COMPUTATION OF AUTOCORRELATIONS OF INTERDEPARTURE TIMES BY NUMERICAL TRANSFORM INVERSION

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April 26, 2004

Abstract

The generating functions of the autocorrelations of the interdeparture times in a stationary M/G/1 system and in a stationary GI/M/1 system involve the probability generating functions of the number of customers served in a busy period. The latter functions are only implicitly determined as solutions to some functional equations. Standard methods for the numerical inversion of generating functions require the values of these functions at many complex arguments. A recently discovered substitution method for contour integrals allows the numerical inversion of implicitly determined generating functions without the numerical solution of the functional equations.

Keywords: Numerical transform inversion; Autocorrelation; Interdeparture time; Generating function; Number served in busy period; M/G/1 queue; GI/M/1 queue.

1 Introduction

The autocorrelations of interdeparture times are important for the analysis of queues in series or more general networks. Burke [5] shows that the interdeparture times are independent and exponentially distributed for stationary M/M/c systems. Reich [15] provides an alternate proof of these results based on reversibility. These facts imply the product-form solution of the stationary joint queue-length distribution for queues in series and acyclic networks with Poisson arrival processes and exponentially distributed service times. Finch [9] shows that successive interdeparture times are in general not independent for M/G/1 systems, except when the service times have an exponential distribution. Jenkins [11] determines the 1- and 2-step autocorrelations for stationary M/E_Ψ/1 systems. Daley [7] obtains the generating function (GF) of the autocorrelations of interdeparture times for stationary M/G/1 systems. This GF is expressed in terms of the probability generating function (PGF) of the distribution of the number of customers served in a busy period which is only implicitly determined as a solution to a functional equation. Daley [7] also proves that the departure process of a stationary GI/M/1 system is a renewal process if and only if the arrival process is a Poisson process. Further, he derives implicit relations for the k-step autocorrelations of interdeparture times in stationary GI/M/1 systems. These relations readily lead with results of Takács [16] to an expression for the GF of these autocorrelations which involves a function which is only implicitly determined as a solution to a functional equation. Pack [14] has found a general formula for the k-step autocorrelation of interdeparture times for stationary M/D/1 systems. The autocorrelations of M/D/1 systems seem to act as upper bounds for the autocorrelations of M/G/1 systems. This was proved for $k = 1$ by Daley [7]. Hu [10] presents MacLaurin series for moments and covariances of interdeparture times for a class of GI/G/1 systems with interarrival time distributions of which the densities are regular at 0. However, these expansions may not converge for all values of the load for which the system is stable and the paper does not indicate when this phenomenon occurs.

The aim of the present paper is to show that the transform results of Daley [7] are suitable for numerical inversion. The numerical inversion of (probability) generating functions has been extensively developed during the last decade, see, e.g., Abate & Whitt [1, 2]. A special case is formed by generating functions that can only be characterized implicitly as solutions to functional equations. Important examples are the PGFs of the distributions of the number of customers served during a busy period in M/G/1 and GI/M/1 systems. Other quantities of interest can be expressed in terms of these PGFs, for instance, the above mentioned GFs of the autocorrelations of interdeparture times. Many algorithms for the numerical inversion of GFs require the values of the involved functions at complex arguments. This means for generating functions which are only characterized implicitly that the related functional equation has to be numerically solved at (many) complex arguments. Abate & Whitt [3] discuss the solution of functional equations for complex arguments and provide conditions for iterative methods to converge. However, this approach is more involved than the basic methods for numerical inversion, and the iterative solution of a functional equation is an additional source of numerical inaccuracy. In Blanc [4] it is shown that alternative inversion formulas can be obtained by simple substitutions in the contour integrals, and possibly an integration by parts, and that upper bounds on the discretization error when applying the trapezoidal rule can be obtained. In this paper it will be shown that this method allows the efficient computation of autocorrelations of interdeparture times for stationary M/G/1 and GI/M/1 systems.

The organization of the rest of this paper is as follows. Section 2 provides a short summary of a standard method for the numerical inversion of generating functions. Section 3 contains some general properties of interdeparture times in stationary GI/G/1 systems and introduces some notations. In Section 4 we will present the derivation of an alternative contour integral for the numerical inversion of the GF of the series of autocorrelations of the interdeparture times in stationary M/G/1 systems. Section 5 is devoted to a similar substitution, but for stationary GI/M/1 systems. The last two sections contain several examples.

2 Numerical inversion of generating functions

The terms of a sequence of real numbers $\{g_k;\ k = 0, 1, 2, \ldots\}$ with $|g_k| \le 1$ for all k can be recovered from its generating function by means of a contour integral in the complex plane over a circle around the origin with radius r, $0 < r < 1$:

$$G(z) \doteq \sum_{k=0}^{\infty} g_k z^k, \quad |z| < 1, \qquad g_k = \frac{1}{2\pi i} \oint_{|z|=r} G(z)\, \frac{dz}{z^{k+1}}, \quad k = 0, 1, 2, \ldots. \tag{1}$$

The contour integral can be converted into an integral over a real interval by means of the substitution $z = re^{iu}$ and by some symmetry properties of the GF: for $0 < r < 1$,

$$g_k = \frac{1}{\pi r^k} \int_0^\pi [\cos(ku)\, \Re G(re^{iu}) + \sin(ku)\, \Im G(re^{iu})]\, du, \quad k = 0, 1, 2, \ldots; \tag{2}$$

here, $i = \sqrt{-1}$ and $\Re z$ ($\Im z$) denotes the real (imaginary) part of a complex number z. The case $k = 0$ is simple: $g_0 = G(0)$. For $k > 0$, Abate & Whitt [2] describe the following method for evaluating the above type of integrals with a prescribed accuracy of, say, ϵ. Application of the trapezoidal rule with a step size of π/k to (2) yields

$$g_k \approx \frac{1}{kr^k} \left[\tfrac{1}{2}\{G(r) + (-1)^k G(-r)\} + \sum_{j=1}^{k-1} (-1)^j \Re G(re^{ij\pi/k}) \right], \quad k = 1, 2, \ldots, \tag{3}$$

while the prescribed accuracy and an upper bound on the discretization error lead to the choice of $r = \sqrt[2k]{\epsilon}$, $k = 1, 2, \ldots$; to avoid roundoff problems, approximately $\frac{3}{2}\gamma$-digit precision is required to obtain $\epsilon = 10^{-\gamma}$ accuracy.

3 Interdeparture times

Consider a stationary GI/G/1 system. The interarrival time distribution will be denoted by $A(.)$, with moments α_m, $m = 1, 2, \ldots$, and LST $\alpha(\zeta)$. The service time distribution will be denoted by $B(.)$, with moments β_m, $m = 1, 2, \ldots$, and LST $\beta(\zeta)$. The load is $\rho = \beta_1/\alpha_1 < 1$. Let the random variable B_k denote the service time of the kth arriving customer after some tagged customer 0, $k = 0, 1, 2, \ldots$, let the random variable A_k denote the interarrival time between the $(k-1)$st and the kth arriving customers, $k = 1, 2, \ldots$. Further, let the random variable D_k denote the interdeparture time between the kth and the $(k+1)$st departing customers, $k = 0, 1, 2, \ldots$. The aim of this paper is the study of the k-step autocorrelations defined by

$$\hat{\rho}_k\{D\} \doteq \left[E\{D_k D_0\} - E^2\{D_0\} \right] \Big/ \sigma^2\{D_0\}, \quad k = 1, 2, \ldots. \tag{4}$$

For the ease of discussion we will assume that customers are served in the order of arrival (FCFS) but the results hold for all work-conserving, nonpreemptive and nonanticipating service disciplines. The kth interdeparture time is equal to the sum of a virtual idle period $\tilde{I}_k$ (which is only nonzero and equal to an actual idle period if the kth customer leaves the system behind empty) and the service time of the $(k+1)$st customer:

$$D_k = \tilde{I}_k + B_{k+1}, \quad k = 0, 1, 2, \ldots. \tag{5}$$

Clearly, $\tilde{I}_k$ and B_{k+1} are independent. Since $\tilde{I}_k > 0$ with probability w_0, the stationary probability that a customer meets an empty system upon arrival, the LST of the distribution of the stationary interdeparture time D is related to the LST of the stationary distribution of an idle period I as

$$E\{e^{-\zeta D}\} = \beta(\zeta)[1 - w_0 + w_0 E\{e^{-\zeta I}\}], \quad \Re\zeta \geq 0. \tag{6}$$

Further, the LST of the stationary distribution of an idle period I is related to the LST of the stationary distribution of the FCFS waiting time W as, cf. Cohen [6, Sect. II.6.7],

$$E\{e^{-\zeta W}\} = \frac{w_0[1 - E\{e^{\zeta I}\}]}{1 - \beta(\zeta)\alpha(-\zeta)}, \quad \Re\zeta = 0; \qquad w_0 \doteq \Pr\{W = 0\}. \tag{7}$$

From (6) and (7) it follows that the mean and the squared coefficient of variation C_D^2 of the stationary interdeparture time distribution are, cf. Marshall [13],

$$E\{D\} = \alpha_1, \quad C_D^2 = C_A^2 + 2\rho^2 C_B^2 - 2\rho(1-\rho)E\{W\}/\beta_1; \tag{8}$$

here, C_A^2 (C_B^2) denotes the squared coefficient of variation of the interarrival (service) time distribution. In the rest of this paper it is understood if the load ρ varies that the mean interarrival time α_1 varies with fixed shape of the interarrival time distribution and with fixed service time distribution. Since $E\{W\} \downarrow 0$ as $\rho \downarrow 0$ and by the heavy-traffic limit $(1-\rho)E\{W\} \to \frac{1}{2}\beta_1[C_A^2 + C_B^2]$, cf. Kingman [12], it holds if $\alpha_2 < \infty$, $\beta_3 < \infty$, that

$$\lim_{\rho\downarrow 0} C_D^2 = C_A^2, \qquad \lim_{\rho\uparrow 1} C_D^2 = C_B^2. \tag{9}$$

The light traffic limit vanishes for D/G/1 systems while the heavy traffic limit vanishes for GI/D/1 systems which will give rise to diverging behavior of the autocorrelations of the interdeparture times for these systems as we will see below. Daley [7] has proved that if $\alpha_2 < \infty$, $\beta_3 < \infty$,

$$\sum_{k=1}^{\infty} \hat{\rho}_k\{D\} = \frac{C_A^2 - C_D^2}{2C_D^2}. \tag{10}$$

This sum has a lower bound of $-\frac{1}{2}$ for all GI/G/1 systems with finite coefficients of variation C_A and C_B. By (9), this sum vanishes in the light traffic limit ($\rho \downarrow 0$) except for D/G/1 systems for which

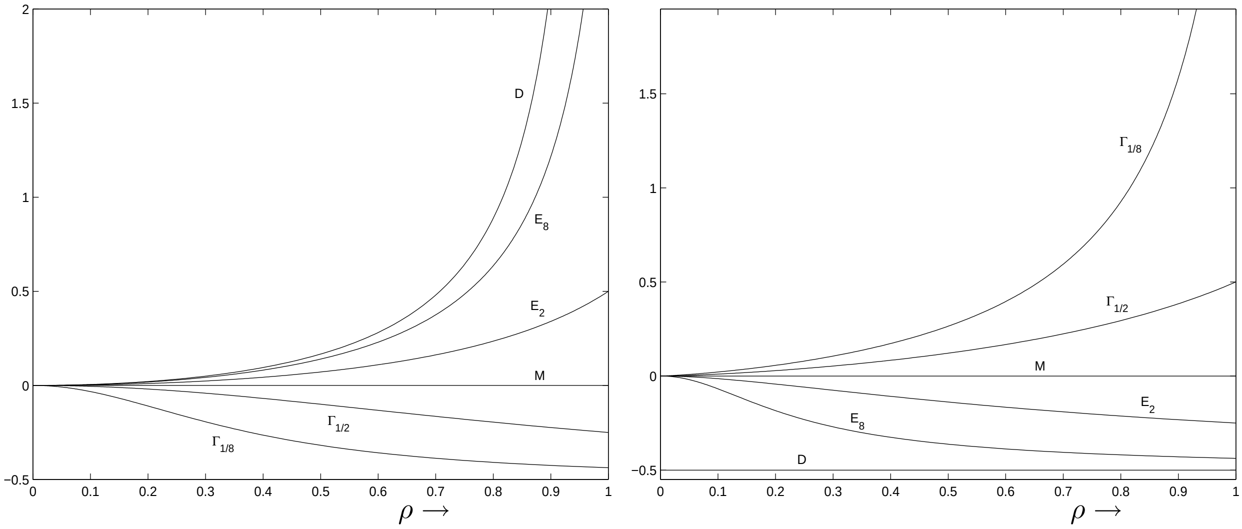

Figure 1: Sum over interdeparture time autocorrelations for M/Γ_Ψ/1 (l.) and Γ_Ψ/M/1 systems (r.).

this sum is equal to $-\frac{1}{2}$ for all ρ. The heavy traffic limit ($\rho \uparrow 1$) has the finite value $\frac{1}{2}[C_A^2 - C_B^2]/C_B^2$ except for GI/D/1 systems.

Figure 1 shows the sum over all autocorrelations of the interdeparture times for M/Γ_Ψ/1 and Γ_Ψ/M/1 systems as a function of ρ for gamma Γ_Ψ distributions with various values of the shape parameter Ψ, including the limiting cases of M/D/1 and D/M/1 systems ($\Psi \to \infty$). For D/M/1 systems this sum is constant and equal to $-\frac{1}{2}$ for all ρ. The heavy traffic limits ($\rho \uparrow 1$) of these sums are all finite except that of the M/D/1 system.

Next, consider the 1-step autocorrelation of the interdeparture times. Repeated use of (5) yields

$$E\{D_1 D_0\} = E\{\tilde{I}_1 D_0\} + E\{B_2 D_0\} = E\{\tilde{I}_1 \tilde{I}_0\} + E\{\tilde{I}_1 B_1\} + \beta_1 E\{D_0\}. \qquad (11)$$

If $\tilde{I}_0 > 0$ then the waiting time W_1 of customer 1 is zero and $\tilde{I}_1 = [A_2 - B_1]^+$ is independent of the length of the idle period I_0. Hence, with (7) it follows that

$$E\{\tilde{I}_1 \tilde{I}_0\} = w_0 E\{I_0\} E\{[A_2 - B_1]^+\} = (\alpha_1 - \beta_1) E\{[A_2 - B_1]^+\}.$$

Further, it holds by a standard relation for GI/G/1 systems that

$$E\{\tilde{I}_1 B_1\} = E\{B_1 [A_2 - W_1 - B_1]^+\}.$$

From the above it follows with (8) that

$$E\{D_1 D_0\} = (\alpha_1 - \beta_1) E\{[A_2 - B_1]^+\} + E\{B_1 [A_2 - W_1 - B_1]^+\} + \beta_1 \alpha_1, \qquad (12)$$

with A_2, B_1 and W_1 mutually independent random variables so that

$$E\{[A_2 - B_1]^+\} = \int_0^\infty \int_u^\infty (v-u)\,dA(v)\,dB(u), \tag{13}$$

$$E\{B_1[A_2 - W_1 - B_1]^+\} = \int_0^\infty \int_0^\infty u \int_{u+s}^\infty (v-u-s)\,dA(v)\,dB(u)\,dW(s); \tag{14}$$

here, $W(s)$ denotes the stationary FCFS waiting time distribution. These relations allow the determination of $\hat{\rho}_1\{D\}$ in many cases without series expansions as in Hu [10]. We only mention here that if the interarrival time distribution is a mixture of exponential distributions,

$$A(v) = \sum_{j=1}^H r_j \left[1 - e^{-\lambda_j v}\right], \;\; v \geq 0; \quad \alpha(\zeta) = \sum_{j=1}^H \frac{r_j \lambda_j}{\zeta + \lambda_j}, \;\; \text{with} \;\; \sum_{j=1}^H r_j = 1; \tag{15}$$

then straightforward manipulations lead from (12) with (13) and (14) to

$$E\{D_1 D_0\} = \beta_1 \alpha_1 + (\alpha_1 - \beta_1) \sum_{j=1}^H \frac{r_j}{\lambda_j}\,\beta(\lambda_j) - \sum_{j=1}^H \frac{r_j}{\lambda_j}\,\beta'(\lambda_j) E\{e^{-\lambda_j W}\}, \tag{16}$$

provided all λ_j, $j = 1, \ldots, H$, are distinct (note that not all r_j are required to be positive for $A(v)$ to represent a distribution). The LST of the stationary FCFS waiting time distribution is in this case given by, cf. Cohen [6, Sect. II.5.11]:

$$E\{e^{-\zeta W}\} = \frac{(\alpha_1 - \beta_1)\zeta}{1 - \beta(\zeta)\alpha(-\zeta)}\,\frac{\lambda_1}{\zeta - \lambda_1} \prod_{j=2}^H \frac{\lambda_j(\zeta - \phi_j)}{\phi_j(\zeta - \lambda_j)}, \quad \Re\zeta \geq 0, \tag{17}$$

with ϕ_j, $j = 2, \ldots, H$, the $H-1$ zeros in the right half-plane $\Re\zeta > 0$ of the denominator $1 - \beta(\zeta)\alpha(-\zeta)$. From (16) and (17) the autocorrelation of two consecutive interdeparture times becomes

$$\hat{\rho}_1\{D\} = (1-\rho)\,\frac{-1 + \frac{1}{\alpha_1}\sum_{j=1}^H \frac{r_j}{\lambda_j}\left[\beta(\lambda_j) - \frac{\lambda_j \beta'(\lambda_j)}{\alpha_1(-\lambda_j)\beta(\lambda_j)} \prod_{h=2}^H \frac{\phi_h - \lambda_j}{\phi_h}\right]}{C_A^2 + 2\rho^2 C_B^2 - 2\rho(1-\rho)E\{W\}/\beta_1}. \tag{18}$$

If two or more rates λ_j coincide (in the sense that the LST $\alpha(\zeta)$ has a higher order pole at $-\lambda_j$) the evaluation of (13) and (14) proceeds somewhat differently. For instance, in the case $H = 2$, $\lambda_1 = \lambda_2 = \lambda$, we find that

$$\hat{\rho}_1\{D\} = (1-\rho)\,\frac{-1 + \beta(\lambda) - (\lambda - \frac{1}{\alpha_1})\beta'(\lambda) + \frac{\lambda(\phi_2 - \lambda)}{\phi_2 \alpha_1}\left[\frac{\beta''(\lambda)}{\beta(\lambda)} - \frac{[\beta'(\lambda)]^2}{\beta^2(\lambda)}\right] - \frac{\beta'(\lambda)}{\alpha_1 \beta(\lambda)}}{C_A^2 + 2\rho^2 C_B^2 - 2\rho(1-\rho)E\{W\}/\beta_1}. \tag{19}$$

Similarly, it is possible to evaluate (13) and (14) if the service time distribution is a mixture of exponential or Erlang distributions. In principle, it is possible to determine $\hat{\rho}_k\{D\}$, $k = 2, 3, \ldots$, by repeated application of the waiting time recursion and evaluation of $(2k+1)$-fold integrals like (14) but the expressions become very complicated.

4 The M/G/1 system

Consider an M/G/1 system with arrival rate λ. The distribution of the number of customers served in a busy period, J, satisfies the following functional equation:

$$E\{z^J\} = \nu(z), \qquad \nu(z) = z\beta(\lambda[1-\nu(z)]), \quad |z| \le 1. \tag{20}$$

Clearly, $\Pr\{J=0\} = \nu(0) = 0$ and $\nu(1) = 1$. Differentiation of this functional equation yields

$$\nu'(z) = \beta(\lambda[1-\nu(z)]) - \lambda z \nu'(z)\beta'(\lambda[1-\nu(z)]), \quad |z| \le 1. \tag{21}$$

Daley [7] has derived a relation for the GF of the series of covariances of successive interdeparture times in stationary M/G/1 systems. Since (8) implies with the well-known formula for the M/G/1 mean waiting time that $C_D^2 = 1 + \rho^2[C_B^2 - 1]$, it follows that the GF of the series of autocorrelations is given by

$$\sum_{k=1}^{\infty} \hat{\rho}_k\{D\} z^k = \frac{1}{1-z}\frac{1-\rho}{1+\rho^2[C_B^2-1]}\left[\frac{\nu(z)-z}{1-\nu(z)} + \frac{z\nu'(z)-\nu(z)}{\nu(z)\nu'(z)}\right], \quad |z| \le 1. \tag{22}$$

The factor $1/(1-z)$ represents a convolution with a series consisting of all ones. Hence, the k-step autocorrelation of the interdeparture times can be written as

$$\hat{\rho}_k\{D\} = \frac{1-\rho}{1+\rho^2[C_B^2-1]}\sum_{j=1}^{k} \Delta_j, \quad k = 1, 2, \ldots, \tag{23}$$

where the quantities Δ_k, $k = 1, 2, \ldots$, are given by the contour integrals, cf. (1),

$$\Delta_k = \frac{1}{2\pi i}\oint_{|z|=r}\left[\frac{\nu(z)-z}{1-\nu(z)} + \frac{z\nu'(z)-\nu(z)}{\nu(z)\nu'(z)}\right]\frac{dz}{z^{k+1}}, \quad k = 1, 2, \ldots. \tag{24}$$

To avoid the computation of the PGF $\nu(z)$ at the values $re^{ik\pi/n}$ by iterative solution of (20) when approximating the above integral by the trapezoidal rule, cf. (3), we substitute $w = \nu(z)$ in the contour integral as in Blanc [4]. Since $\nu'(0) = \beta(\lambda) > 0$, cf. (21), this mapping has an inverse in a neighborhood of the origin. Moreover, it follows from (20) that this inverse is explicitly given by $z = w/\beta(\lambda[1-w])$. Further, it follows from (21) that under this mapping

$$\nu'(z) = \frac{\beta^2(\lambda[1-w])}{\beta(\lambda[1-w]) + \lambda w \beta'(\lambda[1-w])}.$$

Hence, this substitution leads after some rearrangements to the representation, for $k = 1, 2, \ldots$,

$$\Delta_k = \frac{1}{2\pi i}\oint_{|w|=r}\left[\frac{\beta(\lambda[1-w])-1}{1-w} - \frac{\lambda\beta'(\lambda[1-w])}{\beta(\lambda[1-w])}\right]\beta^{k-1}(\lambda[1-w])\left[1 + \frac{\lambda w \beta'(\lambda[1-w])}{\beta(\lambda[1-w])}\right]\frac{dw}{w^k}. \tag{25}$$

The image of a circle $|z| = r$ under the mapping $w = \nu(z)$ is not a circle but a contour with the origin in its interior. Since the integrand in the w-plane has no singularities in $\Re w < 1$ other than $w = 0$ this contour can be replaced by a circle $|w| = r$ by Cauchy's Theorem. Observe that the first factor of the integrand in (25) vanishes for M/M/1 systems, in agreement with the result of Burke [5] that $\hat{\rho}_k\{D\} = 0$, $k = 1, 2, \ldots$, for this system. For $k = 1$ the integrand has a first order pole at $w = 0$ so that the contour integral is simply evaluated. Hence, the 1-step autocorrelation follows with (23) as (the case $H = 1$ of (18)):

$$\hat{\rho}_1\{D\} = (1-\rho)\,\frac{\beta(\lambda) - 1 - \lambda\beta'(\lambda)/\beta(\lambda)}{1 + \rho^2(C_B^2 - 1)}. \tag{26}$$

This autocorrelation vanishes as $\rho \downarrow 0$ ($\lambda \downarrow 0$) where $\hat{\rho}_1\{D\} \sim \frac{1}{2}\rho^2[1 - C_B^2]$. It also vanishes as $\rho \uparrow 1$ except in the case of the M/D/1 system when it tends to $\frac{1}{2}e^{-1} \approx 0.1839$; the latter is a consequence of a factor $1 - \rho$ in the denominator when $C_B^2 = 0$. With some more effort the following expression for the 2-step autocorrelation is derived from (25) and (23):

$$\hat{\rho}_2\{D\} = (1-\rho)\,\frac{\beta(\lambda)[\beta(\lambda) - \lambda\beta'(\lambda)] - 1 + \lambda^2\beta''(\lambda) - \lambda\beta'(\lambda)[1 + \lambda\beta'(\lambda)]/\beta(\lambda)}{1 + \rho^2(C_B^2 - 1)}. \tag{27}$$

This autocorrelation vanishes as $\rho \downarrow 0$ where $\hat{\rho}_2\{D\} \sim \rho^3[(\beta_2/\beta_1^2) - \frac{1}{3}(\beta_3/\beta_1^3)]$. It also vanishes as $\rho \uparrow 1$ except for the M/D/1 system when it tends to $e^{-2} \approx 0.1353$.

For higher values of k the exact formulas for $\hat{\rho}_k\{D\}$ become more and more complex. Then, it becomes more efficient to evaluate the contour integrals (25) numerically, for instance, with the aid of the trapezoidal rule. This rule could be directly applied to (25), but for an error analysis, and an appropriate choice of the parameter r, it is more convenient to apply first an integration by parts as discussed in Blanc [4]. We will deal with the two terms of the first factor of the integrand in (25) separately. That is, we write $\Delta_k = \Delta_k^{(1)} + \Delta_k^{(2)}$ and obtain, for $k = 2, 3, \ldots$,

$$\Delta_k^{(1)} = \frac{-\rho}{2\pi i(k-1)} \oint_{|w|=r} \left[\frac{1 - \beta(\lambda[1-w])}{\rho(1-w)}\right]' \beta^{k-1}(\lambda[1-w])\frac{dw}{w^{k-1}}, \tag{28}$$

and

$$\Delta_k^{(2)} = \frac{\rho}{2\pi i(k-1)} \oint_{|w|=r} \left[\frac{-\beta'(\lambda[1-w])}{\beta_1\beta(\lambda[1-w])}\right]' \beta^{k-1}(\lambda[1-w])\frac{dw}{w^{k-1}}. \tag{29}$$

Note that $\Delta_1^{(1)} = \beta(\lambda) - 1 < 0$ and $\Delta_1^{(2)} = -\lambda\beta'(\lambda)/\beta(\lambda) > 0$, cf. (25), do not possess such representations. For $k = 2$ the above integrals are simply evaluated as

$$\Delta_2^{(1)} = -\beta(\lambda)[1 - \beta(\lambda) + \lambda\beta'(\lambda)], \qquad \Delta_2^{(2)} = \lambda^2[\beta''(\lambda) - \{\beta'(\lambda)\}^2/\beta(\lambda)],$$

leading with (23) again to (27). For $k = 3, 4, \ldots$, the function $\beta^{k-1}(\lambda[1-w])$ in the integrands of (28) and (29) is a PGF. Also, the function of which the derivative appears in the integrand of $\Delta_k^{(1)}$ is a PGF, namely, of the distribution of the number of Poisson arrivals during a residual service time, which has a mean of $\frac{1}{2}\lambda^2\beta_2/\rho$. Because the derivative of a PGF divided by the mean of the corresponding distribution is again a PGF it follows that

$$-\frac{\frac{1}{2}\lambda^2\beta_2}{k-1} < \Delta_k^{(1)} < 0, \quad k = 2, 3, \ldots. \tag{30}$$

Moreover, the following upper bound on the discretization error $e_d^{(1)}(k)$ when applying the trapezoidal rule as in (3) to (28) — but with a step size of $\pi/(k-2)$ since $1/w^{k-1}$ plays the role of $1/z^{k+1}$ — is obtained as explained in Blanc [4] and based on earlier derivations in Abate & Whitt [1, 2]:

$$|e_d^{(1)}(k)| \leq \frac{\frac{1}{2}\lambda^2\beta_2}{k-1}\,\frac{r^{2(k-2)}}{1-r^{2(k-2)}}, \quad k = 2, 3, \ldots. \tag{31}$$

Since the LST $\beta(\zeta)$ is completely monotonic the integrands in (29) are nonnegative for real w, cf. Widder [17, Sect. IV.16]. Observe that $\Delta_k^{(2)} = 0$, $k = 2, 3, \ldots$, for M/D/1 systems. For systems such as M/Γ_Ψ/1 and M/PH/1 systems for which $-\beta'(\zeta)/\{\beta_1\beta(\zeta)\}$ represents the LST of a distribution, with mean $C_B^2\beta_1$, we have

$$0 \leq \Delta_k^{(2)} < \rho^2 C_B^2, \quad k = 2, 3, \ldots, \tag{32}$$

and the following upper bound on the discretization error $e_d^{(2)}(k)$ when applying the trapezoidal rule with step size $\pi/(k-2)$ to (28):

$$|e_d^{(2)}(k)| \leq \frac{\rho^2 C_B^2}{k-1}\,\frac{r^{2(k-2)}}{1-r^{2(k-2)}}, \quad k = 2, 3, \ldots. \tag{33}$$

For $k = 3, 4, \ldots$, the upper bounds on the discretization errors can be used to choose the parameter r such that a desired accuracy is achieved. In comparison with other application as discussed in Blanc [4], additional round-off errors are possible for larger values of k due to the summation in (23) of terms with differing signs. Possible inaccuracies can be detected by comparison of partial sums of the series of autocorrelation with the total sum (10). We did not encounter such round-off errors in our numerical experiments with 16-digit precision, with autocorrelations which are in absolute value larger than 10^{-8}, cf. Section 2, and with values of k up to 100.

Figure 2 shows the 1- and 2-step autocorrelations of interdeparture times for stationary M/Γ_Ψ/1 systems as a function of ρ for various values of the shape parameter Ψ, including the limiting case

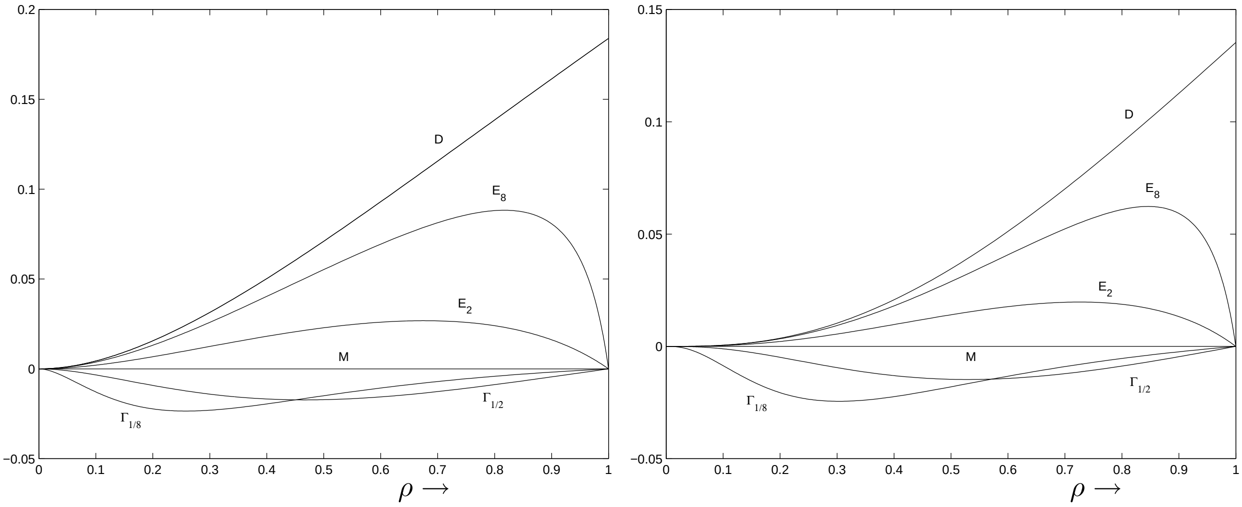

Figure 2: The 1- and 2-step autocorrelations of the interdeparture times for M/Γ_Ψ/1 systems.

of an M/D/1 system ($\Psi \to \infty$). The paper by Jenkins [11] contains similar figures but only with autocorrelations for integer values of Ψ. The autocorrelations turn out to be positive for $\Psi > 1$ ($C_B^2 < 1$) and negative for $\Psi < 1$ ($C_B^2 > 1$). Numerical experiments confirm this phenomenon for k-step autocorrelations with higher values of k. They also confirm that these autocorrelations for M/G/1 systems have as upper bounds the autocorrelations for M/D/1 systems as found by Pack [14]:

$$\hat{\rho}_k\{D\} = \frac{e^{-k\rho}}{1+\rho}\sum_{i=0}^{k-1}\frac{k-i}{k}\frac{(k\rho)^i}{i!} - \frac{1-\rho}{1+\rho}, \quad k = 1, 2, \ldots. \tag{34}$$

Whereas the dependence of $\hat{\rho}_k\{D\}$ on Ψ for ρ and k fixed is monotone for $\Psi > 1$ it clearly is not monotone for $\Psi < 1$. This can also be seen from the heavy-traffic asymptote of $\hat{\rho}_1\{D\}$ for M/Γ_Ψ/1 systems which is readily found from (26) as:

$$\hat{\rho}_1\{D\} \sim (1-\rho)\Psi\left[\left(\frac{\Psi}{\Psi+1}\right)^{\Psi} - \frac{1}{\Psi+1}\right], \quad \rho \uparrow 1. \tag{35}$$

The slope of the autocorrelation near $\rho = 1$ tends to $-\infty$ as $\Psi \to \infty$ where it approximates the singular behavior of the M/D/1 system. This slope has a maximum of about 0.045 at $\Psi \approx 0.48$; it tends to 0 as $\Psi \downarrow 0$ and as $\Psi \uparrow 1$. This and many stated results below have been determined by standard numerical maximization (minimization) procedures executed with several starting values to avoid the risk of local extrema.

In the case $\Psi = \frac{1}{2}$, $\hat{\rho}_1\{D\}$ has a minimum of -0.0173 at $\rho \approx 0.4673$ and $\hat{\rho}_2\{D\}$ has a minimum of -0.0147 at $\rho \approx 0.5294$. In the case $\Psi = \frac{1}{8}$, $\hat{\rho}_1\{D\}$ has a minimum of -0.0235 at $\rho \approx 0.2583$

and $\hat{\rho}_2\{D\}$ has a minimum of -0.0244 at $\rho \approx 0.3031$. The locations of the minima move to lower values of ρ as Ψ decreases. The values of the minima tend to 0 both as $\Psi \uparrow 1$ and as $\Psi \downarrow 0$. The 1-step autocorrelation has an overall minimum in the class of M/Γ_Ψ/1 systems of -0.02455 at $\rho \approx 0.3189$ for $\Psi \approx 0.1948$. The 2-step autocorrelation has a slightly larger overall minimum in this class of -0.02452 at $\rho \approx 0.3223$ for $\Psi \approx 0.1422$. These findings are in contradiction with Daley [7, p. 1019] who states that it is possible to choose the service time distribution so that $\hat{\rho}_1\{D\}$ is arbitrarily close to -1, for instance, through a Γ_Ψ distribution with Ψ sufficiently small (this assertion is repeated in Daley [8, p. 405] without reference to any type of distribution). Also note that the sum over all autocorrelations, cf. (10), tends to $-\frac{1}{2}$ if $C_B^2 \to \infty$ in M/G/1 systems.

For the case of an M/C_2/1 systems with a 2-phase Cox C_2 service time distribution with transition rates μ_1 and μ_2 and LST

$$\beta(\zeta) = \frac{\mu_1\mu_2 + \zeta(\mu_1 + \mu_2 - \beta_1\mu_1\mu_2)}{(\mu_1+\zeta)(\mu_2+\zeta)}, \quad \Re\zeta \geq 0, \tag{36}$$

with squared coefficient of variation and restrictions on the parameter values

$$C_B^2 = 1 - 2\left(\frac{1}{\beta_1\mu_1} - 1\right)\left(\frac{1}{\beta_1\mu_2} - 1\right), \qquad \frac{1}{\mu_1} < \beta_1 \leq \frac{1}{\mu_1} + \frac{1}{\mu_2}, \tag{37}$$

the numerator of the 1-step autocorrelation (26) becomes

$$\beta(\lambda) - 1 - \lambda\frac{\beta'(\lambda)}{\beta(\lambda)} = \frac{\frac{1}{2}[1 - C_B^2]\rho^2\mu_1^2\mu_2^2\beta_1^4}{(\mu_1\beta_1+\rho)(\mu_2\beta_1+\rho)[\mu_1\mu_2\beta_1^2(1-\rho) + \rho(\mu_1+\mu_2)\beta_1]}. \tag{38}$$

Hence, also for M/C_2/1 systems the 1-step autocorrelation $\hat{\rho}_1\{D\}$ is positive for all ρ if $C_B^2 < 1$ and negative for all ρ if $C_B^2 > 1$. Moreover, $\hat{\rho}_1\{D\}$ has an overall minimum in the class of M/C_2/1 systems of -0.03083 at $\rho \approx 0.3443$ for $\mu_1 \to \infty$ and $\mu_2 \approx 0.3562$ ($C_B^2 \approx 4.615$).

However, the foregoing examples are misleading in the sense that there exist service time distributions for which the autocorrelations $\hat{\rho}_k\{D\}$ do not have the same sign for all k, $k = 1, 2, \ldots$, and for which it can occur that $\hat{\rho}_k\{D\}$ does not have a fixed sign for all ρ, $0 < \rho < 1$, for a given k. Examples can be found in the class of M/G/1 systems with $C_B^2 = 1$. This is not surprising since the sum over all autocorrelations (10) vanishes for all ρ for all M/G/1 systems with $C_B^2 = 1$. Consider, for instance, distributions which are mixtures of two Erlang E_2 distributions. These distributions (indicated by ME_2) have LST

$$\beta(\zeta) = \frac{q}{(1+\frac{1}{2}\delta_1\zeta)^2} + \frac{1-q}{(1+\frac{1}{2}\delta_2\zeta)^2}, \quad \Re\zeta \geq 0, \tag{39}$$

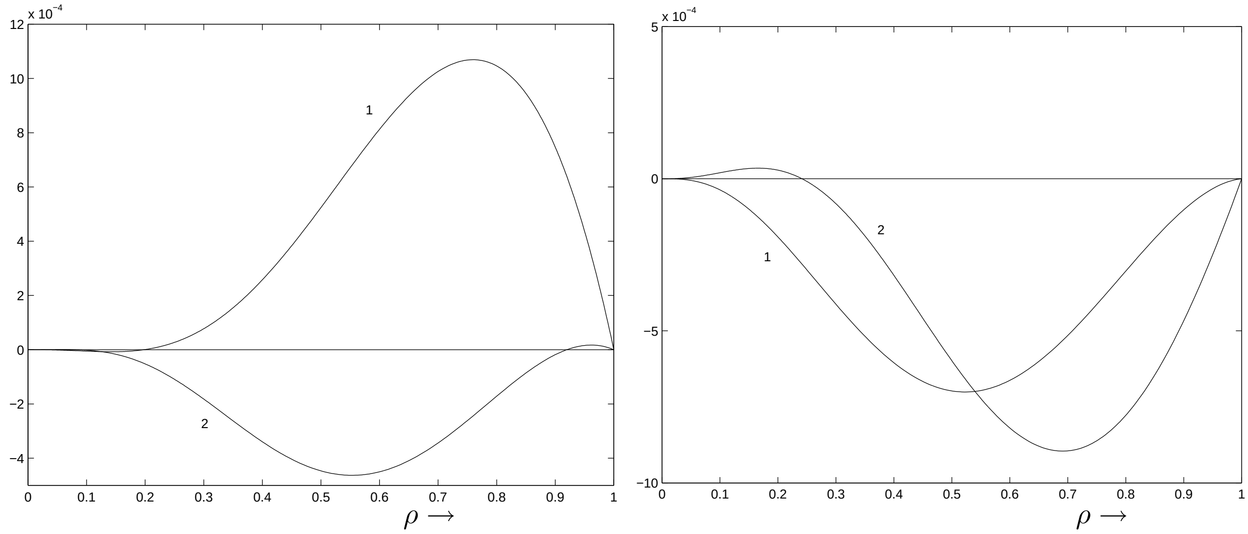

Figure 3: The 1- and 2-step autocorrelations of the interdeparture times for M/ME$_2$/1 systems.

and moments

$$\beta_1 = q\delta_1 + (1-q)\delta_2, \quad \beta_2 = \tfrac{3}{2}[q\delta_1^2 + (1-q)\delta_2^2], \quad \beta_3 = 3[q\delta_1^3 + (1-q)\delta_2^3]. \tag{40}$$

Fixing $\beta_1 = 1$ and $C_B^2 = 1$ leaves one free parameter. For the case $\delta_1 = \frac{2}{5}$ ($\beta_3 \approx 5.956$), see the left graphs in Figure 3, we find that $\hat{\rho}_1\{D\}$ is negative for $0 < \rho < 0.198$, with a minimum of -7.5×10^{-6} at $\rho \approx 0.142$, and positive for $0.198 < \rho < 1$, with a maximum of 1.1×10^{-3} at $\rho \approx 0.760$; and we find that $\hat{\rho}_2\{D\}$ is positive for $0 < \rho < 0.084$, with a maximum of 6.4×10^{-7} at $\rho \approx 0.061$, is negative for $0.084 < \rho < 0.920$, with a minimum of -4.6×10^{-4} at $\rho \approx 0.553$, and is again positive for $0.920 < \rho < 1$, with a maximum of 1.7×10^{-5} at $\rho \approx 0.962$. Also, for fixed load ρ the autocorrelations $\hat{\rho}_k\{D\}$ may have multiple sign changes as function of k. In the foregoing example, with $\rho = 0.7$, $\hat{\rho}_k\{D\}$ is negative for $k = 2, \ldots, 11$, with a minimum of -4.0×10^{-4} at $k = 3$, and is positive for $k = 1$ and for $k \geq 12$, where there is a maximum of 1.8×10^{-5} at $k = 20$. For other values of δ_1 (β_3) quite different behavior may occur; see, for instance, the right graphs in Figure 3, which concern the case $\delta_1 = \frac{1}{3}$ ($\beta_3 \approx 5.833$).

Table 1 contains results of computations based on (28) and (29) for M/Γ_Ψ/1 systems with a load of $\rho = 0.9$. For $\Psi > 1$, the autocorrelations $\hat{\rho}_k\{D\}$ are monotonically decreasing with k. However, they are not monotonically decreasing with k in all cases for $\Psi < 1$. For instance, $\hat{\rho}_k\{D\}$ is minimal at $k = 2$ for $\Psi = \frac{1}{2}$ and at $k = 5$ for $\Psi = \frac{1}{8}$ when $\rho = 0.9$. Further, it turns out that the individual values of $|\hat{\rho}_k\{D\}|$ are not so large, in general, but that these values fade away slowly as

Table 1: Autocorrelations for M/G/1 systems with load $\rho = 0.9$.

	M/D/1	M/E_8/1	M/E_2/1	M/$\Gamma_{1/2}$/1	M/$\Gamma_{1/8}$/1
$\hat{\rho}_1\{D\}$	0.16135	0.08075	0.01619	-0.00447	-0.00182
$\hat{\rho}_5\{D\}$	0.06528	0.03547	0.00865	-0.00373	-0.00274
$\hat{\rho}_{10}\{D\}$	0.04070	0.02249	0.00573	-0.00277	-0.00254
$\hat{\rho}_{100}\{D\}$	0.00395	0.00238	0.00074	-0.00053	-0.00085
$\sum_{k=1}^{100} \hat{\rho}_k\{D\}$	1.79239	0.99629	0.25858	-0.13401	-0.14886
$\sum_{k=1}^{\infty} \hat{\rho}_k\{D\}$	2.13158	1.21674	0.34034	-0.22376	-0.42504

$k \to \infty$, the more so when C_B^2 is larger. For $\Psi = \frac{1}{8}$, the sum over the first 100 autocorrelations only amounts about $\frac{1}{3}$ of the total sum of the series. The decay factor of this series is equal to that of the distribution of J, cf. (22), which implies that for M/Γ_Ψ/1 systems ($\Psi \neq 1$):

$$\lim_{k\to\infty} \frac{\hat{\rho}_{k+1}\{D\}}{\hat{\rho}_k\{D\}} = \rho \left[\frac{\Psi+1}{\Psi+\rho}\right]^{\Psi+1}, \tag{41}$$

which is close to 1 for all ρ when Ψ is small. This decay factor is equal to $\rho e^{1-\rho}$ for M/D/1 systems. Similar observations as for M/Γ_Ψ/1 systems can be made for M/C_2/1 systems.

5 The GI/M/1 system

Consider an GI/M/1 system with service rate μ. The distribution of the number of customers served in a busy period, J, is determined via the following functional equation:

$$E\{z^J\} = \frac{z-\chi(z)}{1-\chi(z)}, \qquad \chi(z) = z\alpha(\mu[1-\chi(z)]), \quad |z| \leq 1. \tag{42}$$

For stationary GI/M/1 systems, Daley [7] expresses the k-step autocorrelation of the interdeparture times in terms of the transient conditional mean waiting time of the k-th customer given that customer 0 found the system empty. The GF of the latter series can be obtained from the GF of the Laplace-Stieltjes transforms of the conditional distributions of these waiting times given in Takács [16, Sect. 1.5]. Combination of these results readily leads to the following relation for the GF of the series of autocorrelations of successive interdeparture times in stationary GI/M/1 systems:

$$\sum_{k=1}^{\infty} \hat{\rho}_k\{D\} z^k = \frac{\rho(1-w_0-\rho)}{(1-w_0)C_D^2} \frac{\chi(z)}{1-\chi(z)}, \quad |z| \leq 1; \tag{43}$$

here, w_0 denotes the stationary probability that an arriving customer does not have to wait. This probability is implicitly determined as the unique solution of $1 - w_0 = \alpha(\mu w_0)$ on the interval $(0, 1)$. Note that $w_0 = 1 - \rho$ for M/M/1 systems so that it is clear that all autocorrelations vanish for this system. For GI/M/1 systems, we have $E\{W\}/\beta_1 = (1 - w_0)/w_0$ in (8). Hence, inversion of the GF in (43) implies that

$$\hat{\rho}_k\{D\} = \frac{\rho w_0}{1 - w_0} \frac{1 - w_0 - \rho}{w_0 C_A^2 - 2\rho(1 - \rho - w_0)} \Xi_k, \quad k = 1, 2, \ldots, \tag{44}$$

with

$$\Xi_k = \frac{1}{2\pi i} \oint_{|z|=r} \frac{\chi(z)}{1 - \chi(z)} \frac{dz}{z^{k+1}}, \quad k = 1, 2, \ldots. \tag{45}$$

As Daley [7] already noted, the quantities Ξ_k, $k = 1, 2, \ldots$, are positive so that the sign of $\hat{\rho}_k\{D\}$ is solely determined by the factor $1 - w_0 - \rho$. This also implies that the autocorrelations $\hat{\rho}_k\{D\}$ have the same sign for all k, $k = 1, 2, \ldots$, for a fixed ρ. If the interarrival time distribution is such that there exists a value of ρ such that $w_0 = 1 - \rho$ then all autocorrelations vanish at this value of ρ. An example for which the latter occurs is a mixture of two Erlang E_2 distributions, cf. (39), with $C_A^2 = 1$ and $\alpha_3/\alpha_1^3 \approx 5.833$.

Since $\chi(1) < 1$, in fact, $\chi(1) = 1 - w_0$, we apply the substitution $w = \chi(z)/\chi(1)$, with inverse $z = w\chi(1)/\alpha(\mu[1 - w\chi(1)])$. This gives, for $k = 1, 2, \ldots$,

$$\Xi_k = \frac{1}{2\pi i} \oint_{|w|=r} \frac{\chi(1)}{1 - w\chi(1)} \left[\frac{\alpha(\mu[1 - w\chi(1)])}{\chi(1)}\right]^k \left[1 + \frac{\mu w\chi(1)\alpha'(\mu[1 - w\chi(1)])}{\alpha(\mu[1 - w\chi(1)])}\right] \frac{dw}{w^k}. \tag{46}$$

As in the case of the M/G/1 system, the image of the circle $|z| = r$ has been replaced by the circle $|w| = r$. In the present case, an integration by parts as discussed in Blanc [4] leads to a simplification:

$$\Xi_k = \frac{1}{2\pi ik} \oint_{|w|=r} \frac{\chi(1)}{[1 - w\chi(1)]^2} \left[\frac{\alpha(\mu[1 - w\chi(1)])}{\chi(1)}\right]^k \frac{dw}{w^k}, \quad k = 1, 2, \ldots. \tag{47}$$

For $k = 1$ we simply have $\Xi_1 = \alpha(\mu)$, so that, cf. (44),

$$\hat{\rho}_1\{D\} = \frac{\rho w_0}{1 - w_0} \alpha(\mu) \frac{1 - \rho - w_0}{w_0 C_A^2 - 2\rho(1 - \rho - w_0)}. \tag{48}$$

This autocorrelation vanishes as $\rho \uparrow 1$ where $\hat{\rho}_1\{D\} \sim (1 - \rho)\alpha(\mu)[C_A^2 - 1]/[C_A^2 + 1]$ since $w_0 \sim 2(1 - \rho)/[C_A^2 + 1]$. The behavior of this autocorrelation as $\rho \downarrow 0$ depends on the shape of the interarrival time distribution. For Γ_Ψ/M/1 systems it holds that $1 - w_0 \sim (\Psi\rho)^\Psi$ as $\rho \downarrow 0$. This implies that

$$\hat{\rho}_1\{D\} \sim (\Psi\rho)^{\Psi+1} \text{ if } \Psi < 1, \quad \hat{\rho}_1\{D\} \sim -\Psi\rho^2 \text{ if } \Psi > 1, \quad \rho \downarrow 0.$$

For the D/M/1 system we have by repeated application of $1 - w_0 = \alpha(\mu w_0) = e^{-w_0/\rho}$:

$$\hat{\rho}_1\{D\} = -\frac{w_0 e^{-1/\rho}}{2(1-w_0)} = -\tfrac{1}{2} w_0 e^{-(1-w_0)/\rho} = -\tfrac{1}{2} w_0 e^{-e^{-w_0/\rho}/\rho}.$$

Note that this autocorrelation tends to $-\frac{1}{2}$ as $\rho \downarrow 0$ since $w_0 \to 1$ and $e^{-1/\rho}/\rho \to 0$. This result can be intuitively explained as follows. When the (mean) interarrival time is much larger than the mean service time there will be hardly any customer who has to wait before service and the interdeparture time is approximately equal to $D_k \approx A_{k+1} + B_{k+1} - B_k$, $k = 0, 1, 2, \ldots$. The latter implies that $\hat{\rho}_1\{D\} \approx -\rho^2 C_B^2/[C_A^2 + 2\rho^2 C_B^2]$ as $\rho \downarrow 0$ and this means that $\hat{\rho}_1\{D\} \approx -\frac{1}{2}$ as $\rho \downarrow 0$ for D/G/1 systems. This reasoning also explains why for Γ_Ψ/M/1 systems $\hat{\rho}_1\{D\} \sim -\Psi\rho^2$ as $\rho \downarrow 0$ if $\Psi > 1$, but it fails for $\Psi \le 1$ ($C_A^2 \ge 1$): when the variance of the interarrival times is high, waiting times cannot be ignored in light traffic. The foregoing argument predicts that $\hat{\rho}_k\{D\} \to 0$, $k = 2, 3, \ldots$, as $\rho \downarrow 0$ for D/G/1 systems.

For $k = 2$ it readily follows from (44) and (47) that

$$\hat{\rho}_2\{D\} = \frac{\rho w_0}{1-w_0}\,[\alpha(\mu) - \mu\alpha'(\mu)]\alpha(\mu)\,\frac{1-\rho-w_0}{w_0 C_A^2 - 2\rho(1-\rho-w_0)}. \tag{49}$$

The factor $\alpha(\mu) - \mu\alpha'(\mu)$ is positive. It behaves like $(1+\Psi)(\Psi\rho)^\Psi$ as $\rho \downarrow 0$ for Γ_Ψ/M/1 systems. It behaves like $e^{-1/\rho}/\rho$ as $\rho \downarrow 0$ for D/M/1 systems which shows that $\hat{\rho}_2\{D\} \to 0$ as $\rho \downarrow 0$ for this system as predicted above.

For general values of k the contour integrals (47) can again be evaluated numerically with the aid of the trapezoidal rule. In fact, it follows with (42) that

$$\Xi_k = \Pr\{J > k\}, \quad k = 1, 2, \ldots. \tag{50}$$

Hence, the upper bound on the discretization error $e_d(k)$ when the trapezoidal rule with step size $\pi/(k-1)$ is applied to (47) is the same as that for $\Pr\{J > k\}$ as derived in Blanc [4]:

$$|e_d(k)| \le \frac{1}{k}\,\frac{r^{2(k-1)}}{1-r^{2(k-1)}}\,\frac{\chi(1)}{[1-\chi(1)]^2}, \quad k = 2, 3, \ldots. \tag{51}$$

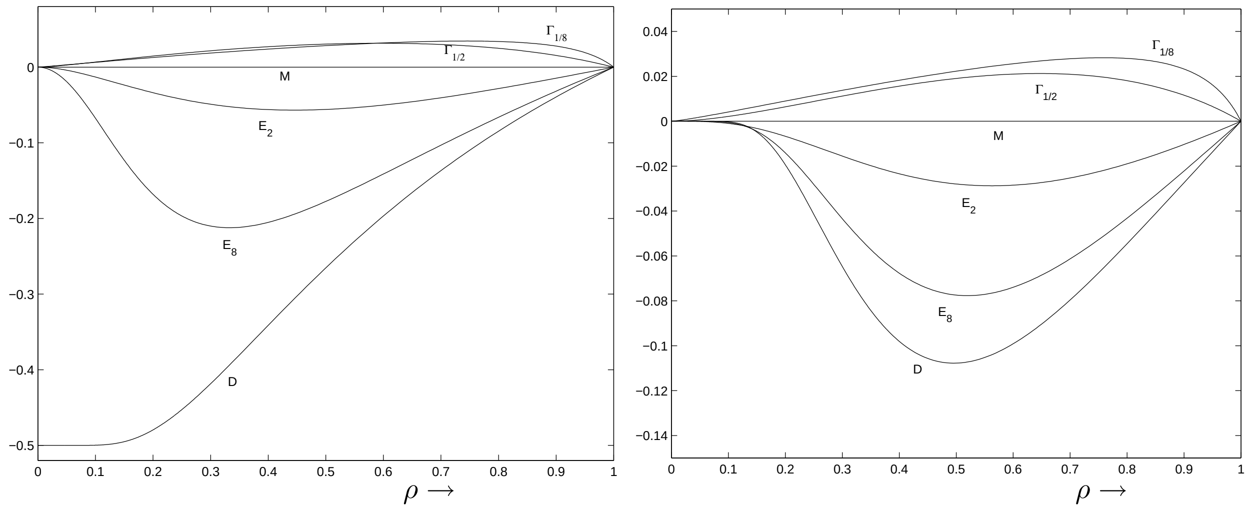

Figure 4: The 1- and 2-step autocorrelations of the interdeparture times for Γ_Ψ/M/1 systems.

Figure 4 shows the 1- and 2-step autocorrelations of interdeparture times for stationary Γ_Ψ/M/1 systems as a function of ρ for various values of the shape parameter Ψ, including the limiting case of a D/M/1 system ($\Psi \to \infty$). In contrast with the M/Γ_Ψ/1 systems, cf. Figure 2, the autocorrelations are negative for $\Psi > 1$ ($C_A^2 < 1$) and positive for $\Psi < 1$ ($C_A^2 > 1$). Numerical experiments confirm this phenomenon for k-step autocorrelations with higher values of k.

In the case $\Psi = \frac{1}{2}$, $\hat{\rho}_1\{D\}$ has a maximum of 0.0316 at $\rho \approx 0.5942$ and $\hat{\rho}_2\{D\}$ has a maximum of 0.0213 at $\rho \approx 0.6467$. In the case $\Psi = \frac{1}{8}$, $\hat{\rho}_1\{D\}$ has a maximum of 0.0345 at $\rho \approx 0.7445$ and $\hat{\rho}_2\{D\}$ has a maximum of 0.0283 at $\rho \approx 0.7585$. The locations of the maxima move to higher values of ρ as Ψ decreases. The values of the maxima tend to 0 both as $\Psi \uparrow 1$ and as $\Psi \downarrow 0$. The overall maximum of $\hat{\rho}_1\{D\}$ in the class of Γ_Ψ/M/1 systems is 0.04006 at $\rho \approx 0.6709$ for $\Psi \approx 0.2538$. The overall maximum of $\hat{\rho}_2\{D\}$ in this class is 0.03044 at $\rho \approx 0.7168$ for $\Psi \approx 0.2077$.

In the case $\Psi = 2$, $\hat{\rho}_1\{D\}$ has a minimum of -0.0568 at $\rho \approx 0.4454$ and $\hat{\rho}_2\{D\}$ has a minimum of -0.0287 at $\rho \approx 0.5629$. In the case $\Psi = 8$, $\hat{\rho}_1\{D\}$ has a minimum of -0.2123 at $\rho \approx 0.3329$ and $\hat{\rho}_2\{D\}$ has a minimum of -0.0777 at $\rho \approx 0.5194$. In the limiting case of a D/M/1 system, $\hat{\rho}_1\{D\}$ has a minimum of $-\frac{1}{2}$ at $\rho = 0$ which is at the same time the overall minimum for this class of systems, and $\hat{\rho}_2\{D\}$ has a minimum of -0.1078 at $\rho \approx 0.4958$ which is again the overall minimum for this class of systems. The influence of the variance of the interarrival time distribution on the autocorrelations is in most cases opposite to and stronger than that of the service time distribution. The latter property is confirmed by the 1-step autocorrelation of the E_2/E_2/1 system, which can be

Table 2: Autocorrelations for GI/M/1 systems with load $\rho = 0.9$.

	D/M/1	E_8/M/1	E_2/M/1	$\Gamma_{1/2}$/M/1	$\Gamma_{1/8}$/M/1
$\hat{\rho}_1\{D\}$	-0.03939	-0.03173	-0.01458	0.01547	0.02782
$\hat{\rho}_5\{D\}$	-0.01576	-0.01286	-0.00612	0.00730	0.01661
$\hat{\rho}_{10}\{D\}$	-0.00977	-0.00804	-0.00392	0.00489	0.01204
$\hat{\rho}_{100}\{D\}$	-0.00090	-0.00081	-0.00048	0.00085	0.00293
$\sum_{k=1}^{100} \hat{\rho}_k\{D\}$	-0.42652	-0.35688	-0.17987	0.23991	0.63703
$\sum_{k=1}^{\infty} \hat{\rho}_k\{D\}$	-0.50000	-0.42934	-0.23237	0.38385	1.58530

derived from (19) as

$$\hat{\rho}_1\{D\} = -\frac{(1-\rho)\rho^2}{(1+\rho)^3} \frac{(1-\rho)^2 - 4\rho + (1+\rho)\sqrt{(1+\rho)^2 + 4\rho}}{3 + 2\rho - 3\rho^2 - (1-\rho)\sqrt{(1+\rho)^2 + 4\rho}}.$$

This correlation is negative for all $\rho < 1$; in particular, $\hat{\rho}_1\{D\} \sim -\rho^2$ as $\rho \downarrow 0$ and $\hat{\rho}_1\{D\} \sim -(1-\rho)\frac{1}{4}(\sqrt{2}-1)$ as $\rho \uparrow 1$. It has a minimum of -0.0226 at $\rho \approx 0.4477$, larger than the minimum of $\hat{\rho}_1\{D\}$ for the E_2/M/1 system.

Table 2 contains some results for Γ_Ψ/M/1 systems computed with a value of r such that an accuracy of about 10^{-8} is achieved. The numerical results confirm the statement of Daley [7] that $|\hat{\rho}_k\{D\}|$ decreases monotonically to 0 as $k \to \infty$ for GI/M/1 systems. Note again the slow decay of $|\hat{\rho}_k\{D\}|$ as $k \to \infty$. The decay factor follows by solving $\chi'(z) = 0$, cf. (43), (42), for Γ_Ψ/M/1 systems ($\Psi \neq 1$) as

$$\lim_{k\to\infty} \frac{\hat{\rho}_{k+1}\{D\}}{\hat{\rho}_k\{D\}} = \frac{1}{\rho} \left[\frac{\rho + \rho\Psi}{1 + \rho\Psi}\right]^{\Psi+1}, \qquad (52)$$

which is again close to 1 for all ρ when Ψ is small. This decay factor is equal to $e^{(\rho-1)/\rho}/\rho$ for D/M/1 systems.

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Table 3: Autocorrelations for GI/M/1 systems with load $\rho = 0.8$.

y	D/M/1	E_8/M/1	E_2/M/1	$\Gamma_{1/2}$/M/1	$\Gamma_{1/8}$/M/1
$\hat{\rho}_1\{D\}$	-0.08463	-0.06625	-0.02831	0.02492	0.03392
$\hat{\rho}_5\{D\}$	-0.02664	-0.02155	-0.00995	0.01057	0.01938
$\hat{\rho}_{10}\{D\}$	-0.01338	-0.01114	-0.00546	0.00645	0.01345
$\hat{\rho}_{100}\{D\}$	-0.00011	-0.00013	-0.00013	0.00044	0.00214
$\sum_{k=1}^{100} \hat{\rho}_k\{D\}$	-0.49708	-0.41512	-0.20772	0.26583	0.63347
$\sum_{k=1}^{\infty} \hat{\rho}_k\{D\}$	-0.50000	-0.41899	-0.21262	0.29448	0.92647

For the case of gamma distributed service times with shape parameter Ψ the autocorrelation can be further specified as (Jenkins [11] for integer Ψ):

$$\hat{\rho}_1\{D\} = \frac{(1-\rho)\Psi}{\Psi + \rho^2(1-\Psi)} \left[\left(\frac{\Psi}{\Psi+\rho} \right)^{\Psi} - 1 + \frac{\Psi\rho}{\Psi+\rho} \right].$$

Examples: M/E_2/1 and M/$\Gamma_{1/2}$/1:

$$\hat{\rho}_1\{D\} = \frac{2(1-\rho)\rho^2}{(2-\rho^2)(2+\rho)^2}, \qquad \hat{\rho}_1\{D\} = -\frac{(1-\rho)[1+\rho-\sqrt{1+2\rho}]}{(1+2\rho)(1+\rho^2)}.$$

E_2/M/1:

$$\hat{\rho}_1\{D\} = -\frac{\rho^2[1-4\rho^2+\sqrt{1+8\rho}]}{(1+2\rho+2\rho^2)(1+2\rho)^2}.$$

M/D/1:

$$\hat{\rho}_1\{D\} = (1-\rho)\,\frac{e^{-\rho}-1+\rho}{1-\rho^2} = \frac{e^{-\rho}-1+\rho}{1+\rho}.$$

$$\hat{\rho}_2\{D\} = (1-\rho)\,\frac{(1+\rho)e^{-2\rho}-1+\rho}{1-\rho^2} = e^{-2\rho} - \frac{1-\rho}{1+\rho}.$$

$$\hat{\rho}_3\{D\} = (1-\rho)\,\frac{(1+2\rho+\frac{3}{2}\rho^2)e^{-3\rho}-1+\rho}{1-\rho^2} = \frac{1+2\rho+\frac{3}{2}\rho^2}{1+\rho}\,e^{-3\rho} - \frac{1-\rho}{1+\rho}.$$

$$\hat{\rho}_4\{D\} = (1-\rho)\,\frac{(1+3\rho+4\rho^2+\frac{8}{3}\rho^3)e^{-4\rho}-1+\rho}{1-\rho^2} = \frac{1+3\rho+4\rho^2+\frac{8}{3}\rho^3}{1+\rho}\,e^{-4\rho} - \frac{1-\rho}{1+\rho}.$$

Optimization of $\hat{\rho}_1\{D\}$ for M/C_2/1: three variables on $(0,1)$: ρ, $1/\mu_1$ and $\mu_1\mu_2/(\mu_1+\mu_2)$.

$$\Delta_k^{(2)} = \frac{\lambda^2}{2\pi i(k-1)} \oint_{|w|=r} \left[\frac{\beta(\lambda[1-w])\beta''(\lambda[1-w]) - [\beta'(\lambda[1-w])]^2}{\beta^2(\lambda[1-w])} \right] \beta^{k-1}(\lambda[1-w]) \frac{dw}{w^{k-1}}.$$