 1}(1 + x)^{-\rho_1 - \rho_2}/B(\rho_1, \rho_2) \quad x > 0 \tag{4}$$

So with an expression for the density g of X for general μ and $\boldsymbol{\Omega}$ it is possible to study the robustness of the F-test for specified probabilities for errors of the first and second kind.

The question of robustness of the F-test is a very old problem. A detailed study for heteroskedasticity and autocorrelation in some special ANOVA-designs can be found in Scheffe (1959). Readers are referred to this book for an overview.

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The best references within the context of the general problem are Lugannani and Rice (1984) and Magnus (1986).

2. Statement of results

Let $(\lambda_j, \boldsymbol{h}_j)$, $j = 1, \ldots, n$, be the eigenvalues and orthogonal eigenvectors of $\boldsymbol{\Omega}$. Set $\alpha_j = \boldsymbol{h}_j'\boldsymbol{A}\boldsymbol{h}_j$, $\beta_j = \boldsymbol{h}_j'\boldsymbol{B}\boldsymbol{h}_j$. Throughout this section it is assumed that $\alpha_j\lambda_j > 0$ for some j, $\beta_j\lambda_j > 0$ for some j and that β_j/α_j is not constant in j. Let $I = (\min \beta_j/\alpha_j, \max \beta_j/\alpha_j)$, where min and max extend over $j = 1, \ldots, n$ with $\lambda_j > 0$ and $(\alpha_j, \beta_j) \neq (0, 0)$. The following theorem 1 precedes the main theorem 2 and is interesting in its own right.

Theorem 1. The density g of X defined by Equation 1 is restricted to the interval I and its value at $x \in I$ is given by

$$g(x) = \frac{e^{-\frac{1}{2}\Sigma\delta_k}}{4\pi i}\sum_{j=1}^{n}\alpha_j\lambda_j\int_{-i\infty}^{i\infty}\{1 - \delta_j/(1 - c_j z)\}e^{\frac{1}{2}\Sigma\delta_k/(1 - c_k z)}$$
$$\times \prod (1 - c_k z)^{-\frac{1}{2} - \delta_{kj}}\,dz \qquad (5)$$

where the Σ and Π operators extend over $k = 1, \ldots, n$ with $\lambda_k > 0$ and with

$$\begin{aligned} \delta_j &= (\boldsymbol{h}_j'\mu)^2/\lambda_j \\ c_j &= \lambda_j(\beta_j - \alpha_j x) \end{aligned} \qquad (6)$$

Example 1. ($\boldsymbol{\Omega} = \sigma^2\boldsymbol{I}_n$, Equation 2, $\mu \in L$): If $\boldsymbol{\Omega} = \sigma^2\boldsymbol{I}_n$ then $\lambda_j = \sigma^2$ for all j. Hence, without loss of generality, it is possible to take $\boldsymbol{h}_j$ such that $L = \mathscr{R}(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)$, $R = \mathscr{R}(\boldsymbol{h}_{l+1}, \ldots, \boldsymbol{h}_{l+r})$. Then $\delta_j = (\boldsymbol{h}_j'\mu)^2/\sigma^2$ for $j = 1, \ldots, l$ and $\delta_j = 0$ elsewhere. This implies $\delta = \Sigma\,\delta_k = |\mu_L|^2/\sigma^2$. Furthermore, $\alpha_j = 1$ for $j = l + 1, \ldots, l + r$, $\beta_j = 1$ for $j = 1, \ldots, l$; other α- and β-values are equal to 0. This gives $I = (0, \infty)$, $c_j = \sigma^2$ for $j = 1, \ldots, l$, $c_j = -\sigma^2 x$ for $j = l + 1, \ldots, l + r$ and $c_j = 0$ for $j = l + r + 1, \ldots, n$. Substitution into Equation 5 leads, for any $x > 0$, to

$$g(x) = \frac{e^{-\delta/2}}{4\pi i}r\sigma^2\int_{-i\infty}^{i\infty}e^{\frac{1}{2}\delta/(1-\sigma^2 z)}(1 - \sigma^2 z)^{-l/2}$$
$$\times (1 + \sigma^2 xz)^{-r/2-1}\,dz$$
$$= e^{-\delta/2}\sum_{k=0}^{\infty}\frac{(\delta/2)^k}{k!}\frac{r}{4\pi i}\int_{-i\infty}^{i\infty}(1 - z)^{-(l/2+k)}$$
$$\times (1 + xz)^{-(r/2+1)}\,dz \qquad (7)$$

The integral in the sum is a variation of Pochhammer's contour integral for the beta-function (see also Lugannani and Rice (1984) p. 487).

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dz}{(z - a)^{\alpha}(b - z)^{\beta}} = \frac{\Gamma(\alpha + \beta - 1)}{(b - a)^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \qquad (8)$$

where $\mathrm{Re}(\alpha + \beta) > 1$ and $a < 0 < b$. This leads to

$$\frac{r}{4\pi i}\int_{-i\infty}^{i\infty}(1 - z)^{-(l/2+k)}(1 + xz)^{-(r/2+1)}\,dz$$
$$= p(x; l/2 + k, r/2) \qquad (9)$$

where p is defined by Equation 4. Hence

$$g(x) = e^{-\delta/2}\sum_{k=0}^{\infty}\frac{(\delta/2)^2}{k!}p(x; l/2 + k, r/2) \quad x > 0 \qquad (10)$$

in agreement with Equation 3.

The following theorem shows that Equation 6 can be written as a proper Riemann-integral on a finite interval.

Theorem 2. (Conditions of theorem 1):

$$g(x) = \frac{1}{4\pi}(a^{-1} + b^{-1})\exp\left(-\frac{1}{2}\sum\delta_k\right)$$
$$\times \sum_{j=1}^{n}\alpha_j\lambda_j I_j\prod(f_k)^{-\frac{1}{2}-\delta_{kj}} \qquad (11)$$

with

$$I_j = \int_0^{\pi/2}B_j(t)\prod(A_k(t))^{-\frac{1}{4}-\frac{3}{2}\delta_{kj}}\exp\left\{\frac{1}{2}\sum\delta_k f_k\cos^2 t/A_k(t)\right\}$$
$$\times \cos^{\frac{1}{2}n-1}t\cos\left[\sum\left\{\left(\frac{1}{2}+\delta_{kj}\right)\arcsin(\gamma_k\sin t/A_k(t)) - S_k(t)\right\}\right.$$
$$\left.+ \arcsin(S_j(t)/C_j(t))\right]dt \qquad (12)$$

where

$$\begin{aligned} a &= \max(\lambda_j\beta_j) \\ b &= x\max(\lambda_j\alpha_j) \\ f_j &= 1 - \tfrac{1}{2}c_j(a^{-1} - b^{-1}) \\ \gamma_j &= \tfrac{1}{2}c_j(a^{-1} + b^{-1})/f_j \end{aligned} \qquad (13)$$

$$\begin{aligned} A_j(t) &= \cos^2 t + \gamma_j^2\sin^2 t \\ C_j(t) &= (1 - \delta_j f_j)\cos^2 t + \gamma_j^2\sin^2 t \\ S_j(t) &= \delta_j f_j\gamma_j\sin t\cos t \\ B_j(t) &= \{C_j^2(t) + S_j^2(t)\}^{\frac{1}{2}} \end{aligned} \qquad (14)$$

Remark. Since $a \geq \max c_j$, $b \geq -\min c_j$ it follows that $f_j > 0$ and $|\gamma_j| \leq 1$.

Corollary. If $\mu = 0$ then $\delta_j = 0$ for all j. Then $C_j(t) = A_j(t) = B_j(t)$ and $S_j(t) = 0$ and so Equation 12 reduces to

$$I_j = \int_0^{\pi/2}\prod(A_k(t))^{-\frac{1}{4}-\frac{1}{2}\delta_{kj}}\cos^{\frac{1}{2}n-1}t$$
$$\times \cos\left[\sum\left(\frac{1}{2}+\delta_{kj}\right)\arcsin(\gamma_k\sin t/A_k(t))\right]dt \qquad (15)$$

Example 2 ($\boldsymbol{\Omega} = \sigma^2 I_n$, Equation 2, $\mu = 0$): Using the results in example 1 it is seen that $\delta_j = 0$ for all j and $a = \sigma^2$, $b = \sigma^2 x$. This leads to $f_j = \frac{1}{2}(1 + 1/x)$, $\gamma_j = 1$ for $j = 1, \ldots, l$; $f_j = \frac{1}{2}(1 + x)$, $\gamma_j = -1$ for $j = l + 1, \ldots, l + r$ and $f_j = 1$, $\gamma_j = 0$ for $j = l + r + 1, \ldots, n$. Substitution into Equations 13–15 leads for any $x \in I = (0, \infty)$ to

$$g(x) = x^{l/2-1}(1+x)^{-(l+r)/2}2^{(l+r)/2}\frac{r}{2\pi}\int_0^{\pi/2}\cos^{(l+r)/2-1} t \times \cos\{(l-r)/2-1\}\,\mathrm{d}t \tag{16}$$

The integral is a variant for the integral expression for the beta-function (see Gradshteyn and Ryzhik (1965) p. 375)

$$\int_0^{\pi/2}\cos^{\alpha+\beta-1} t\cos(\alpha-\beta-1)\,\mathrm{d}t = \pi/\{2^{\alpha+\beta}(\alpha+\beta)B(\alpha, \beta+1)\} \tag{17}$$

where $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > -1$. This leads to $g(x) = p(x; l/2, r/2)$, where p is defined by Equation 4.

3. Proof of theorems

Lemma 1. Let (X_1, X_2) have an absolutely continuous distribution with joint characteristic function φ. If $X_2 \geq 0$ a.s. and $E\{X_2\} < \infty$ then $Y = X_1/X_2$ has a density g given by

$$g(y) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\left(\frac{\partial\varphi(u_1, u_2)}{\partial u_2}\right)\bigg|_{u_2=-yu_1}\mathrm{d}u_1 \tag{18}$$

Proof. See Cramer (1946), exercise 6, p. 317 or Geary (1944) and for the multivariate generalization Phillips (1985).

Lemma 2. Let $X \sim N_n(\mu, \boldsymbol{\Omega})$, $\boldsymbol{\Omega} = \boldsymbol{T}\boldsymbol{T}' > \boldsymbol{0}$ with $\boldsymbol{T} \in \mathbb{R}^{n\times n}$. Let $X_1 = X'\boldsymbol{A}_1 X$, $X_2 = X'\boldsymbol{A}_2 X$ with symmetric $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{n\times n}$. Then the joint characteristic function φ of X_1 and X_2 is given by

$$\varphi(u_1, u_2) = |\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C}|^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\eta'\eta\}\exp\{\tfrac{1}{2}\eta'(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\eta\} \tag{19}$$

where

$$\begin{aligned} \eta &= \boldsymbol{T}^{-1}\mu \\ \boldsymbol{C} &= u_1\boldsymbol{T}'\boldsymbol{A}_1\boldsymbol{T} + u_2\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T} \end{aligned} \tag{20}$$

Proof. See Magnus (1986), lemma 5, p. 102.

Lemma 3. (Conditions of lemma 2 with $\boldsymbol{A}_2 \geq \boldsymbol{0}$): If $\mathrm{vec}(\boldsymbol{A}_1)$ and $\mathrm{vec}(\boldsymbol{A}_2)$ are linearly independent, then the density g of $\boldsymbol{Y} = \boldsymbol{X}_1/\boldsymbol{X}_2$ is given by

$$g(y) = \frac{\mathrm{e}^{-\frac{1}{2}\eta'\eta}}{4\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{e}^{\frac{1}{2}\eta' S^{-1}(y,z)\eta}|S(y,z)|^{-\frac{1}{2}} \times [\mathrm{tr}(S^{-1}(y,z)\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}) + \eta' S^{-1}(y,z)\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}S^{-1}(y,z)\eta]\,\mathrm{d}z \tag{21}$$

where

$$S(y,z) := \boldsymbol{I}_n - z(\boldsymbol{T}'\boldsymbol{A}_1\boldsymbol{T} - y\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}) \tag{22}$$

Proof. Note that $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ has an absolutely continuous distribution iff $\mathrm{vec}(\boldsymbol{A}_1)$ and $\mathrm{vec}(\boldsymbol{A}_2)$ are linearly independent. Lemmas 1 and 2 and the following formula are used:

$$\begin{aligned} \frac{\mathrm{d}A^{-1}}{\mathrm{d}x} &= -\boldsymbol{A}^{-1}\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}x}\boldsymbol{A}^{-1} \\ \frac{\mathrm{d}|\boldsymbol{A}|}{\mathrm{d}x} &= |\boldsymbol{A}|\,\mathrm{tr}\left(\boldsymbol{A}^{-1}\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}x}\right) \quad |\boldsymbol{A}| \neq 0 \end{aligned} \tag{23}$$

Differentiation of Equation 19 leads with Equation 20 and

$$\begin{aligned} \frac{\partial}{\partial u_2}|\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C}|^{-\frac{1}{2}} &= \mathrm{i}|\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C}|^{-\frac{1}{2}}\,\mathrm{tr}\{(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}\} \\ \frac{\partial}{\partial u_2}(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1} &= 2\mathrm{i}(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1} \end{aligned} \tag{24}$$

to

$$\frac{\partial\varphi(u_1,u_2)}{\partial u_2} = \mathrm{i}\varphi(u_1,u_2)[\mathrm{tr}\{(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}\} + \eta'(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}(\boldsymbol{I}_n - 2\mathrm{i}\boldsymbol{C})^{-1}\eta] \tag{25}$$

So with Equation 22

$$\varphi(u_1, -yu_1) = |S(y, 2\mathrm{i}u_1)|^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\eta'\eta\} \times \exp\{\tfrac{1}{2}\eta' S^{-1}(y, 2\mathrm{i}u_1)\eta\} \tag{26}$$

$$\frac{\partial\varphi(u_1,u_2)}{\partial u_2}\bigg|_{u_2=-yu_1} = \mathrm{i}\varphi(u_1, -yu_1)[\mathrm{tr}\{S^{-1}(y, 2\mathrm{i}u_1)\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}\} + \eta' S^{-1}(y, 2\mathrm{i}u_1)\boldsymbol{T}'\boldsymbol{A}_2\boldsymbol{T}S^{-1}(y, 2\mathrm{i}u_1)\eta] \tag{27}$$

Substitution of these expressions into Equation 18 together with $z = 2\mathrm{i}u_1$ leads to Equation 21.

Proof of theorem 1. Suppose $\boldsymbol{\Omega} > \boldsymbol{0}$ or, equivalently, $\lambda_j > 0$ for all j. Use Equations 21 and 22 with $\boldsymbol{A}_1 = \boldsymbol{B}$ and $\boldsymbol{A}_2 = \boldsymbol{A}$. Since $\boldsymbol{\Omega} = \Sigma\,\lambda_j h_j h_j'$ it is possible to take $\boldsymbol{T} = \Sigma\,\lambda_j^{\frac{1}{2}} h_j h_j'$. This gives

$$\begin{aligned} \boldsymbol{T}'\boldsymbol{B}\boldsymbol{T} &= \sum \beta_j\lambda_j\boldsymbol{h}_j\boldsymbol{h}_j' \\ \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T} &= \sum \alpha_j\lambda_j h_j h_j' \\ S = S(y,z) &= \sum (1 - c_j z)\boldsymbol{h}_j\boldsymbol{h}_j' \\ S^{-1} &= \sum (1 - c_j z)^{-1}\boldsymbol{h}_j\boldsymbol{h}_j' \\ |S|^{-\frac{1}{2}} &= \prod (1 - c_j z)^{-\frac{1}{2}} \\ \mathrm{tr}(S^{-1}\boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}) &= \sum (1 - c_j z)^{-1}\alpha_j\lambda_j \\ \eta &= \sum \delta_j^{\frac{1}{2}}\boldsymbol{h}_j \\ \eta'\eta &= \sum \delta_j \\ \eta' S^{-1}\eta &= \sum \delta_j(1 - c_j z)^{-1} \\ \eta' S^{-1}\boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}S^{-1}\eta &= \sum \alpha_j\lambda_j\delta_j(1 - c_j z)^{-2} \end{aligned} \tag{28}$$

Substitution into Equation 21 with the Kronecker symbol $\delta_{kj} = 1$ if $k = j$, $\delta_{kj} = 0$ if $k \neq j$ leads to Equation 5.

For fixed j with $\alpha_j\lambda_j > 0$ the integrand in Equation 5 is $O(|z|^{-\frac{3}{2}})$ for $|z| \to \infty$; furthermore it has singular points in the half plane $\mathrm{Re}\, z > 0$ iff $x < \beta_j/\alpha_j$ for some j and singular points in $\mathrm{Re}\, z < 0$ iff $x > \beta_j/\alpha_j$ for some j. So $g(x) = 0$ if $x > \max(\beta_j/\alpha_j)$ or $x < \min(\beta_j/\alpha_j)$. This concludes the proof of the theorem for $\boldsymbol{\Omega} > \mathbf{0}$. The general case follows by continuity arguments with respect to the eigenvalues λ_j of $\boldsymbol{\Omega}$.

Proof of theorem 2. The substitution $s = (b - a - 2abz)/(b + a)$ and $c = (b - a)/(b + a)$ can be made into Equation 5. Then $1 - c_k z = (1 + \gamma_k s)/f_k$ and so

$$g(x) = \frac{\mathrm{e}^{-\frac{1}{2}\Sigma\delta_k}}{8\pi\mathrm{i}}(a^{-1} + b^{-1}) \sum_{j=1}^{n} \alpha_j\lambda_j \prod (f_k)^{\frac{1}{2} + \delta_{kj}} I_j(c) \quad (29)$$

with

$$I_j(c) = \int_{-\mathrm{i}\infty + c}^{\mathrm{i}\infty + c} \left\{\prod (1 + \gamma_k s)^{-\frac{1}{2} - \delta_{kj}}\right\} \times \{1 - \delta_j f_j/(1 + \gamma_j s)\}\mathrm{e}^{\frac{1}{2}\Sigma\delta_k f_k/(1 + \gamma_k s)}\, \mathrm{d}s \quad (30)$$

The integrand has singular points at $s = -1/\gamma_k$. Since $a \geq \max c_k$ and $b \geq -\min c_k$ then $|\gamma_k| \leq 1$ and so all singular points are outside $\{s: |\mathrm{Re}\, s| < 1\}$. Therefore $I_j(c)$ does not depend on c provided that $|c| < 1$. Since $|b - a|/(b + a) < 1$ it is possible to replace the particular value $c = (b - a)/(b + a)$ by $c = 0$. This gives the intermediate result

$$g(x) = \frac{\mathrm{e}^{-\frac{1}{2}\Sigma\delta_k}}{8\pi}(a^{-1} + b^{-1}) \sum_{j=1}^{n} \alpha_j\lambda_j \prod (f_k)^{\frac{1}{2} + \delta_{kj}} I_j \quad (31)$$

with

$$I_j = I_j(0) = \int_{-\infty}^{\infty} \left\{\prod (1 + \mathrm{i}\gamma_k u)^{-\frac{1}{2} - \delta_{kj}}\right\} \times \{1 - \delta_j f_j/(1 + \mathrm{i}\gamma_j u)\}\mathrm{e}^{\frac{1}{2}\Sigma\delta_k f_k/(1 + \mathrm{i}\gamma_k u)}\, \mathrm{d}u \quad (32)$$

This expression can be rewritten in the form of a Riemann integral on a finite interval. Substitution of $u = \mathrm{tg}\, t$ and $\mathrm{d}u = \cos^2 t\, \mathrm{d}t$ together with

$$\begin{aligned} 1 + \mathrm{i}\gamma_k u &= A_k(t) \cos t \exp\{\mathrm{i} \arcsin(\gamma_k \sin t/A_k(t))\} \\ 1 - \delta_j f_j/(1 + \mathrm{i}\gamma_j u) &= A_j^{-1}(t)B_j(t) \times \exp\{\mathrm{i} \arcsin(S_j(t)/A_j(t))\} \quad (33) \\ \exp\{\tfrac{1}{2}\delta_k f_k/(1 + \mathrm{i}\gamma_k u)\} &= \exp\{\tfrac{1}{2}\delta_k f_k \cos^2 t/A_k(t)\} \times \exp\{-\mathrm{i}S_k(t)\} \end{aligned}$$

leads to Equations 11 and 12.

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