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Classification of Spreads of $PG(3, 4)\backslash PG(3, 2)$

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Abstract. We show that the 800 spreads of $PG(3, 4)\backslash PG(3, 2)$ fall into three orbits of sizes 120, 120 and 560, under the action of its automorphism group.

1. Introduction

A spread in a projective geometry is a set of mutually disjoint lines which partitions the point set. Hirschfeld [3, p. 80] conjectured that for every prime power q there exists a spread of $PG(3, q^2)\backslash PG(3, q)$. Mesner [4] found such a spread for $q = 2$, but Blokhuis and Metsch [1] proved that the conjecture is false for $q > 2$. In this note we shall classify the spreads of $PG(3, 4)\backslash PG(3, 2)$.

Given a fixed coordinatization of $PG(3, 4)$, the nontrivial field automorphism σ of $GF(4)$ acts naturally as a Baer involution of $PG(3, 4)$, fixing a Baer subspace $PG(3, 2)$.

2. Correspondences to Other Systems

Let x be a point in $PG(3, 4)\backslash PG(3, 2)$. Then x and $\sigma(x)$ are on a line in $PG(3, 4)$, of which the other three points are in $PG(3, 2)$. So x (the pair $\{x, \sigma(x)\}$) determines a line in $PG(3, 2)$. Conversely, every line in $PG(3, 2)$ determines a pair $\{x, \sigma(x)\}$ in $PG(3, 4)\backslash PG(3, 2)$. Let l be a line in $PG(3, 4)\backslash PG(3, 2)$, then the five points of l determine five disjoint lines (a spread) in $PG(3, 2)$. Now l and $\sigma(l)$ determine the same spread, and any other line determines another spread. Since there are as many (56) pairs of lines $\{l, \sigma(l)\}$ in $PG(3, 4)\backslash PG(3, 2)$ as there are spreads in $PG(3, 2)$, there is a one-one correspondence between pairs $\{l, \sigma(l)\}$ and spreads of $PG(3, 2)$. So we found bijective functions f and f^*,

$$f: \{\{l, \sigma(l)\}: l \text{ is a line in } PG(3, 4)\backslash PG(3, 2)\} \to \{S: S \text{ is a spread of } PG(3, 2)\},$$

$$f^*: \{\{x, \sigma(x)\}: x \in PG(3, 4)\backslash PG(3, 2)\} \to \{L: L \text{ is a line in } PG(3, 2)\},$$

such that $x \in l$ or $x \in \sigma(l)$ if and only if $f^*(\{x, \sigma(x)\}) \in f(\{l, \sigma(l)\})$, where x is a point and l is a line in $PG(3, 4)\backslash PG(3, 2)$.

Now, a spread of $PG(3, 4)\backslash PG(3, 2)$, which is invariant under σ (seven disjoint pairs of lines $\{l, \sigma(l)\}$) corresponds (one-one) to a resolution (seven disjoint spreads) of $PG(3, 2)$. A spread of $PG(3, 4)\backslash PG(3, 2)$, which has no pair of lines $\{l, \sigma(l)\}$ at all, corresponds—not one-one, for every line l we get a spread $f(\{l, \sigma(l\})$—to a set of fourteen spreads of $PG(3, 2)$, such that for any of the fourteen spreads there are five spreads, which have one line in common with this spread, and eight spreads which are disjoint from it, and such that every line is in two of the fourteen spreads. We shall use these correspondences later.

It is possible to identify the lines of $PG(3, 2)$ with the triples from a set X of seven points, such that intersecting lines correspond to triples having one element in common [5]. So a spread corresponds to five triples which pairwise intersect in zero or two points. If all intersections have size two then the five triples are all triples containing a given pair $\{x, y\} \subset X$ (case 1), otherwise there is a triple $\{u, v, w\}$ disjoint from all others, and the other triples are all triples taken from the remaining four elements of X (case 2).

Case 1

x	y					
1	1	1				
1	1		1			
1	1			1		
1	1				1	
1	1					1

Case 2

u	v	w				
1	1	1				
			1	1	1	
			1	1		1
			1		1	1
				1	1	1

If we take $Y = X \cup \{\infty\}$, then we can identify the spreads of $PG(3, 2)$ with triples from Y. In case 1 we identify the spread with $\{\infty, x, y\}$, in case 2 with $\{u, v, w\}$. So we have a bijective function g,

$$g : \{S: S \text{ is a spread of } PG(3, 2)\} \rightarrow \binom{Y}{3},$$

where $\binom{Y}{3}$ is the set of triples from Y. Now two spreads S and S' will be disjoint if and only if $|g(S) \cap g(S')| = 1$, and S and S' will have one line in common if and only if $|g(S) \cap g(S')| = 2$.

Recall that we identified the lines of $PG(3, 2)$ with the triples from X. By adding ∞ to the triples we find a one-one correspondence between the set of triples $\binom{X}{3}$ and $\frac{1}{2}\binom{Y}{4}$, the set of partitions of Y into two quadruples. Thus we have a bijective function g^*,

$$g^*: \{l: l \text{ is a line in } PG(3, 2)\} \rightarrow \tfrac{1}{2}\binom{Y}{4},$$

such that a line l is in a spread S if and only if $g(S)$ is contained in one of the quadruples of $g^*(l)$. (Then we say that the partition (quadruple) is covered by the triple.)

From this observation it follows that $GL(4, 2)$ is isomorphic to a subgroup of S_8, and since $|GL(4, 2)| = (8!)/2$, we have $GL(4, 2) \simeq A_8$. Furthermore, under the action of A_8, all $56 = \binom{8}{3}$ spreads of $PG(3, 2)$ are equivalent.

By defining $h = g \circ f$ and $h^* = g^* \circ f^*$, we get bijective functions

$$h : \{\{l, \sigma(l)\}: l \text{ is a line in } PG(3, 4)\backslash PG(3, 2)\} \to \binom{Y}{3},$$
$$h^*: \{\{x, \sigma(x)\}: x \in PG(3, 4)\backslash PG(3, 2)\} \to \tfrac{1}{2}\binom{Y}{4},$$

such that $x \in l$ or $x \in \sigma(l)$ if and only if $h(\{l, \sigma(l)\})$ is contained in one of the quadruples of $h^*(\{x, \sigma(x)\})$, where x is a point and l is a line in $PG(3, 4)\backslash PG(3, 2)$.

Furthermore, for lines l and m, such that $l \neq m, \sigma(m)$, we have that

$l, \sigma(l), m$ and $\sigma(m)$ are disjoint if and only if $|h(\{l, \sigma(l)\}) \cap h(\{m, \sigma(m)\})| = 1$

and

l intersects m or $\sigma(m)$, but not both, if and only if

$$|h(\{l, \sigma(l)\}) \cap h(\{m, \sigma(m)\})| = 2.$$

3. Spreads Which Are Invariant Under σ

First, we shall classify the spreads of $PG(3, 4)\backslash PG(3, 2)$ which are invariant under σ. For this, we need an old result. (See [2] for example.)

THEOREM. *There are* 240 *resolutions of* $PG(3, 2)$, *which fall into two orbits of size* 120, *under the action of* $GL(4, 2)$.

Proof. A resolution of $PG(3, 2)$ corresponds to seven triples from Y all of whose mutual intersections have size one. The only way to do this is to make the seven triples the lines of a Fano plane (STS(7)) on seven of the eight points of Y. Since the order of the automorphism group of the Fano plane is 168, there are $240 = 8!/168$ resolutions of $PG(3, 2)$, which fall (under the action of $GL(4, 2)$) into two orbits of size 120. □

Because of the one-one correspondence f, the 240 spreads of $PG(3, 4)\backslash PG(3, 2)$, which are invariant under σ, also fall into two orbits of size 120, under the action of $GL(4, 2)$. Since the spreads are invariant under σ, they also fall into two orbits of size 120 under the action of the automorphism group of $PG(3, 4)\backslash PG(3, 2)$.

4. Spreads Which Are Not Invariant Under σ

Next, we shall classify the spreads which are not invariant under σ. We shall prove that they are all equivalent under the action of the automorphism group of $PG(3, 4)\backslash PG(3, 2)$, and they all contain one pair of lines which is invariant under σ. To do this, we need the following lemma.

LEMMA. *Every spread of* $PG(3, 4)\backslash PG(3, 2)$ *contains at least one pair of lines which is invariant under* σ.

Proof. Suppose there is a spread which contains no pair $\{l, \sigma(l)\}$ of lines, then it follows from the correspondence above—for every line l in the spread we get a triple $h(\{l, \sigma(l)\})$—that there is a set T of fourteen triples from a set Y of eight elements, such that for every triple $t \in T$ there are five triples in T which intersect t in two points, and eight triples in T which intersect t in one point, and such that every partition of Y into two quadruples is covered by two triples in T. (Note that these two triples intersect in two points.) So suppose such a set T exists.

Let $Y = \{1, 2, \ldots, 8\}$. Without loss of generality we may assume that $\{1, 2, 3\} \in T$. We shall call the five triples in T which intersect $\{1, 2, 3\}$ in two points 2-triples and the other eight triples in T, which intersect $\{1, 2, 3\}$ in one point, 1-triples.

Let s and t be 2-triples and suppose s and t intersect in a point x outside $\{1, 2, 3\}$, then the quadruple $\{1, 2, 3, x\}$ is covered by the three triples $\{1, 2, 3\}$, s and t, which is a contradiction. Thus $s \cap t \subset \{1, 2, 3\}$, and so for every $x = 4, 5, 6, 7, 8$, there is one 2-triple which contains x.

Without loss of generality we may assume that $\{1, 4, 5\}$ is one of the 1-triples. Since it must intersect all 2-triples, the 2-triples which contain 6, 7 or 8 also contain 1. Without loss of generality we assume that there are 2-triples $\{1, 2, 7\}$ and $\{1, 2, 8\}$.

Any of the quadruples $\{1, 4, 5, z\}$, $z = 6, 7, 8$, must be covered by another triple, which intersects $\{1, 4, 5\}$ in two points, and so it must be covered by $\{1, y, z\}$, $y = 4$ or 5. Without loss of generality we assume there is a 1-triple $\{1, 4, 8\}$.

Now there is no 1-triple containing 3, because such a triple cannot intersect all three triples $\{1, 2, 7\}$, $\{1, 2, 8\}$ and $\{1, 4, 5\}$.

Now the quadruple $\{1, 3, 4, 8\}$ is covered once, so T must contain $\{1, 3, 4\}$ (the triple $\{3, 4, 8\}$ is a 1-triple containing 3, and $\{1, 3, 8\}$ would be a second 2-triple containing 8). The 2-triple containing 5 cannot be $\{2, 3, 5\}$, since it must intersect $\{1, 4, 8\}$, and it cannot be $\{1, 3, 5\}$, otherwise the quadruple $\{1, 3, 4, 5\}$ would be covered three times, so it must be $\{1, 2, 5\}$.

Recall that we must have a triple $\{1, y, 7\}$, $y = 4$ or 5. If $y = 5$, then the quadruple $\{1, 2, 5, 7\}$ is covered three times, so T contains $\{1, 4, 7\}$.

Now, if there is a 1-triple containing 2, then it must also contain 4, as it must intersect $\{1, 3, 4\}$. Since the quadruples $\{1, 2, 4, z\}$, $z = 5, 7, 8$, are already covered twice, this triple must be $\{2, 4, 6\}$. So the number of 1-triples containing 1 is at least seven. Since the quadruples $\{1, 2, 5, 7\}$, $\{1, 2, 5, 8\}$, $\{1, 2, 7, 8\}$ are already covered twice, a 1-triple containing 1 must contain 4 or 6. But there are only seven of them, so we find 1-triples $\{1, 4, 6\}$, $\{1, 4, x\}$, $\{1, 6, x\}$, $x = 5, 7, 8$. (Note that we already found $\{1, 4, x\}$ before.)

We already knew that the 2-triple containing 6 also contains 1. So we have two possibilities. If T contains $\{1, 2, 6\}$, then the quadruple $\{1, 2, 5, 6\}$ is covered three times, so T must contain $\{1, 3, 6\}$. But now the quadruple $\{1, 3, 4, 6\}$ is covered three times, so we have a contradiction. □

THEOREM. *There are* 560 *spreads of* $PG(3, 4)\backslash PG(3, 2)$, *which are not invariant under* σ, *and they are all equivalent under the action of its automorphism group.*

Proof. Let S be a spread of $PG(3, 4)\backslash PG(3, 2)$, which is not invariant under σ. For the corresponding set T of triples from $Y = \{1, 2, \ldots, 8\}$—for every line l in S we get a triple $h(\{l, \sigma(l)\})$—we have the following properties. Either a triple t intersects all other triples in one point—t corresponds to a pair $\{L, \sigma(L)\}$ of lines in S—or t intersects five other triples in two points and the remaining triples in one point, and then any of the five partitions into two quadruples, which are covered by t, are also covered by one of the five triples which intersect t in two points—t corresponds to a line $M \in S$, such that $\sigma(M) \notin S$.

In the previous lemma it was proven that S contains a pair of lines $\{L, \sigma(L)\}$, and since S is not invariant under σ, there is a line $M \in S$, such that $\sigma(M) \notin S$. Without loss of generality we may assume that $\{L, \sigma(L)\}$ corresponds to the triple $\{1, 2, 3\}$, and that M corresponds to $\{1, 4, 5\}$. Now all other triples of T intersect $\{1, 2, 3\}$ in one point and there are five triples which intersect $\{1, 4, 5\}$ in two points. Each of the partitions containing the quadruple $\{1, x, 4, 5\}$, $x = 2, 3$ must be covered by one of these five triples, so there must be triples $\{2, 4, 5\}$ and $\{3, 4, 5\}$ in T. For any of the triples $\{x, 4, 5\}$, $x = 1, 2, 3$, there must be three other triples which intersect it in two points, and which cover the partitions containing the quadruple $\{x, 4, 5, z\}$, $z = 6, 7, 8$. So for every $x = 1, 2, 3$ and $z = 6, 7, 8$ there must be a triple $\{x, y, z\}$, $y = 4$ or 5. Without loss of generality we may assume that there is a triple $\{1, 4, 6\}$. This triple intersects $\{1, 4, 5\}$ in two points, so there must be four other triples which intersect $\{1, 4, 6\}$ in two points. These triples must be $\{1, 4, 7\}$, $\{1, 4, 8\}$, $\{2, 4, 6\}$ and $\{3, 4, 6\}$. Thus we find triples $\{x, 4, z\}$, $x = 1, 2, 3$, $z = 6, 7, 8$. Now we have proven that every set of triples which satisfies the required properties, is equivalent to T (under the action of A_8), which has the following point-triple incidence matrix. (The triple $\{1, 2, 3\}$ occurs twice because both lines L and $\sigma(L)$ are in S.)

$$
\begin{array}{c|cccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & & \\
2 & 1 & 1 & & & & & 1 & 1 & 1 & 1 & & & & \\
3 & 1 & 1 & & & & & & & & & 1 & 1 & 1 & 1 \\
4 & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & & & 1 & & & & 1 & & & & 1 & & & \\
6 & & & & 1 & & & & 1 & & & & 1 & & \\
7 & & & & & 1 & & & & 1 & & & & 1 & \\
8 & & & & & & 1 & & & & 1 & & & & 1
\end{array}
$$

Now we see that we can form $280 = 5\binom{8}{3}$ different sets of triples which satisfy the required properties, and they are all equivalent under the action of A_8.

Suppose it is possible to construct a spread S of $PG(3, 4)\backslash PG(3, 2)$ which corresponds to the set of triples T. Both lines L and $\sigma(L)$ corresponding to the triple $\{1, 2, 3\}$ are in S. For any of the other triples we have to choose one of the two lines M and $\sigma(M)$ which correspond to the triple. But we only have to make this choice once. Say we choose M for the triple $\{1, 4, 5\}$. Now one of the lines corresponding to $\{1, 4, 6\}$ intersects M, so we must choose the other line. It is easily seen that we can—we supposed it was possible to construct it—complete the spread uniquely (such that it corresponds to T).

So, if it is possible to construct a spread S of $PG(3, 4)\backslash PG(3, 2)$ which corresponds to the set of triples T, then there are two spreads, S and $\sigma(S)$, corresponding to T. Because there are 280 different sets of triples T, we find that there are at most 560 different spreads of $PG(3, 4)\backslash PG(3, 2)$, which are not invariant under σ.

Now the following fourteen lines form a spread of $PG(3, 4)\backslash PG(3, 2)$, which is not invariant under σ. (Here $GF(4) = \{0, 1, a, b\}$.)

$001a$	$1a00$	$1a1a$	$1aab$	$1ab1$
$001b$	$1b00$	$1b1b$	$1bba$	$1ba1$
$100a$	$01a0$	$11aa$	$1aba$	$1b1a$
$100b$	$01ba$	$11b1$	$1a10$	$1baa$
$10a0$	$01aa$	$110a$	$1a1b$	$1bb1$
$10b0$	$011b$	$11ab$	$1a11$	$1b0a$
$10a1$	$011a$	$11bb$	$1a0a$	$1b10$
$10b1$	$010b$	$11ba$	$1ab0$	$1bbb$
$101a$	$01bb$	$11a1$	$1a0b$	$1bb0$
$101b$	$01ab$	$11b0$	$1aaa$	$1b01$
$10ab$	$01b0$	$111b$	$1abb$	$1b0b$
$10ba$	$01b1$	$110b$	$1aa0$	$1b11$
$10aa$	$010a$	$11a0$	$1aa1$	$1bab$
$10bb$	$01a1$	$111a$	$1a01$	$1ba0$

We know that this spread S corresponds to a set of triples T. By letting $GL(4, 2)$ act on S and $\sigma(S)$ (and correspondingly, A_8 on T), we find 560 spreads of $PG(3, 4)\backslash PG(3, 2)$, which are not invariant under σ, and they are all equivalent under the action of its automorphism group. □

From the proof, it follows that any of the 560 spreads contains (exactly) one pair of lines $\{L, \sigma(L)\}$. Combining the results of Sections 3 and 4, we get the following theorem.

THEOREM. *The* 800 *spreads of* $PG(3, 4)\backslash PG(3, 2)$ *fall into three orbits of sizes* 120, 120 *and* 560, *under the action of its automorphism group.*

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