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Doubts and Equilibria*

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Abstract

In real life strategic interactions decision-makers are likely to entertain doubts about the degree of optimality of their play. To capture this feature of real choice-making, we present here a model based on the doubts felt by an agent about how well is playing a game. The doubts are coupled with (and mutually reinforced by) imperfect discrimination capacity, which we model here by means of similarity relations. We assume that each agent builds procedural preferences defined on the space of expected payoffsstrategy frequencies attached to his current strategy. These preferences, together with an adaptive learning process lead to doubt-based selection dynamic systems. We introduce the concepts of Mixed Strategy Doubt Equilibria, Mixed Strategy Doubt-Full Equilibria and Mixed Strategy Doubtless Equilibria and show the theoretical and the empirical relevance of these concepts

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1 Introduction

Doubt, in everyday language, is closely related with the notion of uncertainty. In the **Webster's Encyclopedic Unabridged Dictionary of the English Language**, the entry doubt (verb) is defined as *"to be uncertain in opinion"*. Doubt (noun), on the other hand, is *"a feeling of uncertainty about the truth, reality, or nature of something"*. The **Merriam-Webster Online Dictionary** defines doubt (noun) as *"uncertainty of belief or opinion that often interferes with decision-making".* In the present work we are going to relate doubts with decisions. In real life choice situations, decision-makers feel unsure about the consequences of their choices. Even experienced decision-makers often face doubts when making choices in their domain of expertise. It is thus important to introduce the concept of doubt in the theory of choice, either individual or strategic.

Uriarte (1999) develops a procedure for making choices under risk that introduces a primitive function suggested by the findings of Kahneman and Tversky (1979). The domain of that function is the set of probabilities for winning a prize and the function measured implicitly levels of risk. In the present work, we change the domain of the function and extend its formal possibilities. We now call it the doubt function and we investigate its implications in a strategic environment. It is probably safe to say that strategic environments are more cognitively demanding and raise more doubts than the environment in which one has to choose among simple lotteries (in part because strategic uncertainty is typically of a more ambiguous nature). Thus, we think that embedding the notion of doubt in a strategic context is likely to be a fruitful enterprise.

Let us think of a game played continuously by two player populations. Each population has a set of possible strategies from which to choose We may imagine the following two separate but related stages in which doubts are likely to appear:

First stage: as a member of one population, I feel doubts about the exact consequences of choosing any of the strategies that are available to me, because they depend on the opponent's reactions, which, in most cases, are uncertain.

Second stage: how do I cope with that feeling? Does the environment provide me with information that I might use to construct an acceptable "measure" of my level of doubts?

The first stage depicts the situation in which a subject is directly feeling doubts as a consequence of the decision he is about to make. Let us talk about the second stage. When players make choices continuously, they obtain information about payoffs, but also about the fraction of fellow agents playing each strategy. If an agent observes that many others play a given strategy, it is natural for him to entertain less doubts about whether that strategy is a "good" option.[1] We shall assume that each individual has access to the information about the proportion of agents of his player population playing each strategy. Then each individual will use that information to build a "measure" of the doubts originated by his current strategy. This "measure" will be called the doubt function. Doubts about a strategy which decrease with the number of people using it are the most natural ones[2] and a great deal of the paper is dedicated to them. An additional feature of doubts, which we will also discuss in this paper, is that they are also closely related to imperfect discrimination capacity (of real numbers, such as strategy frequencies and expected payoffs).[3]

The purpose of the present article is to build a procedure, formally represented by a doubt-based selection dynamic model, with which two opposing population of agents choose over time strategies in a game. The players in both populations are assumed to use an adaptive trial an error process of changing their strategies over time. They have an aspiration level for their own welfare and change their strategies when their preferences do not reach their aspiration. A process similar to this one, if preferences were determined by a standard payoff function would yield the replicator dynamics of evolutionary games.[4] In this paper, however, preferences are determined by (typically decreasing) doubt functions and similarity relations.

We sketch now in some more detail the main pieces of the adaptive system that allow an agent with bounded cognitive capacities to work his way in a complex and continuously changing environment.

1. **Doubt functions**. All the agents are endowed with a doubt function that captures their uncertainties about the degree of optimality of the strategy they are currently using.

2. **Similarity relations**. Doubts and imperfect discrimination capacities are closely related. We model imperfect discrimination by means of (correlated) similarity relations.

[1]Economists have understood for a long time that imitation of "common" behavior is a widespread human decision-making strategy (see e.g. Smallwood an Conlisk 1979 or Nelson and Winter 1982).

[2]Even if they are not the only possible type, as we shall see below.

[3]The work of Kahneman and Tversky has plenty of examples about how the human cognitive system copes with such situations of limited capacity for discrimination. See, for instance, Tversky (1977), Kahneman and Tversky (1979), Kahneman (2003) and the references therein.

[4](see e.g. Cabrales 2000).

The doubt function plays a central role in defining these similarity relations.

3. **Preferences**. Each agent uses the similarity relations to build, by means of a choice procedure, a preference relation defined in the space of expected payoffs-strategy frequencies attached to the agent's current strategy.[5] The preference relation will inform the agent about his preferred (or aspiration) set. The resulting procedural preference relation presents thick indifference classes. The (inverse of) distance from the current vector of expected payoff-strategy frequency to the aspiration set (represented by the preferred set) determines the agent's degree of satisfaction with his current strategy.

4. **Adjusting behavior**. The agents' satisficing behavior consists of choosing and switching strategies to minimize the distance to the aspiration set. This adjustment process will give rise to different *doubt-based selection dynamic systems* depending on the type of doubt functions. As mentioned earlier, a special role is given to the assumption that the doubts of an agent decrease with the proportion of agents playing the same strategy as the agent's current one.

We, then, explore the properties of a *doubt-based selection dynamic system* for constant-sum 2x2 games with a unique equilibrium in mixed strategies. We show that the Mixed Strategy Nash Equilibrium is, under some conditions, a rest point for the system. More specifically, let us assume the situation in which all agents operate under the *doubt-full* or *absent* mode of play. We show that the system converges to population frequencies close to the Mixed Strategy Nash Equilibrium when all agents are in the *doubt-full* mode of play. The following interpretation can be given to this result. Agents are aware that the proportions with which each strategy is being played over time are not truly random. Thus, they experience high levels of doubts out of a fear of being exploited by opponents. The high fear and the doubts together with the adaptive choices lead the system to the Mixed Strategy Nash Equilibrium. Once in equilibrium, payoffs are equalized across strategies but the doubt levels continue to be high and equal across strategies. Thus, we show the equilibrium is an asymptotically stable point for the dynamical system in the *doubt-full* mode of play. We also calculate the values of the doubt parameter that would stabilize the Mixed Strategy Nash Equilibrium of 2x2 games, and illustrate this finding with explicit calculations both for

[5]The choice procedure is similar in nature to those described in Rubinstein (1988), Aizpurúa et al.(1993) and Uriarte (1999) (2007)).

the "Penalty Kick Game" of Palacios-Huerta and Volij (2007)[6] and the "Matching Pennies Game."

The dynamics are rather different when agents have a very small level doubts (even if they still decrease in the frequency of play). This is the *doubt-less* or *alert* mode of play, in which agents' doubts are very sensitive to strategy frequencies. In this situation, the only rest point for the doubt-based dynamic system is the center of the simplex (and thus, only in special circumstances it coincides with the Mixed Strategy Nash Equilibrium). However, in this case any perturbation, however small, sends the system away from the equilibrium. An interpretation for the result is that the extreme sensitivity to the "opinions" of others, leads play to a situation where players imitate, whenever doubtful, the current most fashionable action. This creates a tendency to diverge in population behavior. In addition, the doubtless agents are quite satisfied with their current strategies and do not feel the need to experiment with new strategies to exploit the differences in payoffs and strategy proportions. Hence, a low level of imitation and strategy adjustment takes place, and the populations diverges very slowly to a situation where initially popular strategies dominate..

There are also quite interesting intermediate cases, with strictly decreasing doubts that are less extreme than the previous cases, in between the *doubt-full* and *doubt-less* modes of play. In this case, we find a kind of herding behavior, which unlike the one in the *doubt-less* mode, can be stable. The equilibrium of the doubt-based dynamic system is not the Nash equilibrium and has the following feature: the most popular strategy has smaller (expected) payoffs. This is a *general* characteristic of equilibria with decreasing doubt functions. But in the *doubt-full* mode of play it is not so evident since the equilibrium is close to being Nash, and in the *doubt-less* mode, we have unstable dynamics. We believe that this feature of equilibria of *doubt-based selection dynamic system* is a relevant and robust testable implication for our model, and we provide some preliminary evidence to support it.

Finally, we should mention as well the case of *constant* doubts. This means that each agent's hesitations and feelings of uncertainty are not affected by the fraction of fellow agents from his population playing the same strategy. Thus, society does not have any direct influence on this type of agent. Then we show that the adjusting behavior would lead

[6]This interesting paper shows how professional football (soccer) players transfer the skills learnt in the field to the artificial setting of a laboratory and yet play close to the mixed strategy Nash Equilibrium.

us to a doubt-based selection dynamics that is closely related to the replicator dynamics.[7]

How does this work relate with the existing literature? Given the nature of the paper, we think that it is in the realm of the experimental literature that we should look into. Particularly, in the experiments where subjects are given information about the performance of the other participants. In Tang (2001), for instance, the participants in the experiment are given a precise information about the proportion of subjects playing each strategy as well as the average payoffs in the two player populations. This experiment contradicts one of our result, the one that says that the most popular strategy has smaller (expected) payoffs. To our defense, it should be said that it is not very realistic to provide such a precise information, -which, in fact, is only known by the experiment maker-, to subjects who are involved in the experiment. In fact, this kind of information would eliminate the "doubts" that the involved subjects might feel, a feature that plays a central role in the build-up of our model. In Binmore et al. (2001) a subject can compare his performance with the other subjects in the same population by sawing their median payoff. While holding the same critical stand as in the previous case, and taking into account that the information now is about payoffs, we note that the spiral trajectory converging to equilibrium that these authors observe in the experiment has a theoretical counterpart in the *doubt-full* case where the path to equilibrium is shown to be a spiral (sink) as well.

Given the above limitations, we have looked for data coming from field experiment and provide a supportive piece of evidence for our (doubt) equilibrium condition.

To conclude, we think that this paper, by insisting on doubts related with imperfect perception, highlights the need of more evidence from fuzzier, that is, more realistic, experimental environments.

2 A model of doubt-based selection dynamics

2.1 Notation

Consider a noncooperative finite game G in normal form, with $K = \{1, 2, ..., n\}$ denoting the set of players. For each player $k \in K$, let $S_k = \{1, 2, ..., m_k\}$ be her finite set of pure

[7]This result is yet another rationalization for the replicator dynamics. Other foundations for this dynamical system can be found in Binmore, Gale and Samuelson (1995), Weibull (1995), Cabrales (2000) and Schlag (1998), among others.

strategies, for some integer $m_k \geqslant 2$.

Imagine that there exist n large populations, one for each of the n player positions in the game. Members of the n populations chosen at random -one member from each player population- are repeatedly matched to play the game. In what follows, we shall speak of *players* when referring to the game G and we shall speak of *agents* when referring to the members of the populations. Each agent is characterized by a pure strategy. From now on, we shall refer to the agent ki as a member of the player population $k \in K$ who plays pure strategy $i \in S_k$. Let $f_{ki}(t) \in F_{ki} = [0,1]$ be the relative frequency of ki agents at time t, with $f(t)$ being the vector collecting such probabilities. Time index suppressed, $\pi_{ki}(f)$ will denote agent ki's expected payoff given the population state f. Without loss of generality, we may assume that payoffs are strictly positive and do not exceed one; hence, $\pi_{ki}(f) \in \Pi_{ki} = (0,1]$. Finally, $\overline{\pi}_k(f) = \sum_{i=1}^{m_k} f_{ki}(f)\ \pi_{ki}(f)$ is the average payoff in player population $k \in K$. To simplify notation, we shall denote $\pi_{ki}(f)$ as π_{ki}.

2.2 The doubt-based selection dynamics

We will be dealing with boundedly rational players by assuming that they have doubts about how well they are playing the underlying game. Assume that every agent of each player population is endowed with a (primitive) function that we call the "doubt function". This function, denoted d_{ki}, measures the doubts felt by agent ki about how good is his current strategy $i \in S_k$, available to player population $k \in K = \{1, 2, ..., n\}$, as a response to the strategies that the rest of players are using. Agent ki relates the doubts he is feeling with the proportion of individuals who are using the same strategy as his current one. Therefore, $d_{ki}(f_{ki})$ measures how ambiguous agent ki feels about the optimality of strategy $i \in S_k$, given that the proportion of agents of his own population currently playing that strategy is $f_{ki} \in F_{ki}$.

We shall assume in this section that the agents are endowed with a strictly decreasing doubt function. That is, an agent's doubts about how well is playing gradually decrease when he observes (or is informed) a larger number of agents from his player population playing the same strategy as the one he is currently using. In other words, society does have an influence upon this type of agents. Formally,

Assumption 1 *The strictly decreasing doubt functions.*

Each agent ki is endowed with a differentiable doubt function d_{ki} in the set

$$D = \left\{ \begin{array}{c} d_{ki} : F_{ki} \to [0,1] : \text{with } \widehat{f}_{ki} > \widetilde{f}_{ki} \Rightarrow d_{ki}(\widehat{f}_{ki}) < d_{ki}(\widetilde{f}_{ki}) \\ \text{and } d_{ki}(0) = 1,\ d_{ki}(1) = 0 \end{array} \right\}$$

Given a proportion $f_{ki} \in F_{ki}$, -known by the ki agent-, and the $d_{ki} \in D$, $d_{ki}(f_{ki})$ measures the doubts (about how well is playing the game) felt by the agent ki when the proportion of agents in player population k playing strategy $i \in S_k$ at time t is f_{ki}.

Remark 2 *To stress that we are not dealing always with a kind of "herding model of doubts", we highlight the following two types of doubt functions in D which are relevant for the results of section 3:*

1. *Function $d^{\delta} \in D^{\delta} \subset D$ which, for every $f_{ki} \in (0,1)$, $d^{\delta}(f_{ki})$ is "close" to 0 (i.e., $d^{\delta}(f_{ki}) < \delta$ for all $f_{ki} > \delta$, as in figure 1.*

2. *Function $d^{1-\delta} \in D^{1-\delta} \subset D$ which, for every $f_{ki} \in (0,1)$, $d^{1-\delta}(f_{ki})$ is "very close" to 1 (i.e., $d^{\delta}(f_{ki}) > 1 - \delta$ for all $f_{ki} < 1 - \delta$), as in figure 2.*

When $d_{ki} = d^{\delta}$, for sufficiently small δ,we say that the agent ki is in the alert *or* doubt-less *mode and when $d_{ki} = d^{1-\delta}$, for sufficiently small δ, we say the agent is in the* absent *or* doubt-full *mode.*

When the doubt functions of D are in between these two extreme cases, then we may say there is a kind of "herding effect on doubts" that grows stronger as we move away from those cases.

2.2.1 Doubts and Imperfect Discrimination Modeled by (Correlated) Similarity relations

In the present model, doubts are closely related to imperfect discrimination capacity (in the present paper, of real numbers, such as strategy frequencies and expected payoffs). An environment shaped by uncertainty and doubts about the correctness of the choices made is effort demanding for the cognitive system of decision-makers. One way subjects cope with the ambiguous nature of this situation is by simplifying its complexity; for instance, by grouping numbers in intervals of similarity. Inside those intervals, whose size depend

Figure 1: Doubt function of an agent in the alert mode of play. His doubts are almost 0 in the interval (0, 1).

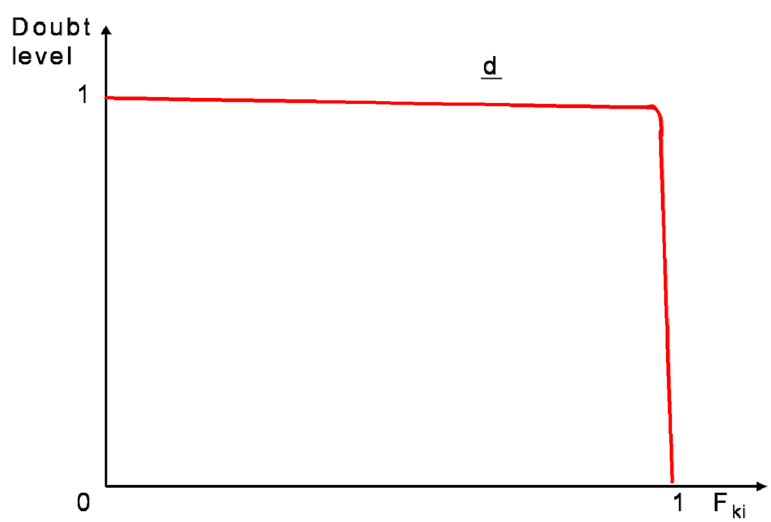

Figure 2: Doubt function of an agent in the alert mode of play. Figure 2: Doubt function of an agent in the absent or doubt-full mode. His doubts are almost 1 in the interval (0, 1).

on threshold levels that change continuously, values - of, say, expected payoffs -, are not distinguished. We thus model subjects' imperfect discrimination by means of correlated similarities. Correlated similarities are an extension of the similarity relations defined by Rubinstein (1988). Rather than being constant, they depend on the value of certain relevant parameters. More specifically, the d_{ki} function defines the correlated similarity relations that will capture agent ki's imperfect discrimination of expected payoffs and strategy frequencies. In the next lines we sketch how this is done (a complete account is given in Appendix A). Let (π_{ki}, f_{ki}) be the vector of expected payoff-proportion of agents of player population k attached to strategy $i \in S_k$ at time t.

1. d_{ki} defines on the space of expected payoffs, Π_{ki}, correlated similarities of the difference-type as follows: given f_{ki}, the similarity interval of π_{ki} is:

$$[\pi_{ki} - d_{ki}(f_{ki}), \pi_{ki} + d_{ki}(f_{ki})]$$

   Thus, given f_{ki}, $d_{ki}(f_{ki})$ defines the threshold level on Π_{ki}. Payoffs inside the similarity interval are not discriminated by the agent. By Assumption 1, if f_{ki} increases, the threshold, $d_{ki}(f_{ki})$, decreases and so the similarity interval of π_{ki} shrinks (giving rise to the vertical cone-shaped form in figure 3). This means that when f_{ki} increases, the discrimination capacity on the space of expected payoffs to strategy i, Π_{ki}, increases (probably because the accumulated experience with strategy i has increased due to the increased number of agents from population k currently playing strategy i). Thus, there is one similarity relation on Π_{ki}, denoted $S\Pi[f_{ki}]$, for each $f_{ki} \in (0, 1)$.

2. d_{ki} builds the λ_{ki} function, which, in turn, is used to define on F_{ki} correlated similarity relations of the ratio-type. This function is defined as follows: given d_{ki} and a specific $f_{ki} \in (0, 1)$, then for all $\pi_{ki} > d_{ki}(f_{ki})$

$$\lambda_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})} > 1$$

Thus, there is one λ_{ki} function for each $f_{ki} \in (0, 1)$, so that, given π_{ki},and f_{ki} attached to agent k's strategy i, with $\pi_{ki} > d_{ki}(f_{ki})$, the similarity interval of f_{ki} is:

$$[f_{ki}/\lambda_{ki}(\pi_{ki}), \;\; f_{ki}.\lambda_{ki}(\pi_{ki})]$$

The properties of the λ_{ki} function that should be kept in mind for the remainder of the paper are the following.

2.2.2 Properties of the λ_{ki} function

1. Given d_{ki} and a proportion $f_{ki} \in (0, 1)$, $\frac{\partial \lambda_{ki}(\pi_{ki})}{\partial \pi_{ki}} < 0$. This means that if the payoffs at stake increase, the similarity interval of f_{ki} shrinks; in other words, the discrimination capacity on $F_{ki} = [0, 1]$ increases if the expected payoffs increase. This property generates the horizontal wedge-shaped form of figure 3.

2. Suppose now that, other things equal, f_{ki} increases (decreases); in other words, since d_{ki} is strictly decreasing, suppose that the doubts of agent ki decreases(increases). Then, we would have a different λ_{ki} function such that, since $d_{ki} \in D$ has not changed, the similarity intervals of f_{ki} will shrink (expand) for a given π_{ki}.

2.2.3 Procedural Preference Relation and Satisficing Behavior

We shall assume that each agent ki compares pairs of vectors in $\Pi_{ki} \times F_{ki}$ with the aid of the of the correlated similarity relations $S\Pi[f_{ki}]$ and $SF[\pi_{ki}, f_{ki}]$, to decide which of the two is preferred. The choice procedure, which is similar in nature to that of Rubinstein (1988), but a bit more sophisticated due to the use of correlated similarity relations, gives rise to the preference relation depicted in figure 3. A detailed description of how the preference is built is given in *Appendix A*.

We assume that every ki agent chooses strategies with the purpose of minimizing the distance to the aspiration set, which, here, is represented by the preferred set relative to vector $(\pi_{ki},\ f_{ki})$, and denoted as $U = U_\alpha \cup U_\beta \cup U_\delta$ in figure 3. In other words, this strategy choice behavior tries to reduce the size of the indifference set $\sim_{ki} [(\pi_{ki},\ f_{ki})]$: the thinner is this set, the closer is $(\pi_{ki},\ f_{ki})$ to its corresponding upper contour set U.

Note that the two properties of λ_{ki} we highlighted above are useful to make it a good measure of the variations in the size of the indifference set, and, therefore, a good measure of the distance to the aspiration set. Thus, the function λ_{ki} could be thought of as an indicator of the degree of satisfaction of agent ki with his current strategy. The smaller the value of λ_{ki}, the happier would feel the agent with his current strategy. Hence, an agent chooses a strategy to reduce the doubt level and/or increase the expected payoffs.

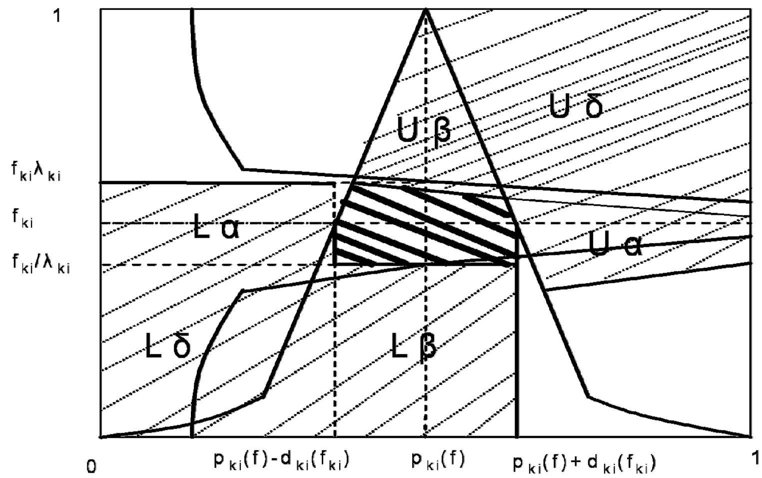

Figure 3: Preference relation derived from a choice procedure based on correlated similarity relations with decreasing doubts.

2.2.4 The Doubt-Based Selection Dynamics

Let

$$\frac{\lambda_{ki}-1}{\sum_{i=1}^{m_k}\lambda_{ki}}=\frac{\lambda_{ki}-1}{\lambda_k}$$

denote the proportion of ki strategists who feel dissatisfied with strategy i at time t. Notice that if λ_{ki} increases this proportion increases .

We assume that time is divided into discrete periods of length τ. In every period, $1-\tau$ is the probability that the agent does retain his current strategy; thus, τ is the probability that each agent does not retain his current strategy. We make now the following assumption to build a selection dynamic model[8]

Assumption 3 *When an agent feels dissatisfied with his current strategy, she will choose a new strategy with a probability that is equal to the proportion of agents playing that strategy.*

¿From Assumption 3, $\tau\frac{(\lambda_{ki}-1)}{\lambda_k}f_{ki}$ will denote the proportion of ki strategists who will choose a new strategy (the *outflow*), and, since a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then $\tau\sum_{j=1}^{m_k}\frac{(\lambda_{kj}-1)}{\lambda_k}f_{kj}f_{ki}=$

[8]For a justification see, for example, Binmore et al. (1995).

$\tau\frac{(\overline{\lambda}_k-1)}{\lambda_k}f_{ki}$ is the proportion of agents who will choose strategy i (the *inflow*), where $\overline{\lambda}_k = \sum_{j=1}^{m_k}\lambda_{kj}f_{kj}$.

Therefore,

$$f_{ki}(t+\tau) = f_{ki}(t) - \tau\frac{(\lambda_{ki}-1)}{\lambda_k}f_{ki} + \tau\frac{(\overline{\lambda}_k-1)}{\lambda_k}f_{ki}$$

As $\tau \to 0$, in the limit we get the *doubt-based* selection dynamic equation:

$$\overset{\bullet}{f}_{ki} = f_{ki}\left[\frac{\overline{\lambda}_k - \lambda_{ki}}{\lambda_k}\right] \tag{1}$$

Remark 4 *Doubts and Strategy Choice behavior*

Note that if λ_{ki} increases -because, other things equal, the doubts of agent ki increase- the ratio of ki strategists who feel dissatisfied with strategy i at time t, $\lambda_{ki} - 1\lambda_k$, increases too, and therefore, the proportion of those agents prone to change strategy will increase too. A similar effect will occur if, given a level of doubts, π_{ki} decreases. *This connection, between doubts and strategy choice behavior, provides an exact meaning to the notion of doubts in this model, that coincides with the intuitive notion of doubts in a continuous decision-making context.* A doubtful agent would be one with a tendency to try new strategies.

To gain some intuition, let us now look at equation (1) in a less compact way. Let G be a two-population constant-sum game with $S_I = \{U, D\}$ and $S_{II} = \{L, R\}$ denoting player I and player II's strategy sets, respectively. Let x denote the probability of playing U, y the probability of playing L and $I = [(x^*, 1-x^*), (y^*, 1-y^*)]$ the Mixed Strategy Nash Equilibrium, with $x^* > 0$ and $y^* > 0$.

To avoid the use of four different doubt parameters, we shall assume that the four doubt functions are the same; that is, $d_{ki} = d \in D$ (where $k = I, II$ and $i = U, D, L, R$).

¿From (1), the *doubt-based* selection dynamics for G is represented by the following system:

$$\dot{x} = \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}(\pi_U d_D - \pi_D d_U) \equiv G_1(x,y)F_1(x,y) \tag{2}$$

$$\dot{y} = \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)}(\pi_L d_R - \pi_R d_L) \equiv G_2(x,y)F_2(x,y) \tag{3}$$

Clearly, a stationary point for the *doubt-based* system (2)-(3), with $x^* > 0$ and $y^* > 0$, requires $\pi_U d_D = \pi_D d_U$ and $\pi_L d_R = \pi_R d_L$. We call this point the Mixed Strategy Doubt Equilibrium (MSDE).

2.2.5 Mixed Strategy Nash Equilibrium (MSNE) and Mixed Strategy Doubt Equilibrium (MSDE)

We should distinguish between the Mixed Strategy Nash Equilibrium (MSNE) and the Mixed Strategy Doubt Equilibrium (MSDE) for the doubt-based dynamic system (1).

1. In a MSNE the requirement is that all strategies in the support of the equilibrium have equal payoffs; that is:

$$\pi_{ki}(f^*) = \pi_{kj}(f^*) \text{ for all } i, j \text{ with } f_i^* > 0 \text{ and } f_j^* > 0 \text{ and all } k.$$

2. From (1) we deduce that for a MSDE the requirement is:

$$\frac{\pi_{ki}(f^*)}{d(f_i^*)} = \frac{\pi_{kj}(f^*)}{d(f_j^*)} \text{for all } i, j \text{ with } f_i^* > 0 \text{ and } f_j^* > 0 \text{ and all } k$$

Note that in this case, the expected payoffs to the strategies in the support of the equilibrium need not be equal, as it is required in the MSNE.

3 Doubt-based selection dynamics in constant sum games

We shall present in this section how subjects with limited cognitive capacities are capable of adapting to the changes of a complex environment, learn interactively to become more skillful in their choices and, eventually, reach, under some conditions, the socially optimal outcome predicted by the theory for rational players.

3.1 Relationship between a MSNE and a MSDE

Let us recall what game theorists say about a MSNE:

"The point of randomizing is to keep the other player(s) just indifferent between the strategies that the other player is randomizing among. One randomizes to keep one's rivals guessing and not because of any direct benefit to oneself." (Kreps 1990, p 408).

The doubt-based model is able to capture that state of players' mutual guessing that characterizes a MSNE. Assume that we are dealing with 2×2 constant sum games having a unique mixed equilibrium with full support. Consider Player I; how would this player interpret different values of (his own probability) x, say 0.2 and 0.6 ? A rational Player I knows that Player II is randomizing to keep him indifferent between the strategies he is randomizing among. Therefore, $x = 0.2$ and $x = 0.6$ will induce in Player I's rational mind the same level of doubts as to which is the best probability distribution, because both of them have the same expected payoff. But, for the same reason, Player I's equilibrium strategy in the game will induce the same level of doubts as 0.2 or 0.6. In other words, Player I does not see, both strategically and in a preference sense, any real difference between different probability distributions in the open unit interval [0,1]. As a consequence, he will have (nearly) equal level of doubts at any x in (0,1). The same will happen to Player II.

Hence, we ask first, *which are the level of doubts embedded in the players' mutual guessing that characterizes steady states close to the the MSNE?*. This is answered in Proposition 5 below, where we show that any interior MSNE converges to a MSDE if all agents are playing in the *doubt-full mode*; that is, a MSNE is a Mixed Strategy Doubt-Full Equilibrium (MSDFE). We also prove that when agents are in the *doubt-less mode* of play, the only rest point of the system (2)-(3) is $[(1/2, 1/2), (1/2, 1/2)]$.

The second issue to deal with is the following: how is the MSNE reached? or,which is the equilibrating process that may lead to the MSNE? This will be answered in Propositions 7 and 8 below.

We shall assume, without loss of generality, that $d_{ki}(f_{ki}) = (1 - f_{ki})^{\alpha}$. Assuming that $\alpha \in (0, \infty)$, we would obtain a large enough subclass of doubt functions in the set D. The convex combinations of elements in this class belong to D as well. Note, in particular, that this class contains the two extreme types of doubt functions introduced in Remark 1: when α is very small, near zero, the doubt parameter characterizing agent ki, denoted as $\Omega = \frac{1}{\alpha}$, is very high for any $f_{ki} \in (0, 1)$. Then the function will have a graph looking like the one of figure 2, and we shall say now that the agent is in the *absent* or *doubt-full mode of play*. When α is very high, the graph of d_{ki} is close to the axes, as in figure 1, and so the doubt parameter, $\Omega = \frac{1}{\alpha}$, is very small, for any $f_{ki} \in (0, 1)$. This is the agent in the *alert* or *doubt-less mode of play.* The results make use of these two modes of play and therefore do not depend on the mathematical form of the doubt functions. On the other hand, with this class of doubt functions we can make numerical calculations in the examples presented below.

Let G be a two-population, two-strategy, constant-sum game with $I = [(x^*, 1 - x^*), (y^*, 1 - y^*)]$, $x^* > 0$, $y^* > 0$, denoting its MSNE.

Proposition 5

1. The (Euclidean) distance between an MSDE and MSNE converges to zero as δ goes to zero if every agent plays with a doubt function in the $D^{1-\delta}$ class; that is, if they play in a *doubt-full* mode. Hence, an MSNE $(x^*,\ y^*) \in (0,1) \times (0,1)$ is an MSDFE.

2. Let $d_{ki}(f_{ki}) = (1 - f_{ki})^{\alpha}$ for all k, i. Then the (Euclidean) distance between an MSDE and the central point of the simplex $C = [(1/2, 1/2), (1/2, 1/2)]$ converges to zero as α goes to infinity. That is, if they play in a a *doubt-less* mode.

Proof: See appendix B

A straightforward corollary of Proposition 5 is the following:

Corollary 6 *Let $C = [(1/2, 1/2), (1/2, 1/2)]$ and d any doubt function in D. Then, if C is the MSNE of G, it is also an MSDE (i.e. a rest point of the* doubt-based *dynamic system (2)-(3)).*

This means that if $C = [(x^*, 1 - x^*), (y^*, 1 - y^*)] = [(1/2, 1/2), (1/2, 1/2)]$ is the MSNE of G, then it is compatible (in the sense of Proposition 5) with agents playing in any of the two modes of play, *doubt-full* or *doubt-less*, as well as with agents endowed with any doubt function of D in between those two extreme doubt functions.

Note that in a Mixed Strategy Doubt-Full Equilibrium (MSDFE), the indifference set will be so thick that it will cover almost the whole space $[0, 1] \times [0, 1]$. In a Mixed Strategy Doubtless Equilibrium (MSDLE) the interior of the indifference set will be almost empty. Even though every $d_{ki} \in D$ is strictly decreasing, the exact values of the mixed strategy equilibrium, (x^*, y^*), with x^* and $y^* > 0$, do not matter since every ki agent is endowed either with a doubt function in the absent mode or in the alert mode. In particular, this means that Proposition 5 does not impose any restriction on the equilibrium values that f_{ki} might take nor it does relate those probability values with their corresponding expected payoff values, π_{ki}, $k = I, II$ and $i = 1, 2$, in a particular manner.

3.2 Learning to Play a Mixed Strategy Nash Equilibrium (MSNE)

The question that we have not answered yet is: how the players do learn to coordinate in the MSNE?

We want now to defend the MSNE concept by some specific adjusting behavior of our rationally bounded players. We know that a fully rational player must avoid being guessed by the opponents and that to achieve this he will behave in such a way so as to create a random sequence of choices. This suggests that a doubtless mode of playing -that implies almost no strategy switching behavior- would be far from being an adjusting process leading to the Nash equilibrium, and indeed for most MSNE that is the case as proposition 5 shows. It seems that, in an equilibrating process, what makes more sense is that players should behave in the doubt-full mode. In our deterministic dynamic model, permanent doubt-full agents will have a tendency to keep trying new strategies and, thus, generating not a truly random sequences of choices, but individual processes of trial-and-error adjustments which could find their way to the MSNE. In Proposition 7 below we show that this is the case: if every agent behaves as if he were constantly with a high level of doubts, the agents' adjusting behavior would lead them to the MSNE and endow the equilibrium with a strong stability property. Proposition 8 shows that the doubtless mode of play has just the opposite consequence.

Proposition 7 *Let G be a two-population, two-strategy, constant-sum game with $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$, $x^* > 0$ and $y^* > 0$, denoting its Mixed Strategy Nash Equilibrium. Then a point close to I^* is asymptotically stable for the doubt-based dynamic system (2)-(3) if every agent plays in the* doubt-full *or* absent *mode of play.*

Proof: See appendix B

Proposition 8 *Let G be a two-population, two-strategy game. If every agent is in the doubtless or* alert *mode of play (i.e. α is arbitrarily large) and the initial conditions of the doubt-based dynamic system (2)-(3) are different from $[(1/2, 1/2), (1/2, 1/2)]$, the system diverges to a corner of the simplex.*

Proof: See appendix B

Remark 9 *Proposition 8 implies that the Mixed Strategy Nash Equilibrium of a constant sum game is unstable.*

One may then ask about why the modes of play of Proposition 7 and 8 would arise. Needless to say, doubts are a subjective feeling and hence it is difficult to ascertain the precise reason why they may arise in each particular case. Proposition 7 suggests that the origin of high level of doubts lies in the fact that every agent seems to be aware that the proportion with which each available strategy is being played and the sequence that the agents, as a player population, are producing is not random. Thus, the high levels of doubts felt by every member of each player population would arise from the fear of being guessed and exploited by the opponent. As a consequence, since agents are very unhappy with their current strategies (measured by a very high valued λ_{ki}) a high proportion of agents will experiment with new strategies in the next period. The fear and the doubts of the agents will continue to be high and, joint with the choices that exploit the variations both in the payoffs and in the strategy proportions, the adjusting behavior would lead the system to the Mixed Strategy Nash Equilibrium. Once in the equilibrium, payoffs are equalized across strategies and the doubt levels continue to be very high and equal across strategies too. Thus, the doubt-full mode of play endow the MSNE with strong stability properties

Proposition 8 suggests that agents seem to be too confident and satisfied with the pure strategies they are currently playing (they have a very low valued λ_{ki}). With almost no doubts, they would just produce small strategy choice changes, not taking care of the randomness of their sequences. Thus, imitation is almost non existent and the resulting dynamics is not sensitive enough to payoff and strategy proportion changes, however small. These features would explain why the dynamics do not converge to equilibrium from any initial point in the state space different from the equilibrium itself.

4 Examples

Example 10 ***The Penalty Kick Game***

Palacios-Huerta (2003) found that the equilibrium theory predictions are observed in the professional players' behavior: (i) their choices follow a random process and (ii) that the probability that a goal will be scored must be the same across each player's strategies and equal to the equilibrium scoring probability (that is, in the Mixed Strategy Nash Equilibrium each player is indifferent among the available strategies). Palacios-Huerta and Volij (2007) extend this result by observing that professional players are capable of transferring their skills from the field to the laboratory, a completely unknown setting for them, and yet behave in a way that is significantly near the Nash equilibrium.

Palacios-Huerta and Volij (2007), from a sample of 2,717 penalty kicks collected from European first division football (soccer) leagues during the period 1995-2004, built the following two player (Player I: the kicker and Player II: goal keeper) two strategy (Left, Right) game.

	(y) L	R
(x)L	0.60, 0.40	0.95 , 0.05
R	0.90, 0.10	0.70, 0.30

where $\pi_I(i,j)$ denotes the kicker's probability of scoring when he chooses i and the goalkeeper chooses j, for $i,j \in \{L,R\}$ .The Mixed Strategy Nash Equilibrium of this game is: $x^* = 0.363\,64, y^* = 0.454\,55$.

Football matches are continuously played and players' game is based on the study of the opponents in the field and watching their play on TV and in videotapes, so that their behavior in the penalty kicks is collected and analyzed. Thus, there is a history of play of each player and, hence, an interactive learning process. Thus, a natural issue is to investigate the type of dynamic process that may lead to the result found by Palacios-Huerta (2003). The *doubt-based* model seems to be a suitable model for this task.

The *doubt-based* selection dynamic system (2)-(3) corresponding to this game is the following:

$$\begin{aligned}
\dot{x} &= \frac{x(1-x)((0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha})}{2(0.95-0.35y)(0.2y+0.7)-(0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha}} \\
\dot{y} &= \frac{y(1-y)((0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha})}{2(0.1+0.3x)(0.3-0.25x)-(0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha}}
\end{aligned}$$

The vector field defining (2)-(3) is

$$\begin{aligned}
F(x,y) &= (\frac{x(1-x)((0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha})}{2(0.95-0.35y)(0.2y+0.7)-(0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha}}, \\
&\quad \frac{y(1-y)((0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha})}{2(0.1+0.3x)(0.3-0.25x)-(0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha}})
\end{aligned}$$

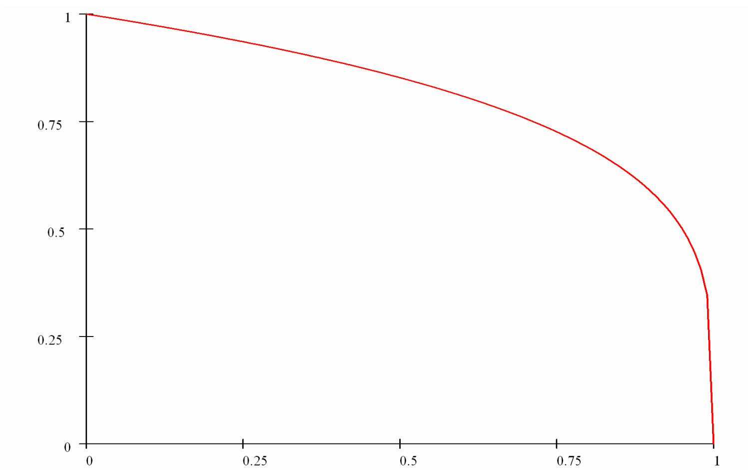

Figure 4: The graph of the doubt function $d(f_{ki}) = (1 - f_{ki})^{0.231}$. The horizontal axis measures the proportion f_{ki} of agents in population k playing the pure strategy i. The vertical axis measures the doubt level associated to each f_{ki}.

We compute first the derivative $DF(x, y)$ and then evaluate $DF(x, y)$ at $(0.363\,64, 0.454\,55)$ to get the following Jacobian matrix:

$$DF(0.363\,64, 0.454\,55) = \begin{bmatrix} \frac{\alpha}{1.\,581\,8 - 2\times 0.363\,64^{\alpha}} & \frac{0.146\,29(-0.2\times 0.636\,36^{\alpha} - 0.35\times 0.363\,64^{\alpha})}{0.790\,91 - 0.363\,64^{\alpha}} \\ 0.592\,88\frac{0.25\times 0.545\,45^{\alpha} + 0.3\times 0.454\,55^{\alpha}}{0.209\,09 - 0.454\,55^{\alpha}} & \frac{\alpha}{0.418\,18 - 2\times 0.454\,55^{\alpha}} \end{bmatrix}$$

It is easy to see that for values of $\alpha \in (0, 0.231\,88)$, all the eigenvalues of $DF(0.363\,64, 0.454\,55)$ have negative real parts and the associated determinants are all positive. Thus, the equilibrium $(0.363\,64, 0.454\,55)$ is a *spiral sink,* for those values of α, and, therefore, it is asymptotically stable. This means that the doubt functions of professional football (soccer) players are in a set which includes one having a graph looking, approximately, like the one of figure 4. The latter would correspond to the player whose performance shows fewer level of doubts, $\delta = 1/0.231\,88 = 4.\,312\,6$, for any frequency level in $(0, 1)$.

Example 11 ***The Matching Pennies Game***

	(y) L	R
(x) U	1, 0.5	0.5 , 1
D	0.5, 1	1, 0.5

The Mixed Strategy Nash equilibrium of this game is $(1/2, 1/2)$, and the doubt-based system (2)-(3) corresponding to the game is the following:

$$\begin{aligned} \dot{x} &= \frac{x(1-x)\left(0.5(1+y)x^{\alpha}-(1-0.5y)(1-x)^{\alpha}\right)}{2\left(0.5+0.25y-0.25y^{2}\right)-0.5(1+y)x^{\alpha}-(1-0.5y)(1-x)^{\alpha}} \\ \dot{y} &= \frac{y(1-y)\left((1.0-0.5x)y^{\alpha}-(0.5x+0.5)(1-y)^{\alpha}\right)}{2\left(0.5+0.25x-0.25x^{2}\right)-(1.0-0.5x)y^{\alpha}-(0.5x+0.5)(1-y)^{\alpha}} \end{aligned}$$

We show now the conditions that makes $(1/2, 1/2)$ asymptotically stable in the above system. More specifically, we show that $(1/2, 1/2)$ is a *spiral sink*.

The vector field defining (8)-(9) is

$$\begin{aligned} F(x,y) &= \Big(\frac{x(1-x)\left(0.5(1+y)x^{\alpha}-(1-0.5y)(1-x)^{\alpha}\right)}{2\left(0.5+0.25y-0.25y^{2}\right)-0.5(1+y)x^{\alpha}-(1-0.5y)(1-x)^{\alpha}}, \\ &\quad \frac{y(1-y)\left((1.0-0.5x)y^{\alpha}-(0.5x+0.5)(1-y)^{\alpha}\right)}{2\left(0.5+0.25x-0.25x^{2}\right)-(1.0-0.5x)y^{\alpha}-(0.5x+0.5)(1-y)^{\alpha}}\Big) \end{aligned}$$

We compute first the derivative $DF(x,y)$ and then evaluate $DF(x,y)$ at $(1/2, 1/2)$ to get the following matrix:

$$DF(1/2,1/2) = \begin{bmatrix} \frac{\alpha}{1.5-2\times 0.5^{\alpha}} & \frac{0.16667}{0.75-0.5^{\alpha}}0.5^{\alpha} \\ -\frac{0.16667}{0.75-0.5^{\alpha}}0.5^{\alpha} & \frac{\alpha}{1.5-2\times 0.5^{\alpha}} \end{bmatrix}$$

We see that the elements $j_{ij}(\alpha)$ of the Jacobian matrix are three functions whose signs depend on the value of the parameter α. Furthermore, these functions are all multiplied by $\frac{1}{0.75-0.5^{\alpha}}$, and $0.75-0.5^{\alpha}=0$ when $\alpha = 0.41504$. Then it is easy to see that only for values of α in $(0, 0.41504)$ all the eigenvalues of the matrix $DF(1/2, 1/2)$ have negative real parts. As in the previous example, the equilibrium $(1/2, 1/2)$ is *a spiral sink*.■

5 Testable implications of doubt-based selection dynamics

Recall that in a Mixed Strategy Doubt Equilibrium (MSDE), the requirement is that for all i, j with $f^*_{ki} > 0$ and $f^*_{kj} > 0$,

$$\frac{\pi_{ki}(f^*)}{d(f^*_{ki})} = \frac{\pi_{kj}(f^*)}{d\left(f^*_{kj}\right)}$$

To satisfy the MSDE condition, we may have the following cases:

1. Agents are in the *absent* or *doubt-full* mode of play: then, for all k, and all i, j with $f^*_{ki} > 0$ and $f^*_{kj} > 0$, $d(f^*_{ki}) \cong d\left(f^*_{kj}\right) \cong 1$ and $\pi_{ki}(f^*) \cong \pi_{kj}(f^*)$. Proposition 5, shows that this happens in the Mixed Strategy Nash Equilibria.

2. Agents are in the *alert* or *doubt-less* mode of play: then, for all k, and all i, j with $f^*_{ki} = 1/2$ and $f^*_{kj} = 1/2$, $d(f^*_{ki}) = d\left(f^*_{kj}\right) \cong 0$. Proposition 5 shows that when the *Mixed Strategy Nash Equilibrium* is at $C = [(1/2, 1/2), (1/2, 1/2)]$ it coincides with the *Mixed Strategy Doubt-Less Equilibrium*.

3. Agents are neither in the *absent* or *doubt-full* mode of play nor in the *alert* or *doubt-less* mode of play: then, for all k and all i, j, with $0 < f^*_{kj} < f^*_{ki} < 1$, since the doubt functions are strictly decreasing, $d(f^*_{ki}) < d\left(f^*_{kj}\right)$, and thus, in order to satisfy equilibrium condition we must have $\pi_{ki}(f^*) < \pi_{kj}(f^*)$.

The third case is clearly distinct from a Nash equilibrium. In words, the more frequent strategies in a MSDE should have lower expected payoffs.

Notice that this condition applies as well as to a pure decision problem than to a non-trivial game situation. So a supportive piece of evidence for our equilibrium condition could come from consumer choice situations. Suppose that several brands of a product are sold (say automobiles). For a particular category of product (a family sedan, a pickup truck), sufficiently narrowly defined so that no horizontal or vertical differentiation of quality is possible, the presence of multiple brands suggests according to standard theory that the consumer should be (close to) indifferent between them *(in our language* $\pi_{ki}(f^*) = \pi_{kj}(f^*)$*)*. Our model, on the other hand, suggests that the quality is lower for brands with higher sales/market share. In our words, when $f^*_{ki} > f^*_{kj}$ we should observe $\pi_{ki}(f^*) < \pi_{kj}(f^*)$. Table

1, compiles statistics of mechanical troubles of cars compiled by the German Automobile Club for 2002 (measured by the number of calls for towing-and-repairing to the Club per thousand vehicles of that kind sold that year), as well as sales in February 2007. It is interesting to note that for the three best kinds of car in all categories, there is a significant correlation between sales of a model and mechanical troubles (a correlation coefficient of 0.65).[9] This is, of course, far from a proof of our result. The overall correlation coefficient is of rather uncertain sign,[10] but we suspect this is not a stable situation and the "worst" cars will eventually exit the market. But it strongly suggestive and it points to an interesting testable implication from our model.

Our conclusions could also be tested in the experimental laboratory. However, subjects in experiments usually do not have information about the proportion of people using each strategy. For example, the only experiment from those surveyed in chapter 3 of Camerer (2003) in which agents are given that information is the one carried out by Tang (2001). In that experiment, and contrary to our predictions, the most frequently played strategies have a higher ex-post average payoff. We suspect, though, that the highly precise (and, we would argue, unnatural) form of the feedback given to subjects eliminates the "doubt" considerations that are important in the build-up of our model. Curiously enough, in the experiment of Tang (2001) only about a fourth of the subjects participating in that experiment used repeatedly this information on frequencies of play. We believe that more evidence, and hopefully, from "fuzzier" (more realistic) environments would be useful to confront some predictions made in this work.

[9]The same computation by category gives a number in excess of 0.75 for each one.

[10]And there are, of course, lots of omitted important variables

Table 1: Quality and sales by category. Source: ADAC 2002

Model	Mechanical problems	Sales
Small cars		
Fiat Punto	34,5	1176
Renault Clio	32,7	1506
Seat Ibiza/ Cordoba	28,3	1399
Opel Corsa	17,9	4983
VW Polo	16,2	4437
Ford Fiesta	15,9	2906
Medium-sized cars		
Renault Mégane	46,2	1508
Ford Escort	27,8	2933
Opel Astra	16,7	5207
VW Golf/Vento/Bora	16,2	11072
Audi A3/S3	15,1	4052
Toyota Corolla	9.8	2236
Large cars		
Volvo S40/V40	27,7	811
BMW 3	17,8	6043
Mercedes C	17,5	2848
VW Passat	16,1	7500
Audi A4/S4	14,0	5267
Mazda 626	10,2	754
Toyota Carina/Avensis	7,6	1025

6 Constant doubt-based selection dynamics

The individual choice model that we are going to use in this section is derived from a choice procedure introduced by Aizpurúa, Ichiishi, Nieto and Uriarte (1993), (referred to as AINU from now on), in the space of simple lotteries. We consider now the case when the level of doubts felt is constant, for any value of $f_{ki} \in F_{ki}$. This means that society has no influence upon the doubt level of the agents. Formally,

Assumption 12 *The Constant Doubt Function. For all $k \in K$, $i \in S_k$ and $f_{ki} \in F_{ki}$, the function $d_{ki} : F_{ki} \to [0,1]$ is constant; i.e.*

$$d_{ki}(f_{ki}) = \epsilon_k \in (0,1)$$

We assume that the constant level of doubts ϵ_k felt by agent ki induces *threshold levels* in both expected payoffs and strategy frequencies and that these threshold levels are described by means of similarity relations.

As in the previous case, it is by means of Assumption 12 about the doubt function that we may define a similarity relation on $\Pi_{ki} = (0,1]$ and correlated similarity relations on $F_{ki} = [0,1]$. Suppose that (π_{ki}, f_{ki}) is the vector of expected payoff-strategy proportion attached to strategy i at time t.

The similarity relation on Π_{ki}, denoted $S\Pi_{ki}$, is assumed to be of the difference type and it is defined as follows

$$\pi_{ki} S\Pi_{ki} \pi'_{ki} \Leftrightarrow |\pi_{ki} - \pi'_{ki}| \leq \epsilon_k$$

On F_{ki}, we define now the correlated similarity relations as follows. First, for all $\pi_{ki}(f) > \varepsilon_k > 0$ we build the function $\phi_{ki} : \Pi_{ki} \to (1, \infty]$ as follows,

$$\phi_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - \epsilon_k} > 1$$

Then, we can establish the following similarity relation (of the ratio-type) between f_{ki} and other proportions in F_{ki}, such as f'_{ki}, given π_{ki}.

$$f_{ki} SF_{ki}(\pi_{ki}) f'_{ki} \Leftrightarrow \frac{1}{\phi_{ki}(\pi_{ki})} \leqq \frac{f_{ki}}{f'_{ki}} \leqq \phi_{ki}(\pi_{ki})$$

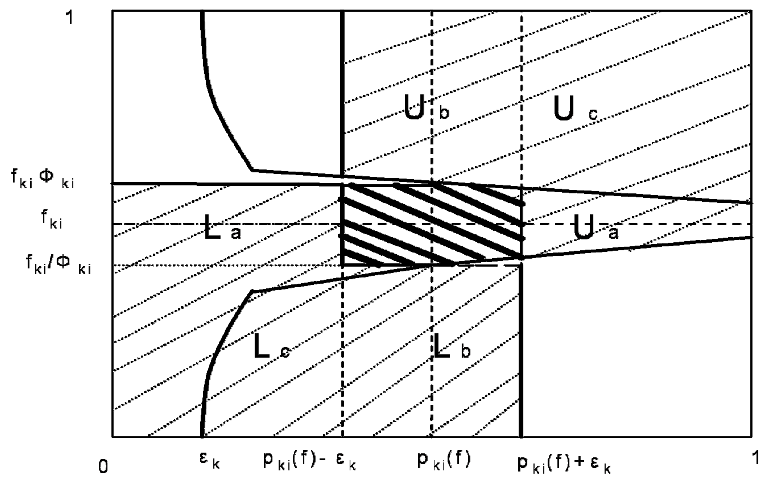

Figure 5: It is depicted the procedural preference $\succsim_{ki}$ when doubts are constant.

We call $SF_{ki}(\pi_{ki})$ a **correlated** similarity relation because the similarity on F_{ki} depends on the level of expected payoff π_{ki} at period t. For values of $\pi_{ki} \leqq \epsilon_k$ the function ϕ_{ki} is not defined and we assume that in that case that $SF_{ki}(\pi_{ki})$ is the degenerate similarity relation (see Rubinstein (1988)).

Remark 13 *The threshold level in the frequency space is inversely related to expected payoffs:* $\frac{\partial \phi_{ki}(\pi_{ki})}{\partial \pi_{ki}} < 0$. *This means that as the expected payoffs at stake increases, the discrimination on the frequency space F_{ki} increases (generating the horizontal wedge type shape of figure 5).*

6.1 The Procedural Preference Relation

As in the previous case, we assume that agents use both $S\Pi$and $SF(\pi_{ki})$ to build a decision procedure (see Appendix 1) that helps them to define at each period of time their preferences on the product space $\Pi_{ki} \times F_{ki}$. The result of this procedure is the preference relation depicted in Figure 5, where the darker part is vector $(\pi_{ki}, f_{ki})'$s indifference set and $U = U_a \cup U_b \cup U_c$ and $L = L_a \cup L_b \cup L_c$ are the upper and lower contour sets, respectively. We assume that the preferred set U represents agent ki's aspiration set.

Assumption 14 *Every agent in a given player position is able to observe the relative frequency of every strategy available to that position. When an agent feels dissatisfied with* *his current strategy, he will choose a new strategy with a probability that is equal to the proportion of agents playing that strategy.*

We proceed as in the previous case and thinking of $\phi_{ki}(\pi_{ki})$ as a "measure" of the distance to the aspiration set or, equivalently, of agent ki's degree of satisfaction with strategy i (for simplicity we shall write ϕ_{ki} instead of $\phi_{ki}(\pi_{ki})$), we define the following ratio

$$\frac{\phi_{ki}-1}{\sum_{i=1}^{m_k}\phi_{ki}}=\frac{\phi_{ki}-1}{\phi_k}$$

We take it as the proportion of ki strategists who feel dissatisfied with strategy i. Note that, everything equal, this function increases with ϕ_{ki}. Hence, an increase in ϕ_{ki}, due to a decrease in the expected payoffs π_{ki}, will increase the proportion of dissatisfied ki strategists.

As before, $\tau\frac{(\phi_{ki}-1)}{\phi_k}f_{ki}$ denotes the proportion of ki strategists who will choose a new strategy at time t (the *outflow*). Since a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then $\tau\sum_{j=1}^{m_k}\frac{(\phi_{kj}-1)}{\phi_k}f_{kj}f_{ki}=\tau\frac{(\overline{\phi}_k-1)}{\phi_k}f_{ki}$ denotes the proportion of agents who choose strategy i ; i.e. the *inflow* (where $\overline{\phi}_k=\sum_{j=1}^{m_k}\phi_{kj}f_{kj}$ is the average perception in player population k at time t).

Therefore

$$f_{ki}(t+\tau)=f_{ki}(t)-\tau\frac{[\phi_{ki}-1]}{\phi_k}f_{ki}+\tau\frac{[\overline{\phi_k}-1]}{\phi_k}f_{ki}. \tag{4}$$

Proposition 15 *As* $\tau\to 0$, *equation (4) becomes*

$$\overset{\bullet}{f}_{ki}=f_{ki}\overline{\phi_k}-\phi_{ki}\phi_k \tag{5}$$

1. (a) If for all player position $k\in K=\{1,2,...,n\}$, the strategy set S_k consists of two elements, i.e. if $m_k=2$ then, equation (5) is just the standard Replicator Dynamics (RD) multiplied by a positive function (i.e. is aggregate monotonic).

   (b) If $m_k>2$, then we obtain a selection dynamics that approximates the RD, but preserves only the positive sign of the RD (i.e. is weakly payoff positive).

Proof: See appendix B

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Appendices

A Procedural preferences

We explain first the procedural preferences based on doubts that are strictly decreasing. The choice procedure is an extension of the one introduced, in the context of simple lotteries, by Uriarte (1999) which, then was used to build a model of evolutionary drift in Uriarte (2007). The constant doubt case is much more simple and would not need additional explanations.

Let (π_{ki}, f_{ki}) be the vector of expected payoff-proportion of agents of player population k attached to strategy $i \in S_k$ at time t.

(b) d_{ki} builds the λ_{ki} function, which is used to define on F_{ki} correlated similarity relations of the ratio-type. This function is defined as follows: given d_{ki} and a specific $f_{ki} \in (0, 1)$, then for all $\pi_{ki} > d_{ki}(f_{ki})$

$$\lambda_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})} > 1$$

Thus, there is one λ_{ki} function for each $f_{ki} \in (0,1)$, so that, given π_{ki},and f_{ki} attached to agent k's strategy i, with $\pi_{ki} > d_{ki}(f_{ki})$, the similarity interval of f_{ki} is:

$$[f_{ki}/\lambda_{ki}(\pi_{ki}),\ \ f_{ki}.\lambda_{ki}(\pi_{ki})]$$

The correlated similarity relation on F_{ki}, denoted $SF[\pi_{ki}, f_{ki}]$, changes with the value of f_{ki} and,- by the property 1of the λ_{ki} function, below-, with the value of π_{ki}.

Decreasing Doubts-Based Correlated Similarity Relations.

Given a pair of vectors, $(\overline{\pi}_{ki}(\overline{f}), \overline{f}_{ki})$ and $(\pi_{ki}(f), f_{ki})$ in $\Pi_{ki} \times F_{ki}$, with $\overline{f}_{ki}$, $f_{ki} \in (0,1)$, we define similarity relations on Π_{ki} and F_{ki} in the following way. To simplify notation, we write $\pi_{ki}(f)$ and $\overline{\pi}_{ki}(\overline{f})$ as π_{ki} and $\overline{\pi}_{ki}$, respectively.

(i) On the space of expected payoffs, Π_{ki}, the doubt function d_{ki} defines correlated similarities of the difference-type as follows: given $\overline{f}_{ki}$ we say that $\overline{\pi}_{ki}$ is similar to π_{ki}, (formally written as $\overline{\pi}_{ki} S\Pi[\overline{f}_{ki}]\pi_{ki}$), if and only if $|\overline{\pi}_{ki} - \pi_{ki}| \leqq d_{ki}(\overline{f}_{ki})$, where $|.|$ stands for absolute value. Thus, there is one similarity relation on Π_{ki}, for each $\overline{f}_{ki} \in (0,1)$

Then the similarity interval of π_{ki}, given $\overline{f}_{ki}$ is:

$$[\pi_{ki} - d_{ki}(\overline{f}_{ki}), \pi_{ki} + d_{ki}(\overline{f}_{ki})]$$

Note that $d_{ki}(\overline{f}_{ki})$, the doubt level felt by $\sum$ agent ki given the proportion $\overline{f}_{ki}$, becomes the threshold level in the definition of this type of similarity relation. By Assumption 1, if $\overline{f}_{ki}$ increases, the threshold, $d_{ki}(\overline{f}_{ki})$, decreases and so the similarity intervals of π_{ki} shrink (giving rise to the vertical cone-shaped form in figure 3).This means that when f_{ki} increases, the discrimination capacity on the space of expected payoffs to strategy i, Π_{ki}, increases (probably because the accumulated experience with strategy i has increased due to the increased number of agents from population k currently playing strategy i). . When $f_{ki} = 0$, the whole set Π_{ki} is similar to π_{ki} and when $f_{ki} = 1$ only π_{ki} is similar to itself.

(ii) On the strategy frequency space, F_{ki}, d_{ki} defines correlated similarity relations of the ratio-type as follows. First, we define the λ_{ki} function: given d_{ki} and a specific $f_{ki} \in (0,1)$, then for all $\pi_{ki} > d_{ki}(f_{ki})$

$$\lambda_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})} > 1$$

Thus, there is one λ_{ki} function for each $f_{ki} \in (0,1)$.

Now we may define on F_{ki} **correlated** similarity relations of the ratio-type, as follows: given $\overline{\pi}_{ki}$ and $\overline{f}_{ki}$, we say that $\overline{f}_{ki}$ is similar to f_{ki},(formally written as, $\overline{f}_{ki}SF[\overline{\pi}_{ki},\ \overline{f}_{ki}]f_{ki}$), if and only if $1/\lambda_{ki} \leqq \overline{f}_{ki}/f_{ki} \leqq \lambda_{ki}$. The similarity intervals are of the following type:

$$[f_{ki}/\lambda_{ki}(\pi_{ki}),\ \ f_{ki}.\lambda_{ki}(\pi_{ki})]$$

These similarity intervals shrink as expected payoffs go from $\pi_{ki} > d_{ki}(f_{ki})$ to 1, giving rise to the horizontal "wedge-shaped" part of figure 3. This means that perception increases if the payoffs at stake increase.

The Procedural Preference on $\Pi_{ki} \times F_{ki}$

We shall assume that each agent ki compares pairs of alternatives in $\Pi_{ki} \times F_{ki}$ with the aid of the above pair of correlated similarity relations, $S\Pi$ and SF, to decide which of the two is preferred. Thus, the agent may define his procedural preference $\succsim_{ki}$on $\Pi_{ki} \times F_{ki}$ and know his aspiration set U at each t (which we identify with the upper contour set of the vector (π_{ki}, f_{ki}) at t). That is, given a pair of vectors $(\overline{\pi}_{ki}, \overline{f}_{ki})$ and (π_{ki}, f_{ki}) in $\Pi_{ki} \times F_{ki}$, the vector $(\overline{\pi}_{ki}, \overline{f}_{ki})$ will be declared to be preferred to (π_{ki}, f_{ki}), i.e. $(\overline{\pi}_{ki}, \overline{f}_{ki}) \succ_{ki} (\pi_{ki}, f_{ki})$, whenever the agent ki perceives that one of the following three conditions is met. Note that since $(\overline{\pi}_{ki}, \overline{f}_{ki})$ is to be preferred, the conditional similarity relation $S\Pi$ on Π_{ki} given $\overline{f}_{ki}$ and the conditional similarity relation SF on F_{ki} given $\overline{\pi}_{ki}$ and $\overline{f}_{ki}$ are to be used.

Condition α : $\overline{\pi}_{ki} > \pi_{ki}$, and *no* $\overline{\pi}_{ki}S\Pi[\overline{f}_{ki}]\pi_{ki}$; while $\overline{f}_{ki}SF[\overline{\pi}_{ki}, \overline{f}_{ki}]f_{ki}$.

In words, $\overline{\pi}_{ki}$ is bigger than π_{ki} and, given $\overline{f}_{ki}$, $\overline{\pi}_{ki}$ is perceived to be *not similar* to π_{ki} ; while , $\overline{f}_{ki}$ is perceived to be *similar* to f_{ki}. U_α in figure 3 is the area implied by this condition.

Condition β : $\overline{f}_{ki} > f_{ki}$ and *no* $\overline{f}_{ki}SF[\overline{\pi}_{ki}, \overline{f}_{ki}]f_{ki}$;while $\overline{\pi}_{ki}S\Pi[\overline{f}_{ki}]\pi_{ki}$.

In words, $\overline{f}_{ki}$ is bigger than f_{ki} and, given $\overline{\pi}_{ki}$ and $\overline{f}_{ki}$, $\overline{f}_{ki}$ is perceived to be *not similar* to f_{ki}; while, given $\overline{f}_{ki}$, $\overline{\pi}_{ki}$ is perceived to be *similar* to π_{ki}.U_β in Figure 3 is the area implied by this condition.

Condition δ : $\overline{\pi}_{ki} > \pi_{ki}$ and *no* $\overline{\pi}_{ki}S\Pi_{ki}[\overline{f}_{ki}]\pi_{ki}$; $\overline{f}_{ki} > f_{ki}$ and *no* $\overline{f}_{ki}SF[\overline{\pi}_{ki}, \overline{f}_{ki}]f_{ki}$.

That is, vector $(\overline{\pi}_{ki}, \overline{f}_{ki})$ is strictly bigger than (π_{ki}, f_{ki}) and no similarity is perceived in both instances. U_δ in figure 3 is the area implied by this condition.

Whenever both expected payoffs and strategy proportions are perceived to be *similar*, then the two vectors will be declared *indifferent*; i.e. when $\overline{\pi}_{ki} S\Pi[\overline{f}_{ki}]\pi_{ki}$, $\pi_{ki} S\Pi[f_{ki}]\overline{\pi}_{ki}$, $\overline{f}_{ki} SF[\overline{\pi}_{ki}, \overline{f}_{ki}] f_{ki}$ and $f_{ki} SF[\pi_{ki}, f_{ki}]\overline{f}_{ki}$, then $(\overline{\pi}_{ki}, \overline{f}_{ki}) \sim_{ki} (\pi_{ki}, f_{ki})$. When none of these four situations takes place, then the two vectors would be non-comparable (see figure 3).

B Proofs of propositions

Let

	(y) L	R
(x)U	a_{11}, b_{11}	a_{12} , b_{12}
D	a_{21}, b_{21}	a_{22}, b_{22}

denote the 2×2 constant-sum game G, and $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$,with $x^* > 0$ and $y^* > 0$, the Mixed strategy Nash Equilibrium of G. We may assume, without loss of generality, that $a_{11} > a_{21}$, then $b_{11} < b_{21}$, $a_{12} < a_{22}$, and $b_{22} < b_{21}$. Recall that payoffs are normalized so that they take values on $(0, 1]$. The doubt-based selection dynamics are represented by the following system:

$$\begin{aligned} \dot{x} &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}(\pi_U d_D - \pi_D d_U) \\ &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}((a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x)) \\ &\equiv G_1(x,y)F_1(x,y) \end{aligned} \tag{B.1}$$

$$\begin{aligned} \dot{y} &= \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)}(\pi_L d_R - \pi_R d_L) \\ &= \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)}((b_{11}x + b_{21}(1-x))d_R(1-y) - (b_{12}x + b_{22}(1-x))d_L(y)) \\ &\equiv G_2(x,y)F_2(x,y) \end{aligned} \tag{B.2}$$

To simplify the proofs, we shall make use of the class of doubt functions $d_{ki}(f_{ki}) = (1 - f_{ki})^\alpha$ with $\alpha \in (0, \infty)$. No generality is lost, since the purpose is just to use the *doubt-full* and *doubt-less modes* by making α close to 0 and ∞, respectively.

Proof of Proposition 5:

1. We must first show that a Mixed Strategy Nash Equilibrium (MSNE) converges to Mixed Strategy Doubt-Full Equilibrium (MSDFE) as δ converges to 0 in $d_{ki} = d^{1-\delta} \in D^{1-\delta} \subset D$ (see Remark 2). Note that, by construction of the λ_{ki} function, the denominators of the system (B.1)-(B.2)) are positive.

   An interior rest point of (B.1)-(B.2),(i.e. a MSDE), satisfies:

$$\begin{aligned}(a_{11}y + a_{12}(1-y))\, d_D(1-x) - (a_{21}y + a_{22}(1-y))\, d_U(x) &= 0\\ (b_{11}x + b_{21}(1-x))\, d_R(1-y) - (b_{12}x + b_{22}(1-x))\, d_L(y) &= 0\end{aligned}$$

   Then, if $d_i \in D^{1-\delta}$ for $i \in \{U, D, L, R\}$,

$$\lim_{\delta\to 0} \frac{d_U(x)}{d_D(1-x)} = \lim_{\delta\to 0} \frac{d_L(y)}{d_R(1-y)} = 1, \text{ for all } (x,y) \in (0,1)\times(0,1)$$

   Now suppose that we are in the MSNE, $(x^*, y^*) \in (0,1)\times(0,1)$, of G and that $d_i \in D^{1-\delta}$. Then, the strategies available to each player get the same expected payoff; that is $a_{11}y^* + a_{12}(1-y^*) = a_{21}y^* + a_{22}(1-y^*)$ and $b_{11}x^* + b_{21}(1-x^*) = b_{12}x^* + b_{22}(1-x^*)$. Thus,

$$\lim_{\delta\to 0} \frac{(a_{11}y^* + a_{12}(1-y^*))\, d_D(1-x^*)}{(a_{21}y^* + a_{22}(1-y^*))\, d_U(x^*)} = \lim_{\delta\to 0} \frac{(b_{11}x^* + b_{21}(1-x^*))\, d_R(1-y^*)}{(b_{12}x^* + b_{22}(1-x^*))\, d_L(y^*)} = 1$$

   This, plus continuity, establishes the result.

2. We show that for all $(x', y') \in (0,1)\times(0,1)$, with $(x', y') \neq (1/2, 1/2)$, there exists an α' large enough that the rest point of (B.1)-(B.2) cannot be $[(x', 1-x'), (y', 1-y')]$ for any $\alpha \geq \alpha'$ and then the result follows.

   An interior rest point of (B.1)-(B.2) must satisfy:

$$\begin{aligned}(a_{11}y + a_{12}(1-y))\, d_D(1-x) - (a_{21}y + a_{22}(1-y))\, d_U(x) &= 0\\ (b_{11}x + b_{21}(1-x))\, d_R(1-y) - (b_{12}x + b_{22}(1-x))\, d_L(y) &= 0\end{aligned}$$

   For interior rest points, this implies that

$$\begin{aligned}(a_{11}y + a_{12}(1-y))\frac{d_D(1-x)}{d_U(x)} - (a_{21}y + a_{22}(1-y)) &= 0\\ (b_{11}x + b_{21}(1-x))\frac{d_R(1-y)}{d_L(y)} - (b_{12}x + b_{22}(1-x)) &= 0\end{aligned}$$

But since

$$\frac{d_D(1-x)}{d_U(x)} = \left(\frac{x}{1-x}\right)^{\alpha}, \frac{d_R(1-y)}{d_L(y)} = \left(\frac{y}{1-y}\right)^{\alpha}$$

Then if $x' > 1/2$, there exists an α' big enough that for all $\alpha \geq \alpha'$

$$\left(\frac{x'}{1-x'}\right)^{\alpha} > \frac{(a_{21}y' + a_{22}(1-y'))}{(a_{11}y' + a_{12}(1-y'))}$$

and thus

$$(a_{11}y' + a_{12}(1-y'))\left(\frac{x'}{1-x'}\right)^{\alpha} - (a_{21}y' + a_{22}(1-y')) > 0$$

If $x' < 1/2$, there exists an α' big enough that for all $\alpha \geq \alpha'$

$$\left(\frac{x'}{1-x'}\right)^{\alpha} < \frac{(a_{21}y' + a_{22}(1-y'))}{(a_{11}y' + a_{12}(1-y'))}$$

and thus

$$(a_{11}y' + a_{12}(1-y'))\left(\frac{x'}{1-x'}\right)^{\alpha} - (a_{21}y' + a_{22}(1-y')) < 0$$

The argument is equivalent for y'.■

Proof of Proposition 7

Let us take into account that in an interior stationary state, $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$, $F_1(x^*, y^*) = 0$ and $F_2(x^*, y^*) = 0$ in (2)-(3), where

$$\begin{aligned} F_1(x,y) &= (a_{11}y + a_{12}(1-y))(x)^{\alpha} - (a_{21}y + a_{22}(1-y))(1-x)^{\alpha} \\ F_2(x,y) &= (b_{11}x + b_{21}(1-x))(y)^{\alpha} - (b_{12}x + b_{22}(1-x))(1-y)^{\alpha} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F_1(x,y)}{\partial x} &= \alpha(x^{\alpha-1}(a_{12}(1-y) + a_{11}y) + (a_{22}(1-y) + a_{21}y)(1-x)^{\alpha-1}) \\ \frac{\partial F_1(x,y)}{\partial y} &= x^{\alpha}(a_{11} - a_{12}) + (a_{22} - a_{21})(1-x)^{\alpha} \\ \frac{\partial F_2(x,y)}{\partial x} &= y^{\alpha}(b_{11} - b_{21}) + (b_{22} - b_{12})(1-y)^{\alpha} \\ \frac{\partial F_2(x,y)}{\partial y} &= \alpha(y^{\alpha-1}(b_{21}(1-x) + b_{11}x) + (b_{22}(1-x) + b_{12}x)(1-y)^{\alpha-1}) \end{aligned}$$

On the other hand, the Jacobian of the dynamic system $J(x, y)$ evaluated at the steady state (x^*, y^*) is:

$$J(x^*, y^*) = \begin{bmatrix} G_1(x^*, y^*) \left.\frac{\partial F_1(x,y)}{\partial x}\right|_{I^*} & G_1(x^*, y^*) \left.\frac{\partial F_1(x,y)}{\partial y}\right|_{I^*} \\ G_2(x^*, y^*) \left.\frac{\partial F_2(x,y)}{\partial x}\right|_{I^*} & G_2(x^*, y^*) \left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*} \end{bmatrix}$$

Noting in equilibrium that $\pi_U d_D = \pi_D d_U$ and $\pi_L d_R - \pi_R d_L$; that is,

$$\begin{aligned} (a_{11}y^* + a_{12}(1 - y^*))(x^*)^\alpha &= (a_{21}y^* + a_{22}(1 - y^*))(1 - x^*)^\alpha \\ (b_{11}x^* + b_{21}(1 - x^*))(y^*)^\alpha &= (b_{12}x^* + b_{22}(1 - x^*))(1 - y^*)^\alpha \end{aligned}$$

Hence,

$$\begin{aligned} G_1(x^*, y^*) &= \frac{x^*(1 - x^*)}{2(a_{11}y^* + a_{12}(1 - y^*))(a_{21}y^* + a_{22}(1 - y^*) - (x^*)^\alpha)} \\ G_2(x^*, y^*) &= \frac{y^*(1 - y^*)}{2(b_{11}x^* + b_{21}(1 - x^*))(b_{12}x^* + b_{22}(1 - x^*) - (y^*)^\alpha)} \end{aligned}$$

Thus, the elements of the Jacobian matrix are the following:

$$\begin{aligned} j_{11} &= G_1(x^*, y^*) \left.\frac{\partial F_1(x, y)}{\partial x}\right|_{I^*} \\ &= \frac{x^*(1 - x^*)\alpha((x^*)^{\alpha-1}\left(a_{12}(1 - y^*) + a_{11}y^*\right) + \left(a_{22}(1 - y^*) + a_{21}y^*\right)(1 - x^*)^{\alpha-1})}{2(a_{11}y^* + a_{12}(1 - y^*))(a_{21}y^* + a_{22}(1 - y^*) - (x^*)^\alpha)} \\ &= \frac{\alpha((a_{12}(1 - y^*) + a_{11}y^*)}{2(a_{11}y^* + a_{12}(1 - y^*))(a_{21}y^* + a_{22}(1 - y^*) - (x^*)^\alpha)} \\ &= \frac{\alpha}{2(a_{21}y^* + a_{22}(1 - y^*) - (x^*)^\alpha)} \end{aligned}$$

$$j_{12} = G_1(x^*, y^*) \left.\frac{\partial F_1(x, y)}{\partial y}\right|_{I^*} = \frac{x^*(1 - x^*)((x^*)^\alpha\left(a_{11} - a_{12}\right) + \left(a_{22} - a_{21}\right)(1 - x^*)^\alpha)}{2(a_{11}y^* + a_{12}(1 - y^*))(a_{21}y^* + a_{22}(1 - y^*) - (x^*)^\alpha)}$$

$$j_{21} = G_2(x^*, y^*) \left.\frac{\partial F_2(x, y)}{\partial y}\right|_{I^*} = \frac{y^*(1 - y^*)((y^*)^\alpha\left(b_{11} - b_{21}\right) + \left(b_{22} - b_{12}\right)(1 - y^*)^\alpha)}{2(b_{11}x^* + b_{21}(1 - x^*))(b_{12}x^* + b_{22}(1 - x^*) - (y^*)^\alpha)}$$

$$
\begin{aligned}
j_{22} &= G_2(x^*, y^*)\left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*} \\
&= \frac{y^*(1-y^*)\alpha((y^*)^{\alpha-1}\left(b_{21}(1-x^*)+b_{11}x^*\right)+\left(b_{22}(1-x^*)+b_{12}x^*\right)(1-y^*)^{\alpha-1})}{2(b_{11}x^*+b_{21}(1-x^*))(b_{12}x^*+b_{22}(1-x^*)-(y^*)^{\alpha})} \\
&= \frac{\alpha\left(b_{11}x^*+b_{21}(1-x^*)\right)}{2(b_{11}x^*+b_{21}(1-x^*))(b_{12}x^*+b_{22}(1-x^*)-(y^*)^{\alpha})} \\
&= \frac{\alpha}{2(b_{12}x^*+b_{22}(1-x^*)-(y^*)^{\alpha})}
\end{aligned}
$$

Hence, the $J(x^*, y^*)$ matrix is

$$
J(x^*, y^*) = \begin{bmatrix} \frac{\alpha}{2(a_{21}y^*+a_{22}(1-y^*)-(x^*)^{\alpha})} & \frac{x^*(1-x^*)((x^*)^{\alpha}(a_{11}-a_{12})+(a_{22}-a_{21})(1-x^*)^{\alpha})}{2(a_{11}y^*+a_{12}(1-y^*))(a_{21}y^*+a_{22}(1-y^*)-(x^*)^{\alpha})} \\ \frac{y^*(1-y^*)((y^*)^{\alpha}(b_{11}-b_{21})+(b_{22}-b_{12})(1-y^*)^{\alpha})}{2(b_{11}x^*+b_{21}(1-x^*))(b_{12}x^*+b_{22}(1-x^*)-(y^*)^{\alpha})} & \frac{\alpha}{2(b_{12}x^*+b_{22}(1-x^*)-(y^*)^{\alpha})} \end{bmatrix}
$$

Recall that the real part of the eigenvalues of $J(x^*, y^*)$ only depends on the sum of the diagonal terms (the trace of the matrix):

$$
\begin{aligned}
\text{Trace of } J(x^*, y^*) &= G_1(x^*, y^*)\left.\frac{\partial F_1(x,y)}{\partial x}\right|_{I^*} + G_2(x^*, y^*)\left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*} \\
&= \frac{\alpha}{2(a_{21}y^*+a_{22}(1-y^*)-(x^*)^{\alpha})} + \frac{\alpha}{2(b_{12}x^*+b_{22}(1-x^*)-(y^*)^{\alpha})} \\
&= \frac{\alpha}{2}\left[\frac{1}{a_{21}y^*+a_{22}(1-y^*)-(x^*)^{\alpha}} + \frac{1}{b_{12}x^*+b_{22}(1-x^*)-(y^*)^{\alpha}}\right]
\end{aligned}
$$

As the doubt parameter α approaches 0, the value of the level of doubts $d(1-x^*) = (x^*)^{\alpha}$ and $d(1-y^*) = (y^*)^{\alpha}$ approaches 1. When α is nearly 0, written as $\alpha \cong 0$, agents are playing in the *absent* or *doubt-full mode* and we can think of $(x^*)^{\alpha}$ and $(y^*)^{\alpha}$ as a constant number very close to 1 (or, rounding up, just 1). Hence, we can rewrite the Trace of $J(x^*, y^*)$ as follows:

$$
\text{Trace of } J(x^*, y^*) = \frac{\alpha}{2}\left[\frac{1}{a_{21}y^*+a_{22}(1-y^*)-1} + \frac{1}{b_{12}x^*+b_{22}(1-x^*)-1}\right]
$$

Note that the expected values $\pi_D = a_{21}y^* + a_{22}(1-y^*)$ and $\pi_R = b_{12}x^* + b_{22}(1-x^*)$, the denominators of the trace, are smaller than 1 because payoffs take values in $(0, 1]$ and we are considering interior mixed equilibria. Thus, $j_{11} < 0$ and $j_{22} < 0$ and so the sign of the trace is negative

$$sign\ \left[G_1(x^*,y^*)\ \frac{\partial F_1(x,y)}{\partial x}\bigg|_{I^*} + G_2(x^*,y^*)\ \frac{\partial F_2(x,y)}{\partial y}\bigg|_{I^*}\right] < 0$$

Without loss of generality, we may assume that $a_{11} > a_{21}$, then $b_{11} < b_{21}$, $a_{12} < a_{22}$, and $b_{22} < b_{21}$. Then, when the agents are playing in the *absent* or *doubt-full mode* the sign of

$$\begin{aligned} j_{21} \times j_{12} &= \left(\frac{y^*(1-y^*)((y^*)^\alpha (b_{11}-b_{21}) + (b_{22}-b_{12})(1-y^*)^\alpha)}{2(b_{11}x^* + b_{21}(1-x^*))(b_{12}x^* + b_{22}(1-x^*) - (y^*)^\alpha)}\right) \\ &\quad \times \left(\frac{x^*(1-x^*)((x^*)^\alpha (a_{11}-a_{12}) + (a_{22}-a_{21})(1-x^*)^\alpha)}{2(a_{11}y^* + a_{12}(1-y^*))(a_{21}y^* + a_{22}(1-y^*) - (x^*)^\alpha)}\right) \\ &= \left(\frac{y^*(1-y^*)((b_{22}-b_{21}) + (b_{11}-b_{12}))}{2(b_{11}x^* + b_{21}(1-x^*))(b_{12}x^* + b_{22}(1-x^*) - 1)}\right) \\ &\quad \times \left(\frac{x^*(1-x^*)((a_{11}-a_{21}) + (a_{22}-a_{12}))}{2(a_{11}y^* + a_{12}(1-y^*))(a_{21}y^* + a_{22}(1-y^*) - 1)}\right) \\ &< 0 \end{aligned}$$

is negative. Hence, the determinant associated to $J(x^*,y^*)$ is Det $J(x^*,y^*) = j_{11} \times j_{22} - j_{21} \times j_{12}$ and its sign is positive. Therefore, when every agent is in the *absent* or *doubt-full mode of play*, the mixed equilibrium $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$ is a *sink* and therefore is an asymptotically stable equilibrium.■

Proof of Proposition 8

In the *doubtless* or *alert mode* of play, α is very high. Therefore, since

$$\begin{aligned} \dot{x} &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)} (a_1 y x^\alpha - a_2(1-y)(1-x)^\alpha) \\ \dot{y} &= \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)} (b_1 x y^\alpha - b_2(1-x)(1-y)^\alpha) \end{aligned}$$

for α large enough

$$sign\left[\dot{x}\right] = sign\left[(a_1 y x^\alpha - a_2(1-y)(1-x)^\alpha)\right] = sign\left[x - 1/2\right]$$

and, thus, if $x(0) > 1/2$, then $\lim_{t\to\infty} x(t) = 1$, whereas if $x(0) < 1/2$, then $\lim_{t\to\infty} x(t) = 0$. The analysis is equivalent for y, thus establishing the result.■

Proof Proposition 15:

(a) Let $S_k = \{1,2\}$ be player population k's strategy set. Without loss of generality, let us refer to the dynamics of strategy 1. Then, by equation (1), we have

$$
\begin{aligned}
\dot{f}_{k1} &= f_{k1}\overline{\phi_k} - \phi_{k1}\phi_k \qquad\qquad (B.3)\\
&= \frac{\epsilon_k}{\pi_{k1}(\pi_{k2}-\epsilon_k)+\pi_{k2}(\pi_{k1}-\epsilon_k)} f_{k1}(\pi_{k1}-\overline{\pi}_k)\\
&= \frac{\epsilon_k}{D(f)} f_{ki}[\pi_{ki}-\overline{\pi}_k]
\end{aligned}
$$

where $D(f) \equiv \pi_{k1}(\pi_{k2}-\epsilon_k)+\pi_{k2}(\pi_{k1}-\epsilon_k) > 0$.

By equation (B.3), the growth rates $\frac{\dot{f}_{ki}}{f_{ki}}$ equal payoff differences $[\pi_{ki}-\overline{\pi}_k]$ multiplied by a (Lipschitz) continuous, positive function $\frac{\epsilon_k}{D(f)}$. This concludes the proof. (Note that, given ϵ_k, a payoff difference $[\pi_{ki}-\overline{\pi}_k]$ will have stronger dynamic effect if $D(f)$ is low than if it is high; if ϵ_k decreases, the dynamic effect of $[\pi_{ki}-\overline{\pi}_k]$ decreases).

(b) Easy.■