



No. 2007–11

**A TAXONOMY OF BEST-REPLY MULTIFUNCTIONS IN
2 X 2 X 2 TRIMATRIX GAMES**

By Carlos González-Alcón, Peter Borm, Ruud Hendrickx,
Kim van Kuijk

January 2007

ISSN 0924-7815

A taxonomy of best-reply multifunctions in $2 \times 2 \times 2$ trimatrix games*

Carlos González-Alcón¹ Peter Borm²
Ruud Hendrickx² Kim van Kuijk²

Abstract

This paper provides an overview of the various shapes the best-reply multifunctions can take in $2 \times 2 \times 2$ trimatrix games. It is shown that, unlike in 2×2 bimatrix games, the best replies to the opponents' pure strategies do not completely determine the structure of the Nash equilibrium set.

Key words: non-cooperative games, best-reply multifunction, Nash equilibrium

JEL code: C71

1 Introduction

Vorobev (1958) introduced an easy graphical way of representing best-reply multifunctions for 2×2 bimatrix games and used this method to find all the Nash equilibria. These best-reply multifunction can take only a few forms. Moreover, the best-replies to the pure strategies of one's opponent

*Part of this research was conducted while the first author was visiting Tilburg University thanks to the support of Dirección General de Universidades e Investigación del Gobierno de Canarias. Financial support from the Ministerio de Ciencia y Tecnología, under project TIC2000-1750-C06-02 is gratefully acknowledged. Ruud Hendrickx acknowledges financial support from the Netherlands Organisation for Scientific Research (NWO).

¹Corresponding author. Departamento de Estadística, I.O. y Computación Universidad de La Laguna, 38071 La Laguna, Spain. E-mail: cgalcon@ull.es.

²CentER and Department of Econometrics and Operations Research, Tilburg University, PO Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: P.E.M.Borm@uvt.nl, ruud@uvt.nl.

completely determine the structure of the set of Nash equilibria. Borm (1987) enumerated all these different shapes.

In $2 \times 2 \times 2$ trimatrix games, the picture is a bit more complicated. Not only are there simply more possibilities, but there is an added complication in that in three dimensions, it is not sufficient any more to know what happens in the extreme points. In this short paper we provide a taxonomy of all the possible shapes of the best-reply multifunctions and give an indication of what the set of Nash equilibria can look like.

2 Best-reply functions of 2×2 games

In this section we briefly review the study of best-reply multifunctions in 2×2 bimatrix games in Borm (1987).

Let (N, S, P) be a 2-person game in strategic form where $S = S_1 \times S_2$ is the space of strategies where each set S_i of pure strategies of player i has only two elements denoted by 1 and 0. Functions $P_i : S \rightarrow \mathbb{R}$ are the payoffs of Player i ($i = 1, 2$). Without loss of generality we develop our exposition for Player 2.

We note by a_{ij} the payoff to Player 2 when Players 1 and 2 use their pure strategies i and j , respectively. Mixed strategies of each player are determined by a value in the $[0, 1]$ interval giving the probability of using the first pure strategy. So, the pure strategies 1 and 0 can be identified with values for p of 1 and 0, respectively.

Given a mixed strategy profile $(p, q) \in [0, 1]^2$, the payoff function of Player 2 can be expressed as

$$P_2(p, q) = (p, 1 - p) \begin{bmatrix} a_{11} & a_{10} \\ a_{01} & a_{00} \end{bmatrix} \begin{pmatrix} q \\ 1 - q \end{pmatrix}.$$

Denoting by U the $[0, 1]$ interval, the general best-reply multifunction of Player 2 against a mixed strategy $p \in U$ is given by

$$B_2(p) = \begin{cases} 0, & \text{if } pa_{11} + (1 - p)a_{01} < pa_{10} + (1 - p)a_{00} \\ U, & \text{if } pa_{11} + (1 - p)a_{01} = pa_{10} + (1 - p)a_{00} \\ 1, & \text{in other case} \end{cases} \quad (1)$$

So, we can say that the shape of B_2 depends on the sign of the linear function

$$p(a_{11} - a_{10}) + (1 - p)(a_{01} - a_{00}). \quad (2)$$

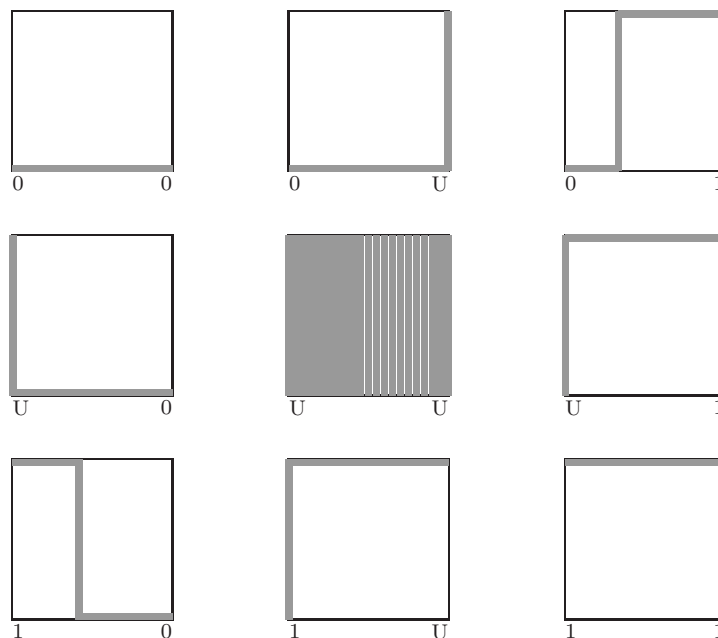


Figure 1: The nine different shapes of best-reply multifunction for the second player in 2×2 games, $B_2(p)$. The best response against pure strategies of Player 1 is indicated in the bottom side of each unit square: on the left against 0, on the right against 1.

Where this function is positive, the best-reply is the pure strategy 1, whereas 0 is the best response where it is negative; where (2) is null any strategy is a best-reply. By linearity of expression (2) the shape of B_2 is given by the signs of $a_{11} - a_{10}$ and $a_{01} - a_{00}$, ie the best response in the extreme points of the $[0,1]$ interval. Since each of these amounts can be positive, negative or zero, we have nine possible shapes for the best-reply functions as Figure 1 shows. Note that in almost all the cases the shape is completely determined. Only in the 0-1 and 1-0 cases a certain point of change must be calculated in which (2) is null.

3 Best-replies in $2 \times 2 \times 2$ games

Now, let (N, S, P) be a 3-person game in strategic form, and as above, $S_i = \{0, 1\}$ ($i = 1, 2, 3$). To simplify the discussion, we make our exposition for Player 3. We note by

$$A_1 = \begin{bmatrix} a_{111} & a_{101} \\ a_{011} & a_{001} \end{bmatrix} \quad \text{and} \quad A_0 = \begin{bmatrix} a_{110} & a_{100} \\ a_{010} & a_{000} \end{bmatrix}$$

the Player 3's payoffs matrix from pure strategies: A_1 for the Player 3's first pure strategy (1) and A_0 for the second pure strategy (0). The subscripts indicate the first, second and third player's pure strategy, respectively.

Given a mixed strategy profile $(p, q, r) \in [0, 1]^3$, the payoff function of Player 3 is

$$\begin{aligned} P_3(p, q, r) &= (p, 1-p)[rA_1 + (1-r)A_0] \begin{pmatrix} q \\ 1-q \end{pmatrix} \\ &= r \cdot (p, 1-p)A_1(q, 1-q)^t + (1-r) \cdot (p, 1-p)A_0(q, 1-q)^t. \end{aligned} \quad (3)$$

Expression (3) gives us an easy way to obtain the best-reply multifunction for Player 3, given the Player 1 and 2's strategies $p, q \in [0, 1]$:

$$B_3(p, q) = \begin{cases} 0, & \text{if } (p, 1-p)(A_1 - A_0)(q, 1-q)^t < 0 \\ U, & \text{if } (p, 1-p)(A_1 - A_0)(q, 1-q)^t = 0. \\ 1, & \text{in other case} \end{cases} \quad (4)$$

The unit square $[0, 1]^2$ in which the pair (p, q) moves can be divided into two regions in which the best reply is 1 and 0, respectively. The border between these regions is given by the points (p, q) satisfying

$$(p, 1-p)(A_1 - A_0) \begin{pmatrix} q \\ 1-q \end{pmatrix} = 0. \quad (5)$$

On these points the best replies are any point in the $[0, 1]$ interval.

Equation $(p, 1-p)(A_1 - A_0)(q, 1-q)^t = 0$ gives a hyperbolic paraboloid in the space (p, q, r) . It is one of the three ruled surfaces: for each point on the surface there are two lines on the surface passing it. In our case, these two lines are parallel to the pq axis (see Figure 2). As a degenerate case, the surface can be a plane (the other ruled surface together with the hyperboloid).

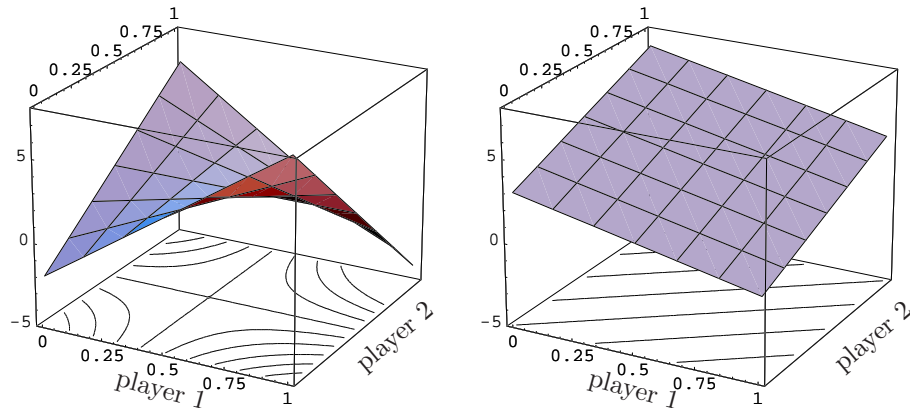


Figure 2: Player 3's payoff function is a section of an hyperbolic paraboloid (left) and or a plane (right). In the bottom, some level curves are shown.

In order to study the best-reply multifunctions we have to determine the sign that the left side of Equation (5) takes for each point (p, q) in the unit square. For points with positive sign, the best reply is the pure strategy 1. If the sign is negative, the best-reply of Player 3 is the second pure strategy, 0. The expression is zero on the indifference points: both pure strategies are equally good response and indeed, each mixed strategy $(r, 1 - r)$ is a best-reply strategy.

The level curves of a hyperbolic paraboloid are hyperbolas. Furthermore, in this case its axes are parallel to the coordinate axes (and the axes are indeed level curves for some level) —see bottom part of the two boxes in Figure 2. In the degenerate case in which the surface is a plane, the level curves are straight lines. Figure 3 shows different possibilities of regions. The different situations of the unit square with respect to the zero-level curves determine the different forms the best-reply multifunction can adopt. However, the trimatrix case is more complicated than the previous 2-person games.

As a starting point, we could have a look at the signs of the four elements of the matrix $A_1 - A_0$, as they give the best response to the four possible combinations of pure strategies for Player 1 and 2: $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. From the three possible signs of these differences (positive, negative or unsigned) we have $3^4 = 81$ cases to consider. Each corner of the strategy

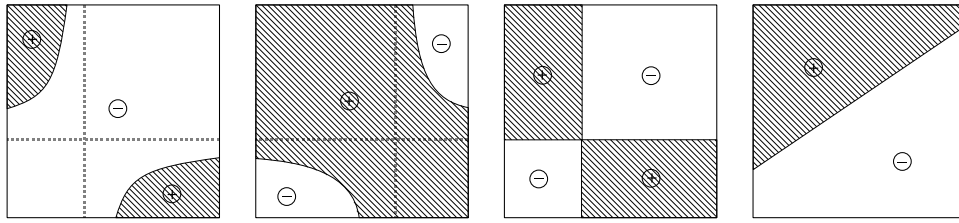


Figure 3: Different possibilities of sign (best replies) regions on the unit square. Plus and minus signs can be interchanged to obtain other four possibilities. Of course, a trivial case with no change of sign could be added.

square can be labelled with 0, 1 or U by the best reply against it. Since many of these cases can be obtained from others by relabelling the strategies of the players, we group them in the thirteen classes of Table 1.

Although we classify the best-reply multifunctions by the best response to the four combinations of pure strategies (the four corners of the unit square), the exact shape of that function can vary. Figure 4 shows an example of three different shapes belonging to the same best-reply pattern against combinations of pure strategies. These three (sometimes four, as in case 4) possibilities are indicated in Table 1 by grey lines.

Table 1: Classification of best-reply multifunctions of $2 \times 2 \times 2$ games. The 4-tuples represent best reply to the (0,0), (0,1), (1,1) and (1,0) pure strategies combinations of the other players. In the square how could go the indifference curves is indicated.

class	best-reply pattern	class	best-reply pattern																		
1	<div style="display: flex; align-items: center; gap: 20px;"> <div style="border: 1px solid black; padding: 5px; text-align: center;"> <table style="border-collapse: collapse;"> <tr><td style="padding: 5px;">0</td><td style="padding: 5px;">0</td></tr> <tr><td style="padding: 5px;">0</td><td style="padding: 5px;">0</td></tr> </table> </div> <div style="text-align: center;"> <table style="border-collapse: collapse;"> <tr><td style="padding: 5px;">0000</td><td style="padding: 5px;">1111</td></tr> </table> </div> </div>	0	0	0	0	0000	1111	2	<div style="display: flex; align-items: center; gap: 20px;"> <div style="border: 1px solid black; padding: 5px; text-align: center;"> <table style="border-collapse: collapse;"> <tr><td style="padding: 5px;">0</td><td style="padding: 5px;">0</td></tr> <tr><td style="padding: 5px;">0</td><td style="padding: 5px;">1</td></tr> </table> </div> <div style="text-align: center;"> <table style="border-collapse: collapse;"> <tr><td style="padding: 5px;">0001</td><td style="padding: 5px;">1110</td></tr> <tr><td style="padding: 5px;">1000</td><td style="padding: 5px;">0111</td></tr> <tr><td style="padding: 5px;">0100</td><td style="padding: 5px;">1011</td></tr> <tr><td style="padding: 5px;">0010</td><td style="padding: 5px;">1101</td></tr> </table> </div> </div>	0	0	0	1	0001	1110	1000	0111	0100	1011	0010	1101
0	0																				
0	0																				
0000	1111																				
0	0																				
0	1																				
0001	1110																				
1000	0111																				
0100	1011																				
0010	1101																				

Table 1: Continuation

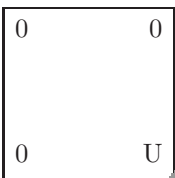
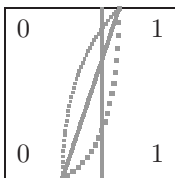
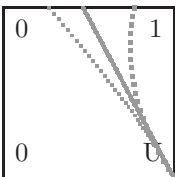
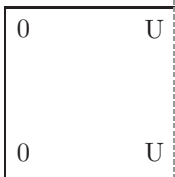
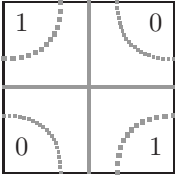
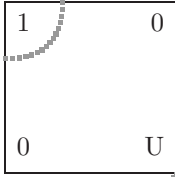
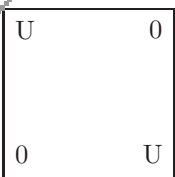
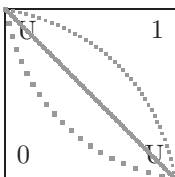
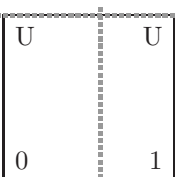
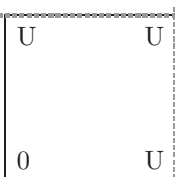
class	best-reply pattern	class	best-reply pattern
3		000U 111U	
		U000 U111	
4		0011 1100	
		1001 0110	
5		001U 110U	
		U001 U110	
6		1U00 0U11	
		01U0 10U1	
7		00U1 11U0	
		100U 011U	
8		U100 U011	
		0U10 1U01	
9		010U 101U	
		U010 U101	
10		00U1 11U0	
		U0U1 U1U0	
11		010U 101U	
		U010 U101	
12		00U1 11U0	
		U0U1 U1U0	

Table 1: Continuation

class	best-reply pattern	class	best-reply pattern				
13	<div style="border: 1px solid black; display: inline-block; padding: 5px;"> <table style="border-collapse: collapse;"> <tr> <td style="padding: 2px 10px;">U</td> <td style="padding: 2px 10px;">U</td> </tr> <tr> <td style="padding: 2px 10px;">U</td> <td style="padding: 2px 10px;">U</td> </tr> </table> </div>	U	U	U	U	UUUU	
U	U						
U	U						

4 Nash equilibria

An interesting aspect of studying the best-reply multifunctions is to obtain the Nash equilibria. We have a Nash equilibrium in each point in which the best replies of all the players coincide. In trimatrix games a point $(p, q, r) \in [0, 1]^3$ constitutes a Nash equilibrium if the best-reply multifunctions satisfy

$$p \in B_1(q, r), \quad q \in B_2(p, r) \quad \text{and} \quad r \in B_3(p, q).$$

Given the three best-reply multifunctions of the players of a game, we are able to obtain the Nash equilibria set. For example, suppose that the three best-reply multifunctions belong to the first case of class 9, which we represent by 0U0U. Figure 5 shows consecutively the best replies of player 1, 2 and 3 and the intersection with the previous player. The last one is the whole Nash equilibria set: four isolated corner points.

Figure 6 shows an example of how the Nash equilibria can adopt different combinations of isolated points, lines and surfaces.

As the interaction among the best-reply functions gives the Nash equilibria set of the game, one could think that the best-reply functions can be used to classify games by their equilibria. This argument works perfectly for 2×2 games, in which the shape of the interaction between the same class of best-reply functions of the two players presents the same structure. For instance, when the players have both a best-reply multifunction of type 1-0, the equilibria set has always three elements (two pure strategy equilibria $(1,0)$ and $(0,1)$, plus a mixed equilibrium) no matter where the multifunctions change

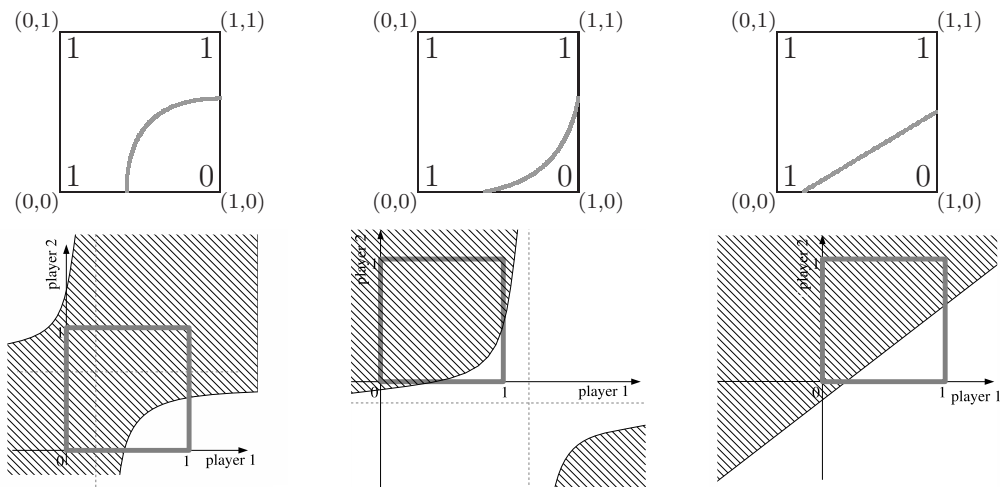


Figure 4: Three different shapes of best-reply function with the same best responses against the pure strategies. It is the case 1110, from class 2.

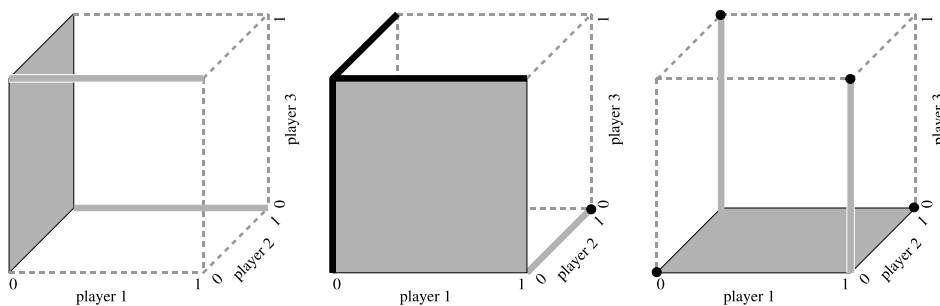


Figure 5: Left to right: best-reply multifunctions of Players 1, 2 and 3 showing a 0U0U pattern. The dark zones in the second graph is the intersection of Player 1 and 2's best-replies; in the third one is the Nash equilibria set.

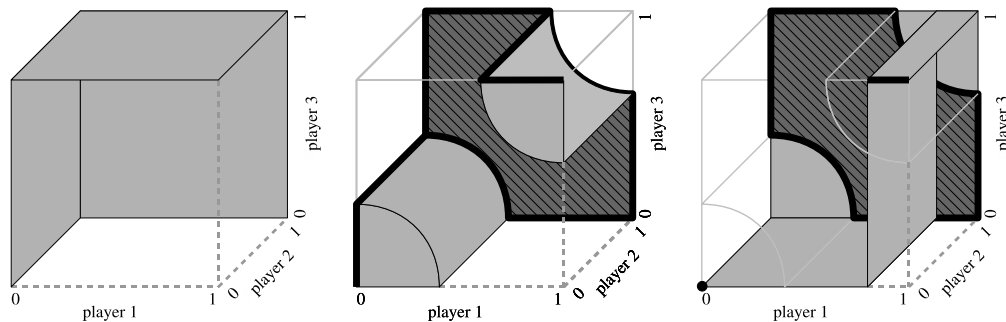


Figure 6: Left to right: best reply multifunctions of Players 1 (0UUU), 2 (0101) and 3 (0UU1). Now, in the third graph can be seen that the Nash equilibria set can be a composition of points, straight segments and surfaces.

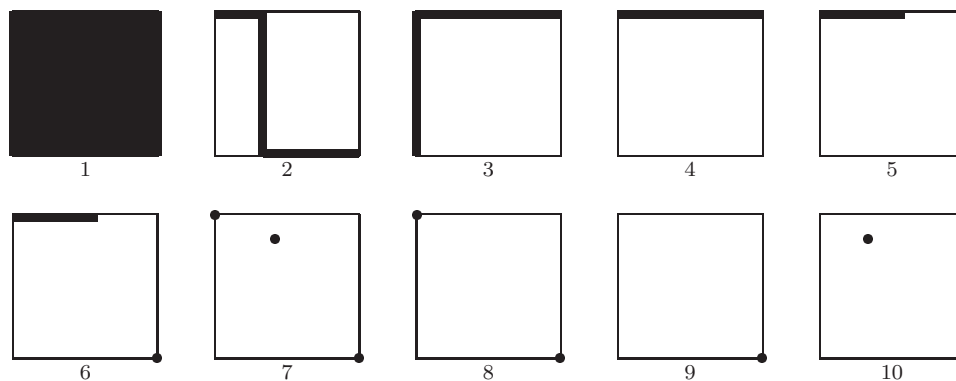


Figure 7: The ten different shapes of Nash equilibria in 2×2 games produced by the 81 combinations of the nine best-reply multifunctions of Figure 1. Each draw represent to the whole group of its symmetries.

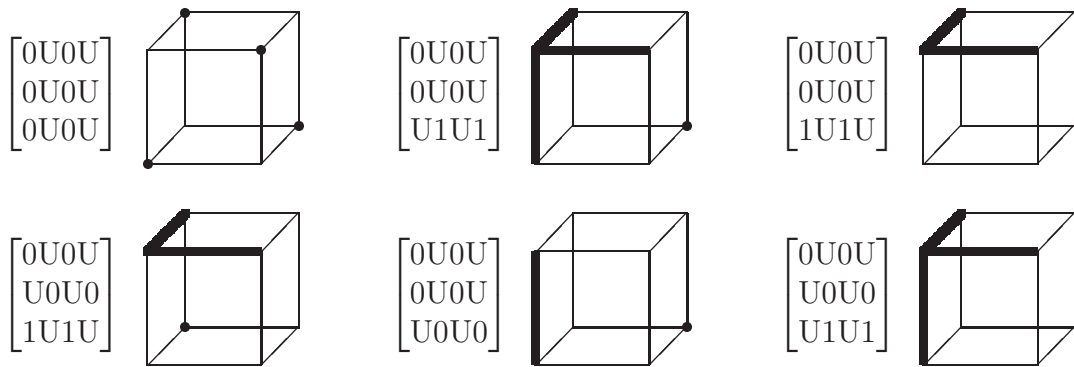


Figure 8: Examples of shapes of the Nash equilibria sets from the interaction of different best replies from class 9.

from 1 to 0. From nine classes of best-reply function, the 81 interactions can be classified in the ten shapes shown by Figure 7.

However, in the $2 \times 2 \times 2$ case, although the classes group best-reply multifunctions showing similar shapes, the interaction among function in the same class can lead to different equilibria sets. Figure 8 shows examples of shapes of the Nash equilibria sets from the interaction of different best replies patterns all belonging to class 9.

Moreover, we can obtain different equilibria set configurations from the interaction of best-reply functions with the same pattern, as Figure 9 shows. The three players have a best-reply multifunction with pattern 010U and indifference curves with similar curvatures, but for slightly different parameters the Nash equilibria set varies.

References

- Borm, P. (1987). A classification of 2×2 bimatrix games. *Cahiers du Centre d'Études de Recherche Opérationnelle*, **29**, 69–84.
- Vorobev, N. (1958). Equilibrium points in bimatrix games. *Theory Probab. Appl.*, **3**, 279–309.

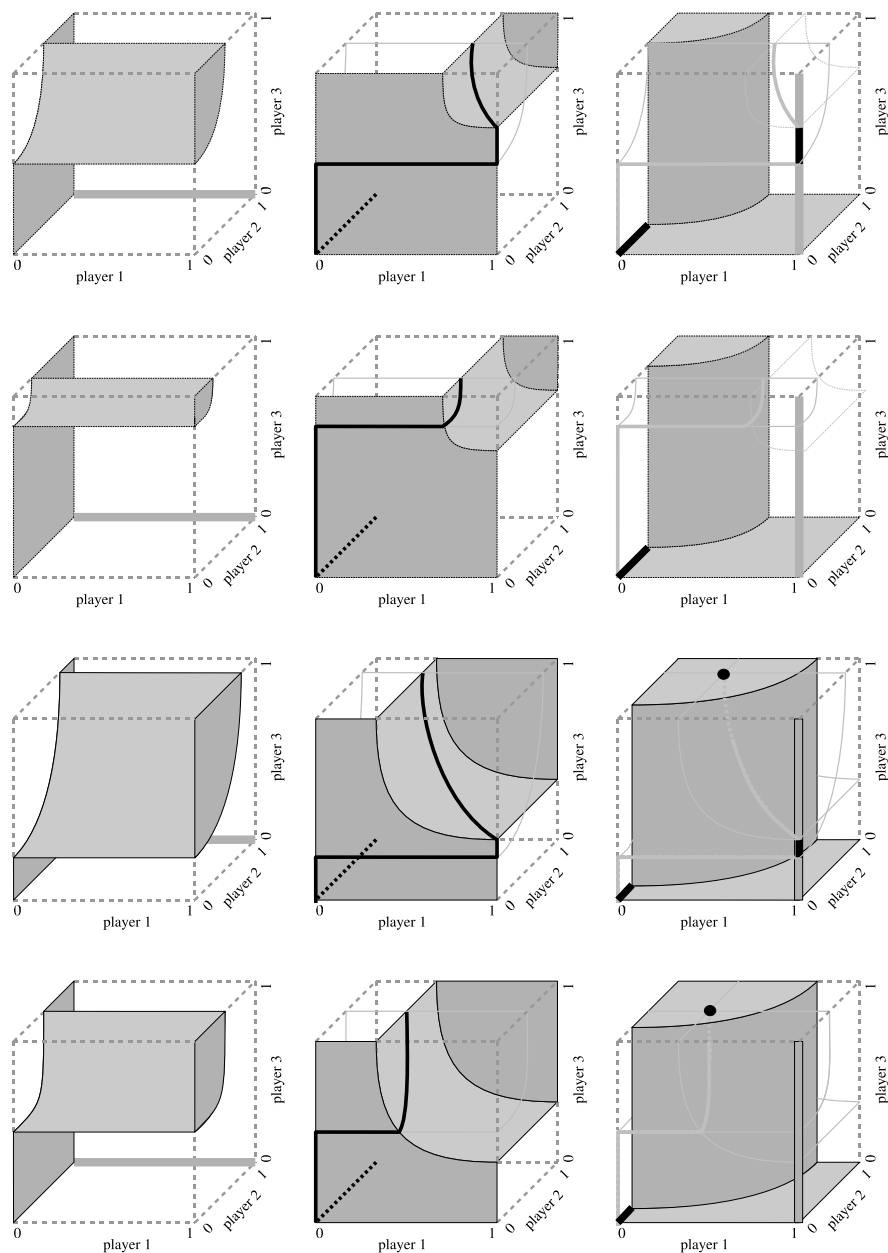


Figure 9: Different Nash equilibrium sets (dark region in the right-hand graphs) from the same best-reply functions combination: a 010U pattern for all players.