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Tullock's rent-seeking contest with a minimum expenditure requirement

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Abstract. We consider a rent-seeking contest of the kind introduced by Tullock (1980) in which two players compete for a monopoly rent. We extend the contest by requiring that if a player puts forward an effort, his expenditures must be larger than or equal to some minimum level. We show that, depending on the model parameters, the number of Nash equilibria of the extended model can be zero, one, two or four. Furthermore, it turns out that the extent of rent dissipation in a Nash equilibrium of the extended model can be larger than, equal to, or smaller than the extent of rent dissipation in the unique Nash equilibrium of the original model.

1. Introduction

Tullock (1980) has introduced a seminal game-theoretic model of a contest in which two players compete for a monopoly rent. In the basic game, the probability that a player wins the rent is given by the ratio of the expenditures of the player himself and the total expenditures of both players. Tullock focuses on the case in which both players attach the same valuation to the rent. Hillman and Riley (1989) have extended Tullock's basic model by allowing that the players might value the rent differently (see also Ellingsen, 1991; and Leininger, 1993). They show that the model thus obtained has a unique Nash equilibrium in which both players have positive expenditures. Furthermore, they demonstrate that the size of the total expenditures of both players in this Nash equilibrium is equal to one half times the harmonic mean of the two players' individual valuations of the monopoly rent. These total expenditures can be considered as a measure of the extent of rent dissipation in the contest.

In this paper we extend the model of Tullock by introducing a minimum expenditure requirement, i.e., we assume that if a player decides to put forward an effort, then his expenditures must be larger than or equal to some given minimum level. Doing so, we allow that the two players attach a different value to the monopoly rent. We demonstrate that, depending on the magnitudes of the players' valuations of the monopoly rent and the size of the minimum

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expenditure level, the extended model either has no Nash equilibrium at all, or has one, two or four Nash equilibria. This contrasts with the unique Nash equilibrium of the model without a minimum expenditure requirement. Moreover, we point out that the extent of rent dissipation in a Nash equilibrium of the extended model can be larger than, equal to, or smaller than the extent of rent dissipation in the unique Nash equilibrium of the original model.

The device of a minimum expenditure requirement can be relevant in practice. For instance, in the Netherlands political parties participating in an election contest for the parliament or the municipal council must pay to the government a (sizable) fixed and equal, legally determined, amount of money (entry fee). A minimum expenditure requirement has been discussed before in the rent-seeking literature by Hillman and Samet (1987). However, the contest analysed by these authors is not of the kind of Tullock (1980), but rather one in which the player with the highest expenditures wins the monopoly rent for sure. Yang (1993) employs a minimum expenditure requirement in a contest which is inspired by the model of Tullock. However, the model of Yang is a multi-period one in which the two players decide alternatingly in a sequential way about their expenditures. We investigate the impact of a minimum expenditure requirement within the original single-period model of Tullock (1980) and Hillman and Riley (1989), in which the players decide simultaneously.

The paper is further organized as follows. In Section 2 we describe the contest with a minimum expenditure requirement and derive the reaction curves of both players. In Section 3 we characterize the Nash equilibria for all feasible configurations of the model parameters. We conclude in Section 4.

2. The model and the reaction curves

Following Tullock (1980), we consider a contest in which two players compete for a monopoly rent. The probability that player i (i = 1,2) wins the contest is given by $p_i(x_1,x_2)$, where x_i denotes the expenditures of player i. Player i has two options regarding the size of his expenditures. First, he can choose $x_i = 0$. Second, he can choose some $x_i > 0$. In that case it is required that x_i is greater than or equal to some minimum level $x_0 > 0$. The size of x_0 is given to both players. The probability that player i wins the contest is given by

$$p_i(x_1,x_2) = \begin{cases} \frac{x_i}{x_1+x_2} & if\ \ x_i \geqq x_0\ \ x_j \geqq x_0 \\ 1 & if\ \ x_i \geqq x_0,\ x_j = 0 \\ 0 & if\ \ x_i = 0,\ \ x_j \geqq x_0 \\ \frac{1}{2} & if\ \ x_i = 0,\ \ x_j = 0 \end{cases} \tag{1}$$

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with $j \neq i$. Notice that $p_1(x_1,x_2)$ and $p_2(x_1,x_2)$ are defined on the set $D = \{(x_1, x_2)|x_1 \geqq x_0\ x_1 = 0 \text{ or } x_2 \geq x_0 \text{ or } x_2 = 0\}$; cf. Baye, Kovenock, and de Vries (1994).[1]

If a player wins the contest, he obtains the monopoly rent. Otherwise, he obtains nothing. We assume that player i ($i = 1,2$) attaches a valuation of $v_i > 0$ to the monopoly rent, and suppose that $v_1 \leqq v_2$. The expected payoff of player i is given by

$$\pi_i(x_1, x_2) = v_i p_i(x_1, x_2) - x_i \tag{2}$$

with $(x_1,x_2) \in D$.

Now, consider the problem faced by player i. We assume that this player chooses x_i in order to maximize his own expected payoff given the expenditures, x_j ($j \neq i$), of his rival. The reaction curve $f_i(x_j)$ gives the optimal choice of player i for all feasible values of x_j. It turns out that we have to distinguish four different cases with respect to $f_i(x_j)$ (see the Appendix for the derivation). In the case $v_i < 2x_0$ we have

$$f_i(x_i) = \begin{cases} 0\ if\ \ x_j = 0 \\ 0\ if\ \ x_j \geqq x_0 \end{cases}. \tag{3}$$

In the case $v_i = 2x_0$ we have

$$f_i(x_j) = \begin{cases} 0\ or\ x_0\ \ if\ \ x_j = 0 \\ 0\ or\ x_0\ \ if\ \ x_j = x_0 \\ 0 \qquad\quad\ \ if\ \ x_j > x_0 \end{cases} \tag{4}$$

whereas in the case $2x_0 < v_i \leqq 4x_0$ we have

$$f_i(x_j) = \begin{cases} x_0 & if\ \ x_j = 0 \\ x_0 & if\ \ x_0 \leqq x_j < v_i - x_0 \\ 0\ or\ x_0 & if\ \ x_j = v_i - x_0 \\ 0 & if\ \ x_j > v_i - x_0 \end{cases} \tag{5}$$

and, finally, in the case $4x_0 < v_i$ we have

$$f_i(x_j) = \begin{cases} x_0 & if\ \ x_j = 0 \\ -x_j + \sqrt{x_j v_i} & if\ \ x_0 \leqq x_j < x_j^* \\ x_0 & if\ \ x_j^* \leqq x_j < v_i - x_0 \\ 0\ or\ x_0 & if\ \ x_j = v_i - x_0 \\ 0 & if\ \ x_j > v_i - x_0 \end{cases} \tag{6}$$

where

$$x_j^* = \frac{(v_i - 2x_0) + \sqrt{v_i(v_i - 4x_0)}}{2}. \tag{7}$$

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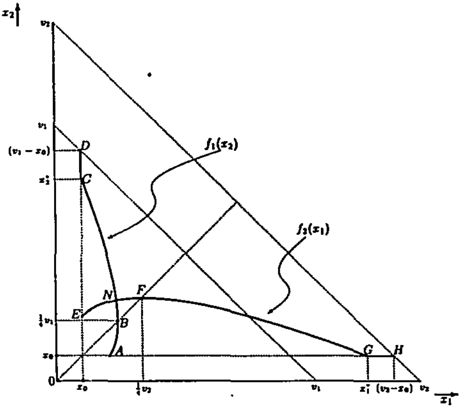

Figure 1.

Note that in case $4x_0 < v_i$, (i) x_j^* is a real number satisfying $x_0 < x_j^* < v_i - x_0$, and (ii) $f_i(x_j) > x_0$ if $x_0 \leqq x_j < x_j^*$.

Figure 1 illustrates the reaction curve of both players for a situation with $4x_0 < v_i < v_2$ (in particular, we have used the values $v_1 = 10$, $v_2 = 14$ and $x_0 = 1$). The reaction curve of player 1 is given by the point $(x_0,0)$, the line ABCD, and the vertical line segment along the x_2-axis with $x_2 \geqq v_1$-x_0. The reaction curve of player 2 is given by the point $(0,x_0)$, the line EFGH, and the horizontal line segment along the x_1-axis with $x_1 \geqq v_2 - x_0$. The line through OBF is given by $x_1 = x_2$. We observe with respect to the reaction curve of player 1 that the curved part ABC corresponds to the part where $f_1(x_2) = -x_2 + \sqrt{x_2 v_1}$. Point B, with coordinates $(\frac{1}{4}v_1, \frac{1}{4}v_1)$, is the point with the maximum x_1-coordinate. Similarly, the part EFG pertains to the part where $f_2(x_1) = -x_1 + \sqrt{x_1 v_2}$. Point F, with coordinates $(\frac{1}{4}v_2, \frac{1}{4}v_2)$, is the point with the maximum x_2-coordinate.

3. The Nash equilibria

In this section we will present two propositions (one for the case with $v_1 < v_2$ and one for the case with $v_1 = v_2$) that characterize the Nash equilibria of our game: i.e., the combinations $(\hat{x}_1, \hat{x}_2) \in D$ such that $\hat{x}_1 = f_1(\hat{x}_2 = f_2(\hat{x}_1)$ (in Figure 1 the Nash equilibrium is given by point N). Before presenting the propositions, it is useful to examine briefly as a benchmark case the situation in which $x_0 = 0$. It then follows from (7) that $x_1^* = v_2$ and $x_2^* = v_1$. As a result, the reaction curve of player i reduces to

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$$f_i(x_j) = \begin{cases} -x_j + \sqrt{x_j v_i} & if \;\; 0 \leqq x_j < v_i \\ 0 & if \;\; x_j \geqq v_i \end{cases} \tag{8}$$

Hillman and Riley (1989) (see also Ellingsen, 1991) have shown that the unique Nash equilibrium in this case is given by

$$\begin{cases} \hat{x}_1^0 = s(1 - \frac{s}{v_1}) \\ \hat{x}_2^0 = s(1 - \frac{s}{v_2}) \end{cases} \tag{9}$$

with $s = 1/\left(\frac{1}{v_1} + \frac{1}{v_2}\right)$. We have attached an index '0' to the Nash equilibrium of the benchmark case. Observe that (i) $\hat{x}_1^0 = \frac{s^2}{v_2} > 0$ and $\hat{x}_2^0 = \frac{s^2}{v_1} > 0$, (ii) $\hat{x}_1^0 + \hat{x}_2^0 = s = \frac{v_1 v_2}{v_1 + v_2}$, (iii) if $v_1 < v_2$, than $\frac{1}{2}v_1 < s < v_1$ and $s < \frac{1}{2}v_2$, and (iv) if $v_1 = v_2 \equiv v$, then $\hat{x}_1^0 = \hat{x}_2^0 = \frac{1}{4}v$ and $s = \frac{1}{2}v < v$. We see from (i) that both players do expend in the Nash equilibrium, and from (ii), (iii) and (iv), respectively, that the total expenditures are given by s and always less than the valuation v_1.

Returning to the case with $x_0 > 0$, we first examine the situation with $v_1 < v_2$. Doing so, recall that we have distinguished four different cases for the reaction curve of player 1 – i.e., $v_1 < 2x_0, v_1 = 2x_0, 2x_0 < v_1 \leqq 4x_0$ and $4x_0 < v_1$ – as well as four different cases for player 2 – i.e., $v_2 < 2x_0, v_2 = 2x_0, 2x_0 < v_2 \leqq 4x_0$ *and* $4x_0 < v_2$. If we combine all cases for the two players together, in principle we expect to obtain sixteen different cases. However, it turns out that seven of these cases cannot occur. For instance, the combination of $v_1 = 2x_0$ with $v_2 < 2x_0$ is not feasible, since we assume now that $v_1 < v_2$. As a result, we actually end up with the following nine different possible cases: case A: $v_1 < v_2 < 2x_0$; case B: $v_1 < 2x_0 = v_2$; case C: $v_1 < 2x_0 < v_2 \leqq 4x_0$; case D: $v_1 < 2x_0 < 4x_0 < v_2$; case E: $v_1 = 2x_0 < v_2 \leqq 4x_0$; case F: $v_1 = 2x_0 < 4x_0 < v_2$; case G: $2x_0 < v_1 < v_2 \leqq 4x_0$; case H: $2x_0 < v_1 \leqq 4x_0 < v_2$; and case I: $4x_0 < v_1 < v_2$. We now present our first proposition.[2]

Proposition 1. Consider the model defined by (1) and (2). Suppose that $0 < x_0 < v_1 < v_2$, so that s = $\frac{v_1, v_2}{v_1 + v_2}$. Depending on the size of the parameters v_1, v_2 and x_0, the number of Nash equilibria of the model is zero, one, or two. In particular, for the cases A to I we have the following results: In order to discuss this proposition, recall that in the benchmark case with $x_0 = 0$, there is a unique Nash equilibrium given by (9). Proposition 1 points out that the introduction of a minimum expenditure requirement may change the outcome completely. Depending on the values of v_1, v_2 and x_0, the model can have one or two Nash equilibria, or no Nash equilibrium at all.

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We can use Proposition 1 to investigate what happens if we keep the values of v_1 and v_2 fixed and take increasing values of x_0 in the (feasible) region $0 < x_0 < v_1$. In particular, we can examine the development in the Nash equilibria and, in order to assess the extent of rent dissipation, the development of the size of the total expenditures in these equilibria compared with the size of the total expenditures s in the benchmark case with $x_0 = 0$. Doing so, we have to distinguish three different situations, i.e., $2v_1 < v_2, 2v_1 > v_2$ *and* $2v_1 = v_2$.

First, take the situation with $2v_1 < v_2$. If we consider now increasing values of x_0 from 0 to v_1, we pass in turn through the cases, I, H, F, D, C, B and A. We note that, depending on the numerical values of v_1, v_2 and x_0, it might happen that from a certain point onwards the last cases in this sequence cannot occur (e.g., case A, or case B and A). A typical example of the present situation is given by $v_1 = 1$ and $v_2 = 5$, which implies that $s = \frac{5}{6}$. Using Proposition 1 we can derive the following results. In this example the cases C, B and A cannot occur. There is either one Nash equilibrium or no one at all. Further, depending on the value of x_0, the size of $\hat{x}_1 + \hat{x}_2$ can be larger than, equal to, or smaller than $s = \frac{5}{6}$. Stated otherwise, the extent of rent dissipation can be larger than, equal to, or smaller than the extent of rent dissipation in the benchmark case with $x_0 = 0$.

Next, consider briefly the situations $2v_1 > v_2$ *and* $2v_1 = v_2$. If we have $2v_1 > v_2$, keep v_1 and v_2 fixed and take increasing values of x_0 from 0 to v_1, we pass through the cases I, H, G, E, C, B and A (as in the situation with $2v_1 < v_2$, possibly only up to a certain point in this sequence). Analogously, in the situation $2v_1 = v_2$, we pass through the cases I, H, E, C, B and A.[3] We remark that if $2v_1 > v_2$ or $2v_1 = v_2$, then the subcases Hii and Iβii cannot occur. This can be seen by noting that $2x_0 < v_1$ and $2v_1 \geqq v_2$ imply that $\sqrt{x_0 v_2} \leqq v_1$.[4]

Proceeding, we turn to the case in which both players have the same valuation of the monopoly rent. Using the reaction curves derived in Section 2 we can easily prove Proposition 2 for this case.

Proposition 2. Consider the model defined by (1) and (2). Suppose that $0 < x_0 < v_1 = v_2 = v$. Depending on the size of the parameters v and x_0, the number of Nash equilibria of the model is one or four. In particular, we have the following different cases: Recall that in the benchmark case with $x_0 = 0$, there is a unique Nash equilibrium with $x_1 = x_2 = \frac{1}{4}v$ *and* $s = \frac{1}{2}v$. We see from Proposition 2 that the presence of a minimum expenditure requirement implies that either there is one Nash equilibrium or there are four Nash equilibria. One can verify that in the model with $x_0 > 0$ the extent of rent dissipation in a Nash equilibrium can be larger than, equal to, or smaller than the extent of rent dissipation in the Nash equilibrium of the model with $x_0 =$

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0. The size of $\hat{x}_1 + \hat{x}_2$ is largest in the Nash equilibrium (x_0, x_0) associated with the case $v = 2x_0$. In that case $\hat{x}_1 + \hat{x}_2 = v$, i.e., there is full rent dissipation.

4. Conclusion

This paper has extended Tullock's game-theoretic model of a contest for a monopoly rent, by incorporating a minimum expenditure requirement. We have shown that such a requirement may have different kinds of effects on the number of Nash equilibria and the extent of rent dissipation. Thus, one must be careful in this respect if such a minimum expenditure requirement is relevant. The precise effects depend on the value of the minimum expenditures as well as on the sizes of the players' valuations of the monopoly rent.

Notes

1. Alternatively, one might also suppose that $p_1(0,0) = p_2(0,0) = 0$, i.e., if both players decide to expend nothing, then the contest stops and nobody wins the rent. In that case we can easily modify the results of this paper. In particular, in the Propositions 1 and 2 of Section 3, the terms '$2x_0$' have to be replaced by 'x_0': e.g., the relevant condition of case C in Proposition 1 becomes $v_1 < x_0 < v_2 \leqq 4x_0$.
2. The proof of Proposition 1 is available from the authors upon request.
3. Typical examples of these two situations are given by $v_1 = 1$ and $v_2 = 1.9$, and $v_1 = 1$ and $v_2 = 2$, respectively. Analysis of these examples will be left to the reader.
4. Remark that if $2v_1 \geqq v_2$, then there exist values of x_0 falling in case C such that the size of $\hat{x}_1 + \hat{x}_2 (= x_0)$ is respectively larger than, equal to, or smaller than s. This can be understood by noting that there always holds $\frac{1}{2}v_1 < s < \frac{1}{2}v_2$, *or* $v_1 < 2s < v_2$. In combination with $2v_1 \geqq v_2$, we obtain from this that $v_1 < 2s < v_2 < 4s$. If we take $x_0 = 2$, we have $v_1 < 2x_0 < v_2 < 4x_0$, i.e., case C holds. Clearly, there also exists $x_0 < s$ *and* $x_0 > s$ satisfying case C.

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Appendix: The derivation of the reaction curves

In this appendix we derive the reaction curve $f_i(x_j)$ of player i. It is convenient to suppose first that $x_j \geqq x_0$ and to require that x_i must satisfy $x_i \geqq x_0$. Doing so, the optimal choice of player i can be determined by solving the Kuhn-Tucker conditions

$$\begin{cases} \frac{v_i x_j}{(x_i+x_j)^2} - 1 + \mu_i = 0 \\ \mu_i(x_i - x_0) = 0 \\ \mu_i \geqq 0,\ x_i - x_0 \geqq 0 \end{cases}$$

where μ_i is the Kuhn-Tucker multiplier associated with the constraint $x_i \geqq x_0$ (see Takayama, 1985). From these conditions it can be derived that the optimal choice of player i is given by the reaction curve

$$\tilde{f}_i(x_j) = \begin{cases} -x_j + \sqrt{x_j v_i} & if\ \ -x_j + \sqrt{x_j v_i} > x_0 \\ x_0 & if\ \ -x_j + \sqrt{x_j v_i} \leqq x_0 \end{cases} \tag{A.1}$$

We have attached a tilde to the reaction curve to indicate that we have limited the attention here to the case $x_j \geqq x_0$ and $x_i \geqq 0$. It can be verified that there exists at least one $x_j \geqq x_0$ satisfying the first subcase of (A.1) if and only if $4x_0 < v_i$.

In case $4x_0 < v_i$, we can reformulate (A.1) as

$$\tilde{f}_i(x_j) = \begin{cases} -x_j + \sqrt{x_j v_i} & if\ \ x_0 \leqq x_j < x_j^* \\ x_0 & if\ \ x_j \geqq x_j^* \end{cases} \tag{A.2}$$

where $x_j^* = ((v_i - 2x_0) + \sqrt{v_i(v_i - 4x_0)})/2$. In case $4x_0 \geqq v_i$, (A.1) reduces to $\tilde{f}_i(x_j) = x_0$ for all $x_j \geqq x_0$.

Next, let us introduce again the possibility that player i can decide to choose an expenditure level equal to zero. Take for instance player 1, and consider in turn case (a) in which his rival's choice satisfies $x_2 \geqq x_0$ and case (b) in which $x_2 = 0$.

In case (a), with $x_2 \geqq x_0$, we have to compare two alternatives for player 1: he can choose either $x_1 = 0$ or $x_1 = \tilde{f}_1(x_2)$. Considering the possible choice $x_1 = 0$, we notice that $\pi_1(0, x_2) = 0$ because $x_2 \geqq x_0$. Considering the alternative choice $x_1 = \tilde{f}_1(x_2)$, we derive from equation (2) of Section 2 that $\pi_1(\tilde{f}_1(x_2), x_2) > 0$ if $\tilde{f}_1(x_2)+x_2 < v_1$, whereas $\pi_1(\tilde{f}_1(x_2), x_2) = 0$ (< 0) if $\tilde{f}_1(x_2)+x_2 = v_1$ ($> v_1$). Now suppose first that $4x_0 < v_1$ and the first subcase of (A.2) holds (i.e., $x_0 \leqq x_2 < x_2^*$). As a result, $\tilde{f}_1(x_2) = -x_2 + \sqrt{x_2 v_1}$, which means that $\tilde{f}_1(x_2)+x_2 = \sqrt{x_2 v_1}$. Because $x_2 < x_2^*$ and $x_2^* < 1$, it follows that $\pi_1(\tilde{f}_1(x_2), x_2) > 0$. Recalling that $\pi_1(0, x_2) = 0$, we conclude from a comparison of the payoffs that player 1 will prefer the choice $x_1 = \tilde{f}_1(x_2)$ now.

Second, suppose that either $4x_0 < v_1$ and the second subcase of (A.2) holds (i.e., $x_2 \geqq x_2^*$) or $4x_0 \geqq v_1$. Considering the possible choice $x_1 = \tilde{f}_1(x_2)$, we see that now $\tilde{f}_1(x_2)$

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$= x_0$, and the resulting expected payoff $\pi_1(x_0, x_2)$ is positive if $x_0 + x_2 < v_1$, equal to zero if $x_0 + x_2 = v_1$, and negative if $x_0 + x_2 > v_1$. Recalling again that $\pi_1(0, x_2) = 0$, we conclude that player 1 will choose $x_1 = \tilde{f}_1(x_2) = x_0$ in case $x_2 < v_1 - x_0$, $x_1 = 0$ in case $x_2 > v_1 - x_0$, and either $x_1 = 0$ or $x_1 = \tilde{f}_1(x_2) = x_0$ in case $x_2 = v_1 - x_0$. (In the latter case he is indifferent between the two choices.)

Next, consider case (b), i.e., the case with $x_2 = 0$. We distinguish the following three alternative choices for player 1: $x_1 = 0$, $x_1 = x_0$, or $x_1 > x_0$. If he chooses $x_1 = 0$, then his (certain) payoff equals $\pi_1(0, 0) = \frac{1}{2}v_1$, whereas his (certain) payoff is $\pi_1(x_0, 0) = v_1 - x_0$ if he chooses $x_1 = x_0$. Finally, player 1 will never choose $x_1 > x_0$, because $\pi_1(x_1, 0) < \pi_1(x_0, 0)$ for all $x_1 > x_0$. From a comparison of the payoffs, we conclude that player 1 will choose $x_1 = 0$ in case $v_1 < 2x_0$, $x_1 = x_0$ in case $v_1 > 2x_0$, and either $x_1 = 0$ or $x_1 = x_0$ in case $v_1 = 2x_0$. (In the latter case he is indifferent between the two choices.)

Concluding, we see that we have to distinguish four different cases with respect to the reaction curve $f_1(x_2)$ of player 1, i.e., $v_1 < 2x_0$, $v_1 = 2x_0$, $2x_0 < v_1 \leqq 4x_0$ and $4x_0 < v_1$. We further remark that we can derive in a completely similar way the reaction curve $f_2(x_1)$ of player 2. Combining results, we obtain the reaction curve $f_i(x_j)$ of player i as given in (3), (4), (5) and (6) in Section 2.

Case	Conditions on v_1, v_2 and x_0	Nash equilibria $(\hat{x}_1, \hat{x}_2)$
A	$v_1 < v_2 < 2x_o$	(0,0)
B	$v_1 < 2x_0 = v_2$	(0,0) and $(0, x_0)$
C	$v_1 < 2x_0 < v_2 \leqq 4x_0$	$(0, x_0)$
D	$v_1 < 2x_0 < 4x_0 < v_2$	$(0, x_0)$
E	$v_1 = 2x_0 < v_2 \leqq 4x_0$	$(0, x_0)$ and (x_0, x_0)
F	$v_1 = 2x_0 < 4x_0 < v_2$	$(0, x_0)$
G	$2x_0 < v_i < v_2 \leqq 4x_0$	(x_0, x_0)
H	$2x_0 < v_1 \leqq 4x_0 < v_2$ *and*	
• subcase Hi	$\sqrt{x_0 v_2} \leqq v_1$	$(x_0, -x_0 + \sqrt{x_0 v_2})$
• subcase Hii	$\sqrt{x_0 v_2} > v_1$	no one
I	$4x_0 < v_1 < v_2$ *and*	
• subcase Iα	$\frac{s^2}{s - x_0} < v_1 < v_2$	$(s(1 - \frac{s}{v_1}), s(1 - \frac{s}{v_2}))$
• subcase Iβi	$v_1 \leqq \frac{s^2}{s - x_0} < v_2$ *and* $\sqrt{x_0 v_2} \leqq v_1$	$(x_0, -x_0 + \sqrt{x_0 v_2})$
• subcase Iβii	$v_1 \leqq \frac{s^2}{s - x_0} < v_2$ *and* $\sqrt{x_0 v_2} > v_1$	no one

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Case	Condition on x_0	Nash equilibria $(\hat{x}_1, \hat{x}_2)$
subcase Iα	$0 < x_0 < \frac{5}{36}$	$(\frac{5}{36}, \frac{25}{36})$
subcase Iβi	$\frac{5}{36} \leqq x_0 < \frac{1}{5}$	$(x_0, -x_0 + \sqrt{5x_0})$
subcase Iβii	$\frac{1}{5} \leqq x_0 < \frac{1}{4}$	no one
subcase Hii	$\frac{1}{4} \leqq x_0 < \frac{1}{2}$	no one
F	$x_0 = \frac{1}{2}$	$0, \frac{1}{2}$
D	$\frac{1}{2} < x_0 < 1$	$(0, x_0)$

Conditions on v and x_0	Nash equilibria $(\hat{x}_1, \hat{x}_2)$
$v < 2x_0$	(0,0)
$v = 2x_0$	(0,0), $(0, x_0)$, $(x_0, 0)$ and (x_0, x_0)
$2x_0 < v \leqq 4x_0$	(x_0, x_0)
$4x_0 < v$	$(\frac{1}{4}v, \frac{1}{4}v)$

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