NORTH-HOLLAND

Interlacing Eigenvalues and Graphs

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Dedicated to J. J. Seidel

Submitted by Aart Blokhuis

ABSTRACT

We give several old and some new applications of eigenvalue interlacing to matrices associated to graphs. Bounds are obtained for characteristic numbers of graphs, such as the size of a maximal (co)clique, the chromatic number, the diameter, and the bandwidth, in terms of the eigenvalues of the standard adjacency matrix or the Laplacian matrix. We also deal with inequalities and regularity results concerning the structure of graphs and block designs.

1. PREFACE

Between 1975 and 1979, under the inspiring supervision of J. J. Seidel, I did my Ph.D. research on applications of eigenvalue techniques to combinatorial structures. It turned out that eigenvalue interlacing provides a handy tool for obtaining inequalities and regularity results concerning the structure of graphs in terms of eigenvalues of the adjacency matrix. After 15 years, my thesis [14] became an obscure reference (I myself have no spare copies left) and, in addition, I came across some new applications. This made me decide to write the present paper, which is an attempt to survey the various kinds of applications of eigenvalue interlacing, and I am very glad to have the opportunity to present it in this issue of *Linear Algebra and Applications*,

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which is dedicated to the person who educated me in combinatorial matrix theory and made me a believer in the power of eigenvalue techniques.

2. INTERLACING

Condider two sequences of real numbers: $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, and $\mu_1 \geqslant \cdots \geqslant \mu_m$ with $m < n$. The second sequence is said to *interlace* the first one whenever

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i}, \quad \text{for } i = 1, \ldots, m.$$

The interlacing is called *tight* if there exist an integer $k \in [0, m]$ such that

$$\lambda = \mu_i \quad \text{for } 1 \leqslant i \leqslant k \quad \text{and} \quad \lambda_{n-m+i} = \mu_i \quad \text{for } k+1 \leqslant i \leqslant m.$$

If $m = n - 1$, the interlacing inequalities become $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m \geqslant \lambda_n$, which clarifies the name. Godsil [13] reserves the name "interlacing" for this particular case and calls it generalized interlacing otherwise. Throughout, the λ_is and μ_is will be eigenvalues of matrices A and B, respectively. Basic to eigenvalue interlacing is Rayleigh's principle, a standard (and easy to prove) result from linear algebra, which can be stated as follows. Let $u_1, \ldots, u_n$ be an orthonormal set of eigenvectors of the real symmetric matrix A, such that u_i is a λ_i-eigenvector (we use this abbreviation for an eigenvector corresponding to the eigenvalue λ_i). Then

$$\frac{u^{\mathsf{T}}Au}{u^{\mathsf{T}}u} \geqslant \lambda_i \quad \text{if } u \in \langle u_1, \ldots, u_i \rangle$$

and

$$\frac{u^{\mathsf{T}}Au}{u^{\mathsf{T}}u} \leqslant \lambda_i \quad \text{if } u \in \langle u_1, \ldots, u_{i-1} \rangle^{\perp}.$$

In both cases, equality implies that u is a λ_i-eigenvector of A.

THEOREM 2.1. *Let S be a real $n \times m$ matrix such that $S^{\mathsf{T}}S = I$ and let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Define $B =$*

$S^{\mathsf{T}}AS$ and let B have eigenvalues $\mu_1 \geqslant \cdots \geqslant \mu_m$ and respective eigenvectors $v_1, \ldots, v_m$.

(i) *The eigenvalues of B interlace those of A.*

(ii) *If $\mu_i = \lambda_i$ or $\mu_i = \lambda_{n-m+i}$ for some $i \in [1, m]$, then B has a μ_i-eigenvector v such that Sv is a μ_i-eigenvector of A.*

(iii) *If for some integer l, $\mu_i = \lambda_i$, for $i = 1, \ldots, l$ (or $\mu_i = \lambda_{n-m+i}$, for $i = l, \ldots, m$), then Sv_i is a μ_i-eigenvector of A for $i = 1, \ldots, l$ (respectively $i = l, \ldots, m$).*

(iv) *If the interlacing is tight, then $SB = AS$.*

Proof. With $u_1, \ldots, u_n$ as above, for each $i \in [1, m]$, take a nonzero vector s_i in

$$\langle v_1, \ldots, v_i \rangle \cap \langle S^{\mathsf{T}}u_1, \ldots, S^{\mathsf{T}}u_{i-1} \rangle^{\perp} . \tag{1}$$

Then $Ss_i \in \langle u_1, \ldots, u_{n-1} \rangle^{\perp}$, hence by Rayleigh's principle,

$$\lambda_i \geqslant \frac{(Ss_i)^{\mathsf{T}} A (Ss_i)}{(Ss_i)^{\mathsf{T}} (Ss_i)} = \frac{s_i^{\mathsf{T}} B s_i}{s_i^{\mathsf{T}} s_i} \geqslant \mu_i,$$

and similarly (or by applying the above inequality to $-A$ and $-B$) we get $\lambda_{n-m+i} \leqslant \mu_i$, proving (i).

If $\lambda_i = \mu_i$, then s_i and Ss_i are λ_i-eigenvectors of B and A, respectively, proving (ii).

We prove (iii) by induction on l. Assume $Sv_i = u_i$ for $i = 1, \ldots, l-1$. Then we may take $s_l = v_l$ in (1), but in proving (ii) we saw that Ss_l is a λ_l-eigenvector of A. (The statement between parentheses follows by considering $-$A and $-$B.) Thus we have (iii).

Let the interlacing be tight. Then by (iii), $Sv_1, \ldots, Sv_m$ is an orthonormal set of eigenvectors of A for the eigenvalues $\mu_1, \ldots, \mu_m$. So we have $SBv_i = \mu Sv_i = ASv_i$, for $i = 1, \ldots, m$. Since the vectors v_i form a basis, it follows that $SB = AS$. ∎

If we take $S = [I \quad O]^{\mathsf{T}}$, then B is just a principal submatrix of A and we have the following corollary.

COROLLARY 2.2. *If B is a principal submatrix of a symmetric matrix A, then the eigenvalues of B interlace the eigenvalues of A.*

Suppose rows and columns of

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{bmatrix}$$

are partitioned according to a partitioning $X_1, \dots, X_m$ of $\{1, \dots, n\}$ with characteristic matrix $\tilde{S}$ [that is, $(\tilde{S})_{i,j} = 1$, if $i \in X_j$, and 0, otherwise]. The *quotient matrix* is the matrix $\tilde{B}$ whose entries are the average row sums of the blocks of A. More precisely,

$$(\tilde{B})_{i,j} = \frac{1}{|X_i|} \underline{1}^{\mathrm{T}} A_{i,j} \underline{1} = \frac{1}{|X_i|} (\tilde{S}^{\mathrm{T}} A \tilde{S})_{i,j}$$

($\underline{1}$ denotes the all-one vector). The partition is called *regular* (or *equitable*) if each block $A_{i,j}$ of A has constant row (and column) sum, that is, $A\tilde{S} = \tilde{S}\tilde{B}$.

COROLLARY 2.3. *Suppose $\tilde{B}$ is the quotient matrix of a symmetric partitioned matrix A.*

(i) *The eigenvalues of $\tilde{B}$ interlace the eigenvalues of A.*
(ii) *If the interlacing is tight, then the partition is regular.*

Proof. Put $D = \operatorname{diag}(|X_1|, \dots, |X_m|)$, $S = \tilde{S}D^{-1/2}$. Then the eigenvalues of $B = S^{\mathrm{T}}AS$ interlace those of A. This proves (i), because B and $\tilde{B} = D^{-1/2}BD^{1/2}$ have the same spectrum. If the interlacing is tight, then $SB = AS$; hence, $A\tilde{S} = \tilde{S}\tilde{B}$. ∎

Theorem 2.1(i) is a classical result; see Courant and Hilbert [6, Vol. 1, Chap. I]. For the special case of a principal submatrix (Corollary 2.2), the result even goes back to Cauchy and is therefore often referred to as Cauchy interlacing. Interlacing for the quotient matrix (Corollary 2.3) is especially applicable to combinatorial structures (as we shall see). Payne (see, for instance, [29]) has applied the extremal inequalities $\lambda_1 \geqslant \mu_i \geqslant \lambda_n$ to finite geometric structures several times. He contributes the method to Higman and Sims and therefore calls it the Higman-Sims technique.

3. GRAPHS AND SUBGRAPHS

Throughout the paper, G is a graph on n vertices (undirected, simple, and loopless) having an adjacency matrix A with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. The size of the largest coclique (independent set of vertices) of G is denoted by $\alpha(G)$. Both Corollaries 2.2 and 2.3 lead to a bound for $\alpha(G)$.

THEOREM 3.1. *$\alpha(G) \leqslant |\{i \mid \lambda_i \geqslant 0\}|$ and $\alpha(G) \leqslant |\{i \mid \lambda_i \leqslant 0\}|$.*

Proof. A has a principal submatrix $B = O$ of size $\alpha = \alpha(G)$. Corollary 2.2 gives $\lambda_\alpha \geqslant \mu_\alpha = 0$ and $\lambda_{n-\alpha+1} \leqslant \mu_1 = 0$. ■

THEOREM 3.2. *If G is regular, then $\alpha(G) \leqslant n(-\lambda_n)/(\lambda_1 - \lambda_n)$, and if a coclique C meets this bound, then every vertex not in C is adjacent to precisely $-\lambda_n$ vertices of C.*

Proof. We apply Corollary 2.3. Let $k = \lambda_1$ be the degree of G and put $\alpha = \alpha(G)$. The coclique gives rise to a partition of A with quotient matrix

$$B = \begin{bmatrix} 0 & k \\ \dfrac{k\alpha}{n-\alpha} & k - \dfrac{k\alpha}{n-\alpha} \end{bmatrix}.$$

B has eigenvalues $\mu_1 = k$ (row sum) and $\mu_2 = -k\alpha/(n-\alpha)(\mathrm{tr}(B) - k)$ and so $\lambda_n \leqslant \mu_2$ gives the required inequality. If equality holds, then $\mu_2 = \lambda_n$, and since $\mu_1 = \lambda_1$, the interlacing is tight and hence the partition is regular. ■

The first bound is due to Cvetković [7]. The second bound is an unpublished result of Hoffman. There are many examples where equality holds. For instance, a 4-coclique in the Petersen graph is tight for both bounds. The second bound can be generalized to arbitrary graphs in the following way:

THEOREM 3.3. *If G has smallest degree δ, then*

$$\alpha(G) \leqslant n \frac{-\lambda_1\lambda_n}{\delta^2 - \lambda_1\lambda_n}.$$

Proof. Now we let k denote the average degree of the vertices of the coclique. Then the quotient matrix B is the same as above, except maybe for the entry $(B)_{2,2}$. Interlacing gives

$$-\lambda_1 \lambda_n \geqslant -\mu_1 \mu_2 = -\det(B) = \frac{k^2\alpha}{n-\alpha} \geqslant \frac{\delta^2\alpha}{n-\alpha},$$

which yields the required inequality. ∎

If G is regular of degree k, then $\delta = \lambda_1 = k$ and Theorem 3.3 reduces to Hoffman's bound (3.2). Lovász [24] proved that Hoffman's bound is also an upper bound for the Shannon capacity of G. This is a concept from information theory defined as follows. Denote by G^l the product of l copies of G. [That is, the graph with vertex set $\{1, \ldots, n\}^l$, where two vertices are adjacent if all of the coordinate places correspond to adjacent or coinciding vertices of G. If we denote the Kronecker product of l copies of a matrix M by $M^{\otimes l}$, then the adjacency matrix of G^l is given by $(A + I)^{\otimes l} - I$.] The number

$$\Theta(G) = \sup_l \sqrt[l]{\alpha(G^l)}$$

is called the Shannon capacity of G. Clearly $\Theta(G) \geqslant \alpha(G)$, so Lovász' bound implies Hoffman's bound. Conversely, Lovász' bound can be proved using Theorem 3.2.

THEOREM 3.4. *Let G be regular of degree k. Then*

$$\Theta(G) \leqslant n\frac{-\lambda_n}{k-\lambda_n}.$$

Proof. First note that the proof of Theorem 3.2 remains valid if the ones in A are replaced by arbitrary real numbers, as long as A remains symmetric with constant row sum. So we may apply Hoffman's bound to $A_l = (A - \lambda_n I)^{\otimes l} - (-\lambda_n)^l I$ to get a bound for $\alpha(G^l)$. It easily follows that A_l has row sum $(k - \lambda_n)^l - (-\lambda_n)^l$ and smallest eigenvalue $-(-\lambda_n)^l$. So we find $\alpha(G^l) \leqslant (n(-\lambda_n)/(k - \lambda_n))^l$. ∎

For the pentagon C_5, we get $\Theta(C_5) \leqslant \sqrt{5}$. This is sharp since C_5^2 has a coclique of size 5. More generally, one can obtain results on the size of induced subgraphs, analogous to Hoffman's bound.

THEOREM 3.5. *Let G be regular of degree k and suppose G has an induced subgraph G' with n' vertices and m' edges. Then*

$$\lambda_2 \geqslant \frac{2m'\dfrac{n}{n'} - n'k}{n - n'} \geqslant \lambda_n.$$

If equality holds on either side, then G' is regular and so is the subgraph induced by the vertices not in G'.

Proof. We now have quotient matrix

$$B = \begin{bmatrix} \dfrac{2m'}{n'} & k - \dfrac{2m'}{n'} \\ \dfrac{n'k - 2m'}{n - n'} & k - \dfrac{n'k - 2m'}{n - n'} \end{bmatrix},$$

with eigenvalues k and $2m'/n' - (n'k - 2m')/(n - n')$, and Theorem 2.3 gives the result. ■

If $m' = 0$, we get Hoffman's bound back. If $m' = \frac{1}{2}n'(n' - 1)$, Theorem 3.5 gives that the size of a clique is bounded above by

$$n\frac{1 + \lambda_2}{n - k + \lambda_2},$$

which is again Hoffman's bound applied to the complement of G. Like in Theorem 3.3, the above result also can be generalized to nonregular graphs. Better bounds can sometimes be obtained if more is known about the structure of G' by considering a refinement of the partition. This is, for example, the case if G' is bipartite. See [14] for details.

4. CHROMATIC NUMBER

A coloring of a graph G is a partition of its vertices into cocliques (color classes). Therefore, the number of color classes, and hence the chromatic number $\chi(G)$ of G, is bounded below by $n/(\alpha(G))$. Thus upper bounds for $\alpha(G)$ give lower bounds for $\chi(G)$. For instance, if G is regular, Theorem 3.2 implies that $\chi(G) \geqslant 1 - \lambda_1/\lambda_n$. This bound, however, remains valid for nonregular graphs (but note that it does not follow from Theorem 3.3).

THEOREM 4.1.

(i) *If G is not the empty graph, then $\chi(G) \geqslant 1 - (\lambda_1/\lambda_n)$.*
(ii) *If $\lambda_2 > 0$, then $\chi(G) \geqslant 1 - (\lambda_{n-\chi(G)+1}/\lambda_2)$.*

Proof. Let $X_1, \dots, X_\chi$ [$\chi = \chi(G)$] denote the color classes of G and let $u_1, \dots, u_n$ be an orthonormal set of eigenvectors of A (where u_i corresponds to λ_i). For $i = 1, \dots, \chi$, let s_i denote the restriction of u_1 to X_i, that is,

$$(s_i)_j = \begin{cases} (u_1)_j, & \text{if } j \in X_i, \\ 0, & \text{otherwise,} \end{cases}$$

and put $\tilde{S} = [s_1 \cdots s_\chi]$ (if some $s_i = 0$, we delete it from $\tilde{S}$ and proceed similarly) and $D = \tilde{S}^{\mathsf{T}}\tilde{S}$, $S = \tilde{S}D^{-1/2}$, and $B = S^{\mathsf{T}}AS$. Then B has zero diagonal (since each color class corresponds to a zero submatrix of A) and an eigenvalue λ_1 ($d = D^{1/2}\underline{1}$ is a λ_1-eigenvector of B). Moreover, interlacing Theorem 2.1 gives that the remaining eigenvalues of B are at least λ_n. Hence

$$0 = \operatorname{tr}(B) = \mu_1 + \cdots + \mu_\chi \geqslant \lambda_1 + (\chi - 1)\lambda_n,$$

which proves (i), since $\lambda_n < 0$. The proof of (ii) is similar, but a bit more complicated. With $s_1, \dots, s_\chi$ as above, choose a nonzero vector s in

$$\langle u_{n-\chi+1}, \dots, u_n \rangle \cap \langle s_1, \dots, s_\chi \rangle^{\perp}.$$

The two spaces have nontrivial intersection since the dimensions add up to n and u_1 is orthogonal to both. Redefine s_i to be the restriction of s to X_i, and let $\tilde{S}$, D, S, and d be analogous to above. Put $A' = A - (\lambda_1 - \lambda_2)u_1u_1^{\mathsf{T}}$. Then the largest eigenvalue of A' equals λ_2, but all other eigenvalues of A are also eigenvalues of A' with the same eigenvectors. Define $B = S^{\mathsf{T}}A'S$. Now B has again zero diagonal (since $u_1^{\mathsf{T}}S = 0$). Moreover, B has smallest eigenvalue $\mu_\chi \leqslant \lambda_{n-\chi+1}$, because

$$\mu_\chi \leqslant \frac{d^{\mathsf{T}}Bd}{d^{\mathsf{T}}d} = \frac{s^{\mathsf{T}}A's}{s^{\mathsf{T}}s} \leqslant \lambda_{n-\chi+1}.$$

So interlacing gives

$$0 = \operatorname{tr}(B) = \mu_1 + \cdots + \mu_\chi \leqslant \lambda_{n-\chi+1} + (\chi - 1)\lambda_2.$$

Since $\lambda_2 > 0$, (ii) follows. ∎

The first inequality is due to Hoffman [20]. The proof given here seems to be due to the author [15] and is a customary illustration of interlacing; see, for example, Lovász [25, Problem 11.21] or Godsil [13, p. 48]. In [14], more inequalities of the above kind are given, but only the two treated here turned out to be useful. The condition $\lambda_2 > 0$ is not strong; only the complete multipartite graphs, possibly extended with some isolated vertices, have $\lambda_2 \leqslant 0$. The second inequality looks a bit awkward, but can be made more explicit if the smallest eigenvalue λ_n has large multiplicity m_n, say. Then (ii) yields $\chi \geqslant \min\{1 + m_n,\ 1 - (\lambda_n/\lambda_2)\}$ (indeed, if $\chi \leqslant m_n$, then $\lambda_n = \lambda_{n-\chi+1}$; hence $\chi \geqslant 1 - (\lambda_n/\lambda_2)$). For strongly regular graphs with $\lambda_2 > 0$, it is shown in [14], by use of Seidel's absolute bound (see Delsarte, Goethals, and Seidel [11]), that the minimum is always taken by $1 - (\lambda_n/\lambda_2)$, except for the pentagon (see Section 7 for more about strongly regular graphs). So we have the next corollary.

COROLLARY 4.2. *If G is a strongly regular graph, not the pentagon or a complete multipartite graph, then*

$$\chi(G) \geqslant 1 - \frac{\lambda_n}{\lambda_2}.$$

For example, if G is the Kneser graph $K(m,2)$ (i.e., the complement of the line graph of K_m), then G is strongly regular with eigenvalues $\lambda_1 = \frac{1}{2}(m-2)(m-3)$, $\lambda_2 = 1$, and $\lambda_n = 3 - m$ (for $m \geqslant 4$). The above bound gives $\alpha(G) \geqslant m - 2$, which is tight, whilst Hoffman's lower bound [Theorem 4.1(i)] equals $\frac{1}{2}m$. On the other hand, if m is even, Hoffman's bound is tight for the complement of G, whilst the above bound is much less.

5. DESIGNS

In case we have a nonsymmetric matrix N (say) or different partitions for rows and columns, we can still use interlacing by considering the matrix

$$A = \begin{bmatrix} O & N \\ N^{\mathrm{T}} & O \end{bmatrix}.$$

Then we find results in terms of the eigenvalues of A, which now satisfy $\lambda_i = -\lambda_{n-i+1}$, for $i = 1, \dots, n$. The positive eigenvalues of A are the singular values of N; they are also the square roots of the nonzero eigenvalues of NN^{T} (and of $N^{\mathrm{T}}N$). In particular, if N is the (0, 1) incidence matrix of some design or incidence structure D, we consider the bipartite incidence graph G. An edge of G corresponds to a flag (an incident point-block pair) of D and D is a 1-(v, k, r) design precisely when g is biregular with degrees k and r.

THEOREM 5.1. *Let D be a* 1-(v, k, r) *design with b blocks and let D' be a substructure with v' points, b' blocks, and m' flags. Then*

$$\left(m'\frac{v}{v'} - b'k\right)\left(m'\frac{b}{b'} - v'r\right) \leqslant \lambda_2^2(v - v')(b - b').$$

Equality implies that all four substructures induced by the point set of D' or its complement and the block set of D' or its complement form a 1-*design (possibly degenerate).*

Proof. We apply Corollary 2.3. The substructure D' gives rise to a partition of A with the following quotient matrix:

$$B = \begin{bmatrix} 0 & 0 & \dfrac{m'}{v'} & r - \dfrac{m'}{v'} \\ 0 & 0 & \dfrac{b'k - m'}{v - v'} & r - \dfrac{b'k - m'}{v - v'} \\ \dfrac{m'}{b'} & k - \dfrac{m'}{b'} & 0 & 0 \\ \dfrac{v'r - m'}{b - b'} & k - \dfrac{v'r - m'}{b - b'} & 0 & 0 \end{bmatrix}$$

We easily have $\lambda_1 = -\lambda_n = \mu_1 = -\mu_4 = \sqrt{rk}$ and

$$\det(B) = rk\left(\frac{m'(v/v') - b'k}{v - v'}\right)\left(\frac{m'(b/b') - v'r}{b - b'}\right).$$

Interlacing gives

$$\frac{\det(B)}{rk} = -\mu_2\mu_3 \leqslant -\lambda_2\lambda_{n-1} = \lambda_2^2,$$

which proves the first statement. If equality holds, then $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$, $\lambda_{n-1} = \mu_3$, and $\lambda_n = \mu_4$, so we have tight interlacing, which implies the second statement. ■

The above result becomes especially useful if we can express λ_2 in terms of the design parameters. For instance, if D is 2-(v, k, λ) design, then $\lambda_2^2 = r - \lambda = \lambda(v - k)/(k - 1)$ (see, for example, Hughes and Piper [21]), and if D is a generalized quadrangle of order (s, t), then $\lambda_2^2 = s + t$ (see, for instance, Payne and Thas [30]). Let us consider two special cases.

COROLLARY 5.2. *If a symmetric* 2-(v, k, λ) *design has a symmetric* 2-(v', k', λ') *subdesign (possibly degenerate), then*

$$(k'v - kv')^2 \leqslant (k - \lambda)(v - v')^2.$$

Proof. Take $b = v$, $r = k$, $b' = v'$, $m' = v'k'$, and $\lambda_2^2 = k - \lambda$ and apply Theorem 5.1. ■

COROLLARY 5.3. *Let X be a subset of the points and let Y be a subset of the blocks of a* 2-(v, k, λ) *design D, such that no point of X is incident with a block of Y. Then $kr|X|\,|Y| \leqslant (r - \lambda)(v - |X|)(b - |Y|)$. If equality holds, then the incidence structure D' formed by the points in X and the blocks not in Y is a 2-design.*

Proof. Take $m' = 0$, $v' = |X|$, $b' = |Y|$, then $\lambda_2^2 = r - \lambda$. Now Theorem 5.1 gives the inequality and that D' is a 1-design, but then D' is a 2-design, because D is. ■

If equality holds in Corollary 5.2, the subdesign is called *tight*. There are many examples of tight subdesigns of symmetric designs; see Haemers and Shrikhande [17] or Jungnickel [22]. Wilbrink used Theorem 5.1 to shorten the proof of Feit's result on the number of points and blocks fixed by an automorphism group of a symmetric design; see Lander [23]. The inequality of the second corollary is, for example, tight for hyperovals and (more generally) maximal arcs in finite projective planes. If we take $|X| = v - k$, we obtain $|Y| \leqslant b/m$, which is Mann's inequality for the number of repeated blocks in a 2-design. This is an unusual approach to Mann's inequality. For bounds concerning the intersection numbers of D (these are the possible intersection sizes of two blocks of D), it is mostly better to consider the matrix N^TN, whose entries are precisely the intersection numbers. We give one illustration.

THEOREM 5.4. *Suppose ρ is an intersection number of a 2-(v, k, λ) design D with b blocks and r blocks through a point.*

(i) $\rho \geqslant k - r + \lambda$.

(ii) *Calling blocks equivalent if they are the same or meet in $k - r + \lambda$ points defines an equivalence relation.*

(iii) *The number of blocks in an equivalence class is at most $b/(b - v + 1)$.*

(iv) *Equality in* (iii) *for all classes imnplies that the intersection size of two distinct blocks only depends on whether these blocks are in the same class or in different classes (that is, D is strongly resolvable).*

Proof. Put $A = N^TN$. Then $\lambda_1 = rk$, $\lambda_2 = r - \lambda$, and A has a principal submatrix

$$B = \begin{bmatrix} k & \rho \\ \rho & k \end{bmatrix}.$$

So $\mu_1 = k + \rho$ and $\mu_2 = k - \rho$ and (i) follows from Cauchy interlacing.

Assume the first two blocks b_1 and b_2 meet in $k - r + \lambda$ points. Then $\mu_2 = \lambda_2$ and by Theorem 2.1(ii), A has a λ_2-eigenvector $[1, -1, 0, \dots, 0]^T$ (since B has μ_2-eigenvector $[1, -1]^T$), which implies that $(A)_{i,1} = (A)_{i,2}$, for $i = 3, \dots, b$. Hence every block $\notin \{b_1, b_2\}$ meets b_1 and b_2 in the same number of points. Therefore, having intersection $k - r + \lambda$ is a transitive relation which proves (ii).

Suppose the first b' blocks of D are equivalent. This gives a partitioning of A with quotient matrix

$$B = \begin{bmatrix} x & rk - x \\ (rk - x)\dfrac{b'}{b - b'} & rk - (rk - x)\dfrac{b'}{b - b'} \end{bmatrix},$$

wherein $x = b'(k - r + \lambda) + r - \lambda$. Then $\mu_1 = kr = \lambda_1$ and $\mu_2 = x - (rk - x)b'/(b - b')$. Now $\mu_2 \geqslant \lambda_b \geqslant 0$ leads to the inequality (iii) (using the 2-design identities $bk = vr$ and $rk - v\lambda = r - \lambda$).

From the proof of (ii) we know that $(A)_{i,1} = (A)_{i,2} = \cdots = (A)_{i,b'}$ for $i = b' + 1, \dots, b$. Equality in (iii) implies tight interlacing; therefore, the row sums of block $A_{2,1}$ are constant and hence all entries of $A_{2,1}$ are equal. This proves (iv). ∎

See [14] for more general and other results on intersection numbers. The above theorem appeared in Beker and Haemers [1], where the intersection number $k - r + \lambda$ is treated in detail. Properties (i) and (ii), however, are much older and due to Majumdar [26].

6. LAPLACE MATRIX

The Laplace matrix L of a graph G is defined by

$$(L)_{i,j} = \begin{cases} \text{the degree of } i, & \text{if } i = j, \\ -1, & \text{if } i \text{ and } j \text{ are adjacent}, \\ 0, & \text{otherwise}. \end{cases}$$

This matrix is singular and positive semidefinite with eigenvalues

$$0 = \theta_1 \leqslant \cdots \leqslant \theta_n,$$

say (Laplace eigenvalues are usually ordered increasingly). If G is regular of degree k with (standard) adjacency matrix A, then $L = kI - A$, so $\theta_i = k - \lambda_i$ and we have an easy one-to-one correspondence between eigenvalues of L and A. For nonregular graphs, there is a different behavior and the Laplace spectrum seems to be the more natural one. For instance, the number of components equals the nullity of L (i.e., the multiplicity of the eigenvalue 0), whilst this number is not deducible from the spectrum of A (indeed, $K_{1,4}$ and C_4 plus an isolated vertex have the same standard spectrum). Notice that the Laplace matrix of a subgraph G' of G is not a submatrix of L unless G' is a component. So the interlacing techniques of Section 3 do not work in such a straightforward manner here. We can obtain results if we consider off-diagonal submatrices of L in a way similar to the previous section.

LEMMA 6.1. *Let X and Y be disjoint sets of vertices of G, such that there is no edge between X and Y. Then*

$$\frac{|X||Y|}{(n - |X|)(n - |Y|)} \leqslant \left(\frac{\theta_n - \theta_2}{\theta_n + \theta_2}\right)^2.$$

Proof. Put $\theta = -\frac{1}{2}(\theta_n + \theta_2)$ and

$$A = \begin{bmatrix} O & L + \theta I \\ L + \theta I & O \end{bmatrix}.$$

Then $-\lambda_1 = \lambda_{2n} = \theta$ and $\lambda_2 = -\lambda_{2n-1} = \frac{1}{2}(\theta_n - \theta_2)$. The sets X and Y give rise to a partitioning of A with quotient matrix

$$B = \begin{bmatrix} 0 & 0 & \theta & 0 \\ 0 & 0 & \theta - \theta\dfrac{|X|}{n - |Y|} & \theta\dfrac{|X|}{n - |Y|} \\ \theta\dfrac{|Y|}{n - |X|} & \theta - \theta\dfrac{|Y|}{n - |X|} & 0 & 0 \\ 0 & \theta & 0 & 0 \end{bmatrix}.$$

Clearly $\mu_1 = \lambda_1 = -\theta$ and $\mu_4 = \lambda_{2n} = \theta$. Using interlacing, we find

$$\theta^2 \frac{|X|\,|Y|}{(n - |X|)(n - |Y|)} = -\mu_2\mu_3 \leqslant -\lambda_2\lambda_{2n-1} = \left(\tfrac{1}{2}(\theta_n - \theta_2)\right)^2,$$

which gives the required inequality. ■

A direct consequence of this lemma is an inequality of Helmberg, Mohar, Poljak, and Rendl [18], concerning the bandwidth of G. A symmetric matrix M is said to have bandwidth w if $(M)_{i,j} = 0$ for all i, j satisfying $|i - j| > w$. The bandwidth $w(G)$ of a graph G is the smallest possible bandwidth for its adjacency matrix (or Laplace matrix). This number (or rather, the vertex order realizing it) is of interest for some combinatorial optimization problems.

THEOREM 6.2. *Suppose G is not the empty graph and define $b = \lceil n(\theta_2/\theta_n)\rceil$. Then*

$$w(G) \geqslant \begin{cases} b, & \text{if } n - b \text{ is even}, \\ b - 1, & \text{if } n - b \text{ is odd}. \end{cases}$$

Proof. Order the vertices of G such that L has bandwidth $w = w(G)$. If $n - w$ is even, let X be the first $\frac{1}{2}(n - w)$ vertices and let Y be the last $\frac{1}{2}(n - w)$ vertices. Then Lemma 6.1 applies and thus we find the first inequality. If $n - w$ is odd, take for X and Y the first and last $\frac{1}{2}(n - w - 1)$ vertices and the second inequality follows. If b and w have different parity, then $w - b \geqslant 1$ and so the better inequality holds. ∎

In case $n - w$ is odd, the bound can be improved a little by applying Lemma 6.1 with $|X| = \frac{1}{2}(n - w + 1)$ and $|Y| = \frac{1}{2}(n - w - 1)$. It is clear that the result remains valid if we consider graphs with weighted edges.

Next we consider an application of interlacing by Van Dam and Haemers [9], which gives a bound for the diameter of G.

LEMMA 6.3. *Suppose G has diameter d and $n \geqslant 2$ vertices, and let P be a polynomial of degree less than d, such that $P(0) = 1$. Then*

$$\max_{i \neq 1} |P(\theta_i)| \geqslant \frac{1}{n-1}.$$

Proof. Assume vertex 1 and n have distance d. Then $(L^l)_{1,n} = 0$ for $0 \leqslant l \leqslant d - 1$; hence, $(P(L))_{1,n} = 0$. Define

$$A = \begin{bmatrix} O & P(L) \\ P(L) & O \end{bmatrix}.$$

Then $(A)_{1,2n} = (A)_{2n,1} = 0$ and A has row sum $P(\theta_1) = P(0) = 1$. This leads to a partition of A with quotient matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 - \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & 1 - \frac{1}{n-1} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of B are $\mu_1 = -\mu_4 = 1$ and $\mu_2 = -\mu_3 = 1/(n-1)$ and those of A are $\pm P(\theta_i)$. Interlacing gives

$$\frac{1}{n-1} = \mu_2 \leqslant \lambda_2 \leqslant \max_{i \neq 1} |P(\theta_i)|,$$

which proves the inequality. ∎

The problem is to find a good choice for the polynomial P. If $\theta_2 > 0$ and L has $l < d$ distinct nonzero eigenvalues, we can take P such that $P(\theta_i) = 0$ for all $i \in [2, n]$. This leads to a contradiction in Lemma 6.3, proving the well-known result that $d \leqslant l - 1$ if G is connected. In general, it turns out that the Chebyshev polynomials are a good choice for P. The Chebyshev polynomial T_l of degree l can be defined by

$$T_l(x) = \tfrac{1}{2}\left(x + \sqrt{x^2 - 1}\right)^l + \tfrac{1}{2}\left(x - \sqrt{x^2 - 1}\right)^l.$$

We need the following properties (see Rivlin [32]):

$$|T_l(x)| \leqslant 1, \quad \text{if } |x| \leqslant 1,$$

$$|T_l(x)| \geqslant 1, \quad \text{if } |x| \geqslant 1,$$

$$T_l\left(\frac{x+y}{x-y}\right) > \frac{1}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}\right)^l, \quad \text{if } x > y \geqslant 0.$$

THEOREM 6.4. *If G is a connected graph with diameter $d > 1$, then*

$$d < 1 + \frac{\log 2(n-1)}{\log\left(\sqrt{\theta_n} + \sqrt{\theta_2}\right) - \log\left(\sqrt{\theta_n} - \sqrt{\theta_2}\right)}.$$

Proof. Define

$$Q(x) = T_{d-1}\left(\frac{\theta_n + \theta_2 - 2x}{\theta_n - \theta_2}\right)$$

($\theta_n \neq \theta_2$, since G is not complete or empty) and put

$$P(x) = \frac{Q(x)}{Q(0)}$$

[$Q(0) \neq 0$, since $(\theta_n + \theta_2)/(\theta_n - \theta_2) \geqslant 1$]. Then $|Q(\theta_i)| \leqslant 1$, for $i = 2, \ldots, n$, and $P(0) = 1$. By use of Lemma 6.3, we have

$$n - 1 \geqslant \min_{i \neq 1} \frac{1}{|P(\theta_i)|} \geqslant Q(0) = T_{d-1}\left(\frac{\theta_n + \theta_2}{\theta_n - \theta_2}\right) > \frac{1}{2}\left(\frac{\sqrt{\theta_n} + \sqrt{\theta_2}}{\sqrt{\theta_n} - \sqrt{\theta_2}}\right)^{d-1}$$

and the bound follows by taking logarithms (the connectivity of G guarantees a nonzero denominator). ■

Computing the diameter of a given graph probably goes faster than computing the above bound, yet the bound can be of use if the graph is not explicitly given, but the involved eigenvalues are. This is, for instance, the case with coset graphs of linear codes with given weights. Then eigenvalues correspond to weights in the dual code and diameter bounds lead to bounds for the covering radius of the code (see Delorme and Solé [10]).

7. REGULARITY

Corollary 2.3(ii) gives a sufficient condition for a partition of a matrix A to be regular. This turns out to be handy for proving various kinds of regularity. In Sections 3 and 5 we mentioned some examples. Here we give a few more. If we apply Theorem 2.3 to the trivial one-class partition of the adjacency matrix of a graph G with n vertices and m edges, we obtain

$$\frac{2m}{n} \leqslant \lambda_1,$$

and equality implies that G is regular. This is a well-known result; see Cvetković, Doob, and Sachs [8]. In fact [since $2m = \operatorname{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2$], it implies that G is regular if and only if $\sum_{i=1}^{n} \lambda_i^2 = n\lambda_1$.

Next we consider less trivial partitions. For a vertex v of G, we denote by $X_i(v)$ the set of vertices at distance i from v. The *neighbor partition* of G with respect to v is the partition into $X_0(v)$, $X_1(v)$, and the remaining vertices. If G is connected, the partition into the $X_i(v)$s is called the *distance partition* with respect to v. We give examples of tight interlacing involving strongly regular and distance-regular graphs. A graph is *distance-regular around* v if the distance partition with respect to v is regular. If G is distance-regular around each vertex with the same quotient matrix, then G is called *distance-regular*. A *strongly regular graph* is a distance-regular graph of diameter 2. A distance-regular graph of diameter d has precisely $d + 1$ distinct eigenvalues, being the eigenvalues of the quotient matrix of the distance partition. See Brouwer, Cohen, and Neumaier [3] for more about distance-regular graphs. For strongly regular graphs there is a nice survey by Seidel [33].

THEOREM 7.1. *Suppose G is regular of degree k $(0 < k < n - 1)$ and let t_v be the number of triangles through the vertex v. Then*

$$nk - 2k^2 + 2t_v \leqslant -\lambda_2 \lambda_n (n - k - 1).$$

If equality holds for every vertex, then G is strongly regular.

Proof. The neighbor partition has the following quotient matrix:

$$B = \begin{bmatrix} 0 & k & 0 \\ 1 & \dfrac{2t_v}{k} & \dfrac{k^2 - k - 2t_v}{k} \\ 0 & \dfrac{k^2 - k - 2t_v}{n - k - 1} & \dfrac{nk - 2k^2 + 2t_v}{n - k - 1} \end{bmatrix}.$$

Interlacing gives

$$k \frac{nk - 2k^2 + 2t_v}{n - k - 1} = -\det(B) = -k\mu_2 \mu_3 \leqslant -k\lambda_2 \lambda_n.$$

This proves the inequality. If equality holds, then $\lambda_2 = \mu_2$ and $\lambda_n = \mu_3$, so (since $k = \lambda_1 = \mu_1$) the interlacing is tight and the neighbor partition is regular with quotient matrix B. By definition, equality for all vertices implies that G is strongly regular. ∎

The average number of triangles through a vertex is

$$\frac{1}{2n} \operatorname{tr}(A^3) = \frac{1}{2n} \sum_{i=1}^{n} \lambda_i^3.$$

So if we replace t_v by this expression, the above inequality remains valid. Equality then means automatically equality for all vertices, so strong regularity. In [16], we looked for similar results for distance-regular graphs of diameter $d > 2$, in order to find sufficient conditions for distance regularity in terms of the eigenvalues. Therefor, one needs to prove regularity of the distance partition. The problem is, however, that in general all eigenvalues $\neq \lambda_1$ of a distance-regular graph have a multiplicity greater than 1, whilst the quotient matrix has all multiplicities equal to 1. So for $d \geqslant 3$ there is not much chance for tight interlacing, but because of the special nature of the partition, we still can conclude regularity.

LEMMA 7.2. *Let A be a symmetric partitioned matrix such that $A_{i,j} = O$ if $|i-j| > 1$ and let B be the quotient matrix. For $i = 1, \dots, m$, let $v_i = [v_{i,1}, \dots, v_{i,m}]^{\mathrm{T}}$ denote a μ_i-eigenvector of B. If $\lambda_0 = \mu_0$, $\lambda_1 = \mu_1$, and $\lambda_n = \mu_m$ and if each triple of consecutive rows of $[v_1 \;\; v_2 \;\; v_m]$ is independent, then the partition is regular.*

Proof. By (iii) of Theorem 2.1, $A\tilde{S}v_i = \mu_i \tilde{S}v_i$, for $i = 1, 2, m$. By considering the lth block row of A, we get

$$v_{i,l-1}A_{l,l-1}\underline{1} + v_{i,l}A_{l,l}\underline{1} + v_{i,l+1}A_{l,l+1}\underline{1} = \mu_i v_{i,l}\,\underline{1}, \quad \text{for } i = 1, 2, m$$

(wherein the undefined terms have to be taken equal to zero). Since, for $i = 1, 2, m$ and $j = l-1, l, l+1$, the matrix $(v_{i,j})$ is nonsingular, we find $A_{l,j}\underline{1} \in \langle \underline{1} \rangle$ for $j = l-1, l, l+1$ (and hence for $j = 1, \dots, m$). Thus the partition is regular. ∎

In [16] it was proved that the independence condition in the above lemma is always fulfilled if we consider the distance partition of a graph. So we have the following theorem:

THEOREM 7.3. *Let G be a connected graph and let B be the quotient matrix of the distance partition with respect to a vertex v. If $\lambda_0 = \mu_0$, $\lambda_1 = \mu_1$, and $\lambda_n = \mu_m$, then G is distance-regular around v.*

Using this result it was proved (among others) that if G has the same spectrum and the same number of vertices at maximal distance from each vertex as a distance regular graph G' of diameter 3, then G is distance-regular (with the same parameters as G').

As a last illustration of tight interlacing we prove a result from [14] used by Peeters [31] in his contribution to this Seidel Festschrift. Suppose G is a strongly regular graph, pseudogeometric to a generalized quadrangle of order (s, t). This means that G has $n = (s+1)(st+1)$ vertices and the following quotient matrix for its neighbor partitions:

$$\begin{bmatrix} 0 & st+s & 0 \\ 1 & s-1 & st \\ 0 & t+1 & st+s-t-1 \end{bmatrix},$$

for integers s and t. So the eigenvalues of G are $\lambda_1 = st + s$, $\lambda_2 = s - 1$, and $\lambda_n = -t - 1$.

PROPOSITION 7.4. *Let C be a component of the graph induced by the neighbors of any vertex v of G. Let C have c vertices. Then c is a multiple of s and every vertex not adjacent to v is adjacent to precisely c/s vertices of C.*

Proof. Assume $c < st + s$ (otherwise the result is immediate). Then C gives rise to a refinement of the neighbor partition with quotient matrix

$$B = \begin{bmatrix} 0 & c & st + s - c & 0 \\ 1 & s - 1 & 0 & st \\ 1 & 0 & s - 1 & st \\ 0 & c/s & t + 1 - (c/s) & st + s - t - 1 \end{bmatrix}.$$

It easily follows that B has eigenvalues $\mu_1 = st + s$, $\mu_2 = \mu_3 = s - 1$, and $\mu_4 = -t - 1$. Thus we have tight interlacing, so the partition is regular and the result follows. ∎

8. MISCELLANEA

Mohar [28] obtained necessary conditions for the existence of a long cycle in a graph G using Cauchy interlacing. In particular, he finds an eigenvalue condition for Hamiltonicity.

THEOREM 8.1. *Let G be regular of degree k. If G has a cycle of length l, then*

$$2\cos\frac{i\pi}{l} \leqslant \sqrt{k + \lambda_{2i+1}}, \quad \text{for } i = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor.$$

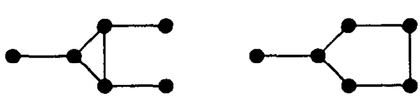
Proof. Let N be the vertex-edge incidence matrix of G. Consider the (bipartite) graph G' with adjacency matrix

$$A' = \begin{bmatrix} O & N \\ N^{\mathrm{T}} & O \end{bmatrix}.$$

(G' is called the subdivision of G; roughly, G' is obtained from G by putting a vertex of degree 2 in the middle of each edge.) If G has a cycle of length l, then G' has a cycle of length $2l$ as an induced subgraph, so the eigenvalues of the $2l$-cycle interlace the eigenvalues of A'. Since G has adjacency matrix $A = NN^{\mathrm{T}} - kI$ (see also the beginning of Section 5), the eigenvalues of G' are $\pm\sqrt{k + \lambda_i}$ and (possibly) zero. The eigenvalues of the $2l$-cycle are $2\cos(i\pi/l)$ ($i = 0, \dots, 2l - 1$) and Cauchy interlacing finishes the proof. ■

For example, if G is the Petersen graph and $l = 10$, then $i = 3$ gives $2\cos(3\pi/10) \leqslant 1$, which is false. This proves that the Petersen graph is not Hamiltonian. The above result has been generalized to arbitrary graphs by Van den Heuvel [19].

Eigenvalues of graphs have application in chemistry via the so-called Hückel theory; see [8]. For instance, a (carbon) molecule is chemically stable if its underlying graph has half of its eigenvalues positive and half of its eigenvalues negative. Manolopoulos, Woodall, and Fowler [27] proved that certain graphs, feasible for a molecule structure, satisfy the desired eigenvalue property and hence provide chemically stable molecules. Their method essentially uses interlacing, although they call it Rayleigh's inequalities. We illustrate their approach by considering the more general question of how to make graphs (on an even number of vertices) with $\lambda_{n/2} \geqslant 0$ and $\lambda_{n/2+1} \leqslant 0$. By Theorem 3.1, graphs with a coclique of size $n/2$ satisfy this property. This includes the bipartite graphs, but there are many more such graphs. We call a graph G an *expanded line graph* of a graph G' if G can be obtained from G' in the following way. The vertices of G are all the ordered pairs (i, j) for which $\{i, j\}$ is an edge in G'. Vertices corresponding to the same edge are adjacent and vertices corresponding to disjoint edges are not adjacent. Of the vertices of G that correspond to intersecting edges $\{i, j\}$ and $\{i, k\}$ (say) of G', either (i, j) is adjacent to (k, i) or (j, i) is adjacent to (i, k), but (i, j) is never adjacent to (i, k) and (j, i) is never adjacent to (k, i). For example, the triangle has the following two expanded line graphs:



THEOREM 8.2. *If G is an expanded line graph of a graph G' with adjacency matrix A, then:*

(i) $\lambda_{n/2} \geqslant 0$ *and equality implies that A has a 0-eigenvector $u = [u_1, \dots, u_n]^{\mathrm{T}}$, where $u_i = u_j$ if i and j correspond to the same edge of G'.*

(ii) $\lambda_{n/2-1} \leqslant 0$ *and equality implies that* A *has a* 0*-eigenvector* $u = [u_1, \dots, u_n]^{\mathrm{T}}$, *where* $u_i = -u_j$ *if* i *and* j *correspond to the same edge of* G'.

Proof. We start with a matrix description of the above construction. Let N be the vertex-edge incidence matrix of G' and let $\tilde{N}$ be a matrix obtained from N by replacing in each column one of the two 1s by -1 (arbitrarily). Consider the matrices $B = N^{\mathrm{T}}N$ and $\tilde{B} = \tilde{N}^{\mathrm{T}}\tilde{N}$ (so that $B - 2I$ is the adjacency matrix of the line graph of G' and $\tilde{B} - 2I$ is a signed version of it). Next we substitute a 2×2 matrix for every entry of $\tilde{B}$ as follows. Replace

$$1 \text{ by} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ or} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$-1 \text{ by} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$2 \text{ by} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } 0 \text{ by} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

but such that the matrix A, thus obtained, is symmetric. Then A is the adjacency matrix of an expanded line graph of G' and every expanded line graph of G' can be obtained like this. By construction we have a partition of A into $m = n/2$ classes of size 2 with quotient matrix $\frac{1}{2}B$. Obviously B has smallest eigenvalue $\mu_m \geqslant 0$, so interlacing gives $\lambda_m \geqslant 0$ and by (ii) of Theorem 2.1 we have the required 0-eigenvector in case $\lambda_m = \mu_m = 0$. To prove (ii), we multiply every odd row and column of A by -1. Then the eigenvalues of A remain the same, but now the partition has quotient matrix $-\frac{1}{2}\tilde{B}$, which has largest eigenvalue at most 0 and (ii) follows by interlacing. ∎

In [27], the authors considered so-called leapfrog fullerenes, which are special cases of expanded line graphs. By use of the second parts of the above statements, they were able to show that leapfrog fullerenes have no eigenvalue 0, and so give rise to stable molecules.

Eigenvalue interlacing has been applied to graphs in many more cases than mentioned in this paper. For instance, Brouwer and Mesner [5] used it to prove that the connectivity of a strongly regular graph equals its degree and in Brouwer and Haemers [4], eigenvalue interlacing is a basic tool for their proof of the uniqueness of the Gewirtz graph. In the present Seidel volume there are also some papers that use eigenvalue interlacing. Brouwer [2] uses it to find bounds for the toughness of a graph, Van den Heuvel [19] applies it in his generalization of Mohar's Hamiltonicity condition (Theorem 8.1), and Ghinelli and Löwe [12] use interlacing to reconstruct generalized quadrangles.

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