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Note

Divisible Designs with $r-\lambda_1=1$

WILLEM H. HAEMERS

Tilburg University, Tilburg, The Netherlands

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We give a classification of divisible designs with $r-\lambda_1=1$. © 1991 Academic Press, Inc.

INTRODUCTION

A (group) divisible design with parameters $(m, n, k, r, \lambda_1, \lambda_2)$ is an incidence structure with constant block size k, and with mn points split into m classes (also called groups) of n points, such that any two points are covered by r, λ_1, or λ_2 blocks, depending on whether these points coincide, belong to the same class, or belong to distinct classes, respectively. In this note we shall classify divisible designs with $r-\lambda_1=1$. A classification in the case that there exists a cyclic divisible difference set was given by Arasu, Jungnickel, and Pott [2]. They also give a construction method for such designs, which was generalised by Arasu, Haemers, Jungnickel, and Pott [1]. This construction method uses a strongly regular graph with $\mu-\lambda=1$, or a skew-symmetric Hadamard matrix. We shall show that apart from these, no other non-trivial constructions exist.

PRELIMINARIES

In terms of the incidence matrix N, the definition of a divisible design reads:

$$N^t\mathbf{1}=k\mathbf{1}, \qquad NN^t=(\lambda_2 J+(\lambda_1-\lambda_2)\,I)\otimes J+(r-\lambda_1)\,I.$$

Herein $\otimes$ denotes the Kronecker (or tensor) product, I denotes the identity matrix, J the all-one matrix, and $\mathbf{1}$ the all-one vector of appropriate

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size. A strongly regular graph with parameters (v, k, λ, μ) is a k-regular graph with v vertices, such that any two distinct vertices have λ or μ common neighbours, depending on whether the vertices are adjacent or non-adjacent, respectively. It is straightforward and well known that a graph with adjacency matrix A is strongly regular with $\mu-\lambda=1$ if and only if

$$A^2+A\in\langle I, J\rangle.$$

A skew-symmetric Hadamard matrix of order v is a $v\times v$ matrix H with entries 1 or -1, such that $H'=2I-H$ and $HH'=vI$. Multiplication of some rows and the corresponding columns by -1 does not affect these properties. So, without loss of generality,

$$H=\begin{pmatrix} 1 & \mathbf{1}' \\ -\mathbf{1} & C \end{pmatrix}.$$

The matrix C is called a core of H. It is not difficult to show that a $(-1, 1)$ matrix C is a core of a skew-symmetric Hadamard matrix if and only if

$$C'=2I-C, \qquad C\mathbf{1}\in\langle\mathbf{1}\rangle, \qquad CC'\in\langle I, J\rangle.$$

CLASSIFICATION

LEMMA 1. *Suppose A is a square $(0, 1)$ matrix of size m with zero diagonal. Let $D_1, ..., D_m$ be the incidence matrices of block designs with parameters $(v', b', k', r', \lambda')$. Put $D=\operatorname{diag}(D_1, ..., D_m)$. Then $N=(A\otimes J)+D$ is the incidence matrix of a divisible design if and only if one of the following holds:*

(i) *$J-2A$ is the core of a skew-symmetric Hadamard matrix.*

(ii) *$b'=2r'$, and A is the adjacency matrix of a strongly regular graph with $\mu-\lambda=1$,*

(iii) *$A=0$, or $A=J-I$.*

Proof. Clearly N has constant column sum whenever A has. Furthermore,

$$\begin{aligned} NN' &= b'AA'\otimes J+r'(A+A')\otimes J+DD' \\ &= (b'AA'+r'(A+A')+\lambda' I)\otimes J+(r'-\lambda')I. \end{aligned}$$

So, by definition, N denotes a divisible design if and only if

$$A'\mathbf{1}\in\langle\mathbf{1}\rangle \qquad \text{and} \qquad b'AA'+r'(A+A')=\alpha J+\beta I, \qquad (*)$$

for some integers α and β. Now, the "if" part of the lemma follows by verification (see [1]). To prove the "only if" part, consider the entries $a=(A)_{ij}$ and $\bar{a}=(A)_{ji}$ for arbitrary i and j ($i \neq j$). We easily have

$$\alpha = \begin{cases} r' \bmod b' & \text{if } a \neq \bar{a}, \\ 0 \bmod b' & \text{if } a = \bar{a} = 0, \\ 2r' \bmod b' & \text{if } a = \bar{a} = 1. \end{cases}$$

Hence, in case $\alpha = r' \bmod b'$, $A + A^t = J - I$. Thus $C = J - 2A$ satisfies $C^t = 2I - C$, and (*) implies $C\mathbf{1} \in \langle \mathbf{1} \rangle$, $CC^t \in \langle I, J \rangle$. This proves *i*. If $\alpha = 0 = 2r' \bmod b'$, then $b' = 2r'$, A is symmetric and (*) becomes $A^2 + A \in \langle I, J \rangle$. This proves ii. Finally, if $\alpha = 0 \neq 2r' \bmod b$, or $\alpha = 2r' \neq 0 \bmod b$, then $A = 0$, or $J - I$, respectively.

LEMMA 2. *Suppose N is the incidence matrix of a divisible design with $r - \lambda_1 = 1$. Then, up to taking complements and after suitable row and column permutations.*

$$N = (A \otimes J) + I,$$

where A is a square (0, 1) *matrix with zero diagonal.*

Proof. For $i = 1, ..., m$ let N_i denote the part of N corresponding to class i. Then $N_i N_i^t = \lambda_1 J + (r - \lambda_1) I = \lambda_1 J + I$. So, any two distinct rows of N_i differ in just two positions. This implies that after a suitable permutation of the columns and, if necessary, complementation N_i takes the form $N_i = [I\, J\, 0]$. With the same column partition, write $N_j = [K\, L\, M]$ for some $j \neq i$. Let $k_1, ..., k_n$ be the columns of K, and let n_x and n_y be any two distinct rows of N_i. Then $n_x - n_y$ has 1 on position x, -1 on position y, and 0 elsewhere; moreover, $N_j n_x^t = N_j n_y^t = \lambda_2 \mathbf{1}$. Hence

$$0 = N_j (n_x - n_y)^t = k_x - k_y.$$

Thus all columns of K are equal, so $K = J$ or $K = 0$. The first column of N, and hence each column, has column sum equal to $1 \bmod n$. So for each column there is precisely one number i ($1 \leqslant i \leqslant m$) for which the column corresponds to the indentity matrix in N_i. By permuting the columns of N such that these indentity matrices are moved to the diagonal, we obtain the desired form for N.

So, by Lemma 2, a divisible design with $r - \lambda_1 = 1$ has the structure of Lemma 1, where $(v', k', b', r', \lambda')$ is one of the trivial parameter sets $(v', 1, v', 1, 0)$ or $(v', v'-1, v', v'-1, v'-2)$. This leads to the main result.

THEOREM. *An incidence structure $\mathscr{D}$ is a divisible design with $r - \lambda_1 = 1$ if* *and only if $\mathscr{D}$ or the complement of $\mathscr{D}$ has an incidence matrix $(A \otimes J) + I$, where one of the following holds:*

(i) *$J - 2A$ is the core of a skew-symmetric Hadamard matrix.*

(ii) *J has size 2×2 and A is the adjacency matrix of a strongly regular graph with $\mu - \lambda = 1$,*

(iii) *$A = 0$, or $A = J - I$.*

REFERENCES

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