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Source: *Operations Research*, Vol. 40, No. 1, (Jan. - Feb., 1992), pp. 157-167

Published by: INFORMS

Stable URL: http://www.jstor.org/stable/171192

Accessed: 25/04/2008 08:08

THE POWER-SERIES ALGORITHM APPLIED TO THE SHORTEST-QUEUE MODEL

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(Received March 1989; revision received February 1990; accepted May 1990)

An iterative numerical technique for the evaluation of queue length distributions is applied to multiserver systems with queues in parallel in which customers join (one of) the shortest queues upon arrival. The technique is based on power-series expansions of the state probabilities as functions of the load of the system. The convergence of the series is accelerated by applying a modified form of the epsilon algorithm. The shortest-queue model lends itself particularly well to a numerical analysis by means of the power-series algorithm due to a specific property of this model. Numerical values for the mean and the standard deviation of the total number of customers and the waiting times in stationary symmetrical systems are obtained for practically all values of the load for systems with up to ten queues and for a load not exceeding 75% for systems with up to 30 queues. Data are also presented for systems with four queues and unequal service rates.

The power-series algorithm is a numerical procedure for computing state probabilities and moments of joint queue-length distributions for systems with more than one waiting line, which can be modeled by multidimensional (quasi-) birth-and-death processes (Blanc 1987b, 1990, Hooghiemstra, Keane and Van de Ree 1988). The algorithm is based on power-series expansions of state probabilities and moments as a function of a parameter of the system, usually the traffic intensity ρ of the system. With the aid of limiting properties of the state probabilities in light traffic, a recursive scheme is obtained for calculating the coefficients of their power-series expansions. It appears, however, that the state probabilities of many queueing models possess singularities inside the unit circle in the complex ρ-plane (it is assumed that the traffic intensity is defined in such a way that the system is stable if and only if $0 < \rho < 1$). In all cases considered, a bilinear transformation of the traffic intensity is suitable for obtaining convergence of the power-series expansions of the state probabilities over the whole range of values of the traffic intensity for which the system is stable, i.e., the interval (0, 1). To accelerate the convergence of the power-series expansions of the state probabilities and the moments of the queue-length distributions for systems in heavy traffic (ρ close to 1) extrapolation methods such as the ϵ-algorithm (Wynn 1966, Brezinski 1977, Blanc 1990) can be applied to partial sums of the series.

Still, the power-series algorithm is an experimental method. It is, in general, not possible to give upper bounds for errors. But experience has taught that the algorithm is more powerful than algorithms based on truncation of the state space, and that it provides more accurate results for moderately sized models than simulations, in less computing time (once the coefficients of the power-series expansions have been computed, queueing characteristics can be calculated for an arbitrary number of values of the traffic intensity in a relatively negligible time).

The power-series algorithm applies in theory to systems with an arbitrary number of queues, but in practice the size of the models is limited by the amount of memory space, which is available in a particular computer for storing the coefficients of the power series. Procedures for the economic use of memory space during the execution of the power-series algorithm have been developed in Blanc (1990). With these procedures it is possible to obtain numerical data for systems with, typically, up to 4–6 queues, depending on the structure of the model, the load of the system, the desired accuracy, and the available memory space.

The power-series algorithm will be applied in this paper to the well known shortest-queue problem. In this model, there are several service units in parallel, each with its own queue in front, and there is one arrival stream of customers who join one of the shortest queues upon arrival and remain in the queue of their choice until they have been served. This model has been investigated by several authors, by means of analytical as well as numerical methods, but usually

Subject classifications: Queues, algorithms: power-series expansions. Queues, multichannel: join the shortest queue.
Area of review: Stochastic Processes and Their Applications.

0030-364X/92/4001-0157 $01.25

only in the case of two service units (e.g., Kingman 1961, Flatto and McKean 1977, Conolly 1984, Gertsbakh 1984, Halfin 1985, Rao and Posner 1987). The shortest-queue model lends itself particularly well to a numerical analysis with the aid of the power-series algorithm, especially when all service units are identical, because many more coefficients of the power-series expansions of the state probabilities vanish in this model than in other models. This is due to the property that a certain queue can only contain n customers if all queues contain at least $n-1$ customers at a previous instant during the current busy period ($n = 2, 3, \ldots$). This paper presents numerical values for the mean and the standard deviation of the number of customers in the system and of the waiting times for symmetrical shortest-queue models with up to ten queues (for $\rho \leq 0.75$ even up to 30 queues). Also, we derive light traffic asymptotes of these quantities from the recursive scheme of the algorithm, and we give estimations of their heavy traffic asymptotes based on numerical data generated by the algorithm. Furthermore, we consider other quantities of interest for these models, such as the amount of work in the system, the difference between the longest and the shortest queue, and the number of servers which are idle but could work if they were able to serve customers who are waiting in other queues. Finally, a system with four queues and unequal service rates is discussed. The primary focus of this paper is a study of the shortest model. We will not discuss details of the implementation of the algorithm and the execution of the computations. The reader is referred to Blanc (1990) for a discussion of these issues in a more general context.

The organization of this paper is as follows. Section 1 contains definitions for the shortest-queue model, the balance equations for the state probabilities in this model, and some general relations between various queueing characteristics. In Section 2 the power-series algorithm is outlined; also, we show that the algorithm requires less computation time and storage capacity for the shortest-queue model than for other models of comparable complexity. Section 3 contains asymptotic expansions in light traffic for several performance measures. Numerical results are presented and discussed in Section 4.

1. GENERAL RELATIONS FOR SHORTEST-QUEUE SYSTEMS

1.1. Description of the Model

The system consists of s servers in parallel. Customers arrive to the system according to a Poisson process with rate λ. They require service from (any) one of the servers, and they form separate queues in front of each server. We will assume that the amount of service that is required by a customer is negative exponentially distributed with mean $1/\mu$. Server j serves the customers who have joined its queue in order of arrival with rate r_j, $j = 1, \ldots, s$. Without loss of generality it may be assumed that the total service capacity of the system is 1, i.e.,

$$\sum_{j=1}^{s} r_j = 1. \tag{1}$$

The foregoing implies that the holding times of customers who join queue j are negative exponentially distributed with mean $1/(r_j\mu)$, $j = 1, \ldots, s$. Arriving customers only observe the lengths of the s queues and have no knowledge about the service rates of the servers. Therefore, they join one of the shortest queues upon arrival. Due to physical circumstances it is not possible for customers to leave one queue and join another queue. Each queue may contain an unbounded number of jobs. When there is not a unique queue that is shorter than all other queues at the instant of an arrival of a new customer, this arriving customer chooses one of the shortest queues according to some probability distribution. This will be described more precisely below.

First, the condition for ergodicity of the system will be considered. The rate of arrivals to the system is λ, and the maximal departure rate of the total system is μ. Hence, the load or traffic intensity ρ of the system is in a natural way defined by $\rho := \lambda/\mu$, and a necessary and sufficient condition for ergodicity of the system is $\rho < 1$. Throughout this paper, it will be assumed that the system is in steady state. Let N_j denote the number of customers in queue j (waiting or being served), $j = 1, \ldots, s$, $\bar{N} := (N_1, \ldots, N_s)$, and let $\bar{n} = (n_1, \ldots, n_s)$ be a vector with nonnegative integer entries. The stationary state probabilities are defined as:

$$p(\rho; \bar{n}) := \Pr\{\bar{N} = \bar{n}; \text{ at load } \rho\} \quad \text{for } 0 \leq \rho < 1. \tag{2}$$

Furthermore, let C be a random variable indicating the number of the queue which an arriving customer joins. Clearly, the variable C is dependent on the vector $\bar{N}$. Because arriving customers join one of the shortest queues, the conditional distribution of C given the vector $\bar{N}$ has the following property:

$$\Pr\{C = j \mid \bar{N} = \bar{n} \text{ with } \exists i\ n_i < n_j\} = 0 \quad \text{for } j = 1, \ldots, s. \tag{3}$$

To simplify the discussion we will restrict the conditional distribution of C given $\bar{N}$. Let $\mathrm{I}\{E\}$ stand for the indicator function of the event E, and ψ_j be a positive weight factor attributed to queue j, $j = 1, \ldots, s$. Then, we assume that

$$\Pr\{C = j \mid \bar{N} = \bar{n} \text{ with } \forall i\ n_i \geq n_j\} = \psi_j \Big/ \left[\sum_{i=1}^{s} \psi_i \mathrm{I}\{n_i = n_j\}\right] \quad \text{for } j = 1, \ldots, s. \tag{4}$$

A relatively large weight factor of a queue means a relatively strong preference of customers to join that queue (provided that it is one of the shortest queues). This preference may be based, for instance, on the physical distance between the entrance of the waiting room and the various service counters.

1.2. Balance Equations

Let $\bar{e}_j$ be the vector with zero entries except an entry of one at the jth position ($j = 1, \ldots, s$). The balance equations for the state probabilities (2) are readily verified to be:

$$\left[\rho + \sum_{j=1}^{s} r_j \mathrm{I}\{n_j > 0\}\right] p(\rho; \bar{n}) = \sum_{j=1}^{s} r_j p(\rho; \bar{n} + \bar{e}_j) + \rho \sum_{j=1}^{s} \mathrm{I}\{n_j > 0\} p(\rho; \bar{n} - \bar{e}_j) \cdot \Pr\{C = j \mid \bar{N} = \bar{n} - \bar{e}_j\}. \tag{5}$$

The following relations, which can be derived from the balance equations (5) by summation over all states $\bar{n}$ with $n_j = m$ for some fixed j and m, and by means of induction, express the balance of flows between the hyperplanes $n_j = m$ and $n_j = m + 1$:

$$\rho \Pr\{C = j, N_j = m\} = r_j \Pr\{N_j = m + 1\} \quad \text{for } j = 1, \ldots, s, \quad m = 0, 1, \ldots. \tag{6}$$

Summation of these relations over m ($m = 0, 1, \ldots$) leads to balance equations for the flows into and out of queue j:

$$\rho \Pr\{C = j\} = r_j \Pr\{N_j > 0\} \quad \text{for } j = 1, \ldots, s. \tag{7}$$

These relations determine the unconditional distribution of C, i.e., the proportion of the customers who join queue j, $j = 1, \ldots, s$, in terms of the queue length distribution. They lead further, with (1), to:

$$\sum_{j=1}^{s} r_j \Pr\{N_j = 0\} = 1 - \rho. \tag{8}$$

This relation is useful for checking the correctness of computations of the state probabilities. Note that (8) and (7) imply that if all servers have the same service rates and all queues have the same weight factors, then:

$$\Pr\{N_j = 0\} = 1 - \rho, \quad \Pr\{C = j\} = 1/s \quad \text{for } j = 1, \ldots, s. \tag{9}$$

Let L denote the total number of customers in the system. The following relations, which can also be deduced from the balance equations (5), express the balance of flows between the hyperplanes $n_1 + \ldots + n_s = m$ and $n_1 + \ldots + n_s = m + 1$:

$$\rho \Pr\{L = m\} = \Pr\{L = m + 1\} - \sum_{j=1}^{s} r_j \Pr\{L = m + 1, N_j = 0\} \quad \text{for } m = 0, 1, \ldots. \tag{10}$$

With induction it follows from (10) that

$$\Pr\{L = m\} = \sum_{i=0}^{m} \rho^i \sum_{j=1}^{s} r_j \Pr\{L = m - i, N_j = 0\} \quad \text{for } m = 0, 1, \ldots. \tag{11}$$

It is readily verified that (11), together with (8), implies

$$\mathrm{E}\{L\} = \frac{1}{1-\rho}\left[\rho + \sum_{j=1}^{s} r_j \mathrm{E}\{L \mid N_j = 0)\} \Pr\{N_j = 0\}\right]. \tag{12}$$

In the case of symmetrical systems (12) reduces with (8) to

$$\mathrm{E}\{L\} = \frac{\rho}{1-\rho} + \mathrm{E}\{L \mid N_j = 0\} \quad \text{for } j = 1, \ldots, s. \tag{13}$$

1.3. Waiting Time, Sojourn Time and Workload

The last part of this section concerns the derivation of general relations between the distributions of W, the waiting time of a customer, H, the holding time of a customer, S, the sojourn time of a customer, Y, the amount of work in the system, and those of C and $\bar{N}$. First, consider the waiting time of a customer. When a customer joins queue j and this queue contains m customers at the instant of arrival, then the waiting time consists of m exponentially distributed phases each with mean length $1/(r_j\mu)$, $j = 1, \ldots, s$, $m = 0, 1, \ldots$. This observation leads to the following expression for the waiting time distribution (with m^* denoting an m-fold convolution):

$$\Pr\{W < t\} = \sum_{j=1}^{s} \sum_{m=0}^{\infty} [1 - e^{-r_j\mu t}]^{m^*} \Pr\{C = j, N_j = m\}. \tag{14}$$

With the aid of relation (6) the distribution of the waiting time can be expressed in terms of the marginal queue-length distributions:

$$\Pr\{W<t\}=\frac{1}{\rho}\sum_{j=1}^{s}\sum_{m=0}^{\infty} r_j[1-e^{-r_j\mu t}]^{m*}\Pr\{N_j=m+1\}. \quad (15)$$

From this relation the probability that a customer does not have to wait and the moments of the waiting time distribution are readily obtained:

$$\Pr\{W=0\}=\Pr\{\exists i\, N_i=0\}=\frac{1}{\rho}\sum_{j=1}^{s} r_j\Pr\{N_j=1\}, \quad (16)$$

$$\mathrm{E}\{W\}=\frac{1}{\rho\mu}\left[\mathrm{E}\{L\}-\sum_{j=1}^{s}\Pr\{N_j>0\}\right], \quad (17)$$

$$\mathrm{E}\{W^2\}=\frac{1}{\rho\mu^2}\sum_{j=1}^{s}\frac{1}{r_j}[\mathrm{E}\{N_j^2\}-\mathrm{E}\{N_j\}]. \quad (18)$$

Similar expressions as for the waiting time distribution, developed in (14) and (15), can be given for the distributions of the holding time and the sojourn time. The first two moments of these distribution are:

$$\mathrm{E}\{H\}=\frac{1}{\rho\mu}\sum_{j=1}^{s}\Pr\{N_j>0\},$$
$$\mathrm{E}\{H^2\}=\frac{2}{\rho\mu^2}\sum_{j=1}^{s}\frac{1}{r_j}\Pr\{N_j>0\}, \quad (19)$$

$$\mathrm{E}\{S\}=\frac{1}{\rho\mu}\mathrm{E}\{L\},$$
$$\mathrm{E}\{S^2\}=\frac{1}{\rho\mu^2}\sum_{j=1}^{s}\frac{1}{r_j}[\mathrm{E}\{N_j^2\}+\mathrm{E}\{N_j\}]. \quad (20)$$

Finally, consider the amount of work Y in the system, i.e., the sum of the holding times of all waiting customers and of the remaining holding times of all customers in service. It is readily verified that the distribution of Y satisfies:

$$\Pr\{Y<t\}=\sum_{n_1=0}^{\infty}\ldots\sum_{n_s=0}^{\infty}p(\rho;\bar{n})\cdot[1-e^{-r_1\mu t}]^{n_1*}*\ldots*[1-e^{-r_s\mu t}]^{n_s*}.$$

From this relation the moments of the distribution of Y are found to be:

$$\mathrm{E}\{Y\}=\sum_{j=1}^{s}\frac{1}{r_j\mu}\mathrm{E}\{N_j\},$$
$$\mathrm{E}\{Y^2\}=\left(\frac{1}{\mu}\right)^2\cdot\left[\sum_{i=1}^{s}\sum_{j=1}^{s}\frac{1}{r_ir_j}\mathrm{E}\{N_iN_j\}+\sum_{j=1}^{s}\left(\frac{1}{r_j}\right)^2\mathrm{E}\{N_j\}\right]. \quad (21)$$

When the system is symmetrical, these expressions reduce to

$$\mathrm{E}\{Y\}=\frac{s}{\mu}\mathrm{E}\{L\},$$
$$\mathrm{E}\{Y^2\}=\left(\frac{s}{\mu}\right)^2[\mathrm{E}\{L^2\}+\mathrm{E}\{L\}]. \quad (22)$$

2. THE POWER-SERIES ALGORITHM

2.1. The Recurrence Relations

The power-series algorithm will be briefly discussed in this section. The reader is referred to Blanc (1987b, 1990), and Hooghiemstra, Keane and Van de Ree (1988) for more details and a motivation of the method. First, we introduce the bilinear mapping of the interval [0, 1] onto itself,

$$\rho=\rho(\theta)=\frac{\theta}{1+G-G\theta}\left(\theta=\frac{(1+G)\rho}{1+G\rho}\right),\quad G\geqslant 0. \quad (23)$$

Here $\theta=(1+G)\rho/(1+G\rho)$ and G is a parameter to be chosen (see the remarks in Section 2.3). Then, we introduce the following power-series expansions:

$$p(\rho(\theta);\bar{n})=\theta^{n_1+\ldots+n_s}\sum_{k=0}^{\infty}\theta^k b(k;\bar{n}). \quad (24)$$

Replace ρ by θ in the balance equations (5) according to (23), and substitute the power series (24) into these equations. Equating the coefficients of corresponding powers of θ in the resulting equations leads to the following iterative scheme for computing the coefficients of the power series (24):

$$\begin{aligned}(1+G)\sum_{j=1}^{s} r_j\mathrm{I}\{n_j>0\}b(k;\bar{n})\\ =\left[-1+G\sum_{j=1}^{s} r_j\mathrm{I}\{n_j>0\}\right]\mathrm{I}\{k>0\}b(k-1;\bar{n})\\ +(1+G)\sum_{j=1}^{s} r_j\mathrm{I}\{k>0\}b(k-1;\bar{n}+\bar{e}_j)\\ -G\sum_{j=1}^{s} r_j\mathrm{I}\{k>1\}b(k-2;\bar{n}+\bar{e}_j)\\ +\sum_{j=1}^{s}\mathrm{I}\{n_j>0\}b(k;\bar{n}-\bar{e}_j)\\ \cdot\Pr\{C=j\mid \bar{N}=\bar{n}-\bar{e}_j\}\quad \text{for } k=0,1,\ldots.\end{aligned} \quad (25)$$

Note that the left-hand side of equation (25) vanishes when $\bar{n}=\bar{0}$. To complete the recursive scheme, the law of total probability is used to determine the coefficients of $p(\rho(\theta);\bar{0})$. Substituting (23) and (24) into the law of total probability gives:

$$b(0; \bar{0}) = 1,$$

$$b(k; \bar{0}) = -\sum_{0<n_1+\ldots+n_s\leq k}\ldots\sum b(k-n_1-\ldots-n_s; \bar{n}),\qquad k = 1, 2, \ldots. \quad (26)$$

There are several ways to compute the coefficients $b(k; \bar{n})$ recursively from (25) and (26) (see Blanc 1987a, 1990).

2.2. Notes on Implementation

A property that can be derived from (25), (26) and (3) by means of induction is:

$$b(k; \bar{n}) = 0,$$

$$\text{if } k + n_1 + \ldots + n_s < s[\max\{n_1, \ldots, n_s\} - 1] + \#\{i; n_i = \max\{n_1, \ldots, n_s\}\}. \quad (27)$$

This property can be explained intuitively by noting that states $\bar{n}$ for which there exist i and j, such that $n_i - n_j > 1$, can only be reached from the empty state by passing through a state in which all coordinates are at least equal to $\max\{n_1, \ldots, n_s\} - 1$ because arriving customers join one of the shortest queues. For example, when $s = 4$, then the state $\bar{n} = (3, 3, 0, 1)$ cannot be reached from the empty state unless there have been at least two customers in each of queues 3 and 4 (otherwise, the length of queues 1 and 2 could not have become 3); this implies that $p(\rho; 3, 3, 0, 1) = 0(\rho^{10})$ as $\rho \downarrow 0$, this order being the minimum number of arrivals which are necessary to reach a state from the empty state; hence, $b(k; 3, 3, 0, 1) = 0$ for $k < 10 - 7 = 3$, using (24).

The number of computations becomes much smaller for the shortest-queue model when compared with other models with the same number of queues (Blanc 1988, 1990). A further reduction of computation time and of the amount of storage capacity, which is needed for the coefficients, can be realized when the model is symmetrical.

Suppose that performance measures for the shortest-queue model need to be calculated up to the Mth power of θ. With the aid of combinatorial arguments we can deduce the number of coefficients $b(k; \bar{n})$ required for this purpose. When the model is symmetrical, then $M + 1 - j$ coefficients are nonvanishing for the number of states, (27):

$$\binom{a+s-k}{a} \quad \text{if } j = as + k \leq M,\qquad a = 0, 1, \ldots, \quad k = 1, \ldots, s.$$

This implies that the total number of coefficients that need to be calculated is equal to,

$$1 + \frac{As}{A+s+1}\binom{A+s+2}{s+1} + c\binom{A+s}{s} + \sum_{k=1}^{c}(c+1-k)\binom{A+s-k}{A}$$

$$\text{for } M = As + c,\quad c = 1, \ldots, s,\quad A = 0, 1, \ldots. \quad (28)$$

When the model is not symmetrical, then $M + 1 - j$ coefficients are nonvanishing for the following number of states, (27):

$$\binom{s}{k}(a+1)^{s-k} \quad \text{if } j = as + k \leq M,\qquad a = 0, 1, \ldots, \quad k = 1, \ldots, s.$$

This implies that the total number of coefficients that need to be calculated is equal to,

$$s\sum_{a=1}^{A} a(a+1)^{s-1} + \sum_{k=0}^{c}(c+1-k)\binom{s}{k}(A+1)^{s-k}$$

$$\text{for } M = As + c,\quad c = 1, \ldots, s,\quad A = 0, 1, \ldots. \quad (29)$$

Table I gives an overview of these numbers of coefficients for some values of M, for the shortest-queue model, as well as for general multidimensional birth-and-death models. It can be seen that the number of coefficients which is required for the shortest-queue model is considerably less than for other models due to property (27). It is remarkable that the required number of coefficients decreases for s beyond some threshold $s_0(M)$ ($s_0(M) \cong \sqrt{M}$) for the symmetrical shortest-queue model. As can be seen from (28), this number is equal to $\frac{1}{2}(M + 1)(M + 2)$, independent of s for $s \geq M$. However, the first s terms of the power-series expansions contain no information on the specific properties of the shortest-queue system. As long as no more than s customers are present in the system, there is no difference between the behavior of this system and that of an M/M/s system. In general, more terms of the power-series expansions are required with an increasing number of queues to reach a certain level of accuracy for a fixed value of the load ρ because the power series converge less quickly. This is related to the fact that the interaction between the queues increases with an increasing number of queues. While storage capacity is the main limiting factor for the application of the algorithm for other models, computation time becomes the main factor for the shortest-queue model.

Table I
The Number of Coefficients $b(k;\, \bar{n})$ Which are Required to Determine Power-Series Expansions up to a Certain Power (M) of θ

	Asymmetrical			Symmetrical		
	General	Shortest Queue		General	Shortest Queue	
s	$M = 48$	$M = 48$	$M = 72$	$M = 48$	$M = 48$	$M = 96$
2	20,825	11,025	35,113	10,725	5,825	41,601
3	270,725	69,785	315,925	50,625	14,365	176,121
4	2,869,685	352,521	2,236,585	157,977	24,193	471,745
5	25,827,165	1,548,460	13,422,565	368,149	32,671	926,234
6	202,927,725	5,676,921	68,191,357	694,497	36,609	1,444,609
7	1.42 E + 09	19,953,444	323,368,848	1,121,693	38,611	1,951,568
8	9.00 E + 09	59,421,537	1.30 E + 09	1,614,444	36,609	2,273,921
9	5.22 E + 10	201,270,006	4.95 E + 09	2,131,737	35,938	2,544,608
10	2.80 E + 11	529,806,725	1.91 E + 10	2,637,924	32,671	2,620,255
11	1.40 E + 12	1.62 E + 09	7.08 E + 10	3,107,797	29,947	2,544,606
12	6.57 E + 12	2.74 E + 09	1.80 E + 11	3,526,974	24,193	2,273,921
13	2.91 E + 13	9.70 E + 09	7.50 E + 11	3,889,825	24,094	2,240,166
14	1.22 E + 14	2.85 E + 10	1.78 E + 12	4,196,781	21,679	1,951,568
15	4.89 E + 14	5.03 E + 10	5.32 E + 12	4,451,881	17,539	1,829,125
16	1.87 E + 15	5.63 E + 10	1.99 E + 13	4,660,969	14,365	1,444,609

2.3. Acceleration of Convergence

Once the coefficients of the power-series expansions of the state probabilities have been determined, those of the moments of the queue length distribution can be obtained as well (Blanc 1987b). To accelerate the convergence of the power-series expansions the modified ϵ-algorithm (Blanc 1990) can be applied. It means that extrapolating terms, which take into account the asymptotic behavior of moments and probabilities as $\rho \uparrow 1$, are added to the partial sums of the power-series expansions; then the ϵ-algorithm, described in Brezinski (1977) and Wynn (1966), can be applied. Because the ϵ-algorithm transforms polynomials into rational functions, it is not necessary to choose the value of G in the transformation (23) so large that the power series are convergent for all required values of θ: The ϵ-algorithm transforms divergent series into convergent series as long as singularities are poles. Numerical experiments have shown that values of G for which the series are divergent, but not too strongly, give the best performance of the power-series algorithm together with the ϵ-algorithm (for $G = 0$, when computations are less than for $G > 0$, (25), the series are in most cases so strongly divergent that numerical instabilities occur). We have used $G = 0.5$ ($s = 2$) up to $G = 1.25$ ($s = 30$) for symmetrical models, and $G = 1.5$ up to $G = 2.5$ for asymmetrical models to obtain the data presented in Section 4.

3. LIGHT TRAFFIC BEHAVIOR

In this section, we shall derive some light traffic limits for symmetrical systems (systems with $r_j = 1/s$, $\psi_j = 1$, $j = 1, \ldots, s$), and for $G = 0$, i.e., $\theta = \rho$, in (23)–(25). First, the following expression holds for coefficients $b(k;\, \bar{n})$ with $k + n_1 + \ldots + n_s \leq s$ and, (27), $|n_i - n_j| \leq 1$ for all i and j, $i, j = 1, \ldots, s$:

$$b(k;\, \bar{n}) = (-1)^k \frac{s^k}{k!} \frac{s^m}{m!} \binom{s}{m}^{-1} \qquad \text{with } m := n_1 + \ldots + n_s.$$

These coefficients are the same as those for the corresponding M/M/s queueing system; as long as there are not more than s customers in the system during a busy period the behavior of the shortest-queue model is the same as that of the M/M/s model. Also, the first coefficient of the state probabilities can easily be obtained for states $\bar{n}$ with, (27), $|n_i - n_j| \leq 1$ for all i and j, $i, j = 1, \ldots, s$:

$$b(0;\, \bar{n}) = \frac{s^s}{s!} \binom{s}{m}^{-1} \qquad \text{with } n_1 + \ldots + n_s = as + m, \quad a = 1, 2, \ldots, \quad 0 \leq m < s.$$

With the above and similar relations the following asymptotical expansions for light traffic can be derived as $\rho \downarrow 0$:

$$P\{L=0\} = \sum_{k=0}^{s+1} \frac{(-s\rho)^k}{k!} - \frac{s^s \rho^{s+1}}{s!} \sum_{j=2}^{s+1} \frac{1}{j} + 0(\rho^{s+2})],$$

$$E\{L\} = s\rho + \frac{(s\rho)^{s+1}}{s!}\left[1 - \frac{s^2-1}{s}\rho + 0(\rho^2)\right],$$

$$\sigma^2\{L\} = s\rho + \frac{(s\rho)^{s+1}}{2(s!)} \cdot \left[s + 3 - \frac{s^3+4s^2-3s-3}{s}\rho + 0(\rho^2)\right],$$

$$P\{W=0\} = 1 - \frac{(s\rho)^s}{s!}\left[1 - \frac{s^2-1}{s}\rho + 0(\rho^2)\right],$$

$$E\{W\} = \frac{s}{\mu}\frac{(s\rho)^s}{s!}\left[1 - \frac{s^2-1}{s}\rho + 0(\rho^2)\right],$$

$$\sigma^2\{W\} = 2\left(\frac{s}{\mu}\right)^2 \frac{(s\rho)^s}{s!}\left[1 - \frac{s^2-1}{s}\rho + 0(\rho^2)\right],$$

$$R = \frac{1}{s-1}\left[\frac{\rho}{1-\rho} + \frac{(s\rho)^s}{2(s!)}\left\{s - 3 - \frac{s^3-3s^2-10s+3}{s}\rho + 0(\rho^2)\right\}\right]. \quad (30)$$

Here R denotes the coefficient of correlation between the lengths of two arbitrarily chosen queues in symmetrical shortest-queue models; it is readily verified that R satisfies:

$$R = \frac{\operatorname{cov}\{N_1, N_2\}}{\sigma^2\{N_1\}} = \frac{1}{s(s-1)}\left[\frac{\sigma^2\{L\}}{\sigma^2\{N_1\}}\right] - \frac{1}{s-1}.$$

From (30), it follows that the mean waiting time for the shortest-queue model is asymptotically s times as large as that for the corresponding M/M/s model in light traffic. This can be explained by noting that the waiting time of the first customer, who has to wait in a busy period, is equal to the minimum of the s remaining holding times of the customers in service for the M/M/s model, while it is equal to the remaining holding time of the customer in front of the waiting customer in the shortest-queue model.

4. NUMERICAL RESULTS

This section contains numerical data for the shortest-queue model which have been obtained by means of the power-series algorithm together with the modified ϵ-algorithm described in Blanc (1990). The moments of the waiting time distributions have been computed for $\mu = 1$ in all tables.

4.1. Symmetrical Systems

Tables II and III show numerical values for the zero-probabilities, averages and standard deviations of the total number L of customers in the system and of the waiting time W, and for the correlation coefficient R in the symmetrical shortest-queue models ($r_j = 1/s$, $\psi_j = 1$, $j = 1, \ldots, s$). It is interesting to compare these data with data for corresponding M/M/s models (Blanc 1987a). While customers are taken into service in the order of arrival in M/M/s systems, this is not the case in systems in which customers join one of the shortest queues upon arrival (although customers are taken into service in order of arrival in each individual queue). This feature gives rise to a larger standard deviation of the waiting times in systems with the "join-the-shortest-queue" discipline. Furthermore, it may happen in the latter systems that some servers are idle while there are customers waiting in other queues. This inefficiency, on the one hand, causes a larger probability that a customer finds a free server upon arrival, but it leads, on the other hand, to, on the average, a larger number of customers in the system, and as a consequence to a larger mean waiting time than in M/M/s systems.

Table IV concerns, for symmetrical systems, the random variables D, the largest difference between the lengths of the various queues, i.e.,

$$D := \max\{N_1, \ldots, N_s\} - \min\{N_1, \ldots, N_s\},$$

and J, the number of servers who are idle but could work if they were able to serve customers who are waiting in other queues; more formally:

$$J := \#\{j; N_j = 0\} - [s - L]^+.$$

The tables show that in the shortest-queue model both the mean and the standard deviation of the waiting times are decreasing with s for low values of ρ (see Table II, $\rho = 0.1$). For moderate values of ρ the standard deviation is first increasing with s, but after having reached a maximal value (at $s = 6$ when $\rho = 0.5$, at $s = 11$ when $\rho = 0.6$, and at $s = 21$ when $\rho = 0.7$) it again becomes a decreasing function of s, while the mean waiting time still decreases with s (see Table II). Finally, for high values of ρ both the mean and the standard deviation of the waiting time are increasing functions of s as far as they have been computed (see Table III). The behavior of the mean waiting time, as a function of the number of queues, is governed by two opposite forces: On the one hand, the probability that a customer finds a free server upon arrival increases, but, on the other hand, the mean total number of customers in the system also

Table II
Queueing Characteristics for Lightly and Moderately Loaded Symmetrical Systems

ρ	s	$P\{L=0\}$	$E\{L\}$	$\sigma\{L\}$	$P\{W=0\}$	$E\{W\}$	$\sigma\{W\}$	R
0.1	2	0.8175	0.203543	0.4571	0.98246906	0.03542858	0.37675	0.0998
0.1	3	0.7405	0.301049	0.5506	0.99650615	0.01049253	0.25082	0.0553
0.1	4	0.6703	0.400296	0.6333	0.99925981	0.00296105	0.15389	0.0371
0.1	5	0.6065	0.500081	0.7073	0.99983770	0.00081149	0.09008	0.0278
0.1	6	0.5488	0.600022	0.7747	0.99996364	0.00021818	0.05117	0.0222
0.1	7	0.4966	0.700006	0.8367	0.99999173	0.00005788	0.02847	0.0185
0.1	8	0.4493	0.800002	0.8944	0.99999810	0.00001520	0.01560	0.0159
0.1	9	0.4066	0.900000	0.9487	0.99999956	0.00000396	0.00845	0.0139
0.1	10	0.3679	1.000000	1.0000	0.99999990	0.00000103	0.00453	0.0123
0.5	2	0.315967	1.426	1.541	0.68403	0.85264	1.9653	0.5257
0.5	3	0.197604	1.867	1.665	0.78821	0.73451	2.1327	0.3925
0.5	4	0.122744	2.316	1.782	0.85297	0.63260	2.2451	0.3060
0.5	5	0.075867	2.772	1.891	0.89532	0.54321	2.3112	0.2472
0.5	6	0.046715	3.232	1.995	0.92400	0.46476	2.3384	0.2053
0.5	7	0.028679	3.698	2.095	0.94394	0.39626	2.3334	0.1744
0.5	8	0.017565	4.168	2.190	0.95810	0.33685	2.3028	0.1507
0.5	9	0.010738	4.643	2.283	0.96834	0.28564	2.2523	0.1322
0.5	10	0.006555	5.121	2.374	0.97586	0.24171	2.1868	0.1173
0.5	15	0.000548	7.551	2.799	0.99315	0.10283	1.7534	0.0738
0.5	20	0.000045	10.021	3.188	0.99785	0.04291	1.3093	0.0535
0.5	25	3.7 E − 06	12.509	3.547	0.99929	0.01769	0.9403	0.0420
0.5	30	3.1 E − 07	15.004	3.878	0.99976	0.00723	0.6584	0.0346
0.7	2	0.156485	2.951	2.891	0.4435	2.2164	3.553	0.766
0.7	3	0.080853	3.589	3.000	0.5440	2.1266	3.896	0.656
0.7	4	0.041542	4.232	3.109	0.6198	2.0458	4.209	0.565
0.7	5	0.021259	4.878	3.214	0.6791	1.9685	4.494	0.490
0.7	6	0.010845	5.525	3.315	0.7265	1.8930	4.751	0.430
0.7	7	0.005518	6.173	3.412	0.7652	1.8186	4.982	0.381
0.7	8	0.002802	6.822	3.504	0.7970	1.7450	5.189	0.341
0.7	9	0.001420	7.471	3.593	0.8234	1.6725	5.373	0.307
0.7	10	0.000718	8.121	3.678	0.8456	1.6011	5.535	0.279
0.7	15	0.000023	11.389	4.056	0.9159	1.2693	6.060	0.188
0.7	20	7.4 E − 07	14.694	4.385	0.9505	0.9912	6.222	0.139
0.7	21	3.7 E − 07	15.360	4.447	0.9552	0.9425	6.223	0.132
0.7	22	1.9 E − 07	16.027	4.508	0.9593	0.8959	6.216	0.125
0.7	25	2.3 E − 08	18.038	4.688	0.9693	0.7687	6.152	0.108
0.7	30	5.0 E − 10	21.416	4.978	0.9802	0.5944	5.942	0.088

increases with an increasing number of queues. The mean waiting time stops being a monotonically decreasing function of s between $\rho = 0.86$ and $\rho = 0.87$. For $\rho = 0.87$ it has a local minimum at $s = 4$ and a local maximum at $s = 9$. For $\rho = 0.88$ these extremal values are located at $s = 3$ and at $s = 13$, respectively. Note that the mean and the standard deviation of the waiting times in the M/M/s model are decreasing functions of s for every value of ρ. Furthermore, the mean values of the random variables D and J increase with increasing s for fixed ρ and small values of s (see Table IV). This is due to the effect that increasing the number of queues leads to more stochastic fluctuations between the lengths of the various queues. When the number of queues increases further, however, these fluctuations will fade because the average lengths of the individual queues become smaller and smaller. Asymptotically, as the number of queues tends to infinity, there will be, with probability tending to one, no customers waiting for service (but their mean holding time tends to infinity); this suggests that $E\{D\}$ will tend to one and $E\{J\}$ to zero. We remark that $E\{D\}$ approaches one from below for low values of ρ. The above mentioned features will also contribute to the behavior of the standard deviation of the waiting times.

Table III
Queueing Characteristics for Heavily Loaded Symmetrical Systems

ρ	s	P{L = 0}	E{L}	σ{L}	P{W = 0}	E{W}	σ{W}	R
0.9	2	0.042246	9.855	9.534	0.1578	8.950	10.43	0.965
0.9	3	0.017664	10.753	9.593	0.2049	8.947	10.91	0.940
0.9	4	0.007352	11.667	9.657	0.2457	8.963	11.40	0.914
0.9	5	0.003052	12.589	9.726	0.2822	8.988	11.88	0.885
0.9	6	0.001265	13.515	9.797	0.3153	9.016	12.36	0.856
0.9	7	0.000524	14.441	9.871	0.3456	9.046	12.84	0.826
0.9	8	0.000217	15.368	9.945	0.3737	9.076	13.31	0.796
0.9	9	0.000089	16.294	10.021	0.3997	9.104	13.78	0.767
0.9	10	0.000037	17.218	10.098	0.4241	9.131	14.25	0.738
0.9	11	0.000015	18.141	10.174	0.4468	9.156	14.70	0.710
0.9	12	6.3 E − 06	19.061	10.250	0.4683	9.179	15.16	0.683
0.98	2	0.007707	49.97	49.51	0.03229	48.99	50.49	0.9984
0.98	3	0.002933	51.00	49.52	0.04284	49.04	50.99	0.9973
0.98	4	0.001110	52.07	49.54	0.05243	49.13	51.50	0.9960
0.98	5	0.000419	53.15	49.56	0.06138	49.24	52.01	0.9944
0.98	6	0.000158	54.25	49.58	0.06985	49.36	52.52	0.9927
0.98	7	0.000059	55.36	49.60	0.07794	49.49	53.04	0.9909
0.98	8	0.000022	56.47	49.63	0.08573	49.62	53.56	0.9889
0.98	9	8.4 E − 06	57.59	49.65	0.09325	49.76	54.08	0.9866
0.98	10	3.2 E − 06	58.71	49.68	0.10056	49.91	54.58	0.9856
0.99	2	0.003808	99.98	99.50	0.01619	98.99	100.49	0.9996
0.99	3	0.001431	101.04	99.51	0.02154	99.06	101.00	0.9993
0.99	4	0.000535	102.13	99.52	0.02642	99.16	101.50	0.9991
0.99	5	0.000199	103.23	99.53	0.03101	99.28	102.01	0.9986
0.99	6	0.000074	104.36	99.54	0.03537	99.41	102.51	0.9982
0.99	7	0.000028	105.49	99.55	0.03956	99.56	103.03	0.9977
0.99	8	0.000010	106.63	99.57	0.04360	99.71	103.54	0.9972
0.99	9	3.8 E − 06	107.78	99.58	0.04754	99.87	104.05	0.9966
0.99	10	1.4 E − 06	108.93	99.60	0.05137	100.03	104.56	0.9963

Table IV
Other Queueing Characteristics for Symmetrical Systems

s	ρ	P{D > 1}	E{D}	P{J > 0}	E{J}	ρ	P{D > 1}	E{D}	P{J > 0}	E{J}
2	0.5	0.06597	0.5737	0.05210	0.05210	0.7	0.1235	0.735	0.0680	0.0680
3	0.5	0.09104	0.8008	0.08161	0.08236	0.7	0.1850	0.990	0.1263	0.1284
4	0.5	0.09914	0.9183	0.09405	0.09796	0.7	0.2214	1.111	0.1713	0.1811
5	0.5	0.09858	0.9828	0.09607	0.10351	0.7	0.2445	1.176	0.2050	0.2261
6	0.5	0.09338	1.0178	0.09220	0.10252	0.7	0.2595	1.213	0.2294	0.2638
7	0.5	0.08583	1.0357	0.08529	0.09751	0.7	0.2691	1.234	0.2467	0.2948
8	0.5	0.07730	1.0435	0.07706	0.09024	0.7	0.2747	1.247	0.2584	0.3200
9	0.5	0.06862	1.0454	0.06851	0.08190	0.7	0.2773	1.254	0.2655	0.3398
10	0.5	0.06026	1.0441	0.06022	0.07326	0.7	0.2776	1.257	0.2693	0.3549
11	0.5	0.05250	1.0410	0.05248	0.06482	0.7	0.2762	1.258	0.2702	0.3660
12	0.5	0.04545	1.0371	0.04544	0.05687	0.7	0.2732	1.257	0.2690	0.3735
14	0.5	0.03360	1.0290	0.03360	0.04293	0.7	0.2641	1.251	0.2620	0.3797
16	0.5	0.02451	1.0219	0.02451	0.03182	0.7	0.2519	1.241	0.2510	0.3769
18	0.5	0.01771	1.0161	0.01771	0.02328	0.7	0.2381	1.229	0.2376	0.3677
20	0.5	0.01271	1.0117	0.01271	0.01688	0.7	0.2233	1.215	0.2231	0.3541
25	0.5	0.00542	1.0051	0.00542	0.00733	0.7	0.1861	1.181	0.1860	0.3097
30	0.5	0.00227	1.0022	0.00227	0.00310	0.7	0.1519	1.148	0.1518	0.2611
2	0.9	0.1883	0.905	0.0395	0.0395	0.98	0.2154	0.980	0.0095	0.0095
3	0.9	0.2835	1.191	0.0802	0.0821	0.98	0.3211	1.282	0.0197	0.0202
4	0.9	0.3424	1.323	0.1184	0.1265	0.98	0.3846	1.421	0.0297	0.0318
5	0.9	0.3828	1.396	0.1535	0.1718	0.98	0.4272	1.498	0.0392	0.0439
6	0.9	0.4125	1.441	0.1857	0.2173	0.98	0.4578	1.545	0.0483	0.0565
7	0.9	0.4353	1.471	0.2153	0.2629	0.98	0.4808	1.577	0.0570	0.0694
8	0.9	0.4536	1.492	0.2426	0.3083	0.98	0.4987	1.598	0.0653	0.0826
9	0.9	0.4686	1.507	0.2678	0.3534	0.98	0.5130	1.614	0.0734	0.0960
10	0.9	0.4813	1.519	0.2913	0.3980	0.98	0.5264	1.627	0.0812	0.1097

Table V
Heavy Traffic Limits for Symmetrical Systems

s	η_0	η_1	η_2	ω_0	ω_1	ω_2	α	δ_0	δ_1	ξ_0	ξ_1
2	0.3763	0.00	1.14	1.62	−1.00	−0.86	4.0	0.222	1.00	0.49	0.49
3	0.1397	1.07	2.68	2.17	−0.93	0.36	6.9	0.330	1.31	1.03	1.06
4	0.0516	2.18	4.52	2.66	−0.82	2.64	10.3	0.395	1.45	1.56	1.67
5	0.0190	3.32	6.62	3.13	−0.68	5.96	14.2	0.437	1.53	2.07	2.32
6	0.0070	4.47	8.94	3.58	−0.53	10.3	18.6	0.468	1.57	2.56	2.99
7	0.0026	5.63	11.5	4.01	−0.37	15.7	23.4	0.491	1.61	3.03	3.69
8	0.0009	6.80	14.1	4.44	−0.20	21.9	28.6	0.509	1.63	3.49	4.40

Table VI
Queueing Characteristics for Asymmetrical Systems With Four Queues

ρ		P{$L=0$}	E{L}	σ{L}	P{$W=0$}	E{W}	σ{W}	E{H}	E{D}
0.1	Ψ1	0.5694	0.5433	0.7135	0.998700	0.007814	0.3083	5.426	0.430
0.1	Ψ2	0.6494	0.4294	0.6523	0.999155	0.003665	0.1855	4.291	0.350
0.1	Ψ3	0.7382	0.3098	0.5676	0.999527	0.001387	0.0912	3.096	0.261
0.5	Ψ1	0.07619	2.681	1.847	0.8280	0.9240	3.114	4.439	1.034
0.5	Ψ2	0.10699	2.430	1.806	0.8442	0.7159	2.571	4.144	0.952
0.5	Ψ3	0.15549	2.134	1.752	0.8640	0.5153	1.981	3.752	0.858
0.7	Ψ1	0.02235	4.751	3.187	0.5877	2.560	5.257	4.227	1.262
0.7	Ψ2	0.03383	4.427	3.146	0.6059	2.234	4.728	4.090	1.163
0.7	Ψ3	0.05331	4.046	3.092	0.6293	1.878	4.118	3.901	1.050
0.9	Ψ1	0.00330	12.475	9.725	0.2264	9.792	12.96	4.069	1.530
0.9	Ψ2	0.00541	12.042	9.698	0.2356	9.347	12.51	4.033	1.410
0.9	Ψ3	0.00933	11.519	9.661	0.2482	8.818	11.99	3.982	1.269
0.95	Ψ1	0.00131	22.84	19.64	0.1169	20.01	23.79	4.034	1.612
0.95	Ψ2	0.00220	22.37	19.62	0.1220	19.53	23.39	4.017	1.485
0.95	Ψ3	0.00390	21.79	19.60	0.1289	18.94	22.92	3.993	1.335

Table V contains values of the following limits:

$$\eta_0 = \lim_{\rho\uparrow 1} \frac{\mathrm{P}\{L=0\}}{1-\rho}, \quad \eta_1 = \lim_{\rho\uparrow 1}\left[\mathrm{E}\{L\} - \frac{1}{1-\rho}\right],$$

$$\eta_2 = \lim_{\rho\uparrow 1}\left[\sigma^2\{L\} - \left(\frac{1}{1-\rho}\right)^2 + \frac{1}{1-\rho}\right],$$

$$\omega_0 = \lim_{\rho\uparrow 1} \frac{\mathrm{P}\{W=0\}}{1-\rho}, \quad \omega_1 = \lim_{\rho\uparrow 1}\left[\mathrm{E}\{W\} - \frac{1}{1-\rho}\right],$$

$$\omega_2 = \lim_{\rho\uparrow 1}\left[\sigma^2\{W\} - \left(\frac{1}{1-\rho}\right)^2 - \frac{s-1}{1-\rho}\right],$$

$$\alpha = \lim_{\rho\uparrow 1} \frac{1-R}{(1-\rho)^2}, \quad \delta_0 = \lim_{\rho\uparrow 1} \mathrm{P}\{D>1\},$$

$$\delta_1 = \lim_{\rho\uparrow 1} \mathrm{E}\{D\}, \quad \xi_0 = \lim_{\rho\uparrow 1} \frac{\mathrm{P}\{J>0\}}{1-\rho},$$

$$\xi_1 = \lim_{\rho\uparrow 1} \frac{E\{J\}}{1-\rho}. \tag{31}$$

The limits in Table V have been computed by fitting Laurent series expansions at $\rho = 1$, with the aid of more data than are shown in the Tables III and IV. The most remarkable difference with the heavy traffic behavior of M/M/s systems is the term $(s-1)/(1-\rho)$ in the expansion of $\sigma^2\{W\}$. It should be noted that the forms of this term and of the other terms indicated in (31) have been deduced from numerical data (they have been checked for values of s up to 10). The heavy traffic behavior of E{L} in the case $s = 2$ has been found in Flatto and McKean (1977) by means of an analytical method.

4.2. Asymmetrical Systems

Tables VI and VII present values of performance measures for asymmetrical shortest-queue systems with $s = 4$ queues. The service rates at the queues are: $r_1 = 0.16$, $r_2 = 0.24$, $r_3 = 0.24$, and $r_4 = 0.36$. Three sets of weight factors, (4), have been considered: $\psi_j:\psi_{j+1} = 100:1$ for $j = 1, 2, 3$ (model Ψ1, preference for queues with lower indices, i.e., for the slower servers), $\psi_j:\psi_{j+1} = 1:1$ for $j = 1, 2, 3$ (model Ψ2, no

Table VII
The Distribution of Customers Over the Queues and the Mean Number of Customers in the Various Queues in Asymmetrical Systems

ρ		P{C = 1}	P{C = 2}	P{C = 3}	P{C = 4}	E{N_1}	E{N_2}	E{N_3}	E{N_4}
0.1	Ψ1s	0.7099	0.2348	0.0484	0.0069	0.2842	0.0939	0.0194	0.0028
0.1	Ψ1	0.6125	0.3014	0.0737	0.0124	0.3836	0.1256	0.0307	0.0034
0.1	Ψ2	0.2350	0.2509	0.2509	0.2633	0.1470	0.1046	0.1046	0.0732
0.1	Ψ3	0.0043	0.0335	0.1851	0.7771	0.0027	0.0140	0.0771	0.2160
0.5	Ψ1s	0.3574	0.2908	0.2135	0.1384	0.8743	0.6688	0.4728	0.3004
0.5	Ψ1	0.2605	0.3068	0.2380	0.1947	1.0721	0.7555	0.5588	0.2950
0.5	Ψ2	0.1949	0.2484	0.2484	0.3084	0.7283	0.6048	0.6048	0.4921
0.5	Ψ3	0.1005	0.1871	0.2634	0.4490	0.3423	0.4300	0.6261	0.7354
0.95	Ψ1s	0.2566	0.2531	0.2483	0.2419	5.841	5.598	5.355	5.116
0.95	Ψ1	0.1667	0.2457	0.2422	0.3454	6.304	5.837	5.594	5.103
0.95	Ψ2	0.1639	0.2414	0.2414	0.3533	5.857	5.590	5.590	5.328
0.95	Ψ3	0.1583	0.2370	0.2420	0.3628	5.327	5.315	5.560	5.589

preference for any queue), and ψ_j:ψ_{j+1} = 1:100 for j = 1, 2, 3 (model Ψ3, preference for queues with higher indices, i.e., for the faster servers). The data in Table VI indicate that systems with unequal service rates perform better than the corresponding symmetrical system only if the fastest servers possess sufficiently high weight factors. Table VII also contains data for models with four queues and equal service rates (r_j = 0.25 for j = 1, . . . , 4), with weight factors ψ_j:ψ_{j+1} = 100:1 for j = 1, 2, 3 (model Ψ1s); the distributions of L and W do not depend on the weight factors for models with equal service rates, so that data concerning these distributions can be found in the Tables II and III for this model.

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