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EFFORT AND SYNERGIES IN NETWORK FORMATION[*]

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Abstract

The aim of this paper is to understand the interactions between productive effort and the creation of synergies that are the sources of technological collaboration agreements, agglomeration, social stratification, etc. We model this interaction in a way that allows us to characterize how agents devote resources to both activities. This permits a full-fledged equilibrium/welfare analysis of network formation with endogenous investment efforts and to derive unambiguous comparative statics results. In spite of its parsimony that ensures tractability, the model retains enough richness to replicate a (relatively) broad range of empirical regularities displayed by social and economic networks, and is directly estimable to recover is structural parameters.

Keywords: random network, spillovers, network formation, network topology.
JEL Classification: L22, L51, O31, O38

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1 Introduction

External effects (spillovers) pervade economies and societies in general. Both inter- and intra-industry cross interaction between firms have been the object of studies, since at least the work of Marshall (1890), for the former, and Jacobs (1969), for the latter.[1] Social interactions also have a crucial importance in the determination of individuals' well-being, as pointed early by Becker (1974) and recently emphasized by the literature on social capital.[2]

Given the importance and pervasiveness of these external effects, it is natural that individuals and firms may want to control and manipulate the size and scope of those external benefits to their advantage. For example, regional economists have convincingly shown that economic agents agglomerate in few locations in the economic landscape, precisely in order to reap these localization externalities (Ciccone and Hall 1996, Guiso and Schivardi 2007).[3] In a similar vein, it is difficult to understand technological collaboration agreement between firms ("joint ventures" and other similar contracts) without thinking that these are done to control external effects (d'Aspremont and Jacquemin 1988).[4] Finally, the persistent stratification of social groups among many dimensions (such as income, race, education) is *prima facie* evidence of the desire of social groups to arrange themselves so as to internalize spillovers (Tiebout 1956, Benabou 1993).

In this paper, we endeavour to fathom the interactions between productive effort and the creation of synergies. We model this interaction in a way that allows us to characterize how agents devote resources to both activities optimally. In turn, this permits a full-fledged equilibrium/welfare analysis of individual decisions and to derive unambiguous comparative statics results. The model is also flexible enough to be brought to the data.

The environment Our model has two main ingredients.

First, given a structure of synergies, the model has a simple linear quadratic structure. More precisely, payoffs are linear quadratic in own productive effort, while spillovers are generated by paired agents and are multiplicative in own's and other's productive effort. We allow for two different sources of heterogeneity. On the one hand, agents can differ in their marginal returns to own productive effort. One the other hand, for identical levels of productive efforts, spillovers can vary with the strength of the synergistic linkage across different pairs of agents. It turns out that this payoff structure allows to pin down exactly how the level of productive effort varies with the pattern of external effects exerted on each individual, and with the idiosyncratic characteristics of the agents.[5]

[1] See Duranton and Puga (2004) for a review of this literature.

[2] Coleman (1990) and Putnam (2000) are standard references. Sobel (2002) and Durlauf (2002) offer critical surveys of this literature.

[3] See Rosenthal and Strange (2004) for a survey.

[4] See also Suzumura (1992).

[5] As in Ballester, Calvó-Armengol and Zenou (2006). See also Ballester and Calvó-Armengol (2006).

Second, we assume that agents devote a (joint) amount of resources to building synergies with others, whomever these others are. Socialization is thus captured by a scalar, rather than by a vector of decisions telling how much to interact with every other agent. This socialization or synergistic effort determines the strength of the synergistic linkage across different pairs of agents. More precisely, agents control the total level of synergies they are involved in. How this aggregate synergy value is distributed across different pair-wise interactions depends on the distribution of socialization efforts of the other agents. Finally, recall that the spillover between two given agents is equal to the product of their respective productive efforts weighted by the synergy value of their interaction.

The main innovation of our study is, precisely, that the synergistic effort is generic $-$a scalar decision. Socializing is not equivalent, in our approach, to elaborating a nominal list of intended relationships, as in the literature on network formation surveyed by Jackson (2005). This is realistic in many applications,[6] particularly when networks are so large that keeping track of every participant becomes a burdensome task. As a matter of fact, most of our results are established for large networks. In addition, this shortcut greatly improves the tractability of the model. Unlike with richer models of link formation, we can resort to off-the-shelf Nash equilibrium analysis without being burdened by the extreme (combinatorial) multiplicity problems of the other models.[7] As a result, we can perform a standard type of equilibrium analysis that equates marginal costs and benefits of both production and socialization. Of course, this equilibrium characterization also greatly simplifies welfare and comparative statics analyses.

In some cases, our model with synergies of varying strengths across different pairs of players[8] can also be understood as a (multinomial) model of a random graph, where independent link probabilities correspond to the synergistic values. Albeit simple, this random graph model is flexible enough to encompass many (if not all) of the topological properties of real life networks. Our equilibrium analysis naturally inherits all the descriptive topological features of this random graph model (as well as its limitations, of course). At the same time, our analysis permits to draw a close connection between topological features and welfare and economic implications. We believe

[6] Researchers go to fairs, or congresses to listen, to be listened to, and to meet other investigators in general. More generally, face-to-face meetings among agents that share a common location often result from random encounters among these agents, as the early literature on segregation indexes already points out (Bell 1954).

[7] In a typical game of network formation, players simultaneously announce all the links they wish to form with. The links that form are those that are mutually announced by both partners. The cost of creating and maintaining links are then payed. As a consequence of the large multi-dimensional strategy space, and because link creation requires the mutual consent of the two involved parties, a severe coordination problem arises. As such, the game often displays a multiplicity of Nash equilibria, and very different network geometries can arise endogenously. A partial solution to this problem can be found by allowing pair-wise or coalitional deviations, or by restricting to cooperative-like network stability notions (Jackson and Wolinsky, 1996). Jackson (2005) surveys this literature, while Calvó-Armengol and Ilkiliç (2006) derive the connections between this approach and standard game-theoretic refinements.

[8] See Bloch and Dutta (2005) for a model with endogenous link strength but in a standard framework of non-random network formation.

that this link between topology, equilibrium and welfare is the main value added of the model.

In spite of its parsimony that ensures tractability, the model retains enough richness to replicate a (relatively) broad range of empirical regularities displayed by social and economic networks, and is directly estimable to recover is structural parameters.[9]

Results We first characterize the equilibria of the model, when agents take their decision about their productive effort and their socialization effort simultaneously. We show that there are two interior equilibria and one (partially) corner equilibrium, when a sufficiently large number of individuals is implicated. The (partially) corner equilibrium where agents do not invest at all in building synergies is unstable. Instead, the two interior equilibria are stable. Existence and stability of interior equilibria are obtained when the level of cross synergies as well as the heterogeneity in individual traits are not too high, which amounts to bounding from above a compound index of both payoff parameters.

For large enough populations, equilibrium actions take a particularly simple form. Recall that agents can display different marginal returns to own productive effort. We label "individual type" the value of this marginal return at the origin. We first show that the ratios of productive as well as socialization efforts across different pairs of agents are all equal to the ratio of their individual types. In other words, at equilibrium, the productive and socialization efforts for a given agent are the product of his individual type with some baseline values for the productive and socialization efforts. These baseline values, in turn, are obtained from a system of two equations with two unknowns that admits exactly two positive solutions −hence the two interior equilibria.

This simple equilibrium characterization has a number of interesting implications. In particular, we can show that one of the interior equilibria displays both higher socialization and productive effort than the other, so that we can talk of high-action and low-action equilibrium. It also turns out that the high-actions equilibrium is Pareto superior.

An important question is then how an exogenous change in the returns to production and socialization affect the relative production and socialization efforts at equilibrium. In turns out that, when the returns increase, all equilibrium actions decrease at the Pareto-superior equilibrium, while they increase at the Pareto-inferior equilibrium. In both cases, the percentage change in socialization effort is higher (in absolute value) than that of the productive effort. We think this may provide an explanation, for example, of the large increase in agreements of collaboration in R&D in the recent past (Caloghirou, Ioannides, Vonortas 2003). It could also explain the decline in social capital documented by Putnam (2000).

We then turn to the implications of the model for the topology of networks. When synergy values are all between zero and one, our equilibrium socialization efforts can be interpreted as a (multinomial) random graph with independent link probabilities, where the expected number of

[9] See Ioannides and Soetevent (2006) for an interesting application of random networks to the labor market.

links accruing to each agent (also known as the degree of the corresponding network node), is equal to the socialization effort of this agent.

Models of random graph with given expected degree sequence (here, the equilibrium profile of socialization efforts) have been analyzed by Chung and Lu (2002). They can replicate some (if not all) of the observed features of real-life networks, and our equilibrium model inherits all the descriptive possibilities as well as limitations of this random graph model. For instance, random graph models with given expected degree sequence can replicate any distribution of the number of relationships per person in a population (also known as the *degree distribution*). In our case, the distribution of network degrees has a one-to-one relationship with the distribution of individual traits in the population, the latter shaping the former. So, one can potentially replicate any degree distribution by fine-tuning the distribution of individual types adequately. However, this also implies that heavy-tailed degree distributions, which are sometimes (but not always) encountered in real-life networks,[10] call for a fat-tailed distribution of individual traits. Note, however, that fat tails have a close connection with lognormal distributions, which call for multiplicative (dynamic growth) processes (Mitzenmacher 2004, Jackson and Rogers 2007), whereas our analysis concentrates on a static (one-shot Nash) equilibrium concept.

Random graphs models with given expected degree sequence can also give a good account of the low average network distance usually observed in real-life networks, and so does our equilibrium model. At the same time, it cannot account for the typically high observed clustering (the friends of my friends are typically my friends as well), as links are created independently. However, we discuss how a small modification of our model could deliver moderately high values for this clustering coefficient as well.

It turns out that a close examination of equilibrium payoffs demonstrates that individuals of the same type are better off if they are matched only with others of the same type or higher. This should generate a tendency to observe at least some homogeneous groups, as long as segregating institutions or mechanisms are available. For this reason, and also to check the robustness of previous results, we also examine the model with homogeneous groups, but also with a more general cost structure. We show that only two stable interior equilibria still exist. And, as in the heterogeneous case, the increase in the returns to socialization induce a larger percentage change in socialization effort than in productive effort. Finally, in the homogeneous case we can also characterize the conditions for the emergence of giant components. That is, we can show the parameters for which completely intra-connected subgroups comprising a large majority of the population exist. These giant components are a feature of many real life networks.

To summarize, we propose a methodology that can usefully relate network topology to economic features of the model (and vice versa), which is an advantage with respect to other models that replicate well observed network topology.[11] This, however, is achieved at the cost of losing the

[10] See Table 1 and Figure 2 in Jackson and Rogers (2007).

[11] Kirman (1983), Kirman, Oddou and Weber (1986) and Ioannides (1990) propose and analyze early models relating

ability to describe some observed features.

Another virtue of our approach is that the model is suitable for estimation with readily available data. By recovering the deep parameters of the model, we can make a welfare assessment and comparison of real life networks, and ponder the impact of potential interventions. With an illustrative purpose, we perform one such exercise. Using data from a network of high school friendships, and using the education outcomes of members of the said network, we recover the parameters which would generate the observations (if our model is correct). We then compare the results of the fitted model with the observed network topology and perform comparative statics exercises. We also do some policy experiments by constructing artificial societies with restricted (more homogeneous) subgroups, and we observe the effect of a mean preserving spreads on the distribution of types.

Our model with link intensities and heterogeneous types has also been used to recover network structure from survey questionnaire data asking how many people of a set of types the responders know (Zheng, Salganik, Gelman 2006).

The paper is organized as follows. Section 2 describes the model, and introduces the baseline game as well as the replica game. Section 3 contains the equilibrium and welfare analysis. The comparative statics results are gathered in Section 4. Section 5 discuss the equilibrium and welfare implications for (and from) the network topology. Section 6 analysis the particular case of homogeneous populations with general cost structures, and the emergence of giant components. All proofs are gathered in the appendix.

2 The game

The replica game $N = \{1, ..., n\}$ is a finite set of players, and $T = \{1, \ldots, t\}$ is a finite set of types for these players. We let n be a multiple of t, that is, $n = mt$ for some integer $m \geq 1$, so that there is the same number of players of each type.

More precisely, we refer to the case $n = t$ as *the baseline game*, and to the general case $n = mt$ as *the $m-$replica* of this baseline game. In an $m-$replica game, there are exactly m players of each type $\tau \in T$.

For each player $i \in N$, we denote by $\tau(i) \in T$ his type.

We consider a simultaneous move game of network formation and investment. The returns to the investment are the sum of a private component and a synergistic component. The private returns are heterogeneous across players and depend on their type. We denote by $\mathbf{b} = (b_1, ..., b_t)$ the profile of these private returns, where $0 < b_1 \leq b_2 \leq ... \leq b_t$. Even though each type in the replica game has the same number of individuals, we can match any finite distribution of types in a population by adding multiple copies of an individual type.

The synergistic returns depend on the network formed on account of individual choices, as

random networks of interaction with economic outcomes.

described below.

Network formation Consider some $m-$replica game, $m \geq 1$. Let $n = mt$.

Each player i selects a number $s_i \geq 0$ which corresponds to a level of socialization effort. Let $\mathbf{s} = (s_1, ..., s_n)$ be a profile of socialization efforts. Then, i and j interact with a link intensity given by:

$$g_{ij}(\mathbf{s}) = \rho(\mathbf{s})\, s_i s_j \tag{1}$$

By definition, links are symmetric, that is, $g_{ij} = g_{ji}$. We also allow for self-loops (when $i = j$). The total intensity of the links accruing from a given player i is:

$$g_i(\mathbf{s}) = \sum_{j=1}^{n} g_{ij}(\mathbf{s}) = \rho(\mathbf{s})\, s_i \sum_{j=1}^{n} s_j.$$

We set

$$\rho(\mathbf{s}) = \begin{cases} 1/\sum_{j=1}^{n} s_j, \text{ if } \mathbf{s} \neq \mathbf{0} \\ 0, \text{ if } \mathbf{s} = \mathbf{0} \end{cases} \tag{2}$$

so that $g_i(\mathbf{s}) = s_i$, that is, players decide upon their total interaction intensity. In this model, the exact identity of the interacting partner is not an object of choice. Rather, players choose their total socialization intensity that they devote to each and every possible bilateral interaction in proportion to the socialization effort of these partners.

As a matter of fact, the functional form in (1) and (2) can be tied back to simple properties of the link intensity $g_{ij}(\mathbf{s})$, as established below.

Lemma **1** *Suppose that, for all* $\mathbf{s} \neq \mathbf{0}$*, the link intensity satisfies:*

(A1) symmetry: $g_{ij}(\mathbf{s}) = g_{ji}(\mathbf{s})$*, for all* i, j*;*

(A2) aggregate constant returns to scale: $\sum_{j=1}^{n} g_{ij}(\mathbf{s}) = s_i$*;*

(A3) multiplicative separability: $g_{ij}(\mathbf{s}) = s_i \psi_j(\mathbf{s})$*, where* $\psi_j : \mathbb{R}_+^n \to \mathbb{R}_+$*;*

then, the link intensity is given by the functional form (1) and (2).

Notice that $(A2)$ and $(A3)$ reflect the fact that i controls his total number of contacts s_i, but the actual composition depends on the others' investments.

When $\max_i s_i^2 < 1/\rho(\mathbf{s})$, all link intensities are between 0 and 1. In this case, we can view the network as a random graph where $g_{ij}(s)$ is the probability of having an edge between i and j, and links are independent across different pairs of players. This is the random graph model with given expected degrees $\mathbf{s} = (s_1, ..., s_n)$ described, e.g., in Chung and Lu (2002) that can replicate many of the degree distributions encountered in real-life networks, such as power laws, Poisson distributions, etc.[12]

[12] See Ioannides (2006), Jackson (2007) and Vega-Redondo (2007) for more on random graphs and their connections to economics.

Investment Each player i selects a number $k_i \geq 0$ that corresponds to an investment level. The choices of s_i and k_i are simultaneous. Let $\mathbf{k} = (k_1, ..., k_n)$ be a profile of investments. The individual investment yields both a private and a synergistic return.

The private returns to player i depend only on his investment level k_i and his individual traits, summarized by $b_{\tau(i)}$. They are captured by a simple quadratic expression $b_{\tau(i)}\, k_i - k_i^2/2$.

The synergistic returns depend on both $(\mathbf{k}, \mathbf{s})$. They correspond to a collection of cross effects $k_i k_j$ weighted by a factor reflecting the link intensity between i and j shaped by $\mathbf{s}$. More precisely, we assume that:

$$\frac{\partial^2 u_i(\mathbf{k}, \mathbf{s})}{\partial k_i \partial k_j} = a g_{ij}(\mathbf{s}), \text{ for all } i \neq j, \tag{3}$$

for some parameter $a \geq 0$ capturing the size of the synergistic returns.

Notice that the symmetry $(A1)$ in Lemma 1 is tantamount to payoffs being twice continuously differentiable in the productive effort $\mathbf{k}$.

Payoffs Player i's utility is given by:

$$u_i(\mathbf{k}, \mathbf{s}) = b_{\tau(i)}\, k_i + a \sum_{j=1, j \neq i}^{n} g_{ij}(\mathbf{s})\, k_j k_i - \frac{1}{2} k_i^2 - \frac{1}{2} s_i^2 \tag{4}$$

Payoffs are a linear-quadratic function of k_is with non-negative cross effects (3) reflecting strategic complementarities in investment levels. The size $a g_{ij}(\mathbf{s}) \geq 0$ of these complementarities depends on the profile of socialization efforts, and varies across different pairs of players.

3 Equilibrium analysis and Pareto ranking of equilibria

3.1 Equilibrium analysis

We solve for the interior Nash equilibria in pure strategies $(\mathbf{k}^*; \mathbf{s}^*) = (k_1^*, ..., k_n^*; s_1^*, ..., s_n^*)$ of the $m-$replica game with heterogeneous types $\mathbf{b} = (b_1, ..., b_t)$, and for m large enough.

Under some conditions that we provide, there are exactly three such equilibria. In one (partially corner) equilibrium, the level of socialization effort is null for all players. The two other equilibria are interior. We characterize these interior equilibria when the population gets large.

We first identify the (partially) corner equilibrium of the game.

Lemma **2** *For all $m-$replica game, $(k_i^*, s_i^*) = \left(b_{\tau(i)}, 0\right)$ for all $i = 1, ..., mt$ is a pure strategy Nash equilibrium with corresponding equilibrium payoffs $b_{\tau(i)}^2/2$.*

This is a strict equilibrium, thus it cannot be discarded on the basis of standard refinements. However, this equilibrium is not stable for sufficiently large populations, as we will show later. For this reason, we concentrate on the interior equilibria, which we now characterize.

Define:

$$a(\mathbf{b}) = a\frac{\sum_{\tau=1}^{t} b_\tau^2}{\sum_{\tau=1}^{t} b_\tau}. \tag{5}$$

Holding the average type $\sum_{\tau=1}^{t} b_\tau / t$ constant, the parameter $a(\mathbf{b})$ increases with the heterogeneity in types. More generally, $a(\mathbf{b})$ increases with the ratio $\sum_{\tau=1}^{t} b_\tau^2 / \sum_{\tau=1}^{t} b_\tau$, which many authors refer to as the second-order average type (e.g., Vega-Redondo 2007). When types are all homogeneous, that is, $b_1 = ... = b_t = b$, we have $a(\mathbf{b}) = ab$.

THEOREM **1** *Suppose that $2/3\sqrt{3} > a(\mathbf{b}) > 0$. Then, there exists an m^* such that for all $m-$replica games with $m \geq m^*$, there are exactly two interior pure strategy Nash equilibria. These pure strategy Nash equilibria are such that, for all players i of type τ, the strategies (k_i, s_i) converge to $(k^*_{\tau(i)}, s^*_{\tau(i)})$ as m goes to infinity, where $k^*_{\tau(i)} = b_{\tau(i)}k$, $s^*_{\tau(i)} = b_{\tau(i)}s$, and (k, s) are positive solutions to:*

$$\begin{cases} s = a(\mathbf{b})k^2 \\ k\left[1 - a(\mathbf{b})s\right] = 1 \end{cases}. \tag{6}$$

Under the conditions on $a(\mathbf{b})$ stated in Theorem 1, the system of two equations (6) with two unknowns has exactly two positive solutions. We also show that both solutions get arbitrarily close to an interior pure strategy Nash equilibrium of the $m-$replica game as m gets large. But the equilibrium correspondence is locally continuous at an interior equilibrium. Thus these two positive solutions of (6) are approximate pure strategy Nash equilibria of the $m-$replica game. Finally, Theorem 1 establishes that all interior pure strategy Nash equilibria of the $m-$replica game are in the neighborhood of a positive solution to (6) when m gets larger.

Table 1 shows population size versus accuracy for the homogenous case where all individuals are of type 1.

Table 1: Simulations on Theorem 1 with $a = 2$, $t = 1$ and $b_1 = 0.1$.[13]

m	2	5	10	20	50	100	500	∞
				Low equilibrium				
k^*	1,898	1,195	1,101	1,065	1,049	1,046	1,046	1,046
s^*	2,366	815	458	303	234	222	218	219
				High equilibrium				
k^*	3,346	4,643	4,591	4,508	4,444	4,420	4,400	4,394
s^*	3,506	3,923	3,911	3,891	3,875	3,869	3,864	3,862

[13] Numbers are multiplied by 10^4.

For the case of an homogeneous population with common trait b, one can verify that the exact equilibrium equations are:

$$\begin{cases} s = abk^2\left(1-\frac{1}{m}\right)^2 \\ k\left[1-abs\left(1-\frac{1}{m}\right)\right]=1 \end{cases}$$

If one then expresses this system as a single equation with a third order polynomial in s, it can easily checked that the approximation error is of the order of $m^{-3/2}$. In particular, when $m = 100$, the approximation error is 10^{-3}.

The two equations (6) equalize marginal costs with marginal benefits at equilibrium for each action available to the players. Consider some player i with type $b_{\tau(i)}$. At an approximate equilibrium $(k_i^*, s_i^*) = b_{\tau(i)}(k^*, s^*)$, where (k^*, s^*) are solutions to (6).

The marginal cost corresponding to an investment level k_i^* is equal to k_i^* itself. Given that at equilibrium marginal cost equals marginal benefit, the marginal benefit from this investment is

$$b_{\tau(i)}/(1-a(\mathbf{b})s^*)$$

which we obtain from the second equation in (6).[14] When $a = 0$, this marginal benefit boils down to $b_{\tau(i)}$, which coincides with the private return in (4). When $a \neq 0$, the private return is scaled up by a *synergistic multiplier* $1/(1-a(\mathbf{b})s^*)$, which is homogeneous across players. It is an increasing function of the second order average type $a(\mathbf{b})$, a measure of population heterogeneity.

Similarly, the marginal cost corresponding to a socialization level s_i^* is equal to s_i^* itself. The marginal benefit now is a complicated expression in s_i's and k_i's, but as the population size gets large, it approximately boils down to

$$a\rho(\mathbf{s})\sum_{j=1,j\neq i}^{n} k_i k_j$$

The approximate equilibrium solution equates this marginal benefit to the marginal cost s_i^* for all i. Theorem 1 gives a closed form expression for this fixed point. It is already apparent, nevertheless, that the equilibrium value of s_i^* is of the order of k^2.

Figure 1 plots equations (6).

[*Insert Figure* 1 *here*]

From the graph, it is clear that the system (6) needs not always to have a non-negative solution. The condition $2/3\sqrt{3} > a(\mathbf{b})$ is necessary and sufficient so that the two graphs of the two equations cross in the positive orthant of the space (k, s). To understand this, notice that when $a(\mathbf{b})$ is too large, the synergistic multiplier operates too intensively and both s and k increase without bound.

Remark 1 *When $0 < a(\mathbf{b}) < 2/3\sqrt{3}$, the system of equations (6) has two different non-negative solutions. When $a(\mathbf{b}) = 2/3\sqrt{3}$, there is a unique non-negative solution $(k, s) = \left(3/2, \sqrt{3}/2\right)$. When $a(\mathbf{b}) > 2/3\sqrt{3}$, there is no non-negative solution.*

[14] This is obtained by multiplying the second equation in (6) by $b_{\tau(i)}$.

The socialization effort at equilibrium varies across players in a way that reflects their relative types $b_{\tau(i)}$. Formally,

$$\frac{s_i^*}{s_j^*} = \frac{b_{\tau(i)}}{b_{\tau(j)}}.$$

Therefore, the intensity of a particular link at an approximate equilibrium is:

$$g_{ij}(\mathbf{s}^*) = s^* \frac{b_{\tau(i)} b_{\tau(j)}}{m \sum_{\tau=1}^{t} b_\tau}, \tag{7}$$

which decreases linearly with $1/m$, inversely proportional to the population size mt. For this reason, the overall socialization effort $g_i(\mathbf{s}^*) = s^* b_{\tau(i)}$ is independent of the population size. This kind of population invariance allows us to work with large populations, where we can discard effects of second-order magnitude without being burdened with population size effects.

Since there are two interior equilibria, plus one partially corner equilibrium, it is legitimate to wonder about the stability of these equilibria.

Proposition 1 *For m sufficiently large, the two interior equilibria are stable while the equilibrium with $(k_i^*, s_i^*) = \left(b_{\tau(i)}, 0\right)$ for all $i = 1, ..., mt$ is not stable.*

3.2 Pareto ranking of equilibria

Given an approximate equilibrium $(\mathbf{k}^*, \mathbf{s}^*)$, we denote by $\mathbf{u}(\mathbf{k}^*, \mathbf{s}^*) = (u_1(\mathbf{k}^*, \mathbf{s}^*), ..., u_m(\mathbf{k}^*, \mathbf{s}^*))$ the corresponding equilibrium payoffs.

The previous result shows that, under some conditions on the exogenous parameters of the game, there are exactly two approximate equilibria as the population gets larger. The next result compares equilibrium actions and payoffs across these two approximate equilibria.

Proposition 2 *Let $(\mathbf{k}^*, \mathbf{s}^*)$ and $(\mathbf{k}^{**}, \mathbf{s}^{**})$ be the two different approximate equilibria of an $m-$replica game. Then, without loss of generality, $(\mathbf{k}^*, \mathbf{s}^*) \geq (\mathbf{k}^{**}, \mathbf{s}^{**})$ and $\mathbf{u}(\mathbf{k}^*, \mathbf{s}^*) \geq \mathbf{u}(\mathbf{k}^{**}, \mathbf{s}^{**})$, where $\geq$ is the component-wise ordering.*

In words, the equilibrium actions are ranked component-wisely and the equilibrium payoffs are Pareto-ranked accordingly. So, the equilibrium multiplicity identified in Theorem 1 reflects a coordination problem. In one equilibrium, all players exert a high socialization effort, thereby contributing to building high cross synergies (3) across them. In turn, this induces them to incur high private investments so that they can all reap these high synergistic returns. This is the high-action and Pareto-superior equilibrium. In the other equilibrium, socialization efforts and the resulting cross synergies are low, which hampers the level of private investments. This is the low-action and Pareto-inferior equilibrium.

4 Comparative statics

4.1 Socialization and investment

We enquire how the equilibrium actions respond to changes in exogenous parameters. More precisely we keep track of changes in socialization and investment as $a(\mathbf{b})$ changes. Recall that $a(\mathbf{b})$ is a compound index of the parameter for the synergistic return a, and the second order average type, a measure of population heterogeneity.

Proposition **3** *Suppose that $a(\mathbf{b})$ increases. Then, in both approximate equilibria of the replica game, the percentage change in socialization effort is higher than that of productive effort (in absolute values), for all agents.*

Note that the statement of the proposition boils down to showing that the elasticity of s_i with respect to k_i for changes in $a(\mathbf{b})$ is smaller than one, at all equilibria. However, in equilibrium, the ratio s_i/k_i is constant across all agents. So, if we can establish that socialization is more responsive than productive effort for changes in a synergistic multiplier equal to $a(\mathbf{b})$ in an homogeneous population with types normalized to 1, the result follows. Inspecting the expression for the payoffs in a homogeneous population

$$u_i(\mathbf{k},\mathbf{s}) = k_i + a(\mathbf{b}) \sum_{j=1, j\neq i}^{n} g_{ij}(\mathbf{s})\, k_j k_i - \frac{1}{2}k_i^2 - \frac{1}{2}s_i^2$$

it is now clear that $a(\mathbf{b})$ affects directly the marginal benefit of s_i through the synergistic multiplier.

There are various ways in which $a(\mathbf{b})$ can increase.

First, through an increase in a, that captures the size of synergistic returns.

Second, through an increase in the second-order average type $\sum_{\tau=1}^{t} b_\tau^2 / \sum_{\tau=1}^{t} b_\tau$, a measure of the variability in private returns, which is a source of idiosyncratic heterogeneity in our model. For instance, a mean-preserving spread in private returns (where $\sum_{\tau=1}^{t} b_\tau$ is held constant while $\sum_{\tau=1}^{t} b_\tau^2$ increases) shifts upwards the second-order average type.

Third, through an upward simultaneous shift in all the b_τ's, and the a. These upward shifts can all be of different intensities for different parameters. Else, it can be an homothetic shift, where all parameters are scaled up by the same factor. In particular, consider the following variation of payoffs (4)

$$u_i(\mathbf{k},\mathbf{s}) = b_{\tau(i)}\, k_i + a \sum_{j=1, j\neq i}^{n} g_{ij}(\mathbf{s})\, k_j k_i - \frac{\alpha}{2}k_i^2 - \frac{\alpha}{2}s_i^2, \tag{8}$$

with $\alpha > 0$. The equilibria of the game with these payoffs (8) is equivalent to the game analyzed so far where the exogenous parameters $b_{\tau(i)}$ and a, and thus the compound index $a(\mathbf{b})$, are all scaled by $1/\alpha$.

Proposition 3 encompasses all those cases and many others, and pins down the absolute and relative variation of the s_i's and k_i's for all these multifarious changes in parameters. We now turn to the absolute variations or comparative statics of the equilibrium actions.

PROPOSITION **4** *Suppose that $a(\mathbf{b})$ increases while keeping b_τ constant, for one arbitrary given type τ. Then, at the Pareto-superior approximate equilibrium the equilibrium actions of all player i such that $\tau(i) = \tau$ decrease, while at the Pareto-inferior approximate equilibrium all the equilibrium actions of all player i such that $\tau(i) = \tau$ increase.*

REMARK **2** *Recall that at an approximate equilibrium $(k_i^*, s_i^*) = b_{\tau(i)}(k^*, s^*)$, where (k^*, s^*) are solutions to (6). As a matter of fact, we prove a stronger result establishing that any general arbitrary increase of $a(\mathbf{b})$ results in a co-movement of both s^* and k^* (increasing in the Pareto-inferior equilibrium, and decreasing in the Pareto-superior one). As long as the change in $a(\mathbf{b})$ keeps $b_{\tau(i)}$ constant or moves it in the right direction, the comparative statics of (k_i^*, s_i^*) coincide with those of (k^*, s^*).*

To understand the comparative statics of (k^*, s^*), we can think, again, about the homogeneous case. Then, in Figure 2, at the low equilibrium, strengthening the marginal returns to productive efforts leads to an increase in both socialization and investment. At the high action equilibrium, instead, this increase in synergistic returns allows players to save on costs, thus reducing both socialization and investment.

[*Insert Figure* 2 *here*]

4.2 Equilibrium payoffs

The next result documents the comparative statics of individual and aggregate equilibrium payoffs.

When m gets large, these are given by the following expression:

$$u_i^* = \frac{b_{\tau(i)}^2}{2a(\mathbf{b})}\frac{s}{k} + o(1), \text{ for all } i = 1, ..., mt. \tag{9}$$

$$= \frac{b_{\tau(i)}^2}{2}k + o(1), \text{ for all } i = 1, ..., mt. \tag{10}$$

Note that (10) is deduced from (9) through the first equation of (6).

When only a increases, while all the b_t's remain constant, the comparative statics of equilibrium payoffs are those of productive actions, as can be deduced from (10).

When the exogenous payoff parameters $(a; b_1, ..., b_t)$ are scaled up by some common factor $\lambda \geq 1$, this induces an upward shift from $a(\mathbf{b})$ to $\lambda^2 a(\mathbf{b})$. In a similar vein, $b_{\tau(i)}^2$ increases to $\lambda^2 b_{\tau(i)}^2$. The multiplicative factor λ^2 thus appears both in the numerator and in the denominator of (9), and the change in equilibrium payoffs is driven solely by the change in the ratio s/k, where (k, s) are solutions to (20). It turns out that the monotonicity of s/k is tied to the elasticity of s_i^* with respect to k_i^* and to the monotonicity properties of s_i^* and k_i^*, whose behavior is characterized in Proposition 3.

Finally, adding up equilibrium payoffs in (9) we get the following:

$$\sum_{i=1}^{mt} u_i^* = \frac{m\sum_{\tau=1}^{t} b_\tau}{2a}\frac{s}{k}.$$

Changes in aggregate payoffs following a change in parameters are thus related to changes in s/k and the sum of the productivity parameters divided by a.

These considerations lead to the following result.

Proposition **5** *Let* $(\mathbf{k}^*, \mathbf{s}^*) \geq (\mathbf{k}^{**}, \mathbf{s}^{**})$ *be the two ranked approximated equilibria of an* $m-$*replica game.*

1. *Suppose that either only* a *increases, or* $(a; b_1, ..., b_t)$ *are all scaled up by a common multiplicative factor. Then, at the Pareto-superior approximated equilibrium all the payoffs* $u_i(\mathbf{k}^*, \mathbf{s}^*)$ *decrease, while at the Pareto-inferior approximated equilibrium all payoffs* $u_i(\mathbf{k}^{**}, \mathbf{s}^{**})$ *increase, for all* $i = 1, ..., mt$.

2. *Suppose that the vector* $(b_1, ..., b_t)$ *changes via a mean preserving spread (i.e. a change that holds* $\sum_{\tau=1}^{t} b_\tau$ *constant but increases* $\sum_{\tau=1}^{t} b_\tau^2$*). Then, at the Pareto-superior approximated equilibrium the sum of payoffs* $\sum_{i=1}^{mt} u_i(\mathbf{k}^*, \mathbf{s}^*)$ *decreases, as well as payoffs for types below the average. At the Pareto-inferior approximated equilibrium the sum of payoffs* $\sum_{i=1}^{mt} u_i(\mathbf{k}^{**}, \mathbf{s}^{**})$ *increases.*

The next result characterizes the preferences of each type with respect to the composition mix of the population in individual traits.

Remark **3** *Fix* i *and let* $b'_{-\tau(i)}$ *and* $b_{-\tau(i)}$ *be two different population types (excluding* i*). If* $a(b_{\tau(i)}, \mathbf{b}_{-\tau(i)}) \geq a(b_{\tau(i)}, \mathbf{b}'_{-\tau(i)})$*, then player* i *gets a lower (resp. higher) utility at the Pareto superior approximated equilibrium (resp. at the Pareto inferior approximated equilibrium) under* $(a, b_{\tau(i)}, \mathbf{b}_{-\tau(i)})$ *that under* $(a, b_{\tau(i)}, \mathbf{b}'_{-\tau(i)})$.

The individual preferences over group composition documented in Proposition 3 allow for group comparisons across populations with different number of types as well as different number of individuals for each type, as long as Theorem 1 holds.

Recall that types are ordered as follows: $0 < b_1 \leq b_2 \leq \cdots \leq b_t$. Then, it is readily checked that:

$$a(b_1) \leq a(b_1, b_2) \leq \cdots \leq a(\mathbf{b}_{-t}) \leq a(\mathbf{b}) \leq a(\mathbf{b}_{-1}) \leq a(\mathbf{b}_{-1-2}) \leq \cdots \leq a(b_{t-1}, b_t) \leq a(b_t).$$

So, invoking Remark 3 that characterizes individual preferences over group composition, we can conclude that at the Pareto superior equilibrium, low types prefer to segregate themselves from the rest of the group, while high types prefer to wander around with types lower than themselves. This is because the overinvestment in socialization hits harder the low types, who reap a lower value from this socialization.

Inversely, high types wish to segregate from lower types at the Pareto inferior equilibrium. Again, this is because the underinvestment hits hard higher types.

So, if extreme types have a means to separate themselves from the rest of the group, simple unravelling dynamics induces full segregation by types, and thus homogeneous subpopulations.

5 Topology

5.1 Theory

Large social networks display a number of key empirical regularities, as reported by many studies analyzing an ever increasing number of available data on social networks.[15]

First, the distribution of network connectivities tends to be fat tailed for some networks (see Jackson and Rogers 2007 for amendments to this claim). That is, there is in some networks a much higher proportion of network nodes with a high number of network links than if the network links were created uniformly and independently at random.

Second, the average distance (or shortest path) between two network nodes is very small compared to the network size, and grows very slowly with this size. For instance, the network of actors who have acted together in at least one Hollywood movie comprises 225,226 individuals and has an average path length of 3.65.[16]

Third, the tendency of two given linked nodes to be linked to a common third-party, which is called the clustering coefficient, is much higher than in a purely random network. For instance, the actual clustering coefficient displayed by the movie actor network is almost 3,000 times higher that of a purely random network with identical average connectivity.

Beyond these key empirical features, social networks tend to exhibit an internal community structure, sometimes arranged hierarchically. Also, highly connected nodes tend to be connected with highly connected nodes like themselves, and poorly connected nodes with poorly connected nodes, a feature often referred to as positive assortativity. Etc.

There are a number of mechanisms for network formation that replicate these topological features. The basic ingredients are a population growth process, and a link formation device for new-

[15] See, for instance, Albert and Barábasi (2002), who describe evidence on the topology of the world-wide web, science collaboration graphs, the web of human sexual contacts, and movie actor collaboration. More recent studies include, e.g., the network of email communication (Guimerà *et al.* 2007). Jackson and Rogers (2007) contains detailed references to a number of social networks displaying rich and disparate features.

[16] See Albert and Barábasi (2002).

comers that combines random meetings with network (local) search for the partner. The growth process together with the network search generate preferential attachment dynamics. That is, positive feed-back loops whereby newcomers tend to connect with a handful of super-connectors, that become even more connected, and so on. This leads to a fat tailed connectivity distribution[17] and to a highly clustered network. The random meetings decrease average distance by creating bridges, but mitigate the fat tailed connectivity distribution.

Our model of network formation is static −a simultaneous move game. As such, it cannot be expected to replicate genuinely dynamic features observed in the data, e.g., the high clustering. Yet, it can still deliver some interesting implications for the topology of network links and, more importantly, relate topology to individual incentives.

Recall that the network formed on account of player's socialization decisions defines the synergistic technology available to everyone. This, in turn, determines the returns to private investments, whose levels are also left at the discretion of those who form the network. Our equilibrium analysis thus sheds light on the interplay between network formation and the private economic use of this jointly created device, and connects topological features of the network to individual behavior and payoffs.

For all $\mathbf{x} \in \mathbb{R}_+^t$, define $\overline{\mathbf{x}} = \sum_{\tau=1}^t x_\tau / t$, and $v(\mathbf{x}) = \sum_{\tau=1}^t x_\tau^2 / t - \overline{\mathbf{x}}$. These are, respectively, the average and the empirical variance of the coordinates of $\mathbf{x}$. We extend this definition to any non-negative vector in an Euclidean space of finite arbitrary size.[18]

Consider an approximate equilibrium $(\mathbf{k}^*, \mathbf{s}^*)$ of the $m-$replica game, that corresponds to some solution (k^*, s^*) to (6).

Theorem 1 and (7) imply that the distribution of socialization efforts $g_i(\mathbf{s}^*) = s_i^* = s^* b_{\tau(i)}$ is related, at equilibrium, to the distribution of types $\mathbf{b} = (b_1, ..., b_t)$. When all such link intensities are smaller than one, we can interpret our weighted network as a random graph, where each link ij is formed with independent probability $g_{ij}(\mathbf{s}^*)$. This random graph has an expected connectivity sequence $(s_1^*, ..., s_n^*)$. We can map the population ex ante heterogeneity into the connectivity distribution for the equilibrium random graph.

More precisely, the average connectivity is $\overline{\mathbf{s}^*} = s^* \overline{\mathbf{b}}$, proportional to the average type.

Also, the empirical variance of connectivities is $v(\mathbf{s}^*) = s^{2*} v(\mathbf{b})$. Therefore,

$$\frac{\sqrt{v(\mathbf{s}^*)}}{\overline{\mathbf{s}^*}} = \frac{\sqrt{v(\mathbf{b})}}{\overline{\mathbf{b}}}. \tag{11}$$

The heterogeneity in connectivities in the resulting equilibrium network is thus solely driven

[17] Preferential attachment breeds a multiplicative growth process for connectivities. The log of the connectivity distribution is thus additive through time. By the Central Limit Theorem, we conclude that the log of the connectivity distribution follows a normal distribution. The degree distribution is thus lognormal, a fat tail distribution barely distinguishable from a Pareto distribution. Mitzenmacher (2004) gives an excellent historical overview of generative models for heavy-tailed distributions.

[18] For all $\mathbf{x} \in \mathbb{R}_+^n$, we set $\overline{\mathbf{x}} = \sum_{i=1}^n x_i / n$, and $v(\mathbf{x}) = \sum_{i=1}^n x_i^2 / n - \overline{\mathbf{x}}$, for all $n \geq 1$.

by the heterogeneity in private returns, and is thus the same for both equilibria. Increasing the latter increases the former, and by varying $\mathbf{b}$ adequately we can cover a broad range of random graph topologies, including the heavy tailed connectivity distributions identified in the data.[19] For instance, a mean-preserving spread in $\mathbf{b}$ increases $\sqrt{v(\mathbf{s}^*)}/\overline{\mathbf{s}^*}$, and leads to a connectivity distribution with a fatter tail.

Beyond this comparative statics about ratios (11), Proposition 3 allows to conduct comparative statics directly about the average connectivity $\overline{\mathbf{s}^*}$ and the variance $v(\mathbf{s}^*)$.

More precisely, a mean-preserving spread in private returns $\mathbf{b}$ has an indirect effect on the average connectivity (through the change in s^*, the solution to (6)), and both a direct and an indirect effect on the variance of connectivities (through both the change in s^* and the change in $v(\mathbf{b})$. At the low-action equilibrium, both the variance and the average connectivity increase. At the high-action equilibrium, instead, the average connectivity decreases while the impact on the variance remains ambiguous. Invoking Proposition 5, we can conclude that an increase in both the variance and the average connectivity is concomitant to a decrease in total welfare. Instead, a decrease in average connectivity together with an increase in the variance of connectivities comes together with an increase in total welfare (provided, of course, that the variations in the network topology result from a mean-preserving spread in private returns).

Note that an increase in a alone also affects the variance and the average connectivity through the resulting impact on s.

More generally, an increase in $a(\mathbf{b})$ increases the variance and the average connectivity at the low-equilibrium, and decreases the average connectivity at the high-equilibrium, although it may have an ambiguous impact on $\sqrt{v(\mathbf{s}^*)}/\overline{\mathbf{s}^*}$.

Our static model also allows to draw conclusions on the average distance in the equilibrium random network. Following Chung and Lu (2002) the average distance in a random graph with given expected connectivity $(s_1^*, ..., s_n^*) = s^*\left(b_{\tau(1)}, ..., b_{\tau(n)}\right)$ is given by:[20]

$$(1 + o(1)) \frac{\log(mt)}{\log\left(s^*\overline{\mathbf{b}}\right)}.$$

This average distance increases slowly with the population size mt.

When the level of synergistic returns a increases while private returns $\mathbf{b} = (b_1, ..., b_n)$ remain constant, the average distance decreases at the low equilibrium while equilibrium payoffs increase. Instead, the average distance increases at the high equilibrium together with a decrease of equilibrium payoffs.

Suppose now that a and $\mathbf{b}$ change homothetically, and are all scaled up by a common parameter. We still have a decrease of the average distance coupled with increasing payoffs at the low equilibrium. The impact of this exogenous change of parameters at the high equilibrium is now

[19] Notice, however, that a fat tailed degree distribution requires a fat tailed distribution of population traits.

[20] Provided that link intensities are all smaller than one, and under some additional technical conditions.

ambiguous. However, an increase of the average distance can only happen at this high equilibrium, and is then concomitant with a decrease in payoffs for all agents.

The impact of a mean-preserving spread in private returns on the average distance is similar to a change in the level a of synergies. That is, the average distance goes down and the average payoffs go up at the low equilibrium, and reciprocally at the high equilibrium (average distance up and average payoffs down).

The next table summarizes this discussion.

Table 2. Comparative statics in the low and high equilibrium

	Low equilibrium					High equilibrium				
	$\overline{\mathbf{s}}$	$v(\mathbf{s})$	Clust.	$\overline{\text{dist.}}$	payoffs	$\overline{\mathbf{s}}$	$v(\mathbf{s})$	Clust.	$\overline{\text{dist.}}$	payoffs
a up	+	+	+	−	+	−	−	−	+	−
$(a, \mathbf{b})$ all up	+	+	+	−	+	·	·	·	·	−
$\mathbf{b}$ spread	+	+	+	−	+ (total payoffs)	−	·	·	+	− (total payoffs)

Our static model of network formation does not generate networks with a high clustering level. Indeed, suppose that link intensities are smaller than one so that we have a random graph. Take three nodes i, j, l such that i and j are linked, and so are j and l. Then, the probability that i and l are linked when m is high is roughly $g_{il}(\mathbf{s}^*)$, independent of the links ij and jl and of the order of the inverse of population size. More precisely, an approximate expression for the size of the clustering is:[21]

$$\frac{1}{mt}\frac{s^*}{\overline{\mathbf{b}}}\left(1+\frac{v(\mathbf{b})}{\overline{\mathbf{b}}}\right)^2.$$

The previous expression suggests how a small variation of the model can deliver moderate levels of clustering. Split the population into (even numbered) subpopulations of finite size (smaller replica of the game). Let f be the number of such subpopulations. Consider now f different meeting rooms, and an $(f+1)\,th$ meta-meeting room that encompasses them all. Assign each subpopulation into one meeting room (a one-to-one assignment). Suppose now that a fraction $1-\varepsilon$ of the socialization effort of each player is invested in-home, in the meeting room of the player, while a residual fraction ε is invested in the meta-meeting room that encompasses them all. We get an equilibrium network that consists on small communities of players (formed within each meeting room) with some bridges across communities. When ε is small enough, the equilibrium actions are approximately those characterized in Theorem 1. The smaller the size of each community, the bigger the clustering level (for identical average connectivity). However, this goes against our

[21] This clustering coefficient is computed averaging over all players, the percentage of triangles they are involved in, out of the total potential triangles.

characterization of equilibrium actions in terms of approximate equilibria which are closer to the true Nash equilibria the bigger the population size.

Finally, empirically observed social networks sometimes display a giant component, that is, they contain a subnetwork including a huge share of the population and for which there is a path inside it connecting any two players in this population share.[22] In the section 6.2 below, we provide conditions on the exogenous parameters of the model for the emergence of a giant component for the case of an homogeneous populations, $b_1 = ... = b_t$, and general costs.

5.2 An empirical illustration

Our model is well suited to learn about the driving economic forces behind the observed structure of networks and outcomes. Using data from an acquaintanceship network and the related education outcomes of a set of individuals, we can recover the parameters that generate the observations according to our model. We can then compare the results of our fitted model with existing topology and even perform comparative statics exercises as well as seeing the outcome of relevant policy experiments by recombining types and operating mean spreads on the distribution of types.

Four our analysis, we use data from the National Longitudinal Survey of Adolescent Health (AddHealth). The AddHealth database has been designed to study the impact of the social environment (i.e. friends, family, neighborhood and school) on adolescents' behavior in the United States by collecting data on students in grades 7-12 from a nationally representative sample of roughly 130 private and public schools in years 1994-95. AddHealth contains unique detailed information on friendship relationships, based upon actual friends nominations.[23]

The in-home questionnaire contains detailed information on the grade achieved by each student in mathematics, history and social studies and science, ranging from D or lower to A, the highest grade (re-coded 1 to 4). We calculate a school performance index for each respondent.[24]

By merging the in-home data to the in-school friendship nominations data and by excluding the individuals that report a non-valid answer to the target questions, we obtain a final sample of 11,964 pupils distributed over 199 networks. For our exercise, we take the network comprising the largest number of individuals that has 107 nodes.

For this network, we focus on the following two columns of information:

- the degree connectivity of each node in the network, that we denote s_i, $i = 1, ..., 107$
- the student achievement for each node in the network, that we denote e_i, $i = 1, ..., 107$

[22] However, giant components are not always the rule. For instance, the biggest component of a network of scientific collaborations among academic economists analyzed by Goyal, Moraga-González and van der Leij (2005) comprises 33,027 authors out of a total of 81,217 authors, that is, 40.7 percent of the total population.

[23] Pupils were asked to identify their best friends from a school roster (up to five males and five females).

[24] We then use the Crombach-α measure is to assess the quality of the derived variable. We obtain an α equal to 0.86 ($0 \leq \alpha \leq 1$) indicating that the different items incorporated in the index have considerable internal consistency.

The data, thus, provides a measure of s_i for each individual, and a perhaps imperfect measure of k_i for each one, e_i. We first perform a transformation of the performance measure. We write $e_i = k_i^{\beta}\exp(\varepsilon_i)$. Equilibrium conditions in Theorem 1 imply that $s_i/k_i = s/k$ (a constant). Therefore,

$$e_i = \left(\frac{k}{s}s_i\right)^{\beta}\exp(\varepsilon_i).$$

We then run the following OLS regression:

$$\log(e_i) = \delta + \beta\log(s_i) + \varepsilon_i.$$

We find $\widehat{\delta} = -2.5126$ and $\widehat{\beta} = 1.3264$, both significant at 1% level. We perform the following change in variable: $k_i = e_i^{1/\widehat{\beta}}$, which can be rewritten as $\log k_i = (\log e_i)/\widehat{\beta}$, and thus $\log(k_i/s_i) = \log\left[\widehat{\delta}/\widehat{\beta}\right]$. Given that $s_i/k_i = s/k$, we use $\exp[-\widehat{\delta}/\widehat{\beta}]$ as an estimator for the ratio k/s.

The first equation in (6), $s = a(\mathbf{b})k^2$, together with the fact that $k_i = b_{\tau(i)}k$, implies that

$$k_i = \frac{b_{\tau(i)}}{a(\mathbf{b})}\frac{s}{k}.$$

We do a Maximum Likelihood fit of the following equation:

$$k_i = \frac{b_{\tau(i)}}{a(\mathbf{b})}\exp[-\widehat{\delta}/\widehat{\beta}] + \nu_i,\ i = 1, ..., 107,$$

conditional on $a(\mathbf{b}) < 2/3\sqrt{3}$.

To conduct this empirical fit, we assign players to types in the following way. In the first estimation we allow only for four different types $(b_1, ..., b_4)$ and we assign each agent to the type corresponding to his place in the distribution of k_i by quartiles. In the second specification with ten heterogeneous parameters $(b_1, ..., b_{10})$, we divide agents in deciles.

We obtain the following parameter fits:

$$(a; b_1, b_2, b_3, b_4) = (0.15;\ 1.22,\ 1.43,\ 1.59,\ 1.78) \quad (12)$$

$$(a; b_1, ..., b_{10}) = (0.19; 1.12, 1.17, 1.25, 1.34, 1.46, 1.57, 1.62, 1.75, 1.88, 1.96) \quad (13)$$

Once we have the types, it becomes possible to think about segmentations of the population according to types, and to establish the impact on the welfare of the affected individuals.

For example, suppose we have a population like the one resulting from the quartile estimation (12). Suppose the individuals from such population are recombined in groups with only two types. Then it is easy to check, invoking Remark 3, that individual preferences rank partners in decreasing value of their type for the high equilibrium, and they are ranked in the opposite order for the low equilibrium.

For instance, at the high equilibrium, type 1 players prefer a group with type 2 partners, to one with type 3 partners, themselves preferred to type 4 partners. In this particular case, at both equilibria the only stable pair-wise matching groups types 1 with types 2, and types 3 with types 4. Notice, however, that this stable matching is not the one that maximizes social welfare at the high equilibrium.

Continuing with the same example, and now allowing for groups with more than two types, the (decreasing) order of preference for type 1 player at the high equilibrium is the following: groups only types (1), then with types $(1,2)\,,(1,3)\,,(1,2,3)\,,(1,2,4)\,,(1,2,3,4)\,,(1,4)$ and finally $(1,3,4)$.[25]

6 The case of homogeneous populations

Remark 3 implies that individuals of the either extreme type prefer a society composed only of individuals such as themselves to any other mixture. Provided that the institutional environment allows them to segregate themselves, they will form a separate society. One should thus expect that at least some homogeneous groups would exist in a given society. However, given that at least some of the types within a subgroup would lose from the segregation of the opposite types, it should not be excluded that the society would not leave complete freedom for segregation at all levels, and heterogeneous group may anyway form.[26]

In what follows, we concentrate our attention on homogeneous groups, for which we can conduct some robustness checks on the technology and for which further insights on the topology are possible.

6.1 Equilibrium analysis and comparative statics

We now consider an homogeneous population of players with a single type corresponding to private returns b, but allow for non-linear marginal costs of both socialization and investment. Player i's utility is:

$$u_i(\mathbf{k},\mathbf{s}) = bk_i + a\sum_{j=1,j\neq i}^{n} g_{ij}\left(\mathbf{s}\right)k_jk_i - \frac{1}{c+1}k_i^{c+1} - \frac{1}{c+1}s_i^{c+1}, \tag{14}$$

where $a, b \geq 0$ and $c \geq 1$. The case $c = 1$ corresponds to quadratic costs. As c increases, the cost function becomes steeper.

We focus on symmetric equilibria, where all (homogeneous) players choose the same socialization effort and invest the same amount. Such symmetric equilibria give rise to random graphs where the

[25] We can also use the estimated types to illustrate the comparative statics of a mean preserving spread in the population composition. We divide the 107 nodes into (roughly) 27 agents of each type. Then, we compute the equilibrium payoffs for a population composed of x individuals for each extreme type 1 and 4, and $54-x$ individuals for each central type 2 and 3, and we vary x from 1 to 53. Each increase in x corresponds (roughly) to a mean-preserving spread in population heterogeneity. Consistently with Proposition 5, we find that the utility of types 1 and 2 decrease as well the total utility of the group at the high equilibrium. The monotonicity is reversed at the low equilibrium. Also, for this particular parametrization the utility of types 3 and 4 (not covered in the Proposition 5) changes monotonically in the same direction as that of the other types.

[26] Of course, it is also possible that there are technological restrictions, such as increasing returns at certain levels, which would make very small groups inefficient.

probability of link creation is independent and identical across all links. This Bernoulli process of link formation was first analyzed by Erdös and Rényi (1959) who establish a number of interesting topological properties for large Bernoulli networks, that is, as the number of the nodes tends to infinity (see Jackson (2007), Section 3.1.1 for details). In particular, when the population gets large, the Erdös and Rényi random graph induces a Poisson distribution over network connectivities.

We analyze the topology and welfare properties of Poisson Nash networks for the richer class of cost functions (beyond the quadratic set up) defined in (14).

We first start by noticing that this game always admits a corner equilibrium.

LEMMA **3** *There always exists a pure strategy Nash equilibrium where no player invests in socialization effort, with symmetric equilibrium strategies* $\left(b^{1/c}, 0\right)$ *and payoffs* $cb^{1+1/c}/\left(c+1\right)$*.*

Notice that this equilibrium corresponds to the partially corner equilibrium identified in Lemma 2 for the case of quadratic costs, $c = 1$.

We now introduce some notations. For all $\alpha < \beta$, define:

$$\mathbf{1}_{(\alpha,\beta]}\left(x\right) = \begin{cases} 1, \text{ if } \alpha < x < \beta \\ 0, \text{ otherwise} \end{cases} .$$

We introduce also the following function $\phi : \mathbb{R} \to \mathbb{R}$ given by:

$$\phi\left(x\right) = c^{\frac{1}{c+1}} \left[x^{c+1} - b^{1+\frac{1}{c}}\right]^{\frac{1}{c+1}} .$$

We are now ready to state the main characterization result.

THEOREM **2** *Suppose that* $\frac{c^2}{2+c^2}\left(\frac{2}{2+c^2}\right)^{\frac{2}{c^2}} > a^{1+\frac{1}{c}} b^{\frac{2}{c^2}}$*. Then, there exists an* n^* *such that for all games with* $n \geq n^*$ *players, the pure strategy Nash equilibrium strategies converge as* n *goes to infinity to the solutions to:*

$$\begin{cases} s^c = \mathbf{1}_{(-\infty,\phi(k)]}\left(s\right) ak^2 \\ k^c\left[1 - as\right] = b \end{cases} , \tag{15}$$

which has one, two or three different non-negative solutions.

It is readily checked that $\left(b^{1/c}, 0\right)$ is always a solution to (15), consistent with Lemma 3. At this equilibrium, the network of synergies is empty.

For large enough populations, we can also have up to two *interior symmetric equilibria*. Inspecting (15), these interior equilibria (k^*, s^*) solve:

$$\begin{cases} s^c = ak^2 \\ k^c\left[1 - as\right] = b \end{cases} , \tag{16}$$

which generalizes (6) for arbitrary c, with the added condition that $s^* \leq \phi(k^*)$. After some algebra, this last inequality is equivalent to:

$$u_i(k^*, s^*) = \frac{1}{c+1}\left[ck^{*c+1} - s^{*c+1}\right] \geq \frac{c}{c+1} b^{1+\frac{1}{c}} = u_i\left(b^{1/c}, 0\right). \tag{17}$$

In words, the condition $s^* < \phi(k^*)$ guarantees that $\left(b^{1/c}, 0\right)$ is not a strict best-response by some arbitrary player i to the rest of the players playing (k^*, s^*). On top of that, the conditions on the exogenous parameter values imposed in Theorem 2 guarantee that the first-order conditions for interior equilibria (16) have at least one solution, and that the second-order conditions hold at these points.

The fact that the inequality (17) holds for any interior Nash equilibrium implies that any interior equilibrium Pareto dominates the partially corner equilibrium where players do not socialize and invest the optimum level $b^{1/c}$ that corresponds to their private costs and returns.

Suppose now that the exogenous parameter values are such that two interior equilibria exist. It turns out that their corresponding actions can also be ranked. As in Proposition 2, we can thus speak of a low-action (interior) equilibrium and a high-action (interior) equilibrium for large enough populations. The response to these equilibrium actions to incentives is identical to that documented in Proposition 3.

PROPOSITION **6** *Let $(k^*, s^*) \geq (k^{**}, s^{**})$ be the two ranked interior symmetric approximate equilibria for a large enough population. When a increases, k^{**} and s^{**} increase, while k^* and s^* decrease. In both cases, the percentage change in s is higher than that of k (in absolute values).*

Notice that the previous result implies that s/k is an increasing function of a at equilibrium. Factorizing by k^{c+1} in the expression for equilibrium payoffs given by (17), one can then readily conclude that individual payoffs decrease with a at the high equilibrium.

6.2 The topology of Erdös-Rényi equilibrium networks

In the Erdös-Rényi (Bernoulli) random networks that correspond to the interior and symmetric Nash equilibria of Theorem 2,[27] the expected number of links to each player is s^*, and each potential link in the network is created (approximately) with independent probability s^*/n when the population gets large. The fact that link creation is *i.i.d.* implies, in particular, that the network connectivity (or degree) is not correlated across different nodes.

Beyond this vanishing degree-degree correlation across nodes as the population gets larger, large Erdös-Rényi networks display a number of interesting topological features. For instance, when $s^* < 1$, the networks is composed of a huge number of disjoint small trees. Instead, when $s^* > 1$, a single giant component that encompasses a high fraction of all the network nodes emerges.

[27] Note that the interpretation of our equilibrium network as a random graph requires that all link intensities are smaller than one, which is equivalent to $a < 1$.

The next result ties down the existence of a giant component to conditions on the exogenous parameter values of the model.

Proposition 7 *Let $a < 1$, so that equilibrium networks can be interpreted as random graphs. Suppose that there are two non-empty equilibrium networks. Then, the two equilibrium networks display different topological characteristics (one network with a giant component, one without) if and only if $ab^{2/c} < (1-a)^{2/c}$. If, instead, $ab^{2/c} > (\frac{c}{a+c})^2$, then both equilibrium networks have a giant component.*[28]

In some cases, both the low-action and the high-action equilibrium networks have a giant component. In some other cases, the low-action equilibrium network is fragmented whereas the high-action equilibrium network contains a connected component encompassing a nontrivial fraction of players. In principle, Proposition 7 does not exclude the possibility of coexistence of two fragmented equilibrium networks (for high and low actions). We conjecture, however, that the only two possibilities are those described in Proposition 7.

Holding $b = 1$, Figure 3 displays for values of c ranging from 1 to 10, $a_{\min}$ and $a_{\max}$, respectively the minimum and maximum values of a for which the equilibrium exists.

[*Insert Figure* 3 *here*]

The line in between $a_{\min}$ and $a_{\max}$ represents a phase transition separating two parameter regions for a and c. For values of a above this line, the two equilibrium networks have a giant component. Instead, for values of a below this line, only the equilibrium network with low actions is fragmented.

Notice that this transition is sharp, that is, the low action equilibrium changes discontinuously the topological properties as a function of the synergistic parameter a.

7 Discussion

We have provided a simple operational model of network formation with welfare and topology predictions, and clear-cut comparative statics. In substance, we identify a "too cold" and a "too hot" equilibrium. We show that socialization is more responsive than production to exogenous shocks in the parameters and that individual preferences over group composition hint towards assortative matching. A variation of the model with several groups allows for partially directed socialization within groups more in line with some empirical evidence on clustering and community structure within networks.

We show and state our results with a class of payoffs corresponding to the functional form given in equation (4). However, the thrust of our analysis carries over to some generalizations of this setup.

[28] Notice that $(\frac{c}{a+c})^2 \geq (1-a)^{2/c}$. See the proof for details.

The three main characteristics of this functional form are: the linear-quadratic expression in production efforts, the genericity of socialization decisions (condition $(A3)$ of Lemma 1), and aggregate constant returns to scale in socialization (condition $(A2)$ of lemma 1). The linear-quadratic form plays an important role in the analysis, as it allows to express existence and interiority of the production equilibrium decisions (for a given socialization profile $\mathbf{s}$) as a a function of the spectral radius of $\mathbf{G}(\mathbf{s})$. Combined with conditions $(A2)$ and $(A3)$, this leads to first-order conditions that take relatively manageable closed-form matrix expression. In turn, when population gets large, and because we are able to control the population size effect in our matrix closed-form expression, approximate equilibrium conditions boil down to a simple system of equations (6). On top of its operational virtues, condition $(A3)$ is hard to dispense with as it embodies the central assumption of our approach, the genericity of socialization. Condition $(A2)$, instead, is chosen mainly for its operationally virtues. We could accommodate variations of this condition, and thus alternative expressions for $g_{ij}(\mathbf{s})$ that allow for some aggregate scale effects in socialization, as long as we can still control for population size effects. Essentially, we need that both the spectral radius of $\mathbf{G}(\mathbf{s})$ and the diagonal cells of the matrix $[\mathbf{I}-\mathbf{G}(\mathbf{s})]^{-1}$ are of finite order, while the off-diagonal terms of the same matrix be of order inverse of the population size.

One slightly artificial feature of the model is the fact that the effort variables are unbounded. This creates existence problems and generates the need for the assumption $2/3\sqrt{3} > a(\mathbf{b})$. In addition, this generates a failure of upper-hemicontinuity in the equilibrium correspondence as the high-action equilibrium diverges to infinity as $a(\mathbf{b})$ goes to zero. A simple way to deal with this problem is to assume that the effort of each individual is bounded. That is, $s_i + k_i \leq T$. This is natural when one interprets the sum of efforts of an individual as related to the time at his disposal, or, more generally as activities that consume resources of this sort. It is relatively easy to characterize the equilibria and their topological and welfare properties under this modification. In particular, the Pareto superior equilibrium disappears for $a(\mathbf{b})T$ low enough. A bounded strategy space can introduce upper corner equilibria that may be stable.

Another economically compelling modification of the original setup is to introduce a market for effort resources that are limited in supply. For example, suppose that there is a fixed amount of productive effort $\sum_{i=1}^{n} k_i = k$ sold in a competitive market. If one interprets (4) as the amount of numeraire produced by agent i, the total profit for an agent is:

$$\left(b_{\tau(i)}\, k_i + a \sum_{j=1}^{n} p_{ij}\left(\mathbf{s}\right) k_j k_i - \frac{1}{2} k_i^2 - \frac{1}{2} s_i^2 \right) - p_k k_i \tag{18}$$

which is equivalent to:

$$\left(b_{\tau(i)} - p_k\right)\, k_i + a \sum_{j=1}^{n} p_{ij}\left(\mathbf{s}\right) k_j k_i - \frac{1}{2} k_i^2 - \frac{1}{2} s_i^2 \tag{19}$$

This corresponds to our standard model, where the types are now $b_{\tau(i)} - p_k$. An equilibrium

$(p_k^*, \mathbf{k}^*, \mathbf{s}^*)$ is now simply a solution to (6) with the modified types and the market clearing conditions, for large populations.

It should be clear that, for a fixed (low enough) p_k, the new system (6) has a solution. For the low equilibrium, lowering types makes the k^* smaller coordinatewise (this is Proposition 4). This yields, in fact, a (downward sloping) demand function for k. Supply is a vertical line, so there is always an equilibrium (potentially at a price of zero and excess supply). The shape of the demand function for the high equilibrium is less straightforward because the comparative statics of the high equilibrium as p_k changes are now ambiguous (see again Proposition 4). In some cases, this demand function will actually be upward sloping at the high equilibrium. Because upward sloping demands give rise to unstable market dynamics, the low equilibrium is then uniquely selected.

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Appendix

Proof of Lemma 1: Fix $\mathbf{s}$. From $(A3)$ we have $g_{ij}(\mathbf{s}) = s_i\psi_j(\mathbf{s})$, where $\psi_j : \mathbb{R}^n_+ \to \mathbb{R}_+$, for all i,j. From $(A2)$ we have $\sum_{j=1}^n \psi_j(\mathbf{s}) = 1$. Multiplying the previous expression by s_i gives $\sum_{j=1}^n s_i\psi_j(\mathbf{s}) = s_i$. By $(A1)$ we have $s_i\psi_j(\mathbf{s}) = s_j\psi_i(\mathbf{s})$, and thus $\sum_{j=1}^n s_j\psi_i(\mathbf{s}) = \psi_i(\mathbf{s})\sum_{j=1}^n s_j = s_i$, and the result follows. ∎

Proof of Theorem 1: It follows from the following Lemmata 4, 5, 6 and 7.

Consider an $m-$replica game involving $n = mt$ players, where $m \geq 1$ is fixed for the time being.

Let $\mathbf{G}(\mathbf{s}) = [g_{ij}(\mathbf{s})]_{i,j\in N}$ be the $n-$symmetric adjacency matrix for the network with link intensities in (1).

For all $\mathbf{x} \in \mathbb{R}^t_+$, define $\overline{\mathbf{x}} = \sum_{\tau=1}^t x_\tau/t$, $\overline{\mathbf{x^2}} = \sum_{\tau=1}^t x_\tau^2/t$, and:

$$\lambda(\mathbf{x}) = \frac{a\overline{\mathbf{x}}}{\overline{\mathbf{x}} - a\overline{\mathbf{x^2}}}.$$

We extend this definition to any non-negative vector in an Euclidean space of arbitrary size.

Lemma 4 *Let $s \in \mathbb{R}^n_+$, $\mathbf{s} \neq \mathbf{0}$ such that $1 > a\,\overline{\mathbf{s^2}}/\overline{\mathbf{s}}$. Then, $\mathbf{M}(\mathbf{s}) = [\mathbf{I}-a\mathbf{G}(\mathbf{s})]^{-1}$ is a well-defined and non-negative $n-$square matrix, equal to $\mathbf{M}(\mathbf{s}) = \mathbf{I} + \lambda(\mathbf{s})\mathbf{G}(\mathbf{s})$.*

Proof. When $\mathbf{M}(\mathbf{s}) \in \mathbb{R}^{n^2}$ is well-defined, we have $\mathbf{M}(\mathbf{s}) = \sum_{p=0}^{+\infty} a^p\mathbf{G}(\mathbf{s})^p$. We compute $\mathbf{G}(\mathbf{s})^p$. First, note that (we omit $\mathbf{s}$ when there is no confusion):

$$g_{ij}^{[2]} = \sum_{l=1}^n g_{il}g_{lj} = \frac{s_is_j}{n\overline{\mathbf{s}}}\sum_{h=1}^n \frac{s_h^2}{n\overline{\mathbf{s}}} = \frac{\overline{\mathbf{s^2}}}{\overline{\mathbf{s}}}g_{ij}, \text{ for all } i,j = 1,...,n$$

By a trivial induction on $p = 1,2,...$, we deduce that $g_{ij}^{[p]} = \left(\overline{\mathbf{s^2}}/\overline{\mathbf{s}}\right)^p g_{ij}$, for all i,j and all $p \geq 1$. Therefore:

$$\mathbf{M}(\mathbf{s}) = \mathbf{I} + \sum_{p=1}^{+\infty}\left(a\frac{\overline{\mathbf{s^2}}}{\overline{\mathbf{s}}}\right)^p \mathbf{G} = \mathbf{I} + \lambda(\mathbf{s})\mathbf{G}(\mathbf{s}).$$

We know from Debreu and Herstein (1953) that $\mathbf{M}(\mathbf{s})$ is well-defined and non-negative if and only if $1 > a\rho(\mathbf{G}(\mathbf{s}))$, where $\rho(\mathbf{G}(\mathbf{s}))$ is the modulus of the largest eigenvalue of $\mathbf{G}(\mathbf{s})$ (see also Theorem 1 in Ballester, Calvó-Armengol and Zenou 2006). Let us show that $\rho(\mathbf{G}(\mathbf{s})) = \overline{\mathbf{s^2}}/\overline{\mathbf{s}}$.

First, note that $\overline{\mathbf{s^2}}/\overline{\mathbf{s}}$ is an eigenvalue of $\mathbf{G}$ for the eigenvector $\mathbf{s}$. Indeed, $\mathbf{G}\cdot\mathbf{s} = (\overline{\mathbf{s^2}}/\overline{\mathbf{s}})\,\mathbf{s}$.

Second, let $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$. We have:

$$\mathbf{G}\cdot\mathbf{x} = \frac{\mathbf{s}\cdot\mathbf{x}}{n\overline{\mathbf{s}}}\mathbf{s},$$

where $\mathbf{s}\cdot\mathbf{x} = \sum_{i=1}^n s_ix_i$ is the scalar product, with $|\mathbf{s}\cdot\mathbf{x}| \leq \|\mathbf{s}\| \times \|\mathbf{x}\| \leq \|\mathbf{s}\|$. Therefore, $\|\mathbf{G}\cdot\mathbf{x}\| \leq \|\mathbf{s}\|^2/n\overline{\mathbf{s}} = \overline{\mathbf{s^2}}/\overline{\mathbf{s}}$. Note that, by definition, $\rho(\mathbf{G}) = \sup\left\{\|\mathbf{G}\cdot\mathbf{x}\|/\|\mathbf{x}\| : \|\mathbf{x}\| = 1\right\}$. Altogether, we can conclude that $\rho(\mathbf{G}) = \overline{\mathbf{s^2}}/\overline{\mathbf{s}}$. ∎

Let now $m^1, m^2, m^3, ...$ be an increasing sequence of integers such that $m^h \to +\infty$ as $h \to +\infty$. Each $h \in \mathbb{N}$ defines a m^h-replica game involving $n^h = m^h t$ players. In the m^h-replica game, there are m^h players of each type $(b_1, ..., b_t)$. In each such game, a profile of strategies is $\left(\mathbf{k}^h, \mathbf{s}^h\right) \in \mathbb{R}_+^{n^h} \times \mathbb{R}_+^{n^h}$. Given a player $i = 1, ..., n^h$, recall that $b_{\tau(i)}$ denotes his type, where $\tau(i) \in T$.

Lemma **5** *Let* $\left\{\left(\mathbf{k}^h, \mathbf{s}^h\right)\ \right\}_{h\in\mathbb{N}}$ *be a sequence of Nash equilibria of the* m^h-*replica games such that* $1 > a\overline{\mathbf{s^{h2}}}/\overline{\mathbf{s^h}}$*, for all* $h \in \mathbb{N}$*. Suppose that the system of equations:*

$$\begin{cases} [1 - a(\mathbf{b})s]\, k = 1 \\ s = a(\mathbf{b})k^2 \end{cases} \tag{20}$$

has a solution $(k, s) \in \mathbb{R}_+^2$ *such that* $1 > a(\mathbf{b})s$*. Then, for all* $\varepsilon > 0$*, there exists some* $h_\varepsilon \in \mathbb{N}$ *such that, for all* $h \geq h_\varepsilon$*, we have* $\max\{\left|k_i^h - b_{\tau(i)}k\right|, \left|s_i^h - b_{\tau(i)}s\right|\} < \varepsilon$*, for all* $i = 1, ..., n^h$*, where* (k, s) *is a solution to (20).*

Proof. Let $\left\{\left(\mathbf{k}^h, \mathbf{s}^h\right)\ \right\}_{h\in\mathbb{N}}$ be a sequence of Nash equilibria such that $1 > a(\mathbf{s}^h)$, for all h. Let $diag\left(\mathbf{G}\left(\mathbf{s}^h\right)\right)$ be the diagonal matrix with diagonal terms $g_{ii}\left(\mathbf{s}^h\right)$ and zero off-diagonal terms. For each h, using the expression for $[\mathbf{I} - a\mathbf{G}(\mathbf{s}^h)]^{-1} \in \mathbb{R}^{n^{h2}}$ in Lemma 4, we write the first-order necessary equilibrium conditions for k^h as:

$$\mathbf{k}^h + a\left[\mathbf{I} + \lambda(\mathbf{s}^h)\mathbf{G}\left(\mathbf{s}^h\right)\right] \cdot diag\left(\mathbf{G}\left(\mathbf{s}^h\right)\right) \cdot \mathbf{k}^h = \left[\mathbf{I} + \lambda(\mathbf{s}^h)\mathbf{G}\left(\mathbf{s}^h\right)\right] \cdot \mathbf{b}^h, \tag{21}$$

where $\mathbf{b}^h \in \mathbb{R}_+^{n^h}$ is defined by $b_i^h = b_{\tau(i)}$, for all $i = 1, ..., n^h$. In words, the ith coordinate of $\mathbf{b}^h$ corresponds to the private returns of player i's type. Note that the n^h coordinates of $\mathbf{b}^h$ take t different possible values, $b_1, ..., b_t$, each repeated m^h times.

The first-order conditions for s_i^h are:

$$s_i^h = ak_i^h \frac{\mathbf{s}^h \cdot \mathbf{k}^h}{n^h \overline{\mathbf{s}}^h} - as_i^h k_i^h \frac{\mathbf{s}^h \cdot \mathbf{k}^h}{(n^h \overline{\mathbf{s}}^h)^2} - a\frac{s_i^h k_i^{h2}}{n^h \overline{\mathbf{s}}^h} + a\frac{s_i^{h2} k_i^{h2}}{(n^h \overline{\mathbf{s}}^h)^2}. \tag{22}$$

Given that $1 > a\overline{\mathbf{s}^{h2}}/\overline{\mathbf{s}^h}$, for all h and that $n^h \to +\infty$ as $h \to +\infty$, necessarily, $s_i^h \in O(1)$, for all $i = 1, ..., n^h$ and for all h. Indeed, suppose that $s_i^h \in O\left(n^{h^p}\right)$, $p > 0$, for some j. Let then $q > 0$ such that $s_i^h \in O\left(n^{h^q}\right)$, $q > 0$, for all i. Then, $a\overline{\mathbf{s}^{h2}}/\overline{\mathbf{s}^h} \in O(n^{h^q})$, and the inequality $1 > a\overline{\mathbf{s}^{h2}}/\overline{\mathbf{s}^h}$ is violated for large enough h. Given that $s_i^h \in O(1)$, we have $g_{ij}\left(\mathbf{s}^h\right) = s_i^h s_j^h / \left(\sum_{l=1}^{n^h} s_l^h\right) \in o(1)$ when $h \to +\infty$, for all $i, j = 1, ..., n^h$

The first-order conditions (21) imply that $k_i^h \in O(1)$, for all $i = 1, ..., n^h$ and for all h.

Then, using (22), we deduce that for h high enough, we have

$$s_i^h = ak_i^h \frac{\mathbf{s}^h \cdot \mathbf{k}^h}{n^h \overline{\mathbf{s}^h}} + o(1)\,, \text{ for all } i = 1, ..., n^h \text{ and for all } h.$$

By (21), k_i^h is a continuous function of $\mathbf{s}^h$. Therefore, $s_i^h = \kappa_i^h + o(1)$ and $k_i^h = \sigma_i^h + o(1)$, for all $i = 1, ..., n^h$ and for all h, where $(\boldsymbol{\sigma}^h, \boldsymbol{\kappa}^h)$ are such that:

$$\boldsymbol{\sigma}^h = \left[\mathbf{I} + \lambda(\boldsymbol{\kappa}^h)\mathbf{G}\left(\boldsymbol{\kappa}^h\right)\right] \cdot \mathbf{b}^h, \tag{23}$$

and

$$\kappa_i^h = a\sigma_i^h \frac{\boldsymbol{\kappa}^h \cdot \boldsymbol{\sigma}^h}{n^h \overline{\boldsymbol{\kappa}^h}}, i = 1, ..., n^h. \tag{24}$$

We solve (23) and (24).

Note, first, that (24) implies that $\sigma_i^h / \kappa_i^h = \sigma_j^h / \kappa_j^h$, for all i, j. Without any loss of generality, we can thus write $\kappa_i^h = \theta_i^h s$ and $\sigma_i^h = \theta_i^h k$, for all $i = 1, ..., n^h$ and for some k, s. Then, (24) rewrites as:

$$s = ak^2 \frac{\overline{\boldsymbol{\theta}^{h2}}}{\overline{\boldsymbol{\theta}^h}}, \tag{25}$$

Noting that $g_{ij}\left(\boldsymbol{\kappa}^h\right) = \theta_i^h \theta_j^h s / n^h \overline{\boldsymbol{\theta}^h}$, we rewrite (23) as:

$$\theta_i^h k = b_i^h + \frac{as}{n^h} \frac{\theta_i^h}{\overline{\boldsymbol{\theta}^h} - a\overline{\boldsymbol{\theta}^{h2}} s} \sum_{j=1}^{n^h} \theta_j^h b_j^h, \tag{26}$$

for all $i = 1, ..., n^h$.

Let $\theta_i^h = b_i^h$. Then, (26) becomes:

$$k = 1 + \frac{a\overline{\mathbf{b}^2} s}{\overline{\mathbf{b}} - a\overline{\mathbf{b}^2} s} = \frac{1}{1 - a(\mathbf{b})s},$$

while (25) becomes $s = a(\mathbf{b})k^2$.

Note that the condition $1 > a\overline{\mathbf{s^{h2}}} / \overline{\mathbf{s^h}}$ is then equivalent to $1 > a(\mathbf{b})s$. ∎

Lemma **6** *If $2/3\sqrt{3} > a(\mathbf{b})$, then the system of equations (20) has exactly two solutions $(k, s) \in \mathbb{R}_+^2$ such that $1 > a(\mathbf{b})s$.*

Proof. Plugging the expression for k into the expression for s in (20), one concludes that every solution (k^*, s^*) of (20) is such that $g(s^*) = s^*$, where;

$$g(s) = \frac{a(\mathbf{b})}{(1 - a(\mathbf{b})s)^2}. \tag{27}$$

We establish conditions such that the graph of $g(s)$ crosses (twice) the 45 degree line for some s such that $1 > a(\mathbf{b})s$. Note that $g(0) = a(\mathbf{b})$ and $\lim_{s \uparrow 1/a(\mathbf{b})} g(s) = +\infty$, so that the function $g(\cdot)$ maps $[0, 1/a(\mathbf{b}))$ into $[a(\mathbf{b}), +\infty)$, while $g'(0) = a(\mathbf{b})^2$ and $\lim_{s \uparrow 1/a(\mathbf{b})} g'(s) = +\infty$. If there exists a tangent to the graph of $g(\cdot)$ on $[0, 1/a(\mathbf{b}))$ parallel to the 45 degree line, and if this tangent is strictly below (resp. tangent to) the 45 degree line, the system (20) has exactly two solutions (resp. one solution) on $[0,1/a(\mathbf{b}))$. Such a tangent exists if $a(\mathbf{b})\leq 1$, which we assume from now on. Next, we solve

$$g'(x^*)=1\Leftrightarrow a(\mathbf{b})x^*=1-\left(2a(\mathbf{b})^2\right)^{1/3} \tag{28}$$

Thus, (20) has two solutions (resp. one solution) if and only if $a(\mathbf{b})\leq 1$ and $g(x^*)<x^*$ (resp. $g(x^*)=x^*$), where x^* is defined by (28). The last inequality is equivalent to $a(\mathbf{b})<2/3\sqrt{3}\leq 1$. When $a(\mathbf{b})<2/3\sqrt{3}$ (resp. $a(\mathbf{b})=2/3\sqrt{3}$), the graph of $g(\cdot)$ thus crosses the 45 degree line twice (resp. once) on $[0,1/a(\mathbf{b}))$. ∎

LEMMA **7** *Let* $\left\{\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right)\right\}_{h\in\mathbb{N}}$ *be such that* $k_i^{h*}=kb_{\tau(i)}$ *and* $s_i^{h*}=sb_{\tau(i)}$*, for all* $i=1,...,n^h$*, where* (k,s) *is some given solution to (20). If* $2/3\sqrt{3}>a(\mathbf{b})$*, then there exists some* $\overline{h}\in\mathbb{N}$ *such that, for all* $h\geq\overline{h}$*, the second-order equilibrium conditions for* $\mathbf{u}=(u_1,...,u_{n^h})$ *hold at* $\left(\mathbf{k}^{*h},\mathbf{s}^{*h}\right)$*.*

Proof. First note that Lemma 6 implies that (20) has a solution such that $1>a(\mathbf{b})s$. Consider this solution. We also know from Lemma 5 that both $s_i^h,k_i^h\in O(1)$, for all $i=1,...,n^h$. We now compute the cross partial derivatives of $\mathbf{u}$ at $\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right)$.

First, we have:

$$\frac{\partial u_i}{\partial s_i}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = \frac{a}{n^h\overline{\mathbf{s}}^{h*}}\sum_{j=1,j\neq i}^{n^h}\left(k_i^{h*}k_j^{h*}s_j^{h*}-g_{ij}\left(\mathbf{s}^{h*}\right)k_i^{h*}k_j^{h*}\right)-s_i^{h*} \tag{29}$$

$$\frac{\partial u_i}{\partial k_i}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = b_{\tau(i)}+a\sum_{j=1,j\neq i}^{n^h}g_{ij}\left(\mathbf{s}^{h*}\right)k_j^{h*}-k_i^{h*} \tag{30}$$

Thus

$$\begin{aligned}
\frac{\partial^2 u_i}{\partial s_i^2}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) &= \frac{2a}{(n^h\overline{\mathbf{s}}^{h*})^2}\sum_{j=1,j\neq i}^{n^h}\left(-k_i^{h*}k_j^{h*}s_j^{h*}+g_{ij}\left(\mathbf{s}^{h*}\right)k_i^{h*}k_j^{h*}\right)-1\\
\frac{\partial^2 u_i}{\partial s_i\partial k_i}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) &= \frac{a}{n^h\overline{\mathbf{s}}^{h*}}\sum_{j=1,j\neq i}^{n^h}\left(k_j^{h*}s_j^{h*}-g_{ij}\left(\mathbf{s}^{h*}\right)k_j^{h*}\right)\\
\frac{\partial^2 u_i}{\partial k_i^2}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) &= -1
\end{aligned}$$

So, for h large enough, we get:

$$\frac{\partial^2 u_i}{\partial s_i^2}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = o(1)-1 \tag{31}$$

$$\frac{\partial^2 u_i}{\partial s_i\partial k_i}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = o(1)+ak\frac{\overline{\mathbf{b^2}}}{\overline{\mathbf{b}}}=o(1)+a(\mathbf{b})k \tag{32}$$

$$\frac{\partial^2 u_i}{\partial k_i^2}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = -1 \tag{33}$$

The second-order conditions amount to checking that the principal minors of the Hessian have alternating signs. But the determinant of a matrix is a continuous (polynomial) function of the matrix entries. Given that (31) and (33), are negative, when $h \to +\infty$, we are thus left to check that the sign of the determinant is positive. This amounts to checking:

$$1 - a(\mathbf{b})^2 k^2 > 0$$

and by the second equation in (20) this is equivalent to checking:

$$1 - a(\mathbf{b})s > 0$$

But Lemma 6 shows that this necessarily holds when $2/3\sqrt{3} > a(\mathbf{b})$. ▮

Proof of Proposition 1: Let $\epsilon > 0$. Take h large enough such that Theorem 1 holds for this ϵ. We check stability by looking at the behavior of the gradient system

$$\frac{\partial s_i(t)}{\partial t} = \frac{\partial u_i(\mathbf{s}(t), \mathbf{k}(t))}{\partial s_i(t)} \tag{34}$$

$$\frac{\partial k_i(t)}{\partial t} = \frac{\partial u_i(\mathbf{s}(t), \mathbf{k}(t))}{\partial k_i(t)} \tag{35}$$

around the equilibrium points.

Let us first look at the partially corner equilibrium. By (29) we have that the first derivative with respect to s_i when h is large is

$$o(1) + \frac{a}{n^h \overline{\mathbf{s}}^{h*}} \sum_{j=1}^{n^h} k_i^{h*} k_j^{h*} s_j^{h*} - s_i^{h*}$$

Let a perturbation around the equilibrium $s^\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, with $\underline{b} = \min\{b_1, ..., b_n\}$. Then, the first derivative with respect to s_i is approximately

$$ab_i \frac{\sum_{j=1}^n \varepsilon_j b_j}{\sum_{j=1}^n \varepsilon_j} - \varepsilon_i > ab_i \underline{b} - \varepsilon_i > 0,$$

for ε_i small enough. For any small enough perturbation, s_i would tend to increase for all i, thus negating stability.

If we linearize the dynamic system (34)$-$(35) around the equilibria we get, for all $i = 1, ..., n^h$:

$$\frac{\partial s_i(t)}{\partial t} = \sum_{j=1}^{n^h} \frac{\partial^2 u_i}{\partial s_i \partial s_j}\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right)\left(s_j(t) - s_j^*\right) + \sum_{j=1}^{n^h} \frac{\partial^2 u_i}{\partial s_i \partial k_j}\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right)\left(k_j(t) - k_j^*\right) \tag{36}$$

$$\frac{\partial k_i(t)}{\partial t} = \sum_{j=1}^{n^h} \frac{\partial^2 u_i}{\partial k_i \partial k_j}\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right)\left(k_j(t) - k_j^*\right) + \sum_{j=1}^{n^h} \frac{\partial^2 u_i}{\partial k_i \partial s_j}\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right)\left(s_j(t) - s_j^*\right)$$

For $i \neq j$ we have:

$$\begin{aligned}
\frac{\partial^2 u_i}{\partial s_i \partial s_j}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) &= \frac{a}{n^h\overline{\mathbf{s}}^{h*}}k_i^{h*}k_j^{h*} - \frac{a}{(n^h\overline{\mathbf{s}}^{h*})^2}k_i^{h*}k_j^{h*}s_i^{h*} \\
&\quad + \frac{a}{(n^h\overline{\mathbf{s}}^{h*})^2}\sum_{r=1,r\neq i}^{n^h}\left(2g_{ir}\left(\mathbf{s}^{h*}\right)k_i^{h*}k_r^{h*} - k_i^{h*}k_r^{h*}s_r^{h*}\right) \\
\frac{\partial^2 u_i}{\partial s_i \partial k_j}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) &= \frac{a}{n^h\overline{\mathbf{s}}^{h*}}\left(k_i^{h*}s_j^{h*} - g_{ij}\left(\mathbf{s}^{h*}\right)k_i^{h*}\right) \\
\frac{\partial^2 u_i}{\partial k_i \partial s_j}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) &= \frac{a}{n^h\overline{\mathbf{s}}^{h*}}(k_j^{h*}s_i^{h*} - \sum_{r=1,r\neq i}^{n^h} g_{ir}\left(\mathbf{s}^{h*}\right)k_r^{h*}) \\
\frac{\partial^2 u_i}{\partial k_i \partial k_j}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) &= ag_{ij}\left(\mathbf{s}^{h*}\right)
\end{aligned}$$

Thus, we have when h gets large and for $i \neq j$:

$$\frac{\partial^2 u_i}{\partial s_i \partial s_j}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = o(1) \tag{37}$$

$$\frac{\partial^2 u_i}{\partial s_i \partial k_j}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = o(1) \tag{38}$$

$$\frac{\partial^2 u_i}{\partial k_i \partial s_j}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = o(1) \tag{39}$$

$$\frac{\partial^2 u_i}{\partial k_i \partial k_j}\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) = o(1) \tag{40}$$

The coefficients of the linearized gradient system (36) correspond to the cells of a $2n^h \times 2n^h$ matrix $\mathbf{\Pi}^h\left(\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right)\right)$ which, when h is large enough, gets arbitrarily close to the following matrix $\mathbf{\Pi}^h$:

$$\mathbf{\Pi}^h = \begin{bmatrix} \mathbf{A},\mathbf{B} \\ \mathbf{B},\mathbf{A} \end{bmatrix}$$

where $\mathbf{A},\mathbf{B}$ are the following $n^h \times n^h$ matrices

$$\mathbf{A} = \begin{bmatrix} -1,...,0 \\ 0,...,-1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} a(\mathbf{b})k,...,0 \\ 0,...,a(\mathbf{b})k \end{bmatrix},$$

Now the matrix $\mathbf{\Pi}^h$ has the following eigenvalues:

1. $\lambda_i^1 = -1 + a(\mathbf{b})k$, for $i = 1,2,...,n^h$, corresponding to the eigenvector $\nu_i = [\mathbf{i}^h,\mathbf{i}^h], i = 2,...,n^h$ where $\mathbf{i}^h$ is an $n^h \times 1$ vector containing a 1 in position $i = 1,2,...,n^h$ and 0's in the other $n^h - 1$ positions.

2. $\lambda_i^2 = -1 - a(\mathbf{b})k$, for $i = 1,2,...,n^h$, corresponding to the eigenvector $\nu_i = [\mathbf{i}^h,-\mathbf{i}^h], i = 2,...,n^h$ where $\mathbf{i}^h$ is an $n^h \times 1$ vector containing a 1 in position $i = 1,2,...,n^h$ and 0's in the other $n^h - 1$ positions.

The eigenvalues λ_i^2 are necessarily negative. The eigenvalues λ_i^1 are negative if $1 - a(\mathbf{b})k > 0$. But this is true provided that:

$$1 - a(\mathbf{b})^2 k^2 > 0$$

and by the second equation in (20) this is equivalent to checking

$$1 - a(\mathbf{b})s > 0$$

But Lemma 6 shows that this necessarily holds when $2/3\sqrt{3} > a(\mathbf{b})$. ▮

Proof of Proposition 2: Let $\epsilon > 0$. Take h large enough such that Theorem 1 holds for this ϵ. We denote by $\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right)$ and $\left(\mathbf{k}^{h**}, \mathbf{s}^{h**}\right)$ the corresponding $\epsilon-$equilibria, where $k_i^{h*} = k^* b_{\tau(i)}, s_i^{h*} = s^* b_{\tau(i)}$, and $k_i^{h**} = k^{**} b_{\tau(i)}, s_i^{h**} = s^{**} b_{\tau(i)}$, for all $i = 1, ..., n^h$, and $(k^*, s^*), (k^{**}, s^{**})$ are the two different solutions to (20). Suppose that $k_i^{h*} \geq k_i^{h**}$, for some i. Then, necessarily, $k^* \geq k^{**}$. By (20), we deduce that $s^* \geq s^{**}$. Therefore, both $k_i^{h*} \geq k_i^{h**}$ and $s_i^{h*} \geq s_i^{h**}$, for all $i = 1, ..., n^h$.

To show the welfare ranking of the $\epsilon-$equilibria, we first use the expression for payoffs in (4) and the first-order conditions for k_i^h, to obtain the following expression for $\epsilon-$equilibrium payoffs for $\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right)$:

$$u_i^{h*} = \frac{b_{\tau(i)}^2}{2}\left(k^{*2} - s^{*2}\right) + o\left(1\right), \text{ for all } i = 1, ..., n^h.$$

Next, using the fact that (k^*, s^*) are solutions to (6), we write:

$$k^{*2} - s^{*2} = \frac{s^*}{a(\mathbf{b})} - s^{*2} = \frac{s^*}{a(\mathbf{b})}(1 - a(\mathbf{b})s^*) = \frac{1}{a(\mathbf{b})}\frac{s^*}{k^*} = k^*, \tag{41}$$

and thus:

$$u_i^{h*} = \frac{b_{\tau(i)}^2}{2} k^* + o\left(1\right), \text{ for all } i = 1, ..., n^h,$$

and similarly for the $\epsilon-$equilibrium payoffs u_i^{h**} corresponding to $\left(\mathbf{k}^{h**}, \mathbf{s}^{h**}\right)$. Since, by definition $k^* \geq k^{**}$ the welfare at the equilibrium $\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right)$ is higher than at the equilibrium $\left(\mathbf{k}^{h*}, \mathbf{s}^{h**}\right)$. ▮

Proof of Propositions 3 and 4: Let $\epsilon > 0$. Take h large enough such that Theorem 1 holds for this ϵ. We denote by $\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right)$ and $\left(\mathbf{k}^{h**}, \mathbf{s}^{h**}\right)$ the corresponding $\epsilon-$equilibria, where $\left(\mathbf{k}^{h*}, \mathbf{s}^{h*}\right) \geq \left(\mathbf{k}^{h**}, \mathbf{s}^{h**}\right)$ are computed with two different solutions $(k^*, s^*) \geq (k^{**}, s^{**})$ of (20). On the (k, s) plane, an increase in $a(\mathbf{b})$ results in a downward shift of the graph of:

$$k = \frac{1}{1 - a(\mathbf{b})s}, \tag{42}$$

and an upward shift of the graph of:

$$s = a(\mathbf{b})k^2. \tag{43}$$

Therefore, the equilibrium actions of the Pareto-inferior equilibrium $\left(\mathbf{k}^{h**},\mathbf{s}^{h**}\right)$ all increase, while those of the Pareto-superior equilibrium $\left(\mathbf{k}^{h**},\mathbf{s}^{h**}\right)$ all decrease. The elasticity η that keeps track of the relative changes on s and k when $a(\mathbf{b})$ varies is:

$$\eta = \frac{s}{k}\frac{\frac{\partial k}{\partial a(\mathbf{b})}}{\frac{\partial s}{\partial a(\mathbf{b})}}.$$

Differentiating (42) and (43) with respect to $a(\mathbf{b})$ gives:

$$\begin{aligned}\frac{\partial k}{\partial a(\mathbf{b})} &= sk^2 + ak^2\frac{\partial s}{\partial a(\mathbf{b})}\\ \frac{\partial s}{\partial a(\mathbf{b})} &= k^2 + 2\frac{s}{k}\frac{\partial k}{\partial a(\mathbf{b})}\end{aligned}$$

Solving for the two partial derivatives gives:

$$\frac{\partial k}{\partial a(\mathbf{b})} = \frac{2sk^2}{1-2ask} \tag{44}$$

$$\frac{\partial s}{\partial a(\mathbf{b})} = \frac{k^2}{1-2ask}\left(2\frac{s^2}{k}+1\right) \tag{45}$$

and thus:

$$\eta = \frac{s}{k}\frac{\frac{\partial k}{\partial a(\mathbf{b})}}{\frac{\partial s}{\partial a(\mathbf{b})}} = \frac{2s^2}{k+2s^2} < 1.$$

∎

Proof of Proposition 5: Let $\epsilon > 0$. Take h large enough such that Theorem 1 holds for this ϵ. We denote by $\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right)$ and $\left(\mathbf{k}^{h**},\mathbf{s}^{h**}\right)$ the corresponding $\epsilon-$equilibria, where $\left(\mathbf{k}^{h*},\mathbf{s}^{h*}\right) \geq \left(\mathbf{k}^{h**},\mathbf{s}^{h**}\right)$ are computed with two different solutions $(k^*,s^*) \geq (k^{**},s^{**})$ of (20).

1. From equation (41) in the proof of Proposition 2, we have

$$k^{*2} - s^{*2} = \frac{s^*}{a(\mathbf{b})} - s^{*2} = \frac{s^*}{a(\mathbf{b})}(1-a(\mathbf{b})s^*) = \frac{1}{a(\mathbf{b})}\frac{s^*}{k^*} = k^*,$$

   and thus:

$$u_i^{h*} = \frac{b_{\tau(i)}^2}{2}k^* + o(1) = \frac{b_{\tau(i)}^2}{2a(\mathbf{b})}\frac{s^*}{k^*} + o(1), \qquad \text{for all } i = 1,...,n^h,$$

   and similarly for the $\epsilon-$equilibrium payoffs u_i^{h**} corresponding to $\left(\mathbf{k}^{h**},\mathbf{s}^{h**}\right)$.

   When only a increases, while all the b_t's remain constant, the comparative statics of equilibrium payoffs are those of productive actions. Since we know from equation (44) that $\frac{\partial k}{\partial a(\mathbf{b})} = \frac{2sk^2}{1-2ask}$, the result follows in that case.

Let us then turn to the case when we multiply a and each $b_\tau, \tau \in T$ by a common factor $\delta \geq 1$. Then, $a(\mathbf{b})$ increases to $\delta^2 a(\mathbf{b})$, while $2a(\mathbf{b})/b_\tau^2$ remains unchanged. Therefore, individual $\epsilon-$equilibrium payoffs move in the same direction than s^*/k^* and s^{**}/k^{**}. We know from the proof of Proposition 3 that, following an increase in $a(\mathbf{b})$, the equilibrium actions increase at the Pareto-inferior equilibrium $\left(\mathbf{k}^{h**}, \mathbf{s}^{h**}\right)$ and decrease at the Pareto-superior equilibrium. Graphically, one can immediately conclude that s^*/k^* decreases while s^{**}/k^{**} increases, and the result follows in that case. A more formal argument is the following. It is readily checked that

$$\left.\frac{\partial s/k}{\partial \delta}\right|_{\delta=1} > 0 \Leftrightarrow k \left.\frac{\partial s}{\partial \delta}\right|_{\delta=1} > s \left.\frac{\partial s}{\partial \delta}\right|_{\delta=1}. \tag{46}$$

We know that, at the Pareto-inferior equilibrium, both $\frac{\partial k^{**}}{\partial \delta}\big|_{\delta=1} > 0$ and $\frac{\partial s^{**}}{\partial \delta}\big|_{\delta=1} > 0$. Therefore, $\left.\frac{\partial s^{**}/k^{**}}{\partial \delta}\right|_{\delta=1} > 0$ is equivalent to $\eta < 1$, which is true from Proposition 3. Instead, at the Pareto-superior equilibrium we have both $\frac{\partial k^{*}}{\partial \delta}\big|_{\delta=1} > 0$ and $\frac{\partial s^{*}}{\partial \delta}\big|_{\delta=1} > 0$, so that $\left.\frac{\partial s^{*}/k^{*}}{\partial \delta}\right|_{\delta=1} < 0$ is equivalent to $\eta < 1$ which, again, follows from Proposition 3.

2. From equation (9) we have that

$$\begin{aligned}\sum_{i=1}^{mt} u_i &\simeq \sum_{i=1}^{mt} \frac{b_{\tau(i)}^2}{2a(\mathbf{b})} \frac{s}{k} \\ &= \frac{m}{a} \frac{s}{k} \sum_{\tau=1}^{t} b_\tau\end{aligned}$$

Now notice that $a(\mathbf{b})$ increases by assumption in our case. Thus $\sum_{i=1}^{mt} u_i$ decreases for the high equilibrium since s^*/k^* increases and $\sum_{\tau=1}^{t} b_\tau$ is constant. And $\sum_{i=1}^{mt} u_i$ increases for the low equilibrium since s^{**}/k^{**} increases and $\sum_{\tau=1}^{t} b_\tau$ is constant. Now let and individual i with lower than average $b_{\tau(i)}$. We know that at the low equilibrium s/k decreases. Also, $\frac{1}{2a(\mathbf{b})}$ decreases (since .$\sum_{\tau=1}^{t} b_\tau$ and .$\sum_{\tau=1}^{t} b_\tau^2$ increases). And, since his type is lower than average, $b_{\tau(i)}^2$ must also decrease. Thus,.$\left(b_{\tau(i)}^2/2a(\mathbf{b})\right)$ (s/k) decreases and the result follows. ▮

Proof of Remark 3: The result is immediate from equation (10) and from equation (44) in the proof of Proposition 3. ▮

Proof of Theorem 2: The proof follows *ceteris paribus* from that of Theorem 1.

First, we rewrite Lemma 5 by simply taking to the power of c the left-hand side terms in the first-order conditions (21) and (22), and the approximated first-order conditions (23) and (24). Then, (25) and (26) yields (15).

Second, rewriting equation (27) in Lemma 6 we get:

$$g_c(s) = \frac{a^{\frac{1}{c}} b^{\frac{2}{c^2}}}{(1-as)^{\frac{2}{c^2}}},$$

so that the solution to (28) is such that $(1-ax^*)^{\frac{2}{c^2}+1} = \frac{2}{c^2} a^{1+\frac{1}{c}} b^{\frac{2}{c^2}}$, that is, $ax^* = 1 - \left(\frac{2}{c^2} a^{1+\frac{1}{c}} b^{\frac{2}{c^2}}\right)^{\frac{c^2}{2+c^2}}$. The equilibrium conditions $g_c(x^*) < x^*$ then boils down to $a^{1+\frac{1}{c}} b^{\frac{2}{c^2}} < \frac{c^2}{2+c^2} \left(\frac{2}{2+c^2}\right)^{\frac{2}{c^2}}$ after some simple algebra. ▮

Proof of Proposition 6: Let's multiply both a and b by a common factor $\delta \geq 1$. Equations (15) become:

$$\begin{cases} s^c = \delta a k^2 \\ k^c [1 - \delta a s] = \delta b \end{cases}.$$

Differentiating with respect to δ and letting $\delta = 1$ gives:

$$\begin{aligned} c s^{c-1} \left.\frac{\partial s}{\partial \delta}\right|_{\delta=1} &= a k^2 + 2ak \left.\frac{\partial k}{\partial \delta}\right|_{\delta=1} \\ c k^{c-1} \left.\frac{\partial k}{\partial \delta}\right|_{\delta=1} (1-as) &= b + k^c \left[as + a \left.\frac{\partial s}{\partial \delta}\right|_{\delta=1} \right] \end{aligned}$$

Simplifying gives:

$$\begin{aligned} c \left.\frac{\partial s}{\partial \delta}\right|_{\delta=1} - 2\frac{s}{k} \left.\frac{\partial k}{\partial \delta}\right|_{\delta=1} &= s \\ -ak \left.\frac{\partial s}{\partial \delta}\right|_{\delta=1} + c(1-as) \left.\frac{\partial k}{\partial \delta}\right|_{\delta=1} &= k \end{aligned}$$

Finally, solving for the two partial derivatives yields to

$$\begin{aligned} \left.\frac{\partial s}{\partial \delta}\right|_{\delta=1} &= \frac{2+c}{c^2(1-as) - 2as} s \\ \left.\frac{\partial k}{\partial \delta}\right|_{\delta=1} &= \frac{as + c(1-as)}{c^2(1-as) - 2as} k \end{aligned} \tag{47}$$

Therefore, both partial derivatives are of the same sign, which is positive if and only if $as < c^2/(2+c^2)$. In the same spirit of the proof of Proposition 5, one can check graphically that the equilibrium actions of the low-actions equilibrium increase with δ whereas the equilibrium actions of the high-actions equilibrium decrease with δ. Then, noting from (46) that the elasticity is smaller than one whenever the slope s/k increases, s and k either all increase or all decrease, the result follows. ▮

Proof of Proposition 7: At an interior equilibrium, the strategies (k, s) solve:

$$\begin{cases} s^c = a k^2 \\ k^c [1 - as] = b \end{cases},$$

The first equation is:

$$s = f(k) = a^{1/c} k^{2/c}.$$

It is readily checked that $f(\cdot)$ is increasing, strictly concave (resp. convex) when $c < 2$ (resp. $c > 2$) and a straight line when $c = 2$ on $[0, +\infty)$. Also, $f(0) = 0$ and $\lim_{x \uparrow +\infty} f(x) = +\infty$. Finally, note that:

$$k = f^{-1}(s) = \left(\frac{s}{a^{1/c}}\right)^{c/2}.$$

The second equation is:

$$k = g(s) = b^{1/c} (1 - as)^{-1/c}.$$

It is readily checked that $g(\cdot)$ is increasing and strictly convex on $[0, 1/a)$, with $g(0) = b^{1/c}$ and $\lim_{x \uparrow 1/ab} g(x) = +\infty$.

Suppose that the equilibrium existence conditions hold. Let $(\underline{k}, \underline{s})$ and $(\overline{s}, \overline{k})$ be the two different equilibria, where $\overline{s} > \underline{k}$ (and, thus, $\overline{k} > \underline{s}$). The two corresponding equilibrium networks are such that only one has a giant component if and only if $\overline{s} > 1 > \underline{k}$.

Let $a < 1$, so that we have a random graph.

Note that the graph of $g(\cdot)$ lies below the graph of $f^{-1}(\cdot)$ only when $s \in [\underline{k}, \overline{s}]$. Therefore, $\overline{s} > 1 > \underline{k}$ is equivalent to $g(1) < f^{-1}(1)$. After some algebra, this is equivalent to:

$$ab^{2/c} < (1 - a)^{2/c}. \tag{48}$$

Notice that the right-hand side of this inequality is well-defined given our assumption that $a < 1$.

When (48) holds, we can conclude that the two equilibrium networks display two different topological characteristics: the densely connected network has a giant component whereas the sparsely connected network doesn't.

Reciprocally, when (48) does not hold, we can conclude that either both equilibrium networks have a giant component, or none does. We now provide an additional sufficient condition such that both have giant component when both $a < 1$ and $ab^{2/c} > (1 - a)^{2/c}$.

Consider the line tangent to the graph of $g(\cdot)$ at the point $(0, g(0))$ with equation:

$$h(s) = g'(0)\, s + g(0) = b^{1/c} \left[\frac{a}{c} s + 1\right].$$

The graph of $g(\cdot)$ lies above that of $h(\cdot)$ on $[0, 1/a)$ (recall that $g(\cdot)$ is strictly convex on that half-segment). Consider the region on the (s, k) space delimited to the left by the vertical axis, from above by the graph of $g(\cdot)$, from below by the graph of $f^{-1}(\cdot)$ and to the right by the point $(\underline{k}, \underline{s})$ at the intersection of these two graphs. If the point $(1, h(1))$ lies in this region, then necessarily $1 < \underline{k}$. Analytically, $(1, h(1))$ lies in this region if and only if $h(1) > f^{-1}(1)$. After some algebra, this is equivalent to:

$$ab^{2/c} > (\frac{c}{a + c})^2.$$

When $a < 1$, we can thus conclude that both networks have a giant component when:

$$ab^{2/c} > \max\{(1-a)^{2/c}, (\frac{c}{a+c})^2\}.$$

We now compare the two terms on the right-hand side of the inequality. Let:

$$\xi(x) = \frac{1}{1+ax} \qquad \text{and} \qquad \zeta(x) = (1-a)^x \text{, where } x \in [0,1].$$

Then, $\left(\frac{c}{a+c}\right)^2 > (1-a)^{2/c}$ for some $c \geq 1$ if and only if $\xi(1/c) > \zeta(1/c)$.

Note that $\xi(0) = \zeta(0) = 1$, $\xi(1) = 1/(1+a) \geq \zeta(1) = 1-a$ (with a strict inequality when $a \neq 0$), and $\xi'(0) = -a > \zeta'(0) = \log(1-a)$. Given the strict convexity of both functions on $[0,1]$, we can conclude that $\xi(x) > \zeta(x)$ on $0 < x < 1$. ∎

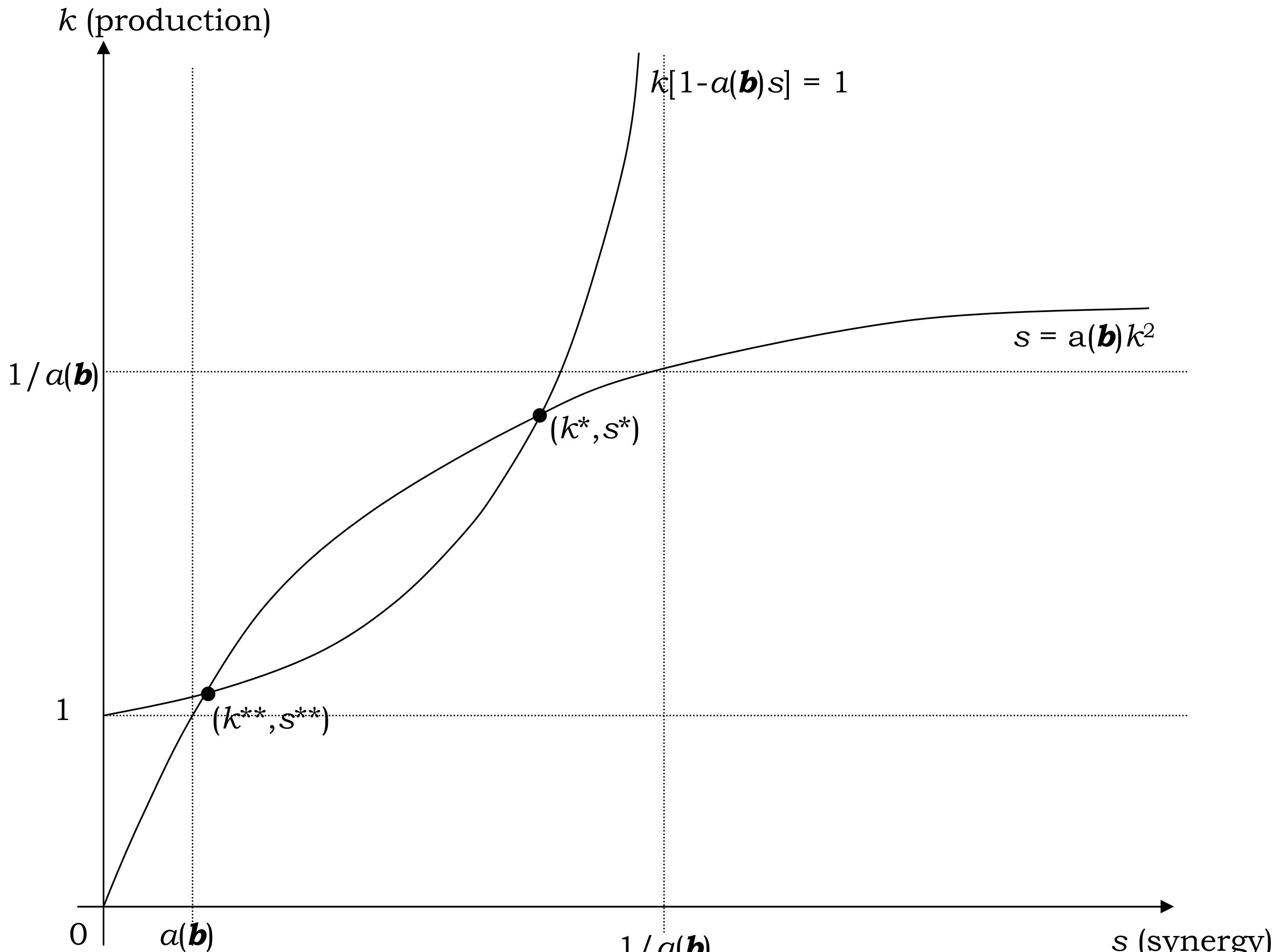


Figure 1

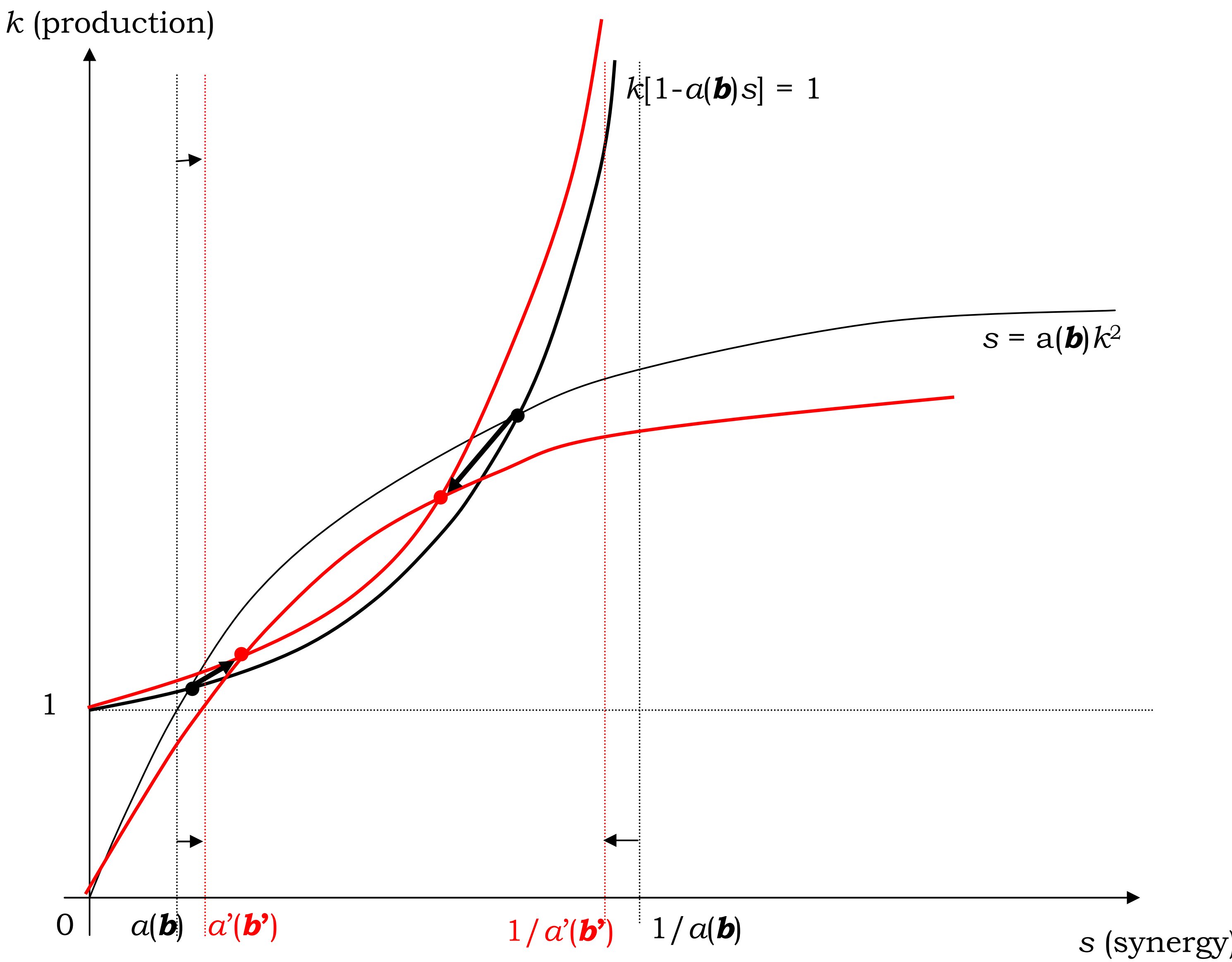
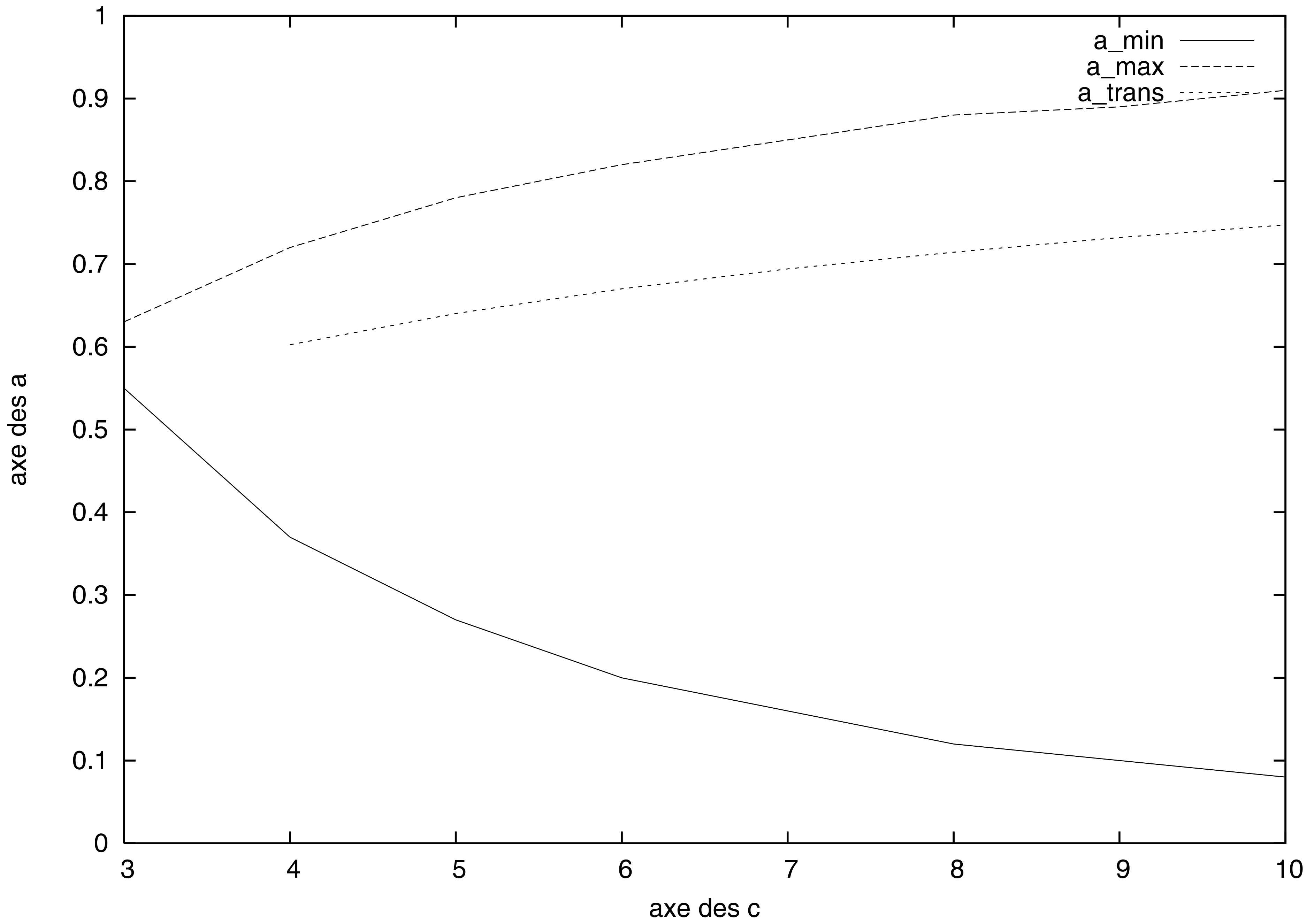


Figure 2


a_min
a_max
a_trans
axe des a
axe des c
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
3
4
5
6
7
8
9
10