NBER WORKING PAPER SERIES

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Working Paper 12567
http://www.nber.org/papers/w12567

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
October 2006

A preliminary version of this paper with the title "On Dynamic Heckscher-Ohlin Models II: Infinitely-Lived Consumers" was circulated in January 2003. We are grateful for helpful comments from participants in conferences and seminars at the 2003 American Economic Association Meeting, the Universidad Carlos III de Madrid, the University of Pennsylvania, the International Monetary Fund, the Centre de Recerca en Economia del Benestar, the Instituto Autonomo Tecnologico de Mexico, the 2004 CNB/CERGE-EI Macro Workshop in Prague, the Universitat Pompeu Fabra, Stanford University, the University of Texas at Austin, and the Banco de Portugal. We also thank Joann Bangs and Jaume Ventura for helpful discussions. Kehoe thanks the National Science Foundation for support. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. The views expressed herein are those of the author(s) and do not necessarily reflect the views of the National Bureau of Economic Research.

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JEL No. F11,F43,O15,O41

ABSTRACT

This paper studies the properties of a dynamic Heckscher-Ohlin model - a combination of a static two-good, two-factor Heckscher-Ohlin trade model and a two-sector growth model - with infinitely lived consumers where international borrowing and lending are not permitted. We obtain two main results: First, even if factor prices are equalized, countries that differ only in their initial endowments of capital per worker may converge or diverge in income levels over time, depending on the elasticity of substitution between traded goods. Divergence can occur for parameter values that would imply convergence in a world of closed economies and vice versa. Second, factor price equalization in a given period does not imply factor price equalization in future periods.

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1. Introduction

In 2004, GDP per capita in the United States was roughly 40,000 U.S. dollars. Using exchange rates to convert pesos to dollars, we calculate GDP per capita in Mexico in 2004 to be roughly 6,500 U.S. dollars. In 1935, the United States had income per capita of about 6,600 2004 U.S. dollars. To predict what will happen in the Mexican economy over the next 70 years, should we study what happened to the U.S. economy since 1935? Or should we take into account that, in 1935, the United States was the country with the highest income in the world, while, in 2004, Mexico had a very large trade relation with the United States — a country with a level of income per capita approximately six times larger? If we use a purchasing power comparison method to calculate Mexican GDP per capita in 2004, we come up with 9,800 U.S. dollars, which was roughly the U.S. level in 1941, but the qualitative nature of our question remains the same.

Much of the discussion of convergence of income levels in traditional growth theory relies on models of closed economies. (See, for example, Barro and Sala-i-Martin 2003.) In this paper we ask: Do the convergence results obtained in closed economy growth models change when we introduce trade? Specifically, we consider a dynamic Heckscher-Ohlin model — a combination of a static two-good, two-factor Heckscher-Ohlin trade model and a two-sector growth model — with infinitely lived consumers where borrowing and lending are not permitted. We find that introducing trade into the growth model radically changes the convergence results: In many environments where income levels converge over time if the countries are closed, for example, they diverge if the countries are open. This is because favorable changes in the terms of trade for poor countries reduce their incentives to accumulate capital.

The model that we use is a special case of the general dynamic Heckscher-Ohlin model studied by Bajona and Kehoe (2006). There are n countries that differ only in their population sizes and their initial endowments of capital. There are two traded goods that are produced using capital and labor; one of the goods is more capital intensive than the other. Time is discrete, and there is a nontraded investment good that is produced using the two traded goods. Consumers have utility functions that are homothetic and identical across countries. They combine the two traded goods to obtain utility in the same manner as firms combine these goods to obtain the investment good in the sense that the period utility function has the form $u(c_{1t}, c_{2t}) = \log f(c_{1t}, c_{2t})$, where f is the production function of the investment good. As we will see, this assumption allows us to reduce the calculation of equilibria in which all countries produce both traded goods in every period to the calculation of an equilibrium of an appropriately specified one-sector model.

The model that we study is both a classic Heckscher-Ohlin model and a classic growth model in the sense that the two factors of production are identified as labor and physical capital. A country that is capital abundant in the terminology of Heckscher-Ohlin theory is rich in the terminology of growth theory. It would be straightforward to redo the analysis for a model in which the two factors of production were labor and human capital. It would be more complicated to extend the analysis to a model with more than two factors of production. Nevertheless, even in models with more than two factors, we would expect the central message of this paper to carry over: Consider a model of closed economies in which countries become richer because they accumulate a factor, or factors, of production. Suppose that convergence in income levels is driven by returns to a factor being higher in countries that are poorer because they have less of the factor. Opening the economies in this model to international trade will reduce the returns to the factor, thereby reducing incentives to accumulate the factor and reducing the tendency towards convergence.

There is a large literature that is at least partially related to the topic studied here. Bardhan (1965) and Oniki and Uzawa (1965) study the patterns of specialization and trade in a Heckscher-Ohlin model in which consumers have fixed savings rates. Deardorff and Hanson (1978) consider a model in which these fixed savings rates differ across countries and show that the country with the higher savings rate will export the capital intensive good in the steady state. Stiglitz (1970) also considers models with fixed savings behavior, in this case a Marxian specification where all labor income is consumed and all capital income is saved. In addition, he considers a model in which there are infinitely lived, utility-maximizing consumers with different discount rates in each country. Stiglitz studies the pattern of trade and specialization in the steady state of this model and studies dynamic equilibrium paths in a small open economy version of the model.

Chen (1992) studies the long-run equilibria of two-country, dynamic Heckscher-Ohlin models with utility-maximizing agents and identical preferences in both countries under the assumption that both countries produce both goods. He finds that there is a continuum of steady states in such models and that, unless initial capital-labor ratios are equal, there is trade in the steady state. Chen also shows that cycles are possible in such models when one good is the consumption good and the other is the investment good if the consumption good is capital intensive. Baxter (1992) considers a model similar to Chen's but in which tax rates differ across countries. She shows that the pattern of trade and specialization in the steady state is determined by these taxes. Brecher, Chen, and Choudhri (2002) consider a model with differences in technologies across countries. Nishimura and Shimomura (2002), Bond, Trask, and Wang (2003), Doi, Nishimura, and Shimomura (2002), and Ono and Shibata (2005) study dynamic Heckscher-Ohlin models with endogenous growth or externalities.

A number of researchers have studied dynamic Heckscher-Ohlin models using the small open economy assumption: Findlay (1970), Mussa (1978), Smith (1984), Atkeson and Kehoe (2000), Chatterjee and Shukayev (2004), and Obiols-Homs (2005). Atkeson and Kehoe and Chatterjee and Shukayev are of particular relevance to our paper. Atkeson and Kehoe study a model in which the rest of the world is in its steady state and the small open economy starts with either a lower or a higher capital-labor ratio. They show that, if the small open economy is outside the rest of the world's cone of diversification, then the country converges to the boundary of this cone. If the small country starts inside the cone of diversification, then it too is in steady state and stays there. This result is in sharp contrast to our result that, for certain parameter values and initial conditions, even if all countries start in the cone of diversification, some necessarily leave it. Contrasting our results with those of Atkeson and Kehoe shows how strong their assumptions are that the rest of the world is in its steady state and that there are no general equilibrium price effects. Chatterjee and Shukayev consider a model similar to that of Atkeson and Kehoe in which there are stochastic productivity shocks and show that, over time, the comparative advantage conferred by different initial endowments can disappear over time.

The paper most closely related to ours is Ventura (1997), who studies trade and growth in a dynamic Heckscher-Ohlin model with utility-maximizing consumers and identical preferences across countries. He assumes that there are two traded goods — one capital intensive and one labor intensive — that are used in consumption and investment. Ventura abstracts away from studying the patterns of specialization by assuming that each good uses only one of the factors in its production process. Under this assumption, all countries produce both goods independently of their relative factor endowments. Ventura studies the evolution of capital stocks over time. Our paper differs from his in that (1) we use discrete time rather than continuous time because it makes it easier to obtain analytical results, although we show how our results can be extended to a continuous-time version of the model, (2) we study the evolution of income levels as well as of capital stocks, (3) we obtain conditions under which countries remain in the cone of diversification and under which they leave it in models with more general production structures, and (4) we study the possibility of equilibria in which one or more countries have zero investment in some periods, a possibility that is present in Ventura's (1997) model, but which is ignored. It is also worth mentioning the work of Cuñat and Maffezzoli (2004), who present numerical experiments using a three-good, two-factor version of the Ventura model.

In this paper we study the patterns of trade, capital accumulation, and income growth over time as a function of the countries' initial relative endowments of capital and labor. We find, as does Ventura (1997), that, if both countries diversify over the entire equilibrium path, the elasticity of substitution between traded goods is crucial in determining convergence behavior. This is no longer true when one of the countries specializes in production in some period. For a given elasticity of substitution, whether countries converge or diverge depends on the pattern of specialization over time. We present an example in which countries' income levels converge in equilibria without factor price equalization for an elasticity of substitution that implies divergence in income for equilibria with factor price equalization along the equilibrium path. We also present an example in which corner solutions in investment cause our convergence results to break down.

2. The general model

There are n countries, $i = 1,...,n$. Each has a continuum of measure L^i of identical, infinitely lived consumer-workers, each of whom is endowed with $\bar{k}_0^i$ units of capital in period 0 and one unit of labor in every period t, $t = 0,1,...$. There are three goods in the economy: an investment good, x, which is not traded, and two traded goods, y_j, $j = 1,2$, which can be consumed or used in the production of the investment good.

Each traded good j, $j = 1,2$, is produced with a constant returns to scale technology that uses capital, k, and labor, ℓ:

$$y_j = \phi_j(k_j, \ell_j) . \tag{1}$$

We assume that good 1 is relatively capital intensive and that the technologies are such that there are no factor intensity reversals. Producers minimize costs taking prices as given and earn zero profits. The first-order conditions from the producers' problems are

$$r \geq p_j \phi_{jK}(k_j, \ell_j), \quad = \quad \text{if} \quad k_j > 0 \tag{2}$$

$$w \geq p_j \phi_{jL}(k_j, \ell_j), \quad = \quad \text{if} \quad \ell_j > 0 \tag{3}$$

for each j, $j = 1,2$, where r is the rental rate, w is the wage, and p_j is the price of good j, $j = 1,2$. Additional subscripts — $\phi_{jK}(k_j, \ell_j)$, $\phi_{jL}(k_j, \ell_j)$ — denote partial derivatives.

The investment good is produced according to the constant-returns production function

$$x = f(x_1, x_2). \tag{4}$$

Letting q be the price of the investment good, the first-order conditions for profit maximization are

$$p_1 \geq qf_1(x_1, x_2), \quad = \quad \text{if} \quad x_1 > 0 \tag{5}$$

$$p_2 \geq qf_2(x_1, x_2), \quad = \quad \text{if} \quad x_2 > 0. \tag{6}$$

In each period, consumers decide how much of each traded good to consume, c_{1t}, c_{2t} and how much capital to accumulate for the next period, k_{t+1}. We assume that there is no international borrowing or lending. Bajona and Kehoe (2006) argue that allowing international borrowing and lending ensures factor price equalization but results in indeterminacy of production and trade in equilibrium.

The representative consumer in country i solves the maximization problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t u(c_{1t}^i, c_{2t}^i) \\ &\text{s.t. } p_{1t}c_{1t}^i + p_{2t}c_{2t}^i + q_t^i x_t^i \leq w_t^i + r_t^i k_t^i \\ &k_{t+1}^i - (1-\delta)k_t^i \leq x_t^i \\ &c_{jt}^i \geq 0, x_t^i \geq 0 \\ &k_0^i \leq \bar{k}_0^i, \end{aligned} \tag{7}$$

where β, $0 < \beta < 1$ is the common discount factor and δ, $0 \leq \delta \leq 1$, is the depreciation rate.

The feasibility condition for good j, $j = 1,2$, is

$$\sum_{i=1}^{n} L^i(c_{jt}^i + x_{jt}^i) = \sum_{i=1}^{n} L^i y_{jt}^i. \tag{8}$$

Labor and capital are perfectly mobile across sectors within a country, but not across countries. Therefore, the feasibility conditions in each country i, $i = 1,...,n$, are

$$k_{1t}^i + k_{2t}^i \le k_t^i \tag{9}$$

$$\ell_{1t}^i + \ell_{2t}^i \le 1. \tag{10}$$

Likewise, the investment good is nontraded and the feasibility condition in each country i is

$$x_t^i = f(x_{1t}^i, x_{2t}^i). \tag{11}$$

It is easy to show that allowing for trade of the investment good would only generate indeterminacy of trade in this model, without otherwise changing the set of equilibria.

Before analyzing the properties of the model described above, we list the main assumptions of the model:

A.1. There are n countries, which are populated by infinitely lived consumers. Countries differ only in their population sizes, $L^i > 0$, and their initial endowments of capital, $\bar{k}_0^i > 0$.

A.2. There are two traded goods, which can be consumed or used in the production of the investment good. The production functions of the traded goods, $\phi_j(k, \ell)$, are increasing, concave, continuously differentiable, and homogeneous of degree one.

A.3. Traded good 1 is relatively capital intensive, and there are no factor intensity reversals: For all $k / \ell > 0$,

$$\frac{\phi_{1L}(k/\ell,1)}{\phi_{1K}(k/\ell,1)} < \frac{\phi_{2L}(k/\ell,1)}{\phi_{2K}(k/\ell,1)}. \tag{12}$$

A.4. Labor and capital are perfectly mobile across sectors but are not mobile across countries.

A.5. There is an investment good in each country, which is not traded. The production function for the investment good, $f(x_1, x_2)$, is increasing, concave, continuously differentiable, and homogeneous of degree one.

A.6. The period utility function $u(c_1, c_2)$ is homothetic, strictly increasing, strictly concave, and twice continuously differentiable, and it satisfies $\lim_{c_j \to 0} u_j(c_1, c_2) = \infty$.

Definition 1: An *equilibrium* of the world economy is sequences of prices, $\{p_{1t}, p_{2t}, q_t^i, w_t^i, r_t^i\}$, consumptions, investments, and capital stocks $\{c_{1t}^i, c_{2t}^i, x_t^i, k_t^i\}$, production plans for the traded goods, $\{y_{jt}^i, k_{jt}^i, \ell_{jt}^i\}$, and production plans for the investment goods $\{x_t^i, x_{1t}^i, x_{2t}^i\}$, such that:

1. Given prices $\{p_{1t}, p_{2t}, q_t^i, w_t^i, r_t^i\}$, the consumptions and capital stocks $\{c_{1t}^i, c_{2t}^i, k_t^i\}$ solve the consumers' problem (7).
2. Given prices $\{p_{1t}, p_{2t}, q_t^i, w_t^i, r_t^i\}$, the production plans $\{y_{jt}^i, k_{jt}^i, \ell_{jt}^i\}$ and $\{x_t^i, x_{1t}^i, x_{2t}^i\}$ satisfy the cost minimization and zero profit conditions (2), (3), (5), and (6).
3. The consumption, capital stock, $\{c_{1t}^i, c_{2t}^i, k_t^i\}$, and production plans, $\{y_{jt}^i, k_{jt}^i, \ell_{jt}^i\}$ and $\{x_t^i, x_{1t}^i, x_{2t}^i\}$, satisfy the feasibility conditions (1), (4), (8), (9), (10), and (11).

Notice that, since trade equalizes the prices of the traded goods across countries, the prices of the investment good are also equal, $q_t^i = q_t$. Since the cost minimization problems are the same across countries, this is true even if some country i does not produce the investment good in period t. The homogeneity of the budget constraints in (7) and the cost minimization and zero profit conditions (2), (3), (5), and (6) in current period prices allow us to impose a numeraire in each period. We set

$$q_t = 1, \; t = 0,1,... \tag{13}$$

It is worth noting that the assumption of no international borrowing and lending implies that trade balance holds:

$$p_{1t}(y_{1t}^i - c_{1t}^i - x_{1t}^i) + p_{2t}(y_{2t}^i - c_{2t}^i - x_{2t}^i) = 0\,. \tag{14}$$

This condition can be derived from the budget constant in the consumer's problem (7) and the cost minimization and zero profit conditions (2), (3), (5), and (6).

Definition 2. A *steady state* of the world economy is consumption levels, an investment level, and a capital stock, $\{\hat{c}_1^i, \hat{c}_2^i, \hat{x}^i, \hat{k}^i\}$, factors of production and output for each traded industry, $\{\hat{y}_j^i, \hat{k}_j^i, \hat{l}_j^i\}$, $j = 1, 2$, factors of production and output for the investment sector $\{\hat{x}^i, \hat{x}_1^i, \hat{x}_2^i\}$, and prices $\{\hat{p}_1, \hat{p}_2, \hat{w}^i, \hat{r}^i\}$, for $i = 1, ..., n$, that satisfy the conditions of a competitive equilibrium for appropriate initial endowments of capital, $\bar{k}_0^i = \hat{k}^i$. Here we set $v_t = \hat{v}$ for all t, where v represents a generic variable.

We say that a steady state is a *nontrivial steady state* if at least one of the countries has a positive level of capital in that steady state: $\hat{k}^i > 0$ for some $i = 1, ..., n$.

The steady state results for general dynamic Heckscher-Ohlin models with infinitely lived consumers derived in Bajona and Kehoe (2006) apply to the Ventura model. The following propositions state them without proof.

Proposition 1: Under assumptions A.1-A.6, in any nontrivial steady state factor prices are equalized.

Proposition 2: Under assumptions A.1-A.6, if there exists a nontrivial steady state, then there exists a continuum of them, which have the same prices and world capital-labor ratio, $\hat{k}$. These steady states are indexed by the distribution of capital-labor ratios across countries, $\hat{k}^1, ..., \hat{k}^n$. Furthermore, international trade occurs in every steady state in which $\hat{k}^i \neq \hat{k}$ for some i.

3. The integrated economy

The characterization and computation of equilibrium of the model described in the previous section is difficult in general, since it involves determining the pattern of specialization in production over an infinite horizon. (See Bajona and Kehoe 2006 for some results on the equilibrium of the general model.) Numerical methods are usually needed to compute equilibrium. The characterization and computation of equilibrium becomes much easier, however, when the model specification is such that we can solve for the equilibrium by disaggregating the equilibrium of the *integrated economy* — a closed economy with initial factor endowments equal to the world endowments — which is equivalent to a two-sector growth model. (See Dixit and Norman 1980 for a description of the methodology.) In this case, the equilibrium prices and aggregate consumption, production, and investment of our economy coincide with the equilibrium prices, consumption, production, and investment of the integrated economy.

Consider the social planner's problem

$$\begin{aligned} &\max \ \sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}) \\ &\text{s.t. } c_{1t} + x_{1t} \le y_{1t} = \phi_1(k_{1t}, \ell_{1t}) \\ &c_{2t} + x_{2t} \le y_{2t} = \phi_2(k_{2t}, \ell_{2t}) \\ &k_{t+1} - (1-\delta)k_t \le x_t = f(x_{1t}, x_{2t}) \\ &k_{1t} + k_{2t} \le k_t \\ &\ell_{1t} + \ell_{2t} \le 1 \\ &c_{jt} \ge 0, x_{jt} \ge 0 \\ &k_0 \le \bar{k}_0, \end{aligned} \tag{15}$$

where $\bar{k}_0 = \sum_{i=1}^{n} L^i \bar{k}_0^i / \sum_{i=1}^{n} L^i$. Notice that assumption A.1 implies that $\bar{k}_0 > 0$.

Proposition 3: Suppose that the allocation $\{c_{1t}, c_{2t}, k_t\}$, $\{y_{1t}, k_{1t}, \ell_{1t}\}$, $\{y_{2t}, k_{2t}, \ell_{2t}\}$, $\{x_t, x_{1t}, x_{2t}\}$ solves the social planner's problem (15). Then this allocation, together with the prices $\{p_{1t}, p_{2t}, q_t, w_t, r_t\}$, is an equilibrium of the integrated economy where $q_t = 1$, $p_{1t} = f_1(x_{1t}, x_{2t})$, $p_{2t} = f_2(x_{1t}, x_{2t})$, $r_t = p_{1t}\phi_{1K}(k_{1t}, \ell_{1t})$, and $w_t = p_{1t}\phi_{1L}(k_{1t}, \ell_{1t})$. Conversely, suppose that $\{p_{1t}, p_{2t}, q_t, w_t, r_t\}$, $\{c_{1t}, c_{2t}, k_t\}$, $\{y_{1t}, k_{1t}, \ell_{1t}\}$, $\{y_{2t}, k_{2t}, \ell_{2t}\}$, $\{x_t, x_{1t}, x_{2t}\}$ is an equilibrium of the integrated economy. Then the equilibrium allocation solves the social planner's problem (15). Furthermore, if the social planner's problem has a solution, then it is the unique equilibrium allocation of the integrated economy.

Proof: The first claim is just the second theorem of welfare economics, and the second claim is the first theorem. In our setting, it is straightforward to prove these claims by showing that the first-order conditions and transversality condition for the social planner's problem are equivalent to the equilibrium conditions in the definition of equilibrium where there is only one country, $n = 1$. If the utility function is bounded on the constraint set of the social planner's problem, there exists a solution to this problem. Since the function u is strictly concave and the functions ϕ_1, ϕ_2, and f are concave, the solution to the planner's problem is unique, which implies that there is a unique equilibrium to the integrated economy. ■

Once we have the equilibrium of the integrated economy, to compute an equilibrium of the world economy, we need to disaggregate the consumption, investment, and production decisions across countries, to find, for example, c_{1t}^i, $i = 1,...,n$, such that

$$\sum_{i=1}^{n} L^i c_{1t}^i / \sum_{i=1}^{n} L^i = c_{1t}. \tag{16}$$

Whether an equilibrium can be solved this way is a guess-and-verify approach.

First consider the disaggregation of production decisions. If capital-labor ratios are very different across countries, assigning nonnegative production plans for both goods to all countries is not consistent with their having the same factor prices, and solving for equilibrium using the integrated approach is not possible. Figure 1, known as the Lerner diagram, shows the endowments of capital and labor that are consistent with using the integrated economy approach to solve for equilibrium for a static Heckscher-Ohlin model. Let p_1, p_2 be the equilibrium prices of the traded goods in the integrated economy. The rays k_1/ℓ_1 and k_2/ℓ_2 represent the capital-labor ratios used in the production of each good in the equilibrium of the integrated economy. The area between both rays is called the *cone of diversification*. If all countries have endowments of capital and labor in the cone of diversification, the equilibrium prices of the integrated economy are consistent with nonnegative production plans for both goods in all countries.

To find the cone of diversification, we solve the problem

$$\begin{aligned} \max\ & p_1\phi_1(k_1,\ell_1) + p_2\phi_2(k_2,\ell_2) \\ \text{s.t. } & k_1 + k_2 \le k \\ & \ell_1 + \ell_2 \le 1 \\ & k_j \ge 0,\ \ell_j \ge 0. \end{aligned} \tag{17}$$

If p_1 and p_2 are the equilibrium prices of the integrated economy, then, since c_{1t}^i and c_{2t}^i are both strictly positive by assumption A.6, the solution to this problem is such that $\phi_1(k_1,\ell_1) > 0$ and $\phi_2(k_2,\ell_2) > 0$. Assumption A.1 implies that $k_1/\ell_1 > k_2/\ell_2$. The cone of diversification is specified by these sector-specific capital-labor ratios, which depend only on the relative price p_2 / p_1, $\kappa_1(p_2 / p_1)$ and $\kappa_2(p_2 / p_1)$. It is the set of country specific capital-labor ratios k^i such that

$$\kappa_1(p_2 / p_1) \geq k^i \geq \kappa_2(p_2 / p_1) . \tag{18}$$

In our dynamic economy, the cone of diversification changes over time since the capital-labor ratio and, consequently, the equilibrium prices of the integrated economy, change over time. Therefore, to solve for an equilibrium using the integrated economy approach, we need to find a way to disaggregate the investment decisions such that countries stay in the corresponding cone of diversification for all time periods.

Given that the period utility function is identical and homothetic across countries, factor price equalization implies that we can use the integrated economy approach to solve for equilibrium in a static model. In our dynamic economy, there is an additional possible complication: If one of the countries has a corner solution in which it chooses zero investment in some period while another country chooses positive investment, then we cannot disaggregate the consumption and investment decisions of the integrated economy. Later, we will show how this possibility makes it difficult to characterize equilibria.

In the rest of the paper, we assume that consumers combine the two traded goods in consumption in the same way that producers of the investment good combine these two goods in production:

$$u(c_1, c_2) = v\left(f(c_1, c_2)\right), \tag{19}$$

where v is a strictly concave, strictly increasing function. This assumption simplifies the dynamics of the model, since it makes the integrated economy equivalent to a one-sector growth model and, therefore, cycles and chaos are ruled out as possible equilibrium behavior of the integrated economy. To simplify the analysis we further assume, as does Ventura (1997), that the function v is logarithmic.

A.7. The period utility function u takes the form $u(c_1, c_2) = \log\left(f(c_1, c_2)\right)$.

Consider the production function defined by solving

$$F(k, \ell) = \max \ f(y_1, y_2)$$

$$\text{s.t. } y_1 \le \phi_1(k_1, \ell_1) \tag{20}$$
$$y_2 \le \phi_2(k_2, \ell_2)$$
$$k_1 + k_2 \le k$$
$$\ell_1 + \ell_2 \le \ell$$
$$k_j \ge 0,\ \ell_j \ge 0.$$

Assumptions A.6 and A.7 imply that f is strictly quasi-concave, which, together with the concavity of ϕ_1 and ϕ_2, implies that for any (k, ℓ) there is a unique solution to this problem. It is straightforward to prove that F is increasing, concave, continuously differentiable, and homogeneous of degree one. Like f, F is strictly quasi-concave.

Assumption A.7 is useful because it allows us to solve the two-sector social planner's problem (15) by solving the related one-sector social planner's problem

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } c_t + x_t \le F(k_t, 1) \tag{21}$$
$$k_{t+1} - (1-\delta)k_t \le x_t$$
$$c_t \ge 0,\ x_t \ge 0$$
$$k_0 \le \bar{k}_0.$$

We state the following proposition without giving a proof because, first, the proof is just a straightforward application of the maximum theorem, and, second, we will not employ the proposition in its general form, but rather will only consider production functions for which we can analytically solve problem (20).

Proposition 4. Let $y_1(k,\ell)$, $y_2(k,\ell)$, $k_1(k,\ell)$, $\ell_1(k,\ell)$, $k_2(k,\ell)$, $\ell_2(k,\ell)$ denote the solution to (20). If $\{c_{1t}, c_{2t}, k_t\}$, $\{y_{1t}, k_{1t}, \ell_{1t}\}$, $\{y_{2t}, k_{2t}, \ell_{2t}\}$, $\{x_t, x_{1t}, x_{2t}\}$ solves the two-sector social planner's problem (15), then $\{c_t, k_t, x_t\}$ solves the one-sector social planner's problem (21) where $c_t = f(c_{1t}, c_{2t})$. Conversely, if $\{c_t, k_t, x_t\}$ solves the one-sector social planner's problem (21), then $\{c_{1t}, c_{2t}, k_t\}$, $\{y_{1t}, k_{1t}, \ell_{1t}\}$, $\{y_{2t}, k_{2t}, \ell_{2t}\}$, $\{x_t, x_{1t}, x_{2t}\}$ solves the two-sector social planner's problem

(15) where $y_{jt} = y_j(k_t, 1)$, $k_{jt} = k_j(k_t, 1)$, $\ell_{jt} = \ell_j(k_t, 1)$, $c_{jt} = \left[c_t / (c_t + x_t)\right] y_j(k_t, 1)$, and $x_{jt} = \left[x_t / (c_t + x_t)\right] y_j(k_t, 1)$.

We first consider a version of the model in which the production function ϕ_j for each traded good uses only one factor of production. Under this assumption, factor prices equalize along the equilibrium path independently of initial conditions. Since this is the assumption made by Ventura (1997), we call this version of the model the *Ventura model.* By disaggregating the equilibrium of the integrated economy, we derive results on the evolution of the world distributions of income and of capital in the Ventura model. We also show by means of an example that, even though factor price equalization holds in every equilibrium of the Ventura model, there may be equilibria in which there is zero investment in some countries and in which our results for the integrated economy do not hold.

We then consider a version of the model in which the more general production functions ϕ_j have the same constant elasticity of substitution as does the production function for the investment good f. In such models, factor prices need not equalize along the equilibrium path, but, if they do, the equilibria have the same properties as those of the Ventura model. We refer to this version of the model as the *generalized Ventura model*. In this model, we derive the cone of diversification analytically, and give conditions under which, if countries are in the cone of diversification, they stay there. We also derive conditions under which, even if countries start in the cone of diversification, they leave it in a finite number of periods. Finally, for the special case of Cobb-Douglas production functions, we analytically solve the model when there is factor price equalization.

4. Ventura model

Following Ventura (1997), we assume that the production function for each of the traded goods uses only one factor of production:

$$y_1 = \phi_1(k_1, \ell_1) = k_1 \tag{22}$$

$$y_2 = \phi_2(k_2, \ell_2) = \ell_2 . \tag{23}$$

This assumption implies that the cone of diversification is the entire nonnegative quadrant, independently of the prices p_{1t} and p_{2t}, and that factor prices equalize along any equilibrium path: $r_t^i = r_t = p_{1t}$ and $w_t^i = w_t = p_{2t}$. Notice that, in this case, $F(k,1) = f(k,1)$.

Furthermore, we assume that the production function of the investment good has a constant elasticity of substitution between the inputs of the two traded goods:

$$f(x_1, x_2) = d\left(a_1 x_1^b + a_2 x_2^b\right)^{1/b} \tag{24}$$

if $b \neq 0$, and f is

$$f(x_1, x_2) = d x_1^{a_1} x_2^{a_2} \tag{25}$$

in the limit where $b = 0$. Here $a_i > 0$ and $a_1 + a_2 = 1$. The elasticity of substitution is $\sigma = 1/(1-b)$. In what follows, we can easily translate statements involving b into statements involving σ.

It is worth pointing out that Ventura (1997) considers a continuous-time version of this model. For completeness, we later sketch out our results for the continuous-time model.

Suppose that we find the equilibrium of the integrated economy by solving the one-sector social planner's problem (21). To disaggregate consumption and investment, we solve the utility maximization of the representative consumer i, (7):

$$\begin{aligned} &\max \ \textstyle\sum_{t=0}^{\infty} \beta^t \log c_t^i \\ &\text{s.t. } c_t^i + x_t^i \leq w_t + r_t k_t^i \\ &k_{t+1}^i - (1-\delta) k_t^i \leq x_t^i \\ &c_{jt}^i \geq 0,\ x_t^i \geq 0 \\ &k_0^i \leq \bar{k}_0^i . \end{aligned} \tag{26}$$

If we solve (7), we can obtain a solution to (26) by setting $c_t^i = f(c_{1t}^i, c_{2t}^i)$, and, if we solve (26), we can obtain a solution to (7) by setting $c_{1t}^i = c_t^i k_t / (w_t + r_t k_t^i)$, $c_{2t}^i = c_t^i / (w_t + r_t k_t^i)$, $x_{1t}^i = x_t^i k_t / (w_t + r_t k_t^i)$, $x_{2t}^i = x_t^i / (w_t + r_t k_t^i)$.

The necessary and sufficient conditions for a sequence of consumption levels and capital stocks to solve (26) are that

$$-\frac{1}{c_t^i}+\beta\frac{1}{c_{t+1}^i}(1-\delta+r_{t+1})\leq 0\,,\ \text{if} =\ \ x_t^i>0 \tag{27}$$

$$c_t^i+k_{t+1}^i-(1-\delta)k_t^i=w_t+r_tk_t^i\,, \tag{28}$$

and that the transversality condition

$$\lim_{t\to\infty}\beta^t\frac{1}{c_t^i}k_{t+1}^i=0 \tag{29}$$

holds. If $x_t^i > 0$ for all i and all t, then we are justified in using the integrated economy approach.

We solve for the integrated economy equilibrium in the Ventura model by solving for the equilibrium of a one-sector growth model. Notice, however, that the two sectors matter a lot for disaggregating the equilibrium. In particular, we cannot solve for the equilibrium values of the variables for one of the countries by solving an optimal growth problem for that country in isolation. Instead, the equilibrium path of a specific country's capital stock and its steady state value depend not only on the country's initial endowment of capital but also — through the interest rate r_t — on the equilibrium path of the world's capital stock, and its steady state value.

If there is positive investment in every period, then the equilibrium path for the integrated economy is determined by the difference equations

$$c_{t+1}=\beta\big(1-\delta+r(k_{t+1})\big)c_t \tag{30}$$

$$c_t+k_{t+1}-(1-\delta)k_t=f(k_t,1)\,, \tag{31}$$

the initial condition $k_0=\bar{k}_0=\sum_{i=1}^n L^i\bar{k}_0^i/\sum_{i=1}^n L^i$, and the transversality condition

$$\lim_{t\to\infty}\beta^t\frac{1}{c_t}k_{t+1}=0\,. \tag{32}$$

Here $r(k_t)$ is the rental rate of capital,

$$r(k_t)=\begin{cases}a_1k_t^{b-1}d(a_1k_t^b+a_2)^{(1-b)/b} & \text{if } b\neq 0\\ a_1dk_t^{a_1-1} & \text{if } b=0\end{cases}\,. \tag{33}$$

Standard results for one-sector models (for example, Rebelo 1991) say that the equilibrium of the integrated economy has sustained growth for some values of the parameters. The existence of steady state depends upon whether the rental rate of capital as a function of initial endowments, $r(k)$, can take the value $1/\beta-1+\delta$ for some $k>0$. If $r(k)<1/\beta-1+\delta$ for all k, then k_t converges to 0. If, however, $r(k)>1/\beta-1+\delta$ for all k, then k_t grows without bound. Consider an economy without labor, where feasible allocations satisfy

$$c_t+k_{t+1}=\left(da_1^{1/b}+1-\delta\right)k_t \tag{34}$$

This economy has a sustained growth path in which $c_t=\left(1-\beta\right)\left(da_1^{1/b}+1-\delta\right)k_t$ and $k_{t+1}=\beta\left(da_1^{1/b}+1-\delta\right)k_t$.

Definition 3. We say that an equilibrium *converges to the sustained growth path* of the corresponding economy without labor if

$$\lim_{t\to\infty} c_t/k_t=\left(1-\beta\right)\left(da_1^{1/b}+1-\delta\right) \tag{35}$$

$$\lim_{t\to\infty} k_{t+1}/k_t=\beta\left(da_1^{1/b}+1-\delta\right) \tag{36}$$

Standard results from, for example, Stokey, Lucas, and Prescott (1989) provide the following characterization of the equilibrium of the integrated economy.

Lemma 1: The behavior of the equilibrium of the integrated economy of the Ventura model depends on parameter values:

1. If $b<0$ and $1/\beta-1+\delta>da_1^{1/b}$, the trivial steady state is the unique steady state, and the unique equilibrium of the integrated economy converges to it.
2. If $b=0$, if $b<0$ and $1/\beta-1+\delta\le da_1^{1/b}$, or if $b>0$ and $1/\beta-1+\delta>da_1^{1/b}$, there is a unique nontrivial stable steady state characterized by the solution of the equation $r(\hat{k})=1/\beta-1+\delta$, and the unique equilibrium of the integrated economy converges to it.
3. If $b>0$ and $1/\beta-1+\delta\le da_1^{1/b}$, there is no nontrivial steady state, and the unique equilibrium of the integrated economy converges to the sustained growth path.

In the case where $b=0$ and $\delta=1$, there is an analytical solution to the one-sector social planner's problem (21) for the integrated economy:

$$k_{t+1} = x_t = \beta a_1 d k_t^{a_1} \tag{37}$$

$$c_t = (1-\beta a_1) d k_t^{a_1} \tag{38}$$

For other parameter values, we need to use numerical methods to solve for the equilibrium. Nevertheless, we can derive analytic results on the evolution of the distribution of income levels over time that depend on the values of variables in the integrated economy equilibrium. Qualitatively characterizing the integrated economy equilibrium then allows us to qualitatively characterize the evolution of income levels. In particular, we can find conditions under which relative income levels converge and conditions under which they diverge. The next proposition derives a formula that compares the level of income per capita in a given country measured in current prices, $y_t^i = w_t + r_t k_t^i$, to the world's average at any given period, $y_t = w_t + r_t k_t$, to the same relative income position in the previous period.

Proposition 5. In the Ventura model, if $x_t^i > 0$ for all i and all t, the income level of country i relative to the world's income level evolves according to the rule

$$\frac{y_t^i - y_t}{y_t} = \frac{s_t}{s_{t-1}} \left(\frac{y_{t-1}^i - y_{t-1}}{y_{t-1}} \right) = \frac{s_t}{s_0} \left(\frac{y_0^i - y_0}{y_0} \right). \tag{39}$$

where $s_t = r_t c_{t-1} / y_t$, $t = 1,2,\ldots$, and $s_0 = r_0 c_0 / \left[\beta(1 + r_0 - \delta) y_0 \right]$. When there is complete depreciation, $\delta = 1$, $s_t = c_t / y_t$, $t = 0,1,\ldots$.

Proof: Subtracting the first-order condition for the consumer's problem in the open economy from the same condition for the integrated economy, we obtain:

$$\frac{c_{t+1}^i - c_{t+1}}{c_{t+1}} = \frac{c_t^i - c_t}{c_t}. \tag{40}$$

It is here that the assumption of no corner solutions in investment is essential, allowing us to impose the first-order conditions (27) and (30) as equalities. Manipulating the first-order conditions (27) and the budget constraints (26), we obtain the familiar demand function for logarithmic utility maximization:

$$c_t^i = (1-\beta)\left[\sum_{s=t}^{\infty}\left(\prod_{\tau=t+1}^{s}\frac{1}{1+r_\tau-\delta}\right)w_s + (1+r_t-\delta)k_t^i\right]. \tag{41}$$

Notice that, since we have factor price equalization,

$$c_t^i - c_t = (1-\beta)(1+r_t-\delta)(k_t^i - k_t). \tag{42}$$

The budget constraint (26) implies that

$$c_t^i - c_t + k_{t+1}^i - k_{t+1} = (1+r_t-\delta)(k_t^i - k_t). \tag{43}$$

Combining (30), (42), and (43), we obtain

$$k_{t+1}^i - k_{t+1} = \frac{c_t}{c_{t-1}}(k_t^i - k_t). \tag{44}$$

The difference between a country's income per worker and the world's income per worker is

$$y_{t+1}^i - y_{t+1} = r_{t+1}(k_{t+1}^i - k_{t+1}). \tag{45}$$

Using the expression for $k_{t+1}^i - k_{t+1}$ in (44), we obtain

$$\frac{y_{t+1}^i - y_{t+1}}{y_{t+1}} = \frac{r_{t+1}c_t / y_{t+1}}{r_t c_{t-1} / y_t}\left(\frac{y_t^i - y_t}{y_t}\right). \tag{46}$$

We can use the first-order condition (30) to rewrite this expression as

$$\frac{y_t^i - y_t}{y_t} = \frac{s_t}{s_{t-1}}\left(\frac{y_{t-1}^i - y_{t-1}}{y_{t-1}}\right) = \frac{s_t}{s_0}\left(\frac{y_0^i - y_0}{y_0}\right), \tag{47}$$

where $s_t = r_t c_{t-1} / y_t$ for $t = 1,2,...$ and $s_0 = r_0 c_0 / \left[\beta(1+r_0-\delta)y_0\right]$. When $\delta = 1$, $c_t / c_{t-1} = \beta r_t$ implies that we can factor β out of the numerator and the denominator of (47) and set $s_t = c_t / y_t$.■

The proof of this proposition relies on factor price equalization occurring in every period and on there never being a corner solution in investment. If factor prices are not equal in some period in the future, the demand functions (41) for each individual country and for the integrated economy would have different prices in that period and, therefore, equation (42) would not hold. Likewise, if a corner solution in investment occurs, equation (40) need not hold. We later provide examples in which lack of factor price equalization and the lack of interior solutions for investment cause the characterization of behavior of relative income in proposition 5 to fail.

Equation (39) in the previous proposition compares a country's income relative to the world average. Whether countries converge or diverge in their income levels depends on whether the ratio $r_{t+1}c_t / y_{t+1}$ decreases or increases over time. If the ratio increases, countries' incomes move further away from the income of the integrated economy and, thus, there is divergence in income levels. If the ratio decreases, countries' income levels become closer to the average income level, and there is convergence in income levels. If the ratio is constant, countries maintain their initial income differences and, therefore, the distribution of income stays constant. We should stress that here convergence means that countries' income levels become more similar over time. It does not mean that they converge to the same level of income: Although the absolute value of $(y_t^i - y_t)/y_t$ can be strictly decreasing over time, it can converge to a constant different from 0.

Using proposition 5, we can reduce the characterization of the convergence properties of equilibria in the case with complete depreciation to a characterization of the behavior of $s_t = c_t / y_t$ in the solution to the one-sector social planner's problem (21).

Lemma 2. In the unique equilibrium of the integrated economy of the Ventura model with complete depreciation, the behavior of $s_t = c_t / y_t$ depends on parameter values:

1. If $b < 0$ and $1/\beta > da_1^{1/b}$, then s_t is a strictly decreasing sequence that converges to $1-\beta$.
2. If $b < 0$ and $1/\beta \le da_1^{1/b}$, then s_t converges to $[f(\hat{k},1)-\hat{k}]/f(\hat{k},1)$ where $\hat{k}$ is the unique nontrivial stable steady state. If $\bar{k}_0 < \hat{k}$, s_t is a strictly increasing sequence; if $\bar{k}_0 > \hat{k}$, s_t is a strictly decreasing sequence.
3. If $b = 0$, then $s_t = 1 - \beta a_1$ is constant.
4. If $b > 0$ and $1/\beta > da_1^{1/b}$, then s_t converges to $[f(\hat{k},1)-\hat{k}]/f(\hat{k},1)$. If $\bar{k}_0 < \hat{k}$, s_t is a strictly decreasing sequence; if $\bar{k}_0 > \hat{k}$, s_t is a strictly increasing sequence.
5. If $b > 0$ and $1/\beta \le da_1^{1/b}$, then s_t is a strictly decreasing sequence that converges to $1-\beta$.

Proof: Since the result for the case 3, where $b = 0$, follows trivially from equation (38), we concern ourselves with the other cases, where $b < 0$ or $b > 0$. Multiplying and dividing the Euler equation, (30), by k_t / y_t and using the feasibility condition, (31), we obtain

$$s_t \left(\frac{1}{s_{t-1}} - 1 \right) = \beta h_t , \tag{48}$$

where $s_t = c_t / y_t$ and $h_t = (r_t k_t) / y_t$. We define the function

$$h(k) = \frac{r(k)k}{f(k,1)} = \frac{a_1 k^b}{a_1 k^b + a_2} . \tag{49}$$

Notice that $h'(k) < 0$ if $b < 0$ and that $h'(k) > 0$ if $b > 0$. Notice that, in the limit where $b = 0$, $h(k) = a_1$ and $h'(k) = 0$. We use the monotonicity of the sequence k_t in any solution to the one-sector social planner's problem (21) to establish monotonicity properties for the sequence h_t. The theorem is then established by showing that the monotonicity properties for the sequence h_t imply the desired monotonicity properties for the sequence s_t.

Consider the different cases enumerated in the statement of the theorem. In case 1, where $b < 0$ and $1/\beta > da_1^{1/b}$, k_t is a strictly decreasing sequence that converges to 0, which implies that h_t is a strictly increasing sequence that converges to 1. In case 2, where $b < 0$ and $1/\beta \le da_1^{1/b}$, and in case 4, where $b > 0$ and $1/\beta > da_1^{1/b}$, k_t is a strictly increasing sequence that converges to $\hat{k}$ if $\bar{k}_0 < \hat{k}$ and a strictly decreasing sequence that converges to $\hat{k}$ if $\bar{k}_0 > \hat{k}$. In case 2, this implies that h_t is a strictly decreasing sequence if $\bar{k}_0 < \hat{k}$ and a strictly increasing sequence if $\bar{k}_0 > \hat{k}$. In case 4, however, h_t is a strictly increasing sequence if $\bar{k}_0 < \hat{k}$ and a decreasing sequence if $\bar{k}_0 > \hat{k}$. In both cases, h_t converges to $a_1 \hat{k}^b / (a_1 \hat{k}^b + a_2)$ no matter what the initial value of k_t. In case 5, where $b > 0$ and $1/\beta \le da_1^{1/b}$, k_t is a strictly increasing sequence that grows without bound, which implies that h_t is an increasing sequence that converges to 1.

We now argue that, if h_t is strictly increasing along a solution path to (21), then s_t is strictly decreasing and, if h_t is strictly decreasing, then s_t is strictly increasing. We begin with the case where h_t is strictly increasing. Suppose, to the contrary, that, although h_t is strictly increasing, s_t is not strictly decreasing, that is, there exists T such that $s_T \geq s_{T-1}$. Since h_t is strictly increasing, equation (48) implies that:

$$s_{T+1}\left(\frac{1}{s_T}-1\right) > s_T\left(\frac{1}{s_{T-1}}-1\right). \tag{50}$$

Since $s_T \geq s_{T-1}$, this implies that

$$s_{T+1}\left(\frac{1}{s_T}-1\right) > s_T\left(\frac{1}{s_T}-1\right), \tag{51}$$

which implies that $s_{T+1} > s_T$. Iterating, we find that, for all $t > T$, the sequence s_t is strictly increasing. Using equation (51), for all $t > T$, we obtain:

$$h_t = \frac{s_{t+1}}{\beta}\left(\frac{1}{s_t}-1\right) > \frac{s_t}{\beta}\left(\frac{1}{s_t}-1\right) = \frac{1-s_t}{\beta}. \tag{52}$$

In the limit, the sequences h_t and $(1-s_t)/\beta$ both converge to the same limit, $(1-\hat{s})/\beta$. Equation (52) implies that

$$h_t > \frac{1-\hat{s}}{\beta}, \tag{53}$$

which contradicts our assumption that h_t is strictly increasing. We prove that, when h_t is strictly decreasing, s_t is strictly decreasing, using the same argument and just reversing the inequalities.■

The next proposition provides our main results for the Ventura model. It follows immediately from proposition 5 and lemma 2.

Proposition 6. (Convergence in relative income levels) In the Ventura model with complete depreciation, if $x_t^i > 0$ for all i and all t:

1. If $b < 0$ and $1/\beta > da_1^{1/b}$, then there is convergence in relative income levels.

2. If $b < 0$ and $1/\beta \leq da_1^{1/b}$, then there is divergence in relative income levels if $\bar{k}_0 < \hat{k}$ and convergence in relative income levels if $\bar{k}_0 > \hat{k}$.

3. If $b = 0$, relative income levels stay constant.

4. If $b > 0$ and $1/\beta > da_1^{1/b}$, then there is convergence in relative income levels if $\bar{k}_0 < \hat{k}$ and divergence in relative income levels if $\bar{k}_0 > \hat{k}$.

5. If $b > 0$ and $1/\beta \leq da_1^{1/b}$, then there is convergence in relative income levels.

We have analyzed all of the cases enumerated in the statement of proposition 6 for the sake of completeness. Case 1 and cases 2 and 4 where $\bar{k}_0 > \hat{k}$ are less interesting than the others. The contrast of the remaining results with the analogous results for a world of closed economies is striking: In cases 2, 3, and 4, if the countries are closed to trade, we know that relative income levels converge over time because all countries have equilibria that converge to the steady state of the integrated economy. If we open the countries to trade, however, relative income levels diverge if $b < 0$ and stay fixed if $b = 0$. Notice that, if $b > 0$, relative income levels converge, but not to the same level as they do in a the world of closed economies. In case 5, if the countries are closed to trade, we know that relative income levels diverge over time because growth accelerates over time and countries that start with lower income levels because they have lower initial capital stocks grow more slowly. If we open these economies to trade, however, income levels converge.

The intuition for the results in proposition 6, at least for the cases where $b < 0$ and $1/\beta \leq da_1^{1/b}$ and where $b = 0$, is obvious: In a world of closed economies, poor countries — that is, countries with lower initial capital stocks — grow faster than rich countries because lower capital stocks lead to higher returns on investment. Trade equalizes the return on capital in poor and rich countries, eliminating the incentive for higher investment in poor countries.

We are left with the question: When are there corner solutions in investment for individual countries, which make the integrated economy approach and the characterization of equilibria in propositions 3, 4, 5, and 6 invalid? The answer is found in the next proposition.

Proposition 7. In the Ventura model with complete depreciation, for the cases enumerated in the statement of lemma 2 where the sequence $s_t = c_t / y_t$ in the equilibrium of the integrated economy is constant or strictly decreasing, there exists an equilibrium where $x_t^i > 0$ for all i and all t. For the cases where s_t is strictly increasing, let $z_t = c_{t-1} / k_t$, $z_0 = c_0 / \left(\beta r_0 k_0\right)$, and

$$\hat{z} = \lim_{t\to\infty} \frac{c_{t-1}}{k_t}. \tag{54}$$

This limit is well defined. Let i_{min} be the country with the lowest initial endowment of capital per worker, $\bar{k}_0^{i_{min}} \le \bar{k}_0^i$, $i = 1,...,n$. If

$$\frac{\hat{z}}{z_0}\left(\frac{\bar{k}_0^{i_{min}} - \bar{k}_0}{\bar{k}_0}\right) \ge -1, \tag{55}$$

then there exists an equilibrium where $x_t^i > 0$ for all i and all t. Otherwise, there is no equilibrium where $x_t^i > 0$ for all i and all t. When there exists an equilibrium with no corner solutions in investment, it is the unique such equilibrium.

Proof: Notice that, since

$$z_t = \frac{s_t}{1 - s_t}, \tag{56}$$

the sequence z_t has the same monotonicity properties as the sequence s_t. In the cases where s_t converges to $1-\beta$, z_t converges to $(1-\beta)/\beta$. In the cases where s_t converges to $\left[f(\hat{k},1) - \hat{k}\right] / f(\hat{k},1)$, z_t converges to $\left[f(\hat{k},1) - \hat{k}\right] / \hat{k}$. Equation (44) implies that

$$\frac{k_t^i - k_t}{k_t} = \frac{z_t}{z_{t-1}}\left(\frac{k_{t-1}^i - k_{t-1}}{k_{t-1}}\right) = \frac{z_t}{z_0}\left(\frac{\bar{k}_0^i - \bar{k}_0}{\bar{k}_0}\right). \tag{57}$$

Since assumption A.1 implies that $\bar{k}_0^i > 0$, we know that $(\bar{k}_0^i - \bar{k}_0)/\bar{k}_0 > -1$. If z_t is weakly increasing, or if z_t is strictly decreasing but condition (55) holds, then $x_t^i = k_t^i > 0$ for all i and all t. If, on the other hand, if z_t is strictly decreasing and condition (55) does not hold, then the investment decisions in the integrated economy equilibrium cannot be disaggregated as in proposition 5 to assign nonnegative investment to each country in every period.

Uniqueness of the disaggregation of the integrated economy equilibrium, if it exists, follows from the uniqueness of the solution to the one-sector social planner's problem (21).■

Uniqueness of the disaggregation of the integrated economy equilibrium is easy to establish because this equilibrium solves a social planner's problem. It is more difficult to say anything about uniqueness of equilibria, if they exist, that involve corner solutions in investment or, in the more general model in the next section, that involve lack of factor price equalization. In such equilibria, Pareto improvements are possible if we allow international borrowing and lending.

5. Generalized Ventura model

Consider a generalization of the Ventura model in which the production functions f, ϕ_1, and ϕ_2 are general constant returns to scale production functions. Propositions 3 and 4 indicate that we can find the integrated economy of the generalized Ventura model by solving the one-sector growth social planner's problem (21).

In this generalized Ventura model, factor price equalization need not occur at any given period of time. Countries can specialize in the production of one of the traded goods, factor prices can differ across countries and, therefore, the equilibrium cannot be solved using the integrated approach in general. In what follows, we characterize the cone of diversification for some specific versions of the model and derive conditions under which factor price equalization in a given period implies factor price equalization in every subsequent period. In such cases, the results of the Ventura model on the evolution of the distribution of income apply to the generalized Ventura model. For situations where factor prices do not equalize after a finite number of periods, the analysis done in the Ventura model is no longer valid. Numerical experiments are needed to determine the behavior of the countries' distribution of income.

5.1. The C.E.S. model

We first consider the model in which

$$y_1 = \phi_1(k_1, \ell_1) = \theta_1 \left(\alpha_1 k_1^b + (1-\alpha_1)\ell_1^b\right)^{1/b} \tag{58}$$

$$y_2 = \phi_2(k_2, \ell_2) = \theta_2 \left(\alpha_2 k_2^b + (1-\alpha_2)\ell_2^b\right)^{1/b} \tag{59}$$

$$f(y_1, y_2) = d\left(a_1 y_1^b + a_2 y_2^b\right)^{1/b}, \tag{60}$$

where $b \le 1$, $b \ne 0$, and $a_2 = 1 - a_1$. Notice that, since we have assumed no factor intensity reversals in the production of the traded goods, the production functions ϕ_j, $j = 1,2$, need to have the same constant elasticity of substitution. Setting this elasticity equal to that of the production function for investment good f allows us to analytically solve for the function F.

We refer to this as the *C.E.S. model*. Here the parameter b determines the common elasticity of substitution $\sigma = 1/(1-b)$, and the production function F defined in (20) is also a C.E.S. production function, with the same elasticity of substitution and with the share parameters that are combinations of the share parameters of the production functions ϕ_1, ϕ_2, and f:

$$F(k, \ell) = D\left(A_1 k^b + A_2 \ell^b\right)^{1/b} \tag{61}$$

$$A_1 = \frac{\left[\left(a_1\alpha_1\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b}}{\left[\left(a_1\alpha_1\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b} + \left[\left(a_1(1-\alpha_1)\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2(1-\alpha_2)\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b}},\ A_2 = 1 - A_1 \tag{62}$$

$$D = d\left\{\left[\left(a_1\alpha_1\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b} + \left[\left(a_1(1-\alpha_1)\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2(1-\alpha_2)\theta_2^b\right)^{\frac{1}{1-b}}\right]^{1-b}\right\}^{\frac{1}{b}}. \tag{63}$$

To determine when factor price equalization occurs and when it does not, we need to characterize the cone of diversification in the integrated economy and how it changes with the world capital-labor ratio. One procedure would be to solve (17) to determine the sector-specific capital-labor ratios as functions of relative prices, $\kappa_1(p_2 / p_1)$ and $\kappa_2(p_2 / p_1)$, then use proposition 4 to determine the prices in the integrated economy equilibrium, $p_1(k_t) = f_1\left(y_1(k_t,1), y_1(k_t,1)\right)$ and $p_2(k_t) = f_2\left(y_1(k_t,1), y_1(k_t,1)\right)$, and, finally, calculate $\kappa_1\left(p_2(k_t)/p_1(k_t)\right)$ and $\kappa_2\left(p_2(k_t)/p_1(k_t)\right)$. In the C.E.S. model, the determination of the cone of diversification of the integrated economy is far simpler than this. Solving the maximization problem that defines F, (20), we find that

$$\kappa_j\left(p_2(k_t)/p_1(k_t)\right) = \bar{\kappa}_j k_t,\ j = 1,2, \tag{64}$$

where the constants $\bar{\kappa}_1$, $\bar{\kappa}_2$ that determine the cone of diversification have the form

$$\bar{\kappa}_j = \left(\frac{\alpha_i}{1-\alpha_i}\right)^{\frac{1}{1-b}} \frac{\left(a_1(1-\alpha_1)\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2(1-\alpha_2)\theta_2^b\right)^{\frac{1}{1-b}}}{\left(a_1\alpha_1\theta_1^b\right)^{\frac{1}{1-b}} + \left(a_2\alpha_2\theta_2^b\right)^{\frac{1}{1-b}}},\ j = 1,2. \tag{65}$$

The next propositions establish conditions under which, for the C.E.S. model, factor price equalization in a given period implies factor price equalization in all subsequent periods. The first proposition gives sufficient conditions for factor prices to equalize along the equilibrium path, given that factor price equalization occurs in a given period T. The key parameter is, once again, b. In particular, when $b > 0$, factor price equalization at T ensures factor price equalization in any subsequent period, at least for the economically interesting cases either where $\bar{k}_0 \le \hat{k}$ and or where k_t goes without bound. The second proposition gives sufficient conditions under which factor price equalization cannot hold forever. It states that, when $b < 0$, and when $1/\beta \le DA_1^{1/b}$ and $\bar{k}_0 < \hat{k}$, if factor price equalization holds at a period T and the capital-labor ratio of one of the countries is close enough to the boundary of the cone of diversification, factor price equalization cannot hold for all subsequent periods. The intuition is simple. If factor price equalization were to occur forever, the analysis in the Ventura model would apply, and the distribution of capital-labor ratios would become more dispersed over time. Since the boundaries of the cone of diversification for the integrated economy are linear functions of the world capital-labor ratio, however, if the capital-labor ratio of one of the countries is close enough to the boundary of the cone, the distribution of capital-labor ratios cannot become more dispersed if all capital-labor ratios are to remain in the cone.

Proposition 8: In the C.E.S. model with complete depreciation, suppose that the sequence $s_t = c_t / y_t$ in the equilibrium of the integrated economy is weakly decreasing. Suppose that factor price equalization occurs in period T. Then there exists an equilibrium in which factor price equalization occurs at all $t \ge T$. Furthermore, this equilibrium is the only such equilibrium.

Proof: Assume that all countries are in the cone of diversification at period T. Define k_t^i, $t > T$, using the formula

$$\frac{k_t^i - k_t}{k_t} = \frac{z_t}{z_T}\left(\frac{k_T^i - k_T}{k_T}\right), \tag{66}$$

where $z_t = c_{t-1} / k_t$ and $z_0 = c_0 /(\beta r_0 k_0)$ are defined as in the statement of proposition 7. We need to show that disaggregating capital this way keeps countries in the cone of diversification and that it solves the equilibrium of our model economy. To prove that the countries remain in the cone, we need to show that, for all $t \geq T$,

$$\bar{\kappa}_2 k_t \leq k_t^i \leq \bar{\kappa}_1 k_t . \tag{67}$$

That is, for all $t \geq T$,

$$\bar{\kappa}_2 - 1 \leq \frac{k_t^i - k_t}{k_t} \leq \bar{\kappa}_1 - 1 . \tag{68}$$

Since we have assumed that sequence s_t is weakly decreasing, we know from the proof of proposition 7 that the sequence is weakly decreasing.

To prove that these sequences of capitals, together with the equilibrium prices of the integrated economy, are a solution to the model economy, we define

$$c_t^i = (1-\beta)\left[\sum_{s=t}^{\infty}\left(\prod_{\tau=t+1}^{s}\frac{1}{1+r_\tau - \delta}\right)w_s + r_t k_t^i\right], \tag{69}$$

where w_s and r_s are equilibrium prices of the integrated economy, and show that consumptions and capital stocks defined this way solve the equilibrium of our model economy.■

Proposition 9: In the C.E.S. model with complete depreciation, suppose that the sequence s_t is strictly increasing. Again let $z_t = c_{t-1} / k_t$, $z_0 = c_0 /(\beta r_0 k_0)$, and

$$\hat{z} = \lim_{t\to\infty} \frac{c_{t-1}}{k_t} . \tag{70}$$

Let i_{min} be the country with the lowest initial endowment of capital per worker, and let i_{max} be the country with the highest, $\bar{k}_0^{i_{min}} \leq \bar{k}_0^i \leq \bar{k}_0^{i_{max}}$, $i = 1,...,n$. If

$$\frac{\hat{z}}{z_0}\left(\frac{\bar{k}_0^{i_{min}} - \bar{k}_0}{\bar{k}_0}\right) \geq \kappa_2 - 1 \tag{71}$$

$$\frac{\hat{z}}{z_0}\left(\frac{\bar{k}_0^{i_{max}} - \bar{k}_0}{\bar{k}_0}\right) \leq \kappa_1 - 1\,, \tag{72}$$

then there exists an equilibrium with factor price equalization in every period. If, however, either of the conditions (71) or (72) is violated, there is no equilibrium with factor price equalization in every period. When there exists an equilibrium with factor price equalization in every period, it is the unique such equilibrium.

Proof: This proof is an obvious generalization of the proof of proposition 7 using the definitions in proposition 8.■

Even though its proof is trivial given the machinery that we have developed, proposition 9 is a powerful result. Under some general conditions, even if factor price equalization occurs at a given period, at some point in the future factor prices will differ across countries. In the case where $b < 0$ and $1/\beta \leq DA_1^{1/b}$, for example, the unique equilibrium of the integrated economy converges to the nontrivial steady state, but, if initial endowments of capital per worker are sufficiently different in the sense that either of the conditions (71) or (72) is violated, then there is no disaggregated equilibrium that corresponds to it. Even if the world economy starts with all countries diversifying in production and factor prices equalized, at some point at least one country necessarily has its capital-labor ratio leave the cone of diversification.

Abstracting away from the patterns of specialization, as Ventura (1997), Chen (1992), and many others do, can cause us to miss out on some important dynamics precisely in the interesting cases, the cases in which there is potentially divergence of income levels. In such cases, we cannot use the integrated economy approach to solve for the equilibrium, and none of the analysis in propositions 3, 4, and 5 applies. Instead, we need to use numerical methods to compute the equilibrium. We briefly explain how to compute equilibrium for the generalized Ventura model in section 6 and present examples of economies for which factor prices are not equalized along the equilibrium path in section 7.

5.2. The Cobb-Douglas model

In this section, we consider the limiting case of the C.E.S. model with complete depreciation and with $b = 0$, that is, with production functions that are Cobb-Douglas.

$$y_1 = \phi_1(k_1, \ell_1) = \theta_1 k_1^{\alpha_1} \ell_1^{1-\alpha_1} \tag{73}$$

$$y_2 = \phi_2(k_2, \ell_2) = \theta_2 k_2^{\alpha_2} \ell_2^{1-\alpha_2} \tag{74}$$

$$f(y_1, y_2) = d y_1^{a_1} y_2^{a_2} . \tag{75}$$

In this case, the function F is also Cobb-Douglas:

$$F(k, \ell) = D k^{A_1} \ell^{A_2} \tag{76}$$

$$A_1 = a_1\alpha_1 + a_2\alpha_2 ,\ A_2 = 1 - A_1 \tag{77}$$

$$D = \frac{d\left[\theta_1 a_1 \alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}\right]^{a_1}\left[\theta_2 a_2 \alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}\right]^{a_2}}{A_1^{A_1} A_2^{A_2}} . \tag{78}$$

The constants that determine the cone of diversification in the Cobb-Douglas case are

$$\bar{\kappa}_j = \left(\frac{\alpha_j}{1-\alpha_j}\right)\frac{A_2}{A_1},\ j = 1, 2 . \tag{79}$$

Proposition 10: In the Cobb-Douglas model with complete depreciation, suppose that factor price equalization occurs in period T. Then factor price equalization occurs at all $t \geq T$. Furthermore,

$$k_t^i = \gamma^i k_t , \tag{80}$$

where $\gamma^i = k_T^i / k_T$ and $k_{t+1} = \beta A_1 D k_t^{A_1}$ for $t \geq T$.

The proof of this proposition is a special case of the proof of proposition 8.

Notice that, when all $\bar{k}_0^i$ are in the cone of diversification, we can use proposition 10 to obtain analytic solutions for all variables. Let $\gamma^i = \bar{k}_0^i / \bar{k}_0$, where $\bar{\kappa}_2 \leq \gamma^i \leq \bar{\kappa}_1$.

$$r_t = A_1 D k_t^{A_1 - 1},\ w_t = A_2 D k_t^{A_1} \tag{81}$$

$$p_{1t}=a_1 d\left(\frac{(\bar{\kappa}_1-1)\theta_2\bar{\kappa}_2^{\alpha_2}}{(1-\bar{\kappa}_2)\theta_1\bar{\kappa}_1^{\alpha_1}}\right)^{a_2} k_t^{-a_2(\alpha_1-\alpha_2)},\ p_{2t}=a_2 d\left(\frac{(1-\bar{\kappa}_2)\theta_1\bar{\kappa}_1^{\alpha_1}}{(\bar{\kappa}_1-1)\theta_2\bar{\kappa}_2^{\alpha_2}}\right)^{a_1} k_t^{a_1(\alpha_1-\alpha_2)} \tag{82}$$

$$x_t^i=k_{t+1}^i=\gamma^i\beta A_1 D k_t^{A_1} \tag{83}$$

$$c_{1t}^i=\frac{D}{d}\left(\frac{(1-\bar{\kappa}_2)\theta_1\bar{\kappa}_1^{\alpha_1}}{(\bar{\kappa}_1-1)\theta_2\bar{\kappa}_2^{\alpha_2}}\right)^{a_2}\left(1-A_1+\gamma^i A_1-\gamma^i\beta A_1\right)k_t^{\alpha_1},\ x_{1t}^i=\frac{D}{d}\left(\frac{(1-\bar{\kappa}_2)\theta_1\bar{\kappa}_1^{\alpha_1}}{(\bar{\kappa}_1-1)\theta_2\bar{\kappa}_2^{\alpha_2}}\right)^{a_2}\gamma^i\beta A_1 k_t^{\alpha_1} \tag{84}$$

$$c_{2t}^i=\frac{D}{d}\left(\frac{(\bar{\kappa}_1-1)\theta_2\bar{\kappa}_2^{\alpha_2}}{(1-\bar{\kappa}_2)\theta_1\bar{\kappa}_1^{\alpha_1}}\right)^{a_1}\left(1-A_1+\gamma^i A_1-\gamma^i\beta A_1\right)k_t^{\alpha_2},\ x_{2t}^i=\frac{D}{d}\left(\frac{(\bar{\kappa}_1-1)\theta_2\bar{\kappa}_2^{\alpha_2}}{(1-\bar{\kappa}_2)\theta_1\bar{\kappa}_1^{\alpha_1}}\right)^{a_1}\gamma^i\beta A_1 k_t^{\alpha_2} \tag{85}$$

$$y_{1t}^i=\frac{\gamma^i-\bar{\kappa}_2}{\bar{\kappa}_1-\bar{\kappa}_2}\theta_1\left(\bar{\kappa}_1 k_t\right)^{\alpha_1},\ k_{1t}^i=\frac{\gamma^i-\bar{\kappa}_2}{\bar{\kappa}_1-\bar{\kappa}_2}\bar{\kappa}_1 k_t,\ \ell_{1t}^i=\frac{\gamma^i-\bar{\kappa}_2}{\bar{\kappa}_1-\bar{\kappa}_2} \tag{86}$$

$$y_{2t}^i=\frac{\bar{\kappa}_1-\gamma^i}{\bar{\kappa}_1-\bar{\kappa}_2}\theta_2\left(\bar{\kappa}_2 k_t\right)^{\alpha_2},\ k_{2t}^i=\frac{\bar{\kappa}_1-\gamma^i}{\bar{\kappa}_1-\bar{\kappa}_2}\bar{\kappa}_2 k_t,\ \ell_{2t}^i=\frac{\bar{\kappa}_1-\gamma^i}{\bar{\kappa}_1-\bar{\kappa}_2} \tag{87}$$

where

$$k_t=\beta A_1 D k_{t-1}^{A_1}=\left(\beta A_1 D\right)^{(1-A_1^t)/(1-A_1)}\bar{k}_0^{A_1^t}. \tag{88}$$

6. Computation of equilibrium

In characterizing the cone of diversification of the integrated economy as the set of capital-labor ratios k_t^i that satisfy $\bar{\kappa}_2 k_t \le k_t^i \le \bar{\kappa}_1 k_t$, we have relied heavily, not just on the assumption of specific functional forms for ϕ_1, ϕ_2, and f, but also on the assumption that all countries produce both goods. Under these assumptions, we derive optimal capital-labor ratios in each industry as functions of the world capital-labor ratio. If capital-labor ratios for all countries are inside the cone of diversification, we are justified in using the integrated economy approach. If not, and at least one of the countries specializes, p_{2t}/p_{1t} does not, in general, equal to its value in the integrated equilibrium. Consequently, we cannot use $\kappa_j\left(p_2(k_t)/p_1(k_t)\right)=\bar{\kappa}_j k_t$ to characterize the cone of diversification and to determine the pattern of specialization. Instead, we must calculate $\kappa_1(p_2/p_1)$ and $\kappa_2(p_2/p_1)$ by solving (17). In the next section, we provide an example that illustrates how $\kappa_j(p_{2t}/p_{1t})$ differs from $\bar{\kappa}_j k_t$ when one of the countries specializes.

In the C.E.S. model, the cone of diversification is determined by the capital-labor ratios

$$\kappa_1(p_2/p_1)=\left(\frac{\alpha_1}{1-\alpha_1}\right)^{1/(1-b)}\left[\frac{(1-\alpha_2)^{1/(1-b)}(\theta_2 p_2/p_1)^{b/(1-b)}-(1-\alpha_1)^{1/(1-b)}\theta_1^{b/(1-b)}}{\alpha_1^{1/(1-b)}\theta_1^{b/(1-b)}-\alpha_2^{1/(1-b)}(\theta_2 p_2/p_1)^{b/(1-b)}}\right]^{1/b} \quad (89)$$

$$\kappa_1(p_2/p_1)=\left[\left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{1-\alpha_1}{\alpha_1}\right)\right]^{1/(1-b)}\kappa_2(p_2/p_1). \quad (90)$$

In Cobb-Douglas model, the limiting case where $b=0$, these become

$$\kappa_1(p_2/p_1)=\left[\frac{\theta_2 p_2/p_1}{\theta_1}\left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{1-\alpha_2}\right]^{1/(\alpha_1-\alpha_2)} \quad (91)$$

$$\kappa_1(p_2/p_1)=\left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{1-\alpha_1}{\alpha_1}\right)\kappa_2(p_2/p_1). \quad (92)$$

If the integrated economy approach is valid because there are no corner solutions in investment and all countries have capital-labor ratios that remain in the cone of diversification, then we need only solve for the sequence of capital stocks for the integrated economy by solving the one-sector social planner's problem (21) to solve for equilibrium. Given this solution, we can then use propositions 3 and 4 calculate the equilibrium prices and the formula

$$\frac{k_t^i-k_t}{k_t}=\frac{z_t}{z_T}\left(\frac{k_T^i-k_T}{k_T}\right) \quad (93)$$

to disaggregate consumption and investment decisions. We disaggregate production decisions by requiring that all countries use capital and labor in the optimal proportions, $k_{jt}^i/\ell_{jt}^i=\bar{\kappa}_j k_t$, and satisfy the feasibility conditions (9) and (10).

If the integrated economy approach is not valid, the situation is far more complicated. To keep our discussion simple, we ignore the possibility of corner solutions in investment and instead focus on the case where complications arise because of specialization in production. The approach that we take is to guess the sequence of capital stocks for all countries and the sequence of prices for good 1. Given the price of good 1, we use the first-order conditions

$$f_1(x_{1t}^i,x_{2t}^i)=f_1(c_{1t}^i,c_{2t}^i)=p_{1t} \quad (94)$$

to determine the ratio $x_{1t}^i / x_{2t}^i = c_{1t}^i / c_{2t}^i$. Given this ratio, we can determine the price of the second good using the first-order condition

$$f_2(x_{1t}^i, x_{2t}^i) = p_{2t}. \tag{95}$$

Given the prices p_{1t} and p_{2t}, we can determine the cone of diversification by solving for $\kappa_1(p_2 / p_1)$ and $\kappa_2(p_2 / p_1)$. Given the cone of diversification, we can solve for the pattern of production and the factor prices in each country. We can divide total income into consumption and investment by fixing investment at the level needed to accumulate capital for the next period.

For computation, we need to reduce the remaining equilibrium conditions to a finite number of equations in the same finite number of unknowns, which are the sequence of capital stocks for all countries and the sequence of prices for good 1 that we have guessed. We do this by truncating the time horizon at some period T by assuming that the equilibrium approximately converges to a steady state — or a sustained growth path, depending on parameter values — by that period. Bajona and Kehoe (2006) argue that any steady state of the generalized Ventura model has factor price equalization across countries. In addition, they show that factor prices equalize as countries converge to a sustained growth path. They also argue that equilibrium cycles are not possible.

In the appendix, we provide details on the algorithm that we use to compute equilibria in the examples in the next section, including an example with corner solutions in investment.

7. Numerical examples

In this section, we present three numerical examples that illustrate different equilibrium patterns for the Ventura model when there are corner solutions in investment and for the generalized Ventura model when countries are not in the cone of diversification along the entire equilibrium path. In each of the examples, we consider a two-country economy. We set $\beta = 0.95$, $\delta = 1$, and $L^1 = L^2 = 10$.

Example 1. (Ventura model with $b < 0$)

$$f(x_1, x_2) = 10\left(0.5x_1^{-0.5} + 0.5x_2^{-0.5}\right)^{-2}. \tag{96}$$

We contrast two different worlds. In the first world, $\bar{k}_0^1 = 5$ and $\bar{k}_0^2 = 3$. Here there is an equilibrium with no corner solutions for investment. The solid lines in figure 2 depict the equilibrium paths for k_t^i in this world. Proposition 5 says that relative income levels diverge over time, and the solid lines in figure 3 depict this divergence, with $(y_t^1 - y_t)/y_t$ increasing from 0.1531 in period 0 to 0.1707 in the steady state. (We show the relative income for country 1 only because symmetry implies that $(y_t^2 - y_t)/y_t = -(y_t^1 - y_t)/y_t$.) In the second world, there is an even larger difference in initial capital stocks, $\bar{k}_0^1 = 6$ and $\bar{k}_0^2 = 2$. In this world, the integrated economy equilibrium is the same as in the first world, but it cannot be disaggregated because there is no equilibrium without corner solutions in investment. The dashed lines in figure 2 depict the equilibrium paths for k_t^i in this world. Country 2 has $x_t^i = k_t^i = 0$ starting in period 3. Proposition 5 gives us no indication of what happens to the distribution of income across countries: $(y_t^1 - y_t)/y_t$ increases from 0.3063 in period 0 to 0.3211 in period 1 and then steadily declines, until it equals 0.2977 in the steady state. Notice that relative incomes converge between periods 1 and 2 even though investment is strictly positive in both countries in both periods.

Example 2. (Cobb-Douglas model)

$$\phi_1(k,\ell) = 10k^{0.6}\ell^{0.4} \tag{97}$$

$$\phi_2(k,\ell) = 10k^{0.4}\ell^{0.6} \tag{98}$$

$$f(x_1, x_2) = x_1^{0.5}x_2^{0.5} \tag{99}$$

We assume that $\bar{k}_0^1 = 4$ and $\bar{k}_0^2 = 0.1$. Figure 4 shows that the initial endowments of capital per worker are different enough for factor prices not to equalize across countries in period 0. The labor abundant country, country 2, specializes in the production of the labor intensive good, and the capital abundant country, country 1, diversifies. This pattern of production is maintained along the equilibrium path, with k_t^2 asymptotically converging to the boundary of the cone of diversification. Notice that, even though the k_t^1 is outside the cone of diversification of the integrated economy, $k_t^1 > \bar{\kappa}_1(k_t^1 + k_t^2)/2$ in periods 0, 1, and 2, it is inside the cone of diversification, $\kappa_1(p_{2t}/p_{1t}) > k_t^1 > \kappa_2(p_{2t}/p_{1t})$. Notice too, in figure 5, that relative incomes converge even though proposition 5 says that relative incomes would stay constant if both countries were to diversify.

Example 3. (C.E.S. model with $b < 0$)

$$\phi_1(k,\ell) = 10\left(0.8k^{-0.5} + 0.2\ell^{-0.5}\right)^{-2} \tag{100}$$

$$\phi_2(k,\ell) = 10\left(0.2k^{-0.5} + 0.8\ell^{-0.5}\right)^{-2} \tag{101}$$

$$f(x_1,x_2) = \left(0.5x_1^{-0.5} + 0.5x_2^{-0.5}\right)^{-2} \tag{102}$$

We assume that $\bar{k}_0^1 = 5$ and $\bar{k}_0^2 = 2$. Here both countries start inside the cone of diversification, but one of them exits after a finite number of periods. As figure 7 — a blowup of the detail in figure 6 — shows, the labor abundant country, country 2, produces both goods in periods 0 and 1. In period 2, k_t^2 jumps outside the cone of diversification and country 2 specializes in the production of the labor intensive good. (We could say that the boundary of the cone of diversification jumps over k_t^2.) Over time, k_t^2 converges back to the boundary of the cone of diversification. Since country 1 diversifies along the equilibrium path, factor prices are equalized in periods 0 and 1, but not afterward. Figure 8 shows that relative incomes converge monotonically over time. This example, again, illustrates the limited scope of proposition 5. Under the assumption that factor prices equalize along the equilibrium path, the model reduces to a Ventura model with $b = -0.5$ and proposition 5 stipulates divergence in income levels. As the example shows, however, even if factor prices equalize in the initial periods, the opposite convergence result can hold if factor prices do not equalize along the entire equilibrium path. The dashed lines in figures 6, 7, and 8 depict the equilibrium of the Ventura model with $f(x_1,x_2) = 5.7328\left(0.5x_1^{-0.5} + 0.5x_2^{-0.5}\right)^{-2}$, a model with the same integrated economy equilibrium as our C.E.S. model. Notice how different are the convergence properties of relative incomes in the two equilibria.

8. Continuous-time Ventura model

In this section, we derive the properties of convergence in income levels for the continuous-time version of the Ventura model. The utility function becomes

$$\int_0^\infty e^{-\rho t} \log f(c_1,c_2)\, dt\,, \tag{103}$$

where $\rho > 0$ is the discount rate. As in the model with discrete time, the production functions of the traded goods are

$$y_1 = k_1 \tag{104}$$

$$y_2 = \ell_2, \tag{105}$$

and the production function for the investment good is

$$x = f(x_1, x_2) \tag{106}$$

where f is specified as in (24) and (25).

We can find the integrated economy equilibrium by solving the social planner's problem

$$\begin{aligned} &\max \int_0^\infty e^{-\rho t} \log c \; dt \\ &\text{s.t. } c + x \leq f(k,1) \\ &\dot{k} + \delta k \leq x \\ &c \geq 0, \; x \geq 0 \\ &k(0) \leq \bar{k}_0 . \end{aligned} \tag{107}$$

Given that the cone of diversification is the whole nonnegative quadrant, factor prices equalize at all equilibrium prices, and, if there are no corner solutions in investment, we can derive formulas relating the equilibrium levels of capital and income to the levels of capital and income in the integrated economy equilibrium. In particular, Ventura (1997) shows that the capital-labor ratios satisfy

$$\frac{k^i(t) - k(t)}{k(t)} = \frac{c(t)/k(t)}{c(0)/k(0)} \left(\frac{k^i(0) - k(0)}{k(0)} \right) = \frac{z(t)}{z(0)} \left(\frac{k^i(0) - k(0)}{k(0)} \right) \tag{108}$$

and draws phase diagrams in (k, z) space to analyze convergence and divergence of k^i and k. Notice that this is not the same as convergence and divergence of y^i and y, where

$$y^i = w + rk^i = f(k^i, 1). \tag{109}$$

Since we are more interested in studying convergence in relative income levels than convergence in capital-labor ratios, we instead study the behavior of

$$\frac{y^i(t)-y(t)}{y(t)}=\frac{s(t)}{s(0)}\left(\frac{y^i(0)-y(0)}{y(0)}\right) \tag{110}$$

where

$$s(t)=\frac{r(t)c(t)}{y(t)}=\frac{(f_K(k,1)-\delta)c(t)}{f(k,1)}=\frac{g'(k)c(t)}{g(k)}, \tag{111}$$

by analyzing phase diagrams in (k, s) space. Here $g(k)=f(k,1)-\delta k$.

We use the first-order conditions and feasibility conditions

$$\frac{\dot{c}}{c}=g'(k)-\rho \tag{112}$$

$$\frac{\dot{k}}{k}=\frac{g(k)}{k}-\frac{c}{k} \tag{113}$$

to obtain a system of differential equations in the (k, s) space:

$$\frac{\dot{s}}{s}=g'(k)-\rho-\left(\frac{g'(k)^2-g(k)g''(k)}{g'(k)^2}\right)\left(g'(k)-s\right) \tag{114}$$

$$\frac{\dot{k}}{k}=\frac{g(k)}{g'(k)k}\left(g'(k)-s\right) \tag{115}$$

Figures 9-11 depict phase diagrams in (k, s) space for different values of the parameter b. Notice that the results we obtain are similar to the results obtained in section 4 for the discrete-time version of the model. Figure 9 shows that, for $b>0$ and $da_1^{1/b}-\delta>\rho$, s decreases over time and relative incomes converge. Figure 10 shows that, for $b<0$ and $da_1^{1/b}-\delta>\rho$, if $\bar{k}_0<\hat{k}$, relative incomes diverge initially for $\bar{k}_0$ small enough, but later converge. Divergence of income everywhere along the equilibrium path is only obtained for very negative values of b and only for cases where there is positive depreciation, where $da_1^{1/b}-\delta>\rho$, and $\bar{k}_0<\hat{k}$, as in figure 11.

Examining equation (108), we see that the continuous-time model can have the same sort of problems with corner solutions in investment as does the discrete time model. If z increases over time, then there are initial endowments of capital per worker sufficiently different so that the equilibrium will necessarily involve corner solutions in investment, making the integrated economy approach — and the corresponding phase diagrams — invalid. As example 1 indicates, even if we allow reversibility of investment and require only that $k^i \geq 0$, we can still have corner solutions where some countries have zero capital stocks..

9. Concluding remarks

This paper presents results that some economists may find surprising.

First, proposition 6 says that introducing international trade into a model that reduces to a one-sector growth model can completely change the results on convergence of income levels. For example, if $b < 0$ and $\delta = 1$, and if the economy is productive enough to converge to a nontrivial steady state and capital starts below this steady state, there is convergence of income in a world of closed economies but divergence in a world of open economies. This example illustrates the danger of using closed economy models to study development in environments where trade may be important. Not only do results change, but they may be completely the opposite of what the closed economy analysis suggests.

Second, proposition 9 says that factor price equalization in a given period does not imply factor price equalization in the future. In fact, for some parameter values and initial conditions, factor price equalization in a given period implies that factor prices cannot be equalized in the future.

Third, our numerical example 3 shows that, unless we can guarantee factor price equalization in all periods, our characterization of equilibria — in particular, the convergence patterns of relative incomes — is not useful even when there is factor price equalization in the current period. In fact, in a situation where we would expect divergence of income levels if we could somehow guarantee factor price equalization in the future, we can get convergence of income levels even while we are in the cone of diversification. This example illustrates the danger of ignoring the possibility of specialization in production. Not only can results change when we allow for specialization, but they may be the opposite of what the analysis that assumes factor price equalization suggests.

There is another aspect of our convergence results worth commenting on. In the case where $b < 0$ and $\delta = 1$, and where the economy is productive enough to converge to a nontrivial steady state and capital starts below this steady state, $\bar{k}_0 < \hat{k}$, opening a closed economy that is poorer than the world average to free trade slows down its growth and leads to convergence to a steady state with a lower capital-output ratio than the country would have reached had it remained closed. It is easy to show, however, that opening to trade leads to a gain in utility. Free trade guarantees higher utility, not higher growth rates.

Appendix: Algorithm used in the numerical examples

We first consider the case where the parameter values are such that the equilibrium of the integrated economy converges to the nontrivial steady state. To solve the model, we choose a truncation period T large enough so that the integrated economy equilibrium numerically converges to this steady state. We solve the system of equations using Newton's method as follows.

1. Compute the steady state of the integrated economy by solving $r(\hat{k}) = 1/\beta - 1 + \delta$.
2. Given initial levels of capital in each country $\bar{k}_0^i$, guess values for the $(n+1)T$ variables k_t^i, $i = 1,...,n$, $t = 1,...,T-1$, p_{1t}, $t = 0,...,T-1$, $\hat{k}^i$, $i = 1,...,n$.
3. For every t, $t = 0,1,...,T-1$
   - Compute x_{1t}/x_{2t} and p_{2t} using equations (94) and (95).
   - Compute the cone of diversification, $\kappa_{1t} = \kappa_1(p_{2t}/p_{1t})$, $\kappa_{2t} = \kappa_2(p_{2t}/p_{1t})$ using equations (89) and (90) or equations (91) and (92).
   - For each country, compute the production of each traded good and the factor prices:
     If $k_t^i \geq \kappa_{1t}$, $k_{1t}^i = k_t^i$, $\ell_{1t}^i = 1$, $y_{1t}^i = \phi_1(k_t^i, 1)$, $k_{2t}^i = 0$, $\ell_{2t}^i = 0$, $y_{2t}^i = 0$, $w_t^i = p_{1t}\phi_{1L}(k_t^i, 1)$, $r_t^i = p_{1t}\phi_{1K}(k_t^i, 1)$.
     If $\kappa_{1t} > k_t^i > \kappa_{2t}$, $k_{1t}^i = \kappa_{1t}^p(k_t^i - \kappa_{2t}^p)/(\kappa_{1t}^p - \kappa_{2t}^p)$, $\ell_{1t}^i = (k_t^i - \kappa_{2t}^p)/(\kappa_{1t}^p - \kappa_{2t}^p)$, $y_{1t}^i = \phi_1(k_{1t}^i, \ell_{1t}^i)$, $k_{2t}^i = \kappa_{2t}^p(\kappa_{1t}^p - k_t^i)/(\kappa_{1t}^p - \kappa_{2t}^p)$, $\ell_{2t}^i = (\kappa_{1t}^p - k_t^i)/(\kappa_{1t}^p - \kappa_{2t}^p)$, $y_{2t}^i = \phi_2(k_{2t}^i, \ell_{2t}^i)$ $w_t^i = p_{1t}\phi_{1L}(\kappa_{1t}^p, 1)$, $r_t^i = p_{1t}\phi_{1K}(\kappa_{1t}^p, 1)$.
     If $k_t^i \leq \kappa_{2t}$, $k_{1t}^i = 0$, $\ell_{1t}^i = 0$, $y_{1t}^i = 0$, $k_{2t}^i = k_t^i$, $\ell_{2t}^i = 1$, $y_{2t}^i = \phi_2(k_t^i, 1)$, $w_t^i = p_{2t}\phi_{2L}(k_t^i, 1)$, $r_t^i = p_{2t}\phi_{2K}(k_t^i, 1)$.
   - For each country, compute the demand for each traded good in each country, $c_{1t}^i + x_{1t}^i$, $c_{2t}^i + x_{2t}^i$ using the budget constraint (7) :

$$c_{1t}^i + x_{1t}^i = \frac{(x_{1t}/x_{2t})(w_t^i + r_t^i k_t^i)}{p_{1t}(x_{1t}/x_{2t}) + p_{2t}} \tag{116}$$

$$c_{2t}^i + x_{2t}^i = \frac{w_t^i + r_t^i k_t^i}{p_{1t}(x_{1t}/x_{2t}) + p_{2t}}. \tag{117}$$

   - For each country, compute the total production of the investment good $x_t^i = k_{t+1}^i - (1-\delta)k_t^i$, where $k_T^i = \hat{k}^i$, and use this to disaggregate $c_{1t}^i + x_{1t}^i$ and $c_{2t}^i + x_{2t}^i$:

$$x_{1t}^i = \frac{(x_{1t}/x_{2t})x_t^i}{f(x_{1t}/x_{2t}, 1)} \tag{118}$$

$$x_{2t}^i = \frac{x_t^i}{f(x_{1t}/x_{2t}, 1)}. \tag{119}$$

- Compute values for the $(n+1)T$ functions:

$$\frac{c_{t+1}^i}{\beta c_t^i} - (1-\delta + r_{t+1}^i)\,,\ t = 0,...,T-2\,,\ i = 1,...,n \tag{120}$$

$$\frac{f(\hat{k}^i,1) - \delta \hat{k}^i}{c_{T-1}^i} - 1\,,\ i = 1,...,n \tag{121}$$

$$\sum_{i=1}^n (y_{1t}^i - c_{1t}^i - x_{1t}^i)\,,\ t = 0,...,T-1 \tag{122}$$

$$\frac{\hat{k}^i}{\hat{k}} - \frac{k_{T-1}^i}{\sum_{j=1}^n L^j k_{T-1}^j / \sum_{j=1}^n L^j}\,,\ i = 1,...,n\,. \tag{123}$$

4. If these functions are not all within some specified distance from 0, adjust the guesses for the $(n+1)T$ variables k_t^i, p_{1t}, $\hat{k}^i$ using Newton's method and go to step 3. Iterate until convergence.

In the cases where k_t converges to 0 or where k_t grows without bound, we define $\gamma = \beta\left(DA_1^{1/b} + 1 - \delta\right)$. We replace $\hat{k}$ and $\hat{k}^i$ in the above algorithm with $\hat{k}_T = \gamma \sum_{i=1}^n L^i k_{T-1}^i$ and $\hat{k}_T^i$, and we replace the functions (121) with

$$\frac{f(\hat{k}_T^i,1) + (1-\delta+\gamma)\hat{k}_T^i}{c_{T-1}^i} - \gamma \tag{124}$$

If T is very large, the Jacobian matrix for Newton's method will be ill conditioned because some of the variables k_t^i will be much larger than others. We can solve this problem by working with the rescaled variables $\tilde{k}_t^i = k_t^i / \gamma^t$, $t = 1,...,T$.

A brute-force way to deal with corner solutions in investment is to guess the pattern of corner solutions and to solve the resulting system of equations. We replace the functions corresponding to first-order conditions,

$$\frac{c_{t+1}^i}{\beta c_t^i} - (1-\delta + r_{t+1}^i) \tag{125}$$

with

$$k_{t+1}^i - (1-\delta)k_t^i\,. \tag{126}$$

When we have solved this system of equations, we can check that the complementary slackness conditions (27) are satisfied:

$$-\frac{c_{t+1}^i}{\beta c_t^i} + (1-\delta + r_{t+1}^i) \le 0 \tag{127}$$

whenever we have guessed that $x_t^i = 0$. If these conditions are satisfied, we can stop. Otherwise, we provide a new guess for the pattern of corner solutions and try again.

There is a limitation to the algorithm that is worth noting: The functions involved in the system of equations are not everywhere continuously differentiable. In particular, production plans and factor prices change in a continuous, but not in a continuously differentiable, manner as changes in prices cause a country's capital-labor ratio to pass through the boundary of the cone of diversification. This can cause Newton's method to be less stable than it is when working with continuously differentiable functions. Nonetheless, by keeping the step sizes in the Newton's method small, we can still compute equilibria.

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Figure 1

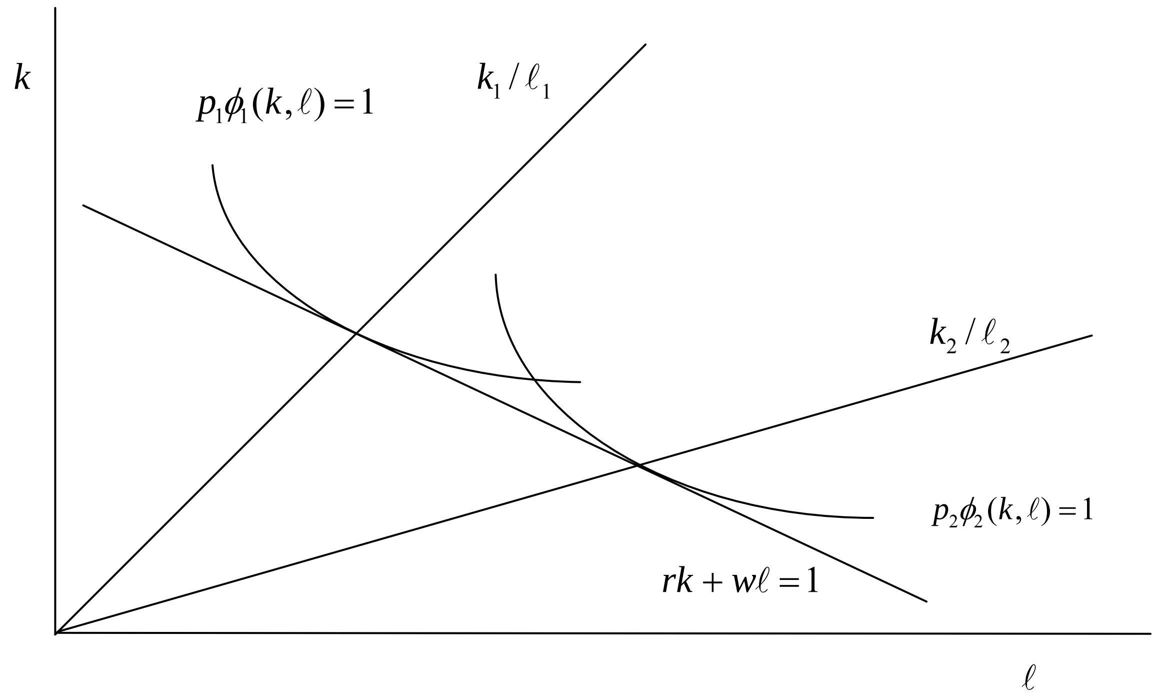

Figure 2

Example 1: Capital-labor ratios

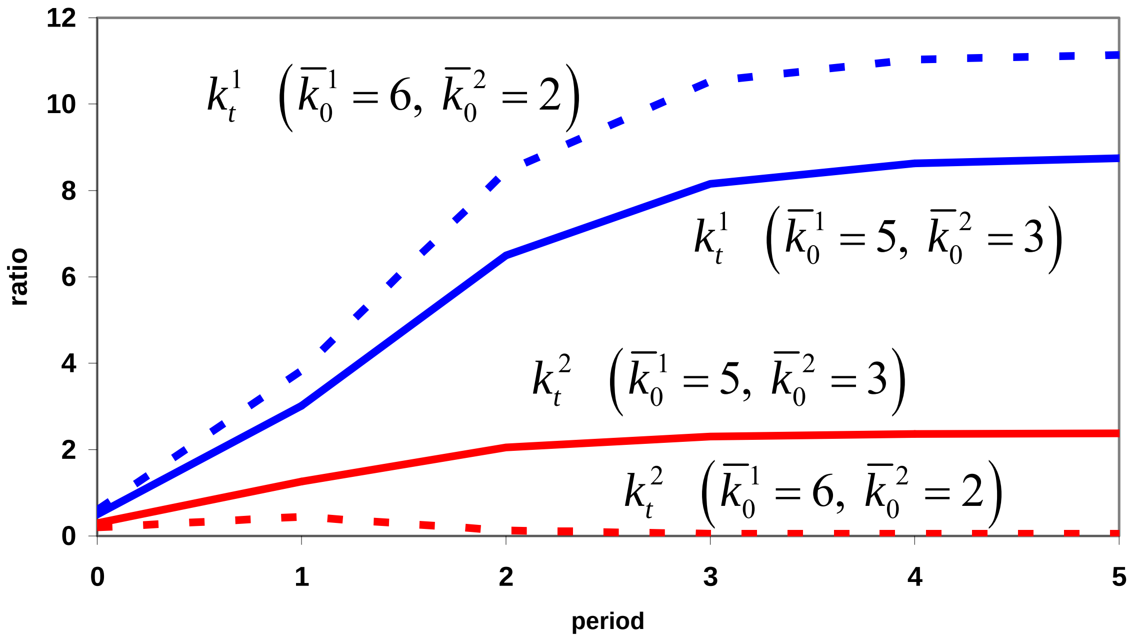

Figure 3

Example 1: Relative income in country 1

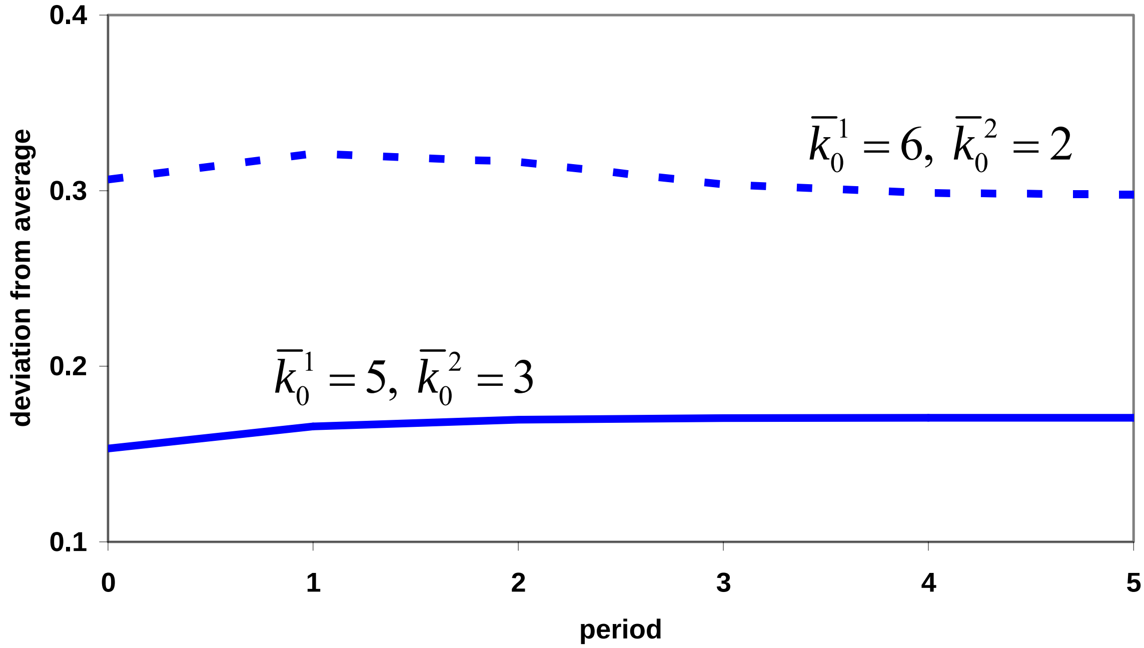

Figure 4

Example 2: Capital-labor ratios

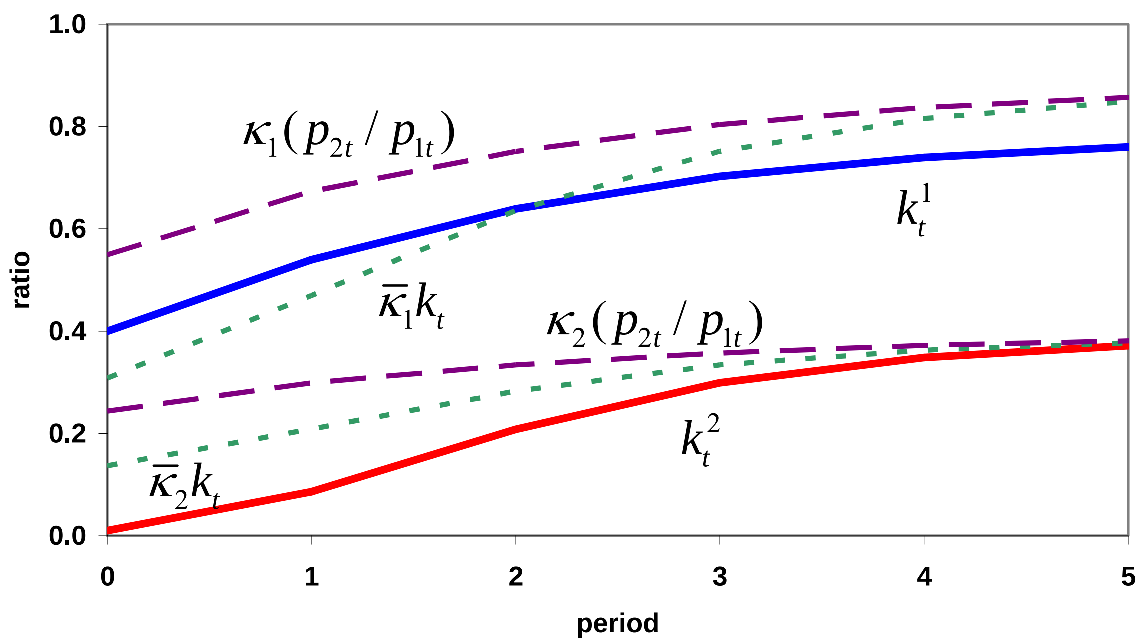

Figure 5

Example 2: Relative income in country 1

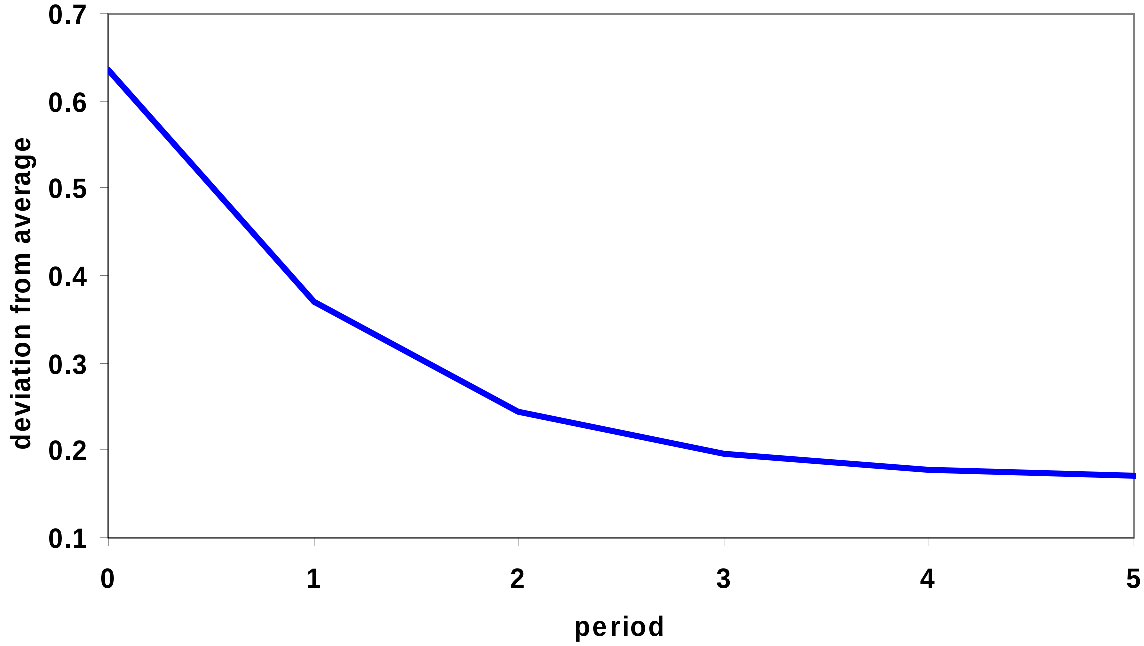

Figure 6

Example 3: Capital labor ratios

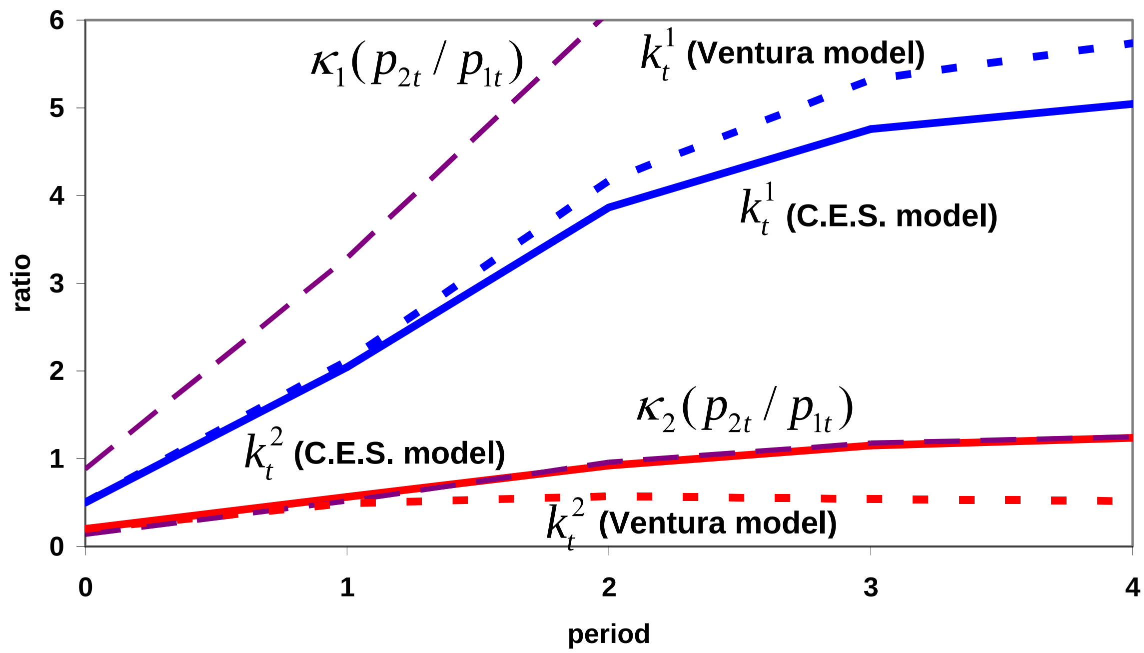

Figure 7

Example 3: Capital labor ratios (detail)

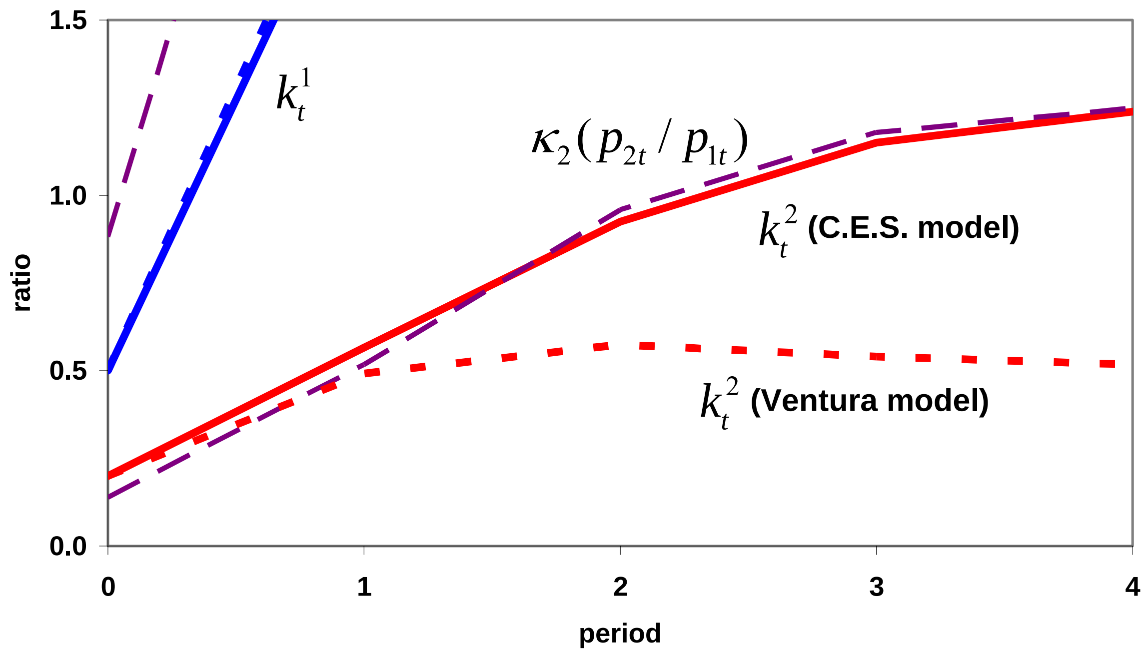

Figure 8

Example 3: Relative income in country 1

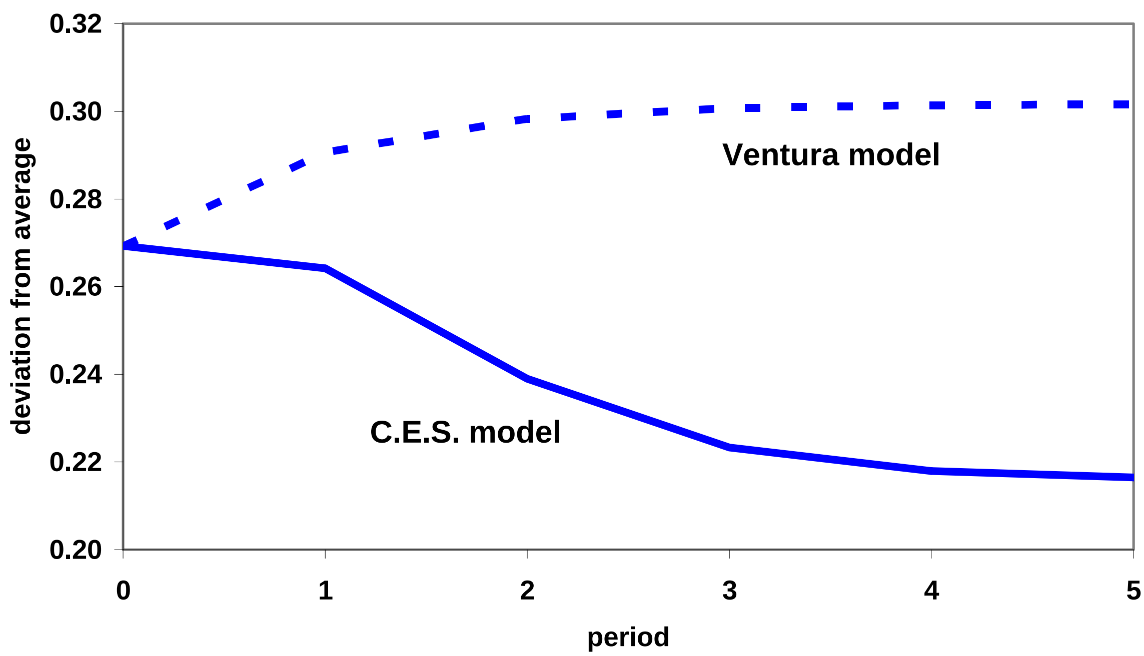

Figure 9

$b > 0$

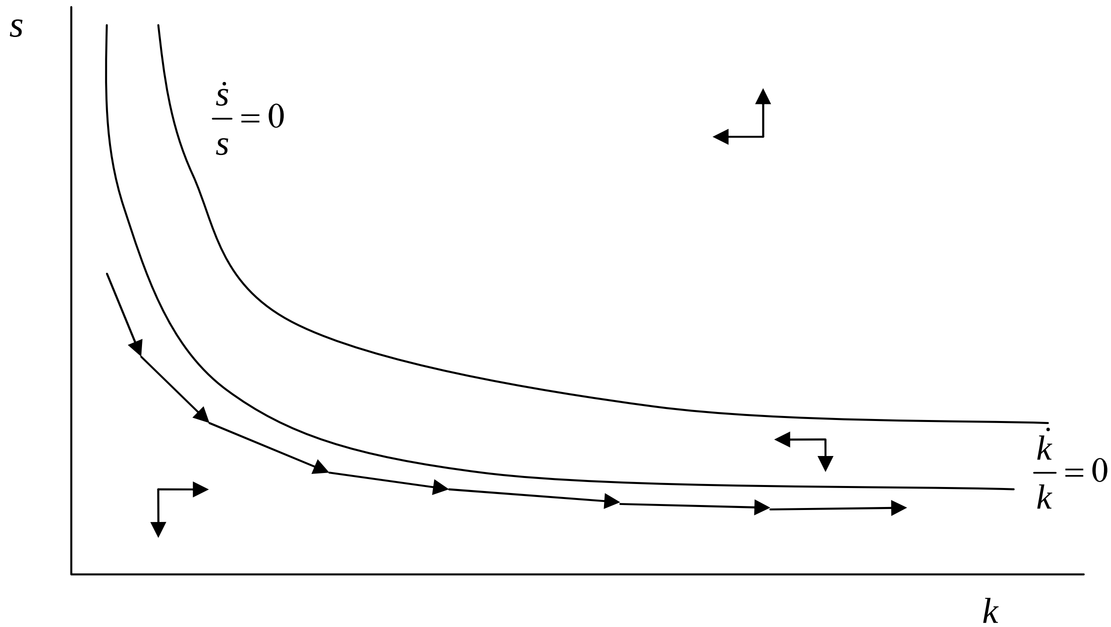

Figure 10

$b \leq 0$

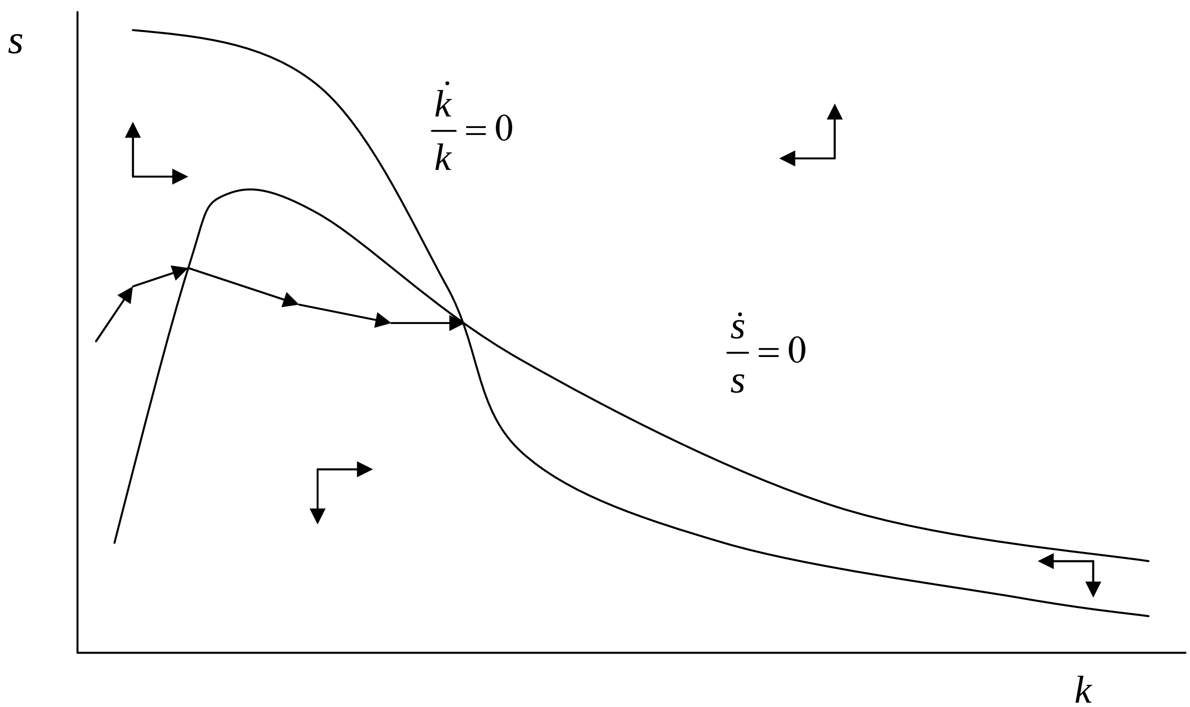

Figure 11

$b < 0$ and $\delta > 0$