Multiple fund investment situations and related games

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Abstract

This paper deals with interactive multiple fund investment situations, in which investors can invest their capital in a number of funds. The investors, however, face some restrictions. In particular, the investment opportunities of an investor depend on the behaviour of the other investors. Moreover, the individual investment returns may differ. We consider this situation from a cooperative game theory point of view. Based on different assumptions modelling the gains of joint investment, we consider three corresponding games and analyse their properties. We propose an allocation process for the maximal total investment revenues.

1 Introduction

Of the many decisions that a firm has to make, none is likely to have more impact than the decision to invest capital, which often involves large, extended commitments of money and management time. Such investment decisions determine the company's future course and, hence, its market value. It is not surprising, therefore, that firms devote much time and effort to planning capital expenditure.

The importance of investment decisions is also reflected in the enormous amount of attention that is devoted to it in the economic literature. In most of this literature on investment, firms are modelled as individually acting agents, ie, *cooperation* between firms is not taken into account. Another assumption that is predominant in the literature on investment, is that the agents face investment opportunities that are exogenously given. That is, the investment opportunities of an agent are

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not influenced by the investments of other agents; the *strategic* aspects of investment are often overlooked. In this paper, we analyse situations in which investment opportunities of an agent depend on the behaviour of other investors. Moreover, the situations will be analysed by taking into account the consequences of possible cooperative behaviour.

In this paper, we introduce a new class of cooperative situations, called *multiple fund investment (MFI)* situations. In an MFI situation, agents can invest their capital in a certain number of funds. There are restrictions on the funds such that there is a maximum number of capital units that can be invested in each of them. The agents (players) in an MFI situation are characterised by the amount of capital they can invest and by their individual returns on the different funds. That is, we consider the possibility that the return of an investment project depends on the player (eg, firm) that is involved in this project. Furthermore, investment opportunities are limited; we assume that the total capital available exceeds the total investment opportunities for which expected returns are satisfactory to the players.

Our model of MFI situations resembles the model of *transportation situations* as introduced in Sánchez-Soriano et al. (2001). A transportation situation can be seen as an MFI situation in which also the funds are (controlled by) players and as a result, these funds should also share in the revenues of the grand coalition.

In case there is only one fund available and all individual returns on this fund are equal (say, 1), then the resulting situation boils down to a bankruptcy situation (cf. O'Neill (1982)), where the fund restriction represents the estate and the individual capital restrictions are the claims. As a result, our model can be seen as an extension of the model of bankruptcy situations.

Associated with each MFI situation, we define three cooperative MFI games in characteristic function form, which for the bankruptcy case all coincide with the corresponding bankruptcy game. These games are based on three possible assumptions on the coalitional expectations of the return on their joint investments. These coalitional expectations relate to the behaviour of the players outside the coalition. To actually calculate the coalitional values of the MFI games, one has to solve linear programs. These turn out to be transportation problems, allowing for a fairly quick calculation of these values.

The central question in an MFI situation is how to divide, in an acceptable way, the maximal total investment revenues of the players if they all cooperate and coordinate their investment plans in an optimal way. In this context, we study properties of the associated cooperative games, in particular convexity and (total) balancedness. We also propose a two-stage allocation rule for MFI situations. In the first stage, an *allotment* is made, which gives each player investment rights in the various funds. In the second stage, the players are thought of as facing a linear production situation (cf. Owen (1975)) in which their investment rights and capital stock are resources. Owen vectors of this linear production situation are then seen as solutions of the original MFI situation. Stability of these solutions is shown.

This paper is organised as follows. Section 2 introduces MFI situations and the three corresponding MFI games. In section 3, the properties of convexity and (total) balancedness of these games are studied. In section 4, we introduce the concept of allotment and propose our two-stage solution method for MFI situations. In section 5, we elaborate on how our analysis can be extended when some of the assumptions are modified.

2 The MFI model

A *multiple fund investment* or *MFI* situation is a tuple (N, M, e, A, d), where $N = \{1, \ldots, n\}$ is the set of players, $M = \{1, \ldots, m\}$ denotes the set of available funds and $e \in \mathbb{R}^M_{++}$ is the vector of fund restrictions. An element e_j denotes the maximum number of capital units that can be invested in fund $j \in M$. Furthermore, $A \in \mathbb{R}^{N \times M}_+$ is the return matrix, where an element A_{ij} denotes the revenue player i obtains when he invests one unit of his capital in fund j. Finally, $d \in \mathbb{R}^N_{++}$ is the vector of individual investment capital. We assume that $\sum_{j \in M} e_j < \sum_{i \in N} d_i$ [1].

Let (N, M, e, A, d) be an MFI situation. In order to define corresponding MFI games, we first state the program that determines the maximum revenue a coalition $S \subset N, S \neq \emptyset$ can obtain when the fund restrictions are given by a vector $z \in \mathbb{R}^M_+$. These direct revenues are denoted by $DR(S, z)$ and defined by

$$DR(S,z) = \max_{X \in \mathbb{R}^{S \times M}} \sum_{j \in M} \sum_{i \in S} A_{ij} X_{ij} \tag{2.1}$$

$$\text{such that} \quad \sum_{j \in M} X_{ij} \leq d_i \text{ for all } i \in S,$$

[1]This assumption will be elaborated upon in section 5.

$$\sum_{i \in S} X_{ij} \leq z_j \text{ for all } j \in M,$$
$$X_{ij} \geq 0 \text{ for all } i \in S, j \in M.$$

For simplicity, we define $DR(\emptyset, z) = 0$ for all $z \in \mathbb{R}_+^M$. By introducing a dummy fund or player in order to obtain equality restrictions, this problem is translated into a *balanced transportation problem* (cf. Hitchcock (1941)), which can be solved very efficiently.

Facing fund restrictions $z \in \mathbb{R}_+^M$, the players in S will construct an optimal plan $X^S \in \mathbb{R}^{S \times M}$ according to this program in order to maximise their total revenue. The set of all feasible plans is given by

$$FP(S,z) = \{X^S \in \mathbb{R}_+^{S \times M} \,|\, \forall_{i \in S} : \sum_{j \in M} X_{ij}^S \leq d_i, \forall_{j \in M} : \sum_{i \in S} X_{ij}^S \leq z_j\}.$$

For a plan $X \in \mathbb{R}^{S \times M}$, the corresponding revenues are given by the *direct payoff* vector $O(X) \in \mathbb{R}^S$, where $O_i(X) = \sum_{j \in M} A_{ij} X_{ij}$ for all $i \in N$. The set of all optimal feasible plans X^S is denoted by $OP(S,z)$:

$$OP(S,z) = \{X^S \in FP(S,z) \,|\, \sum_{i \in S} O_i(X) = DR(S,z)\}.$$

Once the members of a coalition S have decided upon a particular plan X^S, they will invest their capital accordingly, thereby tightening the fund restrictions z for the remaining players. The resulting fund restrictions $z(X^S)$ are given by

$$z_j(X^S) = z_j - \sum_{i \in S} X_{ij}^S$$

for all $j \in M$.

Using this notation, we now introduce three TU games that correspond to an MFI situation. A *TU (transferable utility) game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, assigning to every coalition $S \subset N$ a value $v(S)$, representing the total monetary payoff the members of S can guarantee themselves if they cooperate. By convention, $v(\emptyset) = 0$.

Depending on how the "guarantee" in the last paragraph is interpreted, an MFI situation gives rise to three TU games, which will be denoted by v^1, v^2 and v^3. The common feature is that first the players outside S can invest their capital and afterwards the members of S invest optimally given the resulting (tightened) fund restrictions. The difference between the games lies in the way the players outside S are assumed to behave in the first stage.

Let (N, M, e, A, d) be an MFI situation. The game v^1 is defined by

$$v^1(S) = \min\{DR(S, e(X^{N\backslash S})) \,|\, X^{N\backslash S} \in FP(N\backslash S, e)\}$$

for all $S \subset N$. That is, the players outside S, facing fund restrictions e, are assumed to choose that feasible plan $X^{N\backslash S}$ for which the resulting revenue for S, facing fund restrictions $e(X^{N\backslash S})$, is minimal. This pessimistic maxmin approach is standard practice in cooperative game theory. In this interpretation, the "guarantee" to coalition S is taken literally.

For our second game, we again take a pessimistic approach, but with the assumption that the choice of the players in $N\backslash S$ is restricted to plans that maximise their own total revenue. This implicitly assumes that the players outside of S do act rationally. They choose an investment plan in $OP(N\backslash S, e)$:

$$v^2(S) = \min\{DR(S, e(X^{N\backslash S})) \,|\, X^{N\backslash S} \in OP(N\backslash S, e)\}$$

for all $S \subset N$.

For the third game, the players outside S also choose an optimal plan for themselves, giving them a revenue of $DR(N\backslash S, e)$. Next, we assume that the players in S can persuade the members of $N\backslash S$ to change their investment plan as long as those members still receive $DR(N\backslash S, e)$. Of course, coalition S will persuade them to choose a plan in such a way that the two coalitions together generate a total revenue of $DR(N, e)$. After giving up the promised $DR(N\backslash S, e)$ to the members of $N \setminus S$, the net revenue of coalition S equals

$$v^3(S) = DR(N, e) - DR(N\backslash S, e).$$

So, v^3 is the dual of the "direct revenue" game which assigns value $DR(S, e)$ to any coalition $S \subset N$.

Example 2.1 Consider the MFI situation (N, M, e, A, d) with three players (rows) and two funds (columns):

$$\begin{array}{cc|c} 3 & 3 & \\ \hline 10 & 9 & 1 \\ 1 & 4 & 4 \\ 4 & 10 & 3 \end{array}$$

So, $N = \{1, 2, 3\}$, $M = \{1, 2\}$, $e = (3, 3)$, $A = \begin{bmatrix} 10 & 9 \\ 1 & 4 \\ 4 & 10 \end{bmatrix}$ and $d = (1, 4, 3)$.

The unique optimal plan for the grand coalition is

$$X^N = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}$$

with total payoff 42 and direct payoff $O(X^N) = (10, 2, 30)$.

Next, take $S = \{2, 3\}$. In order to compute $v^1(S)$, we have to determine where player 1 should invest his single unit of capital so that the resulting optimal payoff to S is minimal. If player 1 invests his unit in fund 1 ($X^1 = [\ 1 \quad 0\ ] \in FP(N \backslash S, e)$), then $e(X^1) = (2, 3)$ and coalition S can obtain 32 with plan $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \in OP(S, e(X^1))$. If player 1 invests in fund 2, coalition S can get 26 with plan $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$. Hence, $v^1(S) = 26$.

For our second game, player 1 has to invest in fund 1, which is optimal for him. As a result, $v^2(S) = 32$.

For the third game, we first determine $DR(N \backslash S, e)$, which equals 10 with plan $[\ 1 \quad 0\ ]$ for player 1. Hence, $v^3(S) = DR(N, e) - DR(N \backslash S, e) = 42 - 10 = 32$.

In the following table, we list the direct revenues and the three coalitional values of each coalition:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$DR(S,e)$	10	13	30	23	40	33	42
$v^1(S)$	0	2	4	12	14	26	42
$v^2(S)$	0	2	4	12	14	32	42
$v^3(S)$	9	2	19	12	29	32	42

$\triangleleft$

The first game is the most pessimistic, whereas the third game is the most optimistic, as is shown in the following proposition.

Proposition 2.1 *Let (N, M, e, A, d) be an MFI situation. Then for the three corresponding games we have that $v^1(N) = v^2(N) = v^3(N) = DR(N, e)$ and $v^1 \leq v^2 \leq v^3$.*

Proof:

The first part of the proposition follows immediately from the definitions. The relation between v^1 and v^2 is obvious. It remains to show that $v^2 \leq v^3$. Let $S \subset N, S \neq \emptyset$

and let $X^{N\backslash S} \in OP(N\backslash S, e)$, resulting in the revenue of $DR(N\backslash S, e)$ for coalition $N\backslash S$. Let $X^S \in OP(S, e(X^{N\backslash S}))$, resulting in the revenue of $DR(S, e(X^{N\backslash S}))$ for coalition S. If we combine the plans $X^{N\backslash S}$ and X^S, we obtain a feasible plan for the grand coalition. Therefore,

$$v^2(S) + DR(N\backslash S, e) \leq DR(S, e(X^{N\backslash S})) + DR(N\backslash S, e) \leq DR(N, e).$$

Hence, $v^2(S) \leq v^3(S)$. □

Note that when $|M| = 1$ and $A_{i1} = 1$ for all $i \in N$, then we basically have a bankruptcy situation (cf. O'Neill (1982)). In this case, all three MFI games coincide with the corresponding bankruptcy game, defined by $v_{e_1,d}(S) = \max\{e_1 - \sum_{i \in N\backslash S} d_i, 0\}$ for all $S \subset N$.

3 Properties of MFI games

In this section, we analyse some properties of our three MFI games. In particular, we consider convexity and (total) balancedness.

A TU game (N, v) is called convex if (cf. Shapley (1971))

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$$

for all $S, T \subset N$.

In the following theorem, we prove that our most optimistic game v^3 is convex.

Theorem 3.1 *Let (N, M, e, A, d) be an MFI situation. Then the corresponding game v^3 is convex.*

Proof: We prove convexity of v^3 by showing that its dual, the direct revenue game is concave. It suffices to show that

$$DR(S \cup \{t, l\}, e) + DR(S, e) \leq DR(S \cup \{t\}, e) + DR(S \cup \{l\}, e) \tag{3.1}$$

for all $S \subset N, t, l \notin S$. Taking the dual program of 2.1, it follows that $DR(S, e)$ is the minimal value of the following linear program

$$\begin{aligned} DR(S, e) \quad &= \quad \min_{y \in \mathbb{R}^S, v \in \mathbb{R}^M} OD^S(y, v) \\ \text{such that} \quad & \quad y_i + v_j \geq A_{ij} \text{ for all } i \in S, j \in M, \end{aligned} \tag{3.2}$$

$$y_i \geq 0 \text{ for all } i \in S,$$
$$v_j \geq 0 \text{ for all } j \in M,$$

where $OD^S(y,v) = \sum_{i \in S} d_i y_i + \sum_{j \in M} e_j v_j$.

Now, consider the (dual) linear programs that are associated with $DR(S,e), DR(S \cup \{t\},e), DR(S \cup \{l\},e)$ and $DR(S \cup \{t,l\})$. Suppose that $(y^*, v^*) \in \mathbb{R}^{S \cup \{t\}} \times \mathbb{R}^M$ is an optimal solution for the program associated with $DR(S \cup \{t\},e)$ and that $(y^{**}, v^{**}) \in \mathbb{R}^{S \cup \{l\}} \times \mathbb{R}^M$ is an optimal solution for the program associated with $DR(S \cup \{l\},e)$. So we have that $OD^{S \cup \{t\}}(y^*, v^*) = DR(S \cup \{t\},e)$ and $OD^{S \cup \{l\}}(y^{**}, v^{**}) = DR(S \cup \{l\},e)$.
From the structure of the linear programs, it readily follows that $(\widehat{y}, \widehat{v}) \in \mathbb{R}^{S \cup \{t,l\}} \times \mathbb{R}^M$ with $\widehat{y}_i = y_i^* \wedge y_i^{**}$ for all $i \in S, \widehat{y}_t = y_t^*, \widehat{y}_l = y_l^{**}$ and $\widehat{v} = v_j^* \vee v_j^{**}$ for all $j \in M$ is a feasible solution for the linear program associated with $DR(S \cup \{t,l\})$. Similarly, $(\widetilde{y}, \widetilde{v}) \in \mathbb{R}^S \times \mathbb{R}^M$ with $\widetilde{y}_i = y_i^* \vee y_i^{**}$ for all $i \in S$ and $\widetilde{v} = v_j^* \wedge v_j^{**}$ for all $j \in M$ is a feasible solution for the program associated with $DR(S,e)$. Since we have that

$$\begin{cases} (y_i^* \wedge y_i^{**}) + (y_i^* \vee y_i^{**}) = y_i^* + y_i^{**} \\ (v_i^* \wedge v_i^{**}) + (v_i^* \vee v_i^{**}) = v_i^* + v_i^{**}, \end{cases}$$

it follows that

$$OD^{S \cup \{t,l\}}(\widehat{y}, \widehat{v}) + OD^S(\widetilde{y}, \widetilde{v}) = DR(S \cup \{k\}, e) + DR(S \cup \{l\}, e).$$

Now, since $(\widehat{y}, \widehat{v})$ and $(\widetilde{y}, \widetilde{v})$ are only feasible and not necessarily optimal, it follows that

$$DR(S \cup \{t,l\}, e) + DR(S,e) \leq OD^{S \cup \{t,l\}}(\widehat{y}, \widehat{v}) + OD^S(\widetilde{y}, \widetilde{v}).$$

Combining the last two equations we obtain (3.1) and hence, v^3 is convex. □

We proved convexity of v^3 by showing that the direct revenues are concave in the player set. That is, smaller coalitions benefit more from an additional player (in terms of direct revenues) than do larger coalitions. On the other hand, the direct revenues are convex in the fund restrictions; larger coalitions benefit more from an increase in the fund restrictions (in terms of direct revenues) than do smaller coalitions. This is stated in the following theorem, which follows from Theorem 3.4.1(a) in Topkis (1998).[2]

[2]We would like to thank an anonymous referee for pointing this out.

Theorem 3.2 *Let (N, M, e, A, d) be an MFI situation. Then for all $e' \in \mathbb{R}^M$ such that $e' \geq e$, we have*

$$DR(T, e') - DR(T, e) \geq DR(S, e') - DR(S, e)$$

for all $S \subset T \subset N, S \neq \emptyset$.

From convexity of v^3 and from Proposition 2.1 it follows that all three games are balanced, ie, that their respective cores are nonempty, where the core of a game (N, v) is defined by

$$C(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \forall_{S \subset N} : \sum_{i \in S} x_i \geq v(S)\}.$$

The games v^1 and v^2 need not be convex. However, the game v^1 is totally balanced, ie, for each $S \subset N, S \neq \emptyset$, the subgame (S, v^1_S) defined by $v^1_S(T) = v^1(T)$ for all $T \subset S$ is balanced.

Proposition 3.3 *Let (N, M, e, A, d) be an MFI situation. Then the corresponding game v^1 is totally balanced.*

Proof: Let $S \subset N, S \neq \emptyset$. In calculating $v^1(S)$, first the players in $N \backslash S$ use their investment capital to lower the fund restrictions in such a way that the revenues that thereafter can be obtained by S are as low as possible. Let $X^{N \backslash S} \in FP(N \backslash S, e)$ denote an investment plan that is chosen by $N \backslash S$ for that reason. Consider the MFI situation $(S, M, e(X^{N \backslash S}), A_S, d_S)$ with $A_S = (A_{ij})_{i \in S, j \in M}$ and $d_S = (d_i)_{i \in S}$. Denote the corresponding most pessimistic game by $v^{1,S}$. Trivially, we have that

$$v^{1,S}(S) = v^1_S(S).$$

Moreover, we have that

$$v^{1,S}(T) \geq v^1_S(T)$$

for all $T \subset S$. To see this, notice that the fund restrictions faced by a coalition T in calculating $v^{1,S}(T)$ are equal to the restrictions faced by T in calculating $v^1_S(T)$. Since in the case of $v^1_S(T)$ this sum is distributed over the funds in the most pessimistic way for coalition T, we have the stated inequality. Since $v^{1,S}$ is balanced, we have that the subgame v^1_S is balanced and hence, v^1 is totally balanced. □

4 MFI solutions: a linear production approach

In this section, we present a procedure for solving MFI situations, ie, we propose a method of dividing $DR(N, e)$ among the players. This procedure consists of two stages. In the first stage, a division of the investment rights in the available funds (an allotment) is made. An important aspect of this first stage is that an allotment can be assigned without knowing the return matrix A. In the second stage, this allotment is used as an input vector of a related linear production process and the eventual allocation for the grand coalition is an Owen vector of this process. Interestingly, it turns out that such an Owen vector is in the core of our most pessimistic game v^1, irrespective of the allotment that is made in the first stage.

Let (N, M, e, A, d) be an MFI situation. An *allotment* is an investment plan $Y \in FP(N, e)$ satisfying

$$\sum_{i \in N} Y_{ij} = e_j$$

for all $j \in M$.

An element Y_{ij} is interpreted as the amount that player i is allowed to invest in fund j. When the players individually invest in the funds according to the investment rights they receive from an allotment Y, a payoff vector $O(Y) \in \mathbb{R}^N_+$ results.

Example 4.1 One way to construct an allotment is simply to divide the investment rights of each fund proportional to the investment capital of the players, ie,

$$Y_{ij} = \frac{d_i}{\sum_{k \in N} d_k} e_j$$

for all $i \in N, j \in M$. For the MFI situation in Example 2.1, this yields

$$Y = \frac{3}{8} \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \end{bmatrix}.$$

Note that the corresponding direct payoff $O(Y) = \frac{1}{8}(57, 60, 126)$ is not efficient with respect to $DR(N, e) = 42$. ◁

The payoff $O_i(Y)$ to player $i \in N$ according to Y can be viewed as the direct revenue of coalition $\{i\}$ with fund restrictions $(Y_{ij})_{j \in M}$, ie,

$$O_i(Y) = DR(\{i\}, (Y_{ij})_{j \in M})$$

for all $i \in N$.

The players may decide to merge their investment rights and thereafter maximise their joint revenues. Suppose a coalition $S \subset N$ of players decides to work together. Define

$$Y^S = (\sum_{i \in S} Y_{ij})_{j \in M}.$$

The joint revenues that coalition S can obtain when working together is then given by $DR(S, Y^S)$. So, after an allotment Y is made, a new situation arises, which can be modelled as a TU game. This game, denoted by v_Y, is defined by

$$v_Y(S) = DR(S, Y^S)$$

for all $S \subset N$.

This process of joining the investment rights according to an allotment turns out to be a linear production process (cf. Owen (1975)). Let (N, M, e, A, d) be an MFI situation and let Y be an allotment. Each player i is allowed to invest Y_{ij} units of his capital d_i in fund j, resulting in revenues of A_{ij} per invested unit. This is equivalent with saying that each player i can produce one unit of a product p_{ij} by using one unit of his "capital resource" (of which he possesses d_i) and one unit of his "investment right in fund j resource" (of which he possesses Y_{ij}), with a market price for one unit of p_{ij} equal to A_{ij}. So the situation that arises after making the allotment Y can be characterised as a linear production process $L = (N, R, P, Q, B, c)$ (for notation, we refer to Van Gellekom et al. (2000)), where the resource set R consists of $|M|$ "fund" resources $\{r_1^f, \dots, r_m^f\}$ and $|N|$ "capital" resources $\{r_1^c, \dots, r_n^c\}$. Each player makes $|M|$ different products corresponding to the resources, so we define $|N||M|$ products $P = (p_{ij})_{i \in N, j \in M}$. For $n = 3, m = 2$, the technology matrix Q then looks as follows:

$$\begin{array}{c} \\ r_1^f \\ r_2^f \\ r_1^c \\ r_2^c \\ r_3^c \end{array} \begin{array}{c} \begin{array}{cccccc} p_{11} & p_{12} & p_{21} & p_{22} & p_{31} & p_{32} \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{array}$$

and the resource matrix B looks like this:

$$\begin{array}{c} \\ r_1^f \\ r_2^f \\ r_1^c \\ r_2^c \\ r_3^c \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} Y_{11} & Y_{21} & Y_{31} \\ Y_{12} & Y_{22} & Y_{32} \\ d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \end{array}$$

Finally, the price vector c is derived from A:

$$c = [A_{11}, \ldots, A_{1m}, A_{21}, \ldots, A_{2m}, \ldots, A_{n1}, \ldots, A_{nm}].$$

A linear production situation $L = (N, R, P, Q, B, c)$ gives rise to a corresponding linear production game v_L, defined by

$$v_L(S) = \max_{x \in F(S)} c^\top x$$

for every $S \subset N$, where $F(S) = \{x \in \mathbb{R}_+^P \,|\, Qx \leq (\sum_{i \in S} B_{ri})_{r \in R}\}$. The Owen set of L is defined by

$$Owen(L) = \{y^\top B \in \mathbb{R}^N \,|\, y \in F^*, v_L(N) = \sum_{r \in R} \sum_{i \in N} y_r B_{ri}\},$$

where $F^* = \{y \in \mathbb{R}_+^R \,|\, y^\top Q \geq c^\top\}$ is the dual feasible set for the grand coalition and the vector y reflects the shadow prices of the resources. An element of the Owen set is called an Owen vector. Every Owen vector is an element of the core of the corresponding linear production game:

$$Owen(L) \subset C(v_L).$$

In particular, this implies that every linear production game is balanced. Also, since every subgame corresponds in a natural way to a linear production situation which is a subsituation of the original one, every linear production game is totally balanced.

So, when an allotment Y is made, the situation that arises can be viewed as a linear production process. We refer to this process as $L(Y)$. It is easily verified that the corresponding linear production game $v_{L(Y)}$ coincides with v_Y.

Theorem 4.1 *Let (N, M, e, A, d) be an MFI situation and let $Y \in FP(N, e)$ be an allotment. Then $v_{L(Y)} = v_Y$.*

As a consequence, v_Y is totally balanced for every allotment Y.

Given an allotment Y, we propose $Owen(L(Y))$ as solution for the MFI situation, where every Owen vector is an efficient division of $DR(N, e)$ $(= v_{L(Y)}(N))$. Irrespective of the allotment that is chosen, the resulting allocation lies in the core of the most pessimistic MFI game v^1, as is shown in the following theorem.

Theorem 4.2 *Let (N, M, e, A, d) be an MFI situation and let $Y \in FP(N, e)$ be an allotment. Then $Owen(L(Y)) \subset C(v^1)$.*

Proof: Let $S \subset N$. Then

$$v_{L(Y)}(S) = v_Y(S) = DR(S, Y^S)$$

and

$$v^1(S) = DR(S, e(X^{N \setminus S}))$$

for some $X^{N \setminus S}$ such that the resulting direct revenue for coalition S is minimal. So, by construction, we have

$$\sum_{j \in M} e_j(X^{N \setminus S}) = \max\{0, \sum_{j \in M} e_j - \sum_{i \in N \setminus S} d_i\}.$$

Also, we have that

$$\sum_{j \in M} Y_j^S \geq \max\{0, \sum_{j \in M} e_j - \sum_{i \in N \setminus S} d_i\},$$

and so,

$$\sum_{j \in M} Y_j^S \geq \sum_{j \in M} e_j(X^{N \setminus S}).$$

Moreover, we have that given the total of fund restrictions $\sum_{j \in M} e_j(X^{N \setminus S})$, the division of this sum over the funds is such that $DR(S, e(X^{N \setminus S}))$ is as low as possible. Hence,

$$v_{L(Y)}(S) \geq v^1(S).$$

Now, since any Owen vector of $L(Y)$ is in the core of the corresponding linear production game $L(Y)$ and, trivially, $v_{L(Y)}(N) = v^1(N)$, the statement follows. □

Example 4.2 Consider the MFI situation (N, M, e, A, d) of Example 2.1. Solving the (dual) linear production program for the grand coalition yields a solution set with two extreme points: $(1, 7, 9, 0, 3)$ and $(1, 4, 9, 0, 6)$, where the first two coordinates correspond to the "fund" resources and the other three to the "capital" resources. Using the resource matrix corresponding to the proportional allotment of Example 4.1,

$$B = \begin{bmatrix} \frac{3}{8} & \frac{12}{8} & \frac{9}{8} \\ \frac{3}{8} & \frac{12}{8} & \frac{9}{8} \\ 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

this yields

$$Owen(L(Y)) = Conv\{(12, 12, 18), (\frac{87}{8}, \frac{60}{8}, \frac{189}{8})\}.$$

Note that $Owen(L(Y)) \subset C(v^1)$. However, the Owen solution $(12, 12, 18)$ is not an element of $C(v^2)$, since $12 + 18 = 30 < 32 = v^2(\{2, 3\})$.

Computing the Owen set for *all* possible allotments yields the following allocations: $Conv\{(9, 3, 30), (9, 22, 11), (10, 22, 10), (16, 16, 10), (16, 2, 24), (10, 2, 30)\}$[3], where $Conv$ denotes the convex hull. The core of v_1 is given by $C(v_1) = Conv\{(16, 4, 22), (10, 28, 4), (0, 28, 14), (0, 12, 30), (10, 2, 30), (16, 2, 24)\}$. Note that the three faces that are most beneficial to players 1 and 3 coincide for the two sets, while the core elements that are best for player 2 cannot be reached as an Owen vector. ◁

The previous example shows that the allocations that can be obtained from our procedure form a proper subset of $C(v_1)$. For bankruptcy situations, the two sets coincide.

It follows from the construction of the linear production process that when a fund restriction increases, the corresponding shadow price decreases, since the feasible set of the dual program does not change. The impact of such a shift on the resulting allocation, however, is not *a priori* determined, since it is unclear to what extent the extra investment opportunities for this fund are offset by their lower shadow price.

[3]These allocations correspond to the dual optimal point (1,7,9,0,3). The other optimal point, (1,4,9,0,6), does not yield additional allocations.

Suppose that for an allotment Y the corresponding direct division of revenues is already efficient with respect to $DR(N, e)$, ie,

$$\sum_{i \in N} O_i(Y) = DR(N, e),$$

or equivalently,

$$Y \in OP(N, e).$$

Then the corresponding direct division of the revenues $O(Y)$ coincides with the allocation that is proposed by the Owen set of the corresponding linear production game. So according to this process there is no need to redistribute the allocation of revenues as given by $O(Y)$. This is the result of the following theorem.

Theorem 4.3 *Let (N, M, e, A, d) be an MFI situation. Let $Y \in OP(N, e)$. Then*

$$Owen(L(Y)) = \{O(Y)\}.$$

Proof: Consider the linear production process $L(Y)$ and let $S \subset N$. For the corresponding linear production game we have

$$v_{L(Y)}(S) = \sum_{i \in S} O_i(Y),$$

because $Y \in OP(N, e)$. Let $y \in Owen(L(Y))$. Since the Owen vector is in the core of the linear production game, we have

$$\sum_{i \in S} y_i \geq v_{L(Y)}(S) = \sum_{i \in S} O_i(Y).$$

Efficiency of both y and $O(Y)$ with respect to $DR(N, e)$ implies

$$\sum_{i \in S} y_i = \sum_{i \in S} O_i(Y)$$

for all $S \subset N$ and hence,

$$Owen(L(Y)) = \{O(Y)\}.$$

□

For an optimal allotment $Y \in OP(N, e)$, the resulting allocation $O(Y)$ belongs to the core of the most optimistic game v^3, as is shown in the next theorem.

Theorem 4.4 *Let (N, M, e, A, d) be an MFI situation and let $Y \in OP(N, e)$. Then $O(Y) \in C(v^3)$.*

Proof: Let $S \subset N$. Then

$$\begin{aligned} v^3(S) &= DR(N, e) - DR(N \backslash S, e) \\ &\leq DR(N, e) - \sum_{i \in N \backslash S} O_i(Y) \\ &= \sum_{i \in S} O_i(Y). \end{aligned}$$

Together with efficiency, we obtain $O(Y) \in C(v^3)$. □

Because of Proposition 2.1, $O(Y)$ also belongs to $C(v^1)$ and $C(v^2)$. Note that for a bankruptcy situation, every allotment is optimal and the resulting payoff vector is in the core of the corresponding bankruptcy game.

To compare the various solutions, consider again the MFI situation of Example 2.1. As we saw in Example 4.1, the direct payoff corresponding to the proportional allotment is not efficient and hence, not an element of any of the three cores. After constructing the corresponding linear production game and applying the Owen procedure we obtained a solution set which is part of the core of v^1, although not of v^2 (and hence, v^3). According to Theorem 4.4, the direct division corresponding to the optimal plan X^N in Example 2.1 should be in all three cores, which is indeed the case.

5 Extensions

One of the assumptions in our MFI model is that the total capital available is larger than the sum of the fund restrictions, ie, $\sum_{j \in M} e_j < \sum_{i \in N} d_i$. Note that this assumption is common in the bankruptcy literature (cf. O'Neill (1982)), where the total amount of the claims (capital) exceeds the available estate (investment opportunities). If we do not impose this assumption, we can still compute the three corresponding games in the same way and the results of section 2 still hold.

The problem with dropping this assumption, however, lies in the concept of allotment. An allotment is a feasible plan which is efficient with respect to the fund restrictions. If the sum of the fund restrictions is larger than the total capital, such a feasible plan does not exist. If we drop the requirement of feasibility of an allotment and allow a player to have more investment rights than his total capital, Theorem 4.2 no longer holds. However, an allotment Y for which the direct division of revenues $O(Y)$ is efficient with $DR(N, e)$ is always in the core of all three games, regardless whether we require an allotment to be feasible or not.

Another (implicit) assumption in our MFI model is that when a coalition S of players decides to cooperate, they can coordinate their investment actions, but they cannot pool their capital. If we allow capital to be transferable, the direct revenues of S would be given by

$$\begin{aligned} \widetilde{DR}(S,z) \quad &= \quad \max_{X \in \mathbb{R}^{S\times M}} \sum_{j\in M}\sum_{i\in S} A_{ij}X_{ij} \\ \text{such that} \quad & \quad \sum_{j\in M}\sum_{i\in S} X_{ij} \leq \sum_{i\in S} d_i, \\ & \quad \sum_{i\in S} X_{ij} \leq z_j \text{ for all } j\in M, \\ & \quad X_{ij} \geq 0 \text{ for all } i\in S, j\in M. \end{aligned}$$

This maximisation problem is quite trivial to solve. For each fund $j \in M$, the players in S determine $\tilde{A}_{Sj} = \max_{i\in S} A_{ij}$ and invest their capital in those funds with the highest $\tilde{A}_{Sj}$, taking the fund restrictions into account.

For this transferable capital case, we can define the same three corresponding games as for the nontransferable capital case. Again, we have that $v^1 \leq v^2 \leq v^3$ and that v^1 is totally balanced. However, the game v^3 need not be convex (or even balanced).

Allotments can be defined in the same way as for nontransferable MFI situations, but the constructions of the corresponding linear production game is different and involves the introduction of an additional resource representing "total capital". With this adjusted linear production situation, the analysis of Section 3 can be fully translated to the transferable capital setting.

A more detailed discussion of these and other extensions can be found in Wintein (2002).

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