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Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions[1]

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Abstract

We consider private value auctions where bidders' types are dependent, a case usually treated by assuming affiliation. We show that affiliation is a restrictive assumption in three senses: topological, measure-theoretic and statistical (affiliation is a very restrictive characterization of positive dependence). We also show that affiliation's main implications do not generalize for alternative definitions of positive dependence. From this, we propose new approaches to the problems of pure strategy equilibrium existence in first-price auctions (PSEE) and the characterization of the revenue ranking of auctions. For equilibrium existence, we slightly restrict the set of distributions considered, without loss of economic generality, and offer a complete characterization of PSEE. For revenue ranking, we obtain a characterization of the expected revenue differences between second and first price auctions with general dependence of types.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: affiliation, dependence of types, auctions, pure strategy equilibrium, revenue ranking.

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1 Introduction

What auction format gives higher expected revenue when bidders' types are dependent? This is an important question, set in a realistic context. Dependence of private information is more realistic because there are many sources of correlation of opinions and assessments in the real world: education, culture, evolution, common sources of information, etc. Moreover, independence is a "knife-edge" assumption. The importance of the revenue ranking of auctions can be seen by realizing that it is a practical version of the main problem of mechanism design — what is the best mechanism for a decision maker (in this case, the seller)? We say that it is a "practical version" because it differs from the general one in restricting the set of considered mechanisms to the auctions most observed in practice (English auction and first price auctions).

In fact, the revenue ranking of auctions is a problem as old as auction theory itself. Among the many questions addressed by Vickrey (1961), this was one of most centrality. His celebrated Revenue Equivalence Theorem (RET) says that the two above cited auctions (and his second price auction) give exactly the same expected revenue. Nevertheless, RET requires independence of private information (types). Thus, the revenue ranking of auctions with dependent types remained unknown.

The best contribution to this matter was made through a remarkable insight of Milgrom and Weber (1982a), which introduced the concept of affiliation in auction theory.[1]

Affiliation is a generalization of independence — see the definition in subsection 3.2 below — that was explained through the appealing notion that our assessments of values are positively dependent: "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely." (Milgrom and Weber (1982a), p. 1096.)

Among the many interesting results in Milgrom and Weber (1982a), we underline the following:

- affiliation implies the existence of a symmetric, increasing, pure strategy equilibrium for first price auctions;[2]
- under affiliation, the English and the second price auction have higher expected revenue than the first price auction.[3]

[1]In two previous papers, Paul Milgrom presented results that use a particular version of the same concept, under the name "monotone likelihood ratio property" (MLRP): Milgrom (1981a, 1981b). Before Milgrom's works, Wilson (1969 and 1977) made important contributions that may be considered the foundations of the affiliated value model. Nevertheless, the concept is fully developed and the term affiliation first appears in Milgrom and Weber (1982a). See also Milgrom and Weber (1982b). When there is a density function, the property had been previously studied by statisticians under different names. Lehmann (1966) calls it Positive Likelihood Ratio Dependence (PLRD), Karlin (1968) calls it Total Positivity of order 2 (TP-2), and others use the name Monotone Likelihood Ratio Property (for the bivariate case).

[2]They also proved the existence of equilibrium for second price auctions with interdependent values. In our set-up (private values), the second price auction always has an equilibrium in weakly dominant pure strategies, which simply consists of bidding the private value. Although equilibria in mixed strategies always exist (Jackson and Swinkels 2005), first price auctions may fail to possess a pure strategy equilibrium when types are dependent.

[3]For private value auctions, which we consider in this paper, English and second price auctions are equivalent — see Milgrom and Weber (1982a). Thus, we restrict our analysis to the latter.

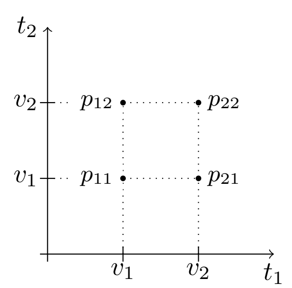
In the face of these results, it is possible to cite at least three reasons for the profound influence of the aforementioned paper in auction theory: (i) its theoretical depth and elegance; (ii) the plausibility of the hypothesis of affiliation, as explained by a clear economic intuition; (iii) the fact that it implies that English auctions are better for sellers, which is a good explanation for the fact that English auctions are more common than first price auctions.

Nevertheless, since we were originally interested in the general problem of dependence of types, it is important to have an assessment of how strong as assumption affiliation is and how far its implications go.

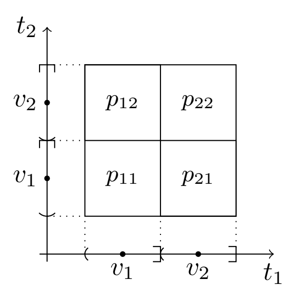
In section 3 we show that affiliation is quite restrictive in three senses. It is restrictive in a *topological* sense: the set of no affiliated probability density functions (p.d.f.'s) is open and dense in the set of continuous p.d.f.'s. It is also restrictive in a *measure-theoretic* sense: if μ is a probability measure over the set of joint probabilistic density functions (p.d.f.'s), and if μ satisfies some reasonable conditions, then then μ puts zero measure in the set of affiliated p.d.f.'s.

Finally, we argue that affiliation is restrictive in a *statistical* sense. Recall that affiliation was introduced through the intuition of positive dependence. In Statistics, there are many definitions of positive dependence, and affiliation is just one of the most restrictive. Moreover, in sections 4 and 5 we show that the main implications of affiliation — equilibrium existence and the revenue ranking of auctions — do not extend for other (still restrictive) definitions of positive dependence.

From this, it seems important to reconsider the problems of equilibrium existence and revenue ranking of auctions with dependent types. For solving the problem of symmetric increasing pure strategy equilibrium existence, we restrict the set of distributions considered. The idea is so simple that it can be explained graphically (see Figure 1).



(a) $p_{ij} = \Pr(t_1 = v_i, t_2 = v_j)$



(b) $p_{ij} = f(v_i, v_j)$

Figure 1: Discrete values, such as in (a), capture the relevant economic possibilities in a private value model, but preclude the use of calculus. We use continuous variables, but consider only simple density functions (constant in squares), such as in (b).

Let us expand the above explanation. Consider the setting of symmetric private value auctions with two risk neutral players, but general dependence of types. Since we are analyzing auctions of single objects, it would be sufficient to consider the case where bidders' types are distributed according to a finite number of values (the values can be specified only up to cents and are obviously bounded). Nevertheless, to work with discrete values precludes us from using the convenient tools of differential calculus, which allow, for instance, a complete characterization of equilibrium strategies. Maintaining the advantage of continuum variables, but without requiring unnecessary richness in the set of distributions, we focus on the set of densities which are constant in some squares around fixed values. This imposes no economic restriction on the cases considered, but allows a complete characterization of symmetric increasing pure strategy equilibrium (PSE) existence (see subsection 4.2).

It is easy to see that, as we take arbitrarily small squares, we can approximate any p.d.f. (including non-continuous ones). Thus, even if the reader insists on mathematical generality, that is, to include other distributions, our results are still meaningful because they cover a dense set.

For this set of simple p.d.f.'s, we are able to provide an algorithm, implementable by a computer, that completely characterizes whether or not a pure strategy equilibrium exists. Theoretical results are also available.

The results in section 4 reveal that the set of affiliated distribution is small even in the set of distributions with a PSE, sharpening the results of restrictiveness of affiliation. On the other hand, the proportion of p.d.f.'s which have PSE is also small. We offer a mathematical proof of this fact. This suggests that most of the cases only have equilibria in mixed strategies.

From this, we consider the characterization of the revenue ranking of auctions. The standard approach in the literature (see Milgrom and Weber 1982 for affiliated distributions and Maskin and Riley 2000 for asymmetrical independent distributions) is to give conditions on the set of distributions that imply such or such ranking. Unfortunately, the conditions are usually very restrictive.

Numerical simulations, made possible by the results of sections 4 and 5, suggest that a complete characterization is very difficult. Even p.d.f.'s with positive dependence may often present a revenue ranking contrary to that implied by affiliation.

From this, we see two options for approaching the problem of revenue ranking. One is to obtain (experimental or empirical) information related to the specific situation and restrict the set of p.d.f.'s to analyze. Then, with this restriction, use our model to run simulations and determine what auction format gives higher expected revenue in the specific environment. Although this "engineering"-type approach was recently proposed by Roth (2002) as a tool for economists, it will be useful to also have a general, theoretical model. This is the other part of the proposal.

We construct a "natural" measure over our proposed set of p.d.f.'s — this is natural in the sense that it comes from the limit of Lebesgue measure over finite-dimensional sets and also has some (partial) characteristics of Lebesgue measure. Using this measure, we obtain an "expected value" of the expected revenue difference of the second-price and the first-price auction.

We illustrate this theoretical (and specification-free) method in section 5. This method suggests that the English auction gives higher revenue than the first price auction "on average". Surprisingly, this conclusion seems to hold in general, not only for positive dependent distributions.[4]

[4]The reader should not be confused: the conclusion is in a weak sense: just "on average". For specific

Thus, our paper makes the following contributions: it illustrates the restrictiveness of affiliation; proposes a convenient and sufficiently general set of distributions; offers a method to numerically test equilibrium existence and the revenue ranking of auctions for non-affiliated distributions; and shows that the revenue ranking implied by affiliation is valid, in a weak sense, for a bigger set of distributions.

The paper is organized as follows. Section 2 gives a brief exposition of the standard auction model. Section 3 compares affiliation and other definitions of positive dependence and shows that affiliation is a restrictive condition. Section 4 presents the equilibrium existence results. Section 5 describes the proposed methods for approaching the problem of revenue ranking in auctions. Section 6 contains a comparison with related literature and concluding remarks. The more important and short proofs are given in an appendix, while lengthy constructions are presented in a separate supplement to this paper.

2 Basic model and definitions

Our model and notations are standard. There are n bidders, $i = 1, ..., n$. Bidder i receives private information $t_i \in \left[\underline{t}, \overline{t}\right]$ which is the value of the object for himself. The usual notation $t = (t_i, t_{-i}) = (t_1, ..., t_n) \in \left[\underline{t}, \overline{t}\right]^n$ is adopted. The values are distributed according to a p.d.f. $f : \left[\underline{t}, \overline{t}\right]^n \to \mathbb{R}_+$ which is symmetric, that is, if $\pi : \{1, ..., n\} \to \{1, ..., n\}$ is a permutation, $f(t_1, ..., t_n) = f\left(t_{\pi(1)}, ..., t_{\pi(n)}\right)$.[5] Let $\overline{f}(x) = \int f(x, t_{-i})\, dt_{-i}$ be a marginal of f. Our main interest is the case where f *is not* the product of its marginals, that is, the case where the types are dependent. We denote by $f(t_{-i} \mid t_i)$ the conditional density $f(t_i, t_{-i}) / \overline{f}(t_i)$. After knowing his value, bidder i places a bid $b_i \in \mathbb{R}_+$. He receives the object if $b_i > \max_{j \neq i} b_j$. We consider both first and second price auctions. As Milgrom and Weber (1982a) argue, second price and English auctions are equivalent in the case of private values, as we assume here. In a first price auction, if $b_i > \max_{j \neq i} b_j$, bidder i's utility is $u(t_i - b_i)$ and is $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. In a second price auction, bidder i's utility is $u(t_i - \max_{j \neq i} b_j)$ if $b_i > \max_{j \neq i} b_j$ and $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. For both auctions, ties are randomly broken.

By reparametrization, we may assume, without loss of generality, $\left[\underline{t}, \overline{t}\right] = [0, 1]$. It is also useful to assume $n = 2$, but this is not needed for most of the results, although some (especially in sections 4 and 5) may require non-trivial adaptations for $n > 2$. For most of the paper, we assume risk neutrality, that is, $u(x) = x$. Thus, unless otherwise stated, the results will be presented under the following set-up:

BASIC SETUP: *There are $n = 2$ risk neutrals bidders, that is, $u(x) = x$, with private values distributed according to a symmetric density function $f : [0, 1]^2 \to \mathbb{R}_+$.*

A pure strategy is a function $b : [0, 1] \to \mathbb{R}_+$, which specifies the bid $b(t_i)$ for each type t_i. The interim payoff of bidder i, who bids β when his opponents follow

distributions, as we said, the rank can be in any direction.

[5]For the reader familiar with Mertens and Zamir (1986)'s construction of universal type spaces: we make the usual assumption in auction theory that the model is "closed" at the first level, that is, all higher level beliefs are consistently given by (and collapse to) f.

$b : [0,1] \to \mathbb{R}_+$ is given by

$$\Pi_i\left(t_i, \beta, b\right) = u\left(t_i - \beta\right) F\left(b^{-1}\left(\beta\right) \mid t_i\right) = u\left(t_i - \beta\right) \int_{\underline{t}}^{b^{-1}(\beta)} f\left(t_{-i} \mid t_i\right) dt_{-i},$$

if it is a first price auction and

$$\Pi_i\left(t_i, \beta, b\right) = \int_{\underline{t}}^{b^{-1}(\beta)} u\left(t_i - b\left(t_{-i}\right)\right) f\left(t_{-i} \mid t_i\right) dt_{-i},$$

if it is a second price auction.

We focus attention on symmetric increasing pure strategy equilibrium (PSE), which is defined as a function $b : [0,1] \to \mathbb{R}_+$ such that $\Pi_i\left(t_i, b\left(t_i\right), b\right) \geq \Pi_i\left(t_i, \beta, b\right)$ for all β and t_i. The usual definition requires this inequality to be true only for almost all t_i. This stronger definition creates no problem and makes some statements simpler, as those about the differentiability and continuity of the equilibrium bidding function (otherwise, such properties should be always qualified by the expression "almost everywhere"). Finally, under our assumptions, the second price auction always has a PSE in a weakly dominant strategy, which is $b\left(t_i\right) = t_i$.

3 Restrictiveness of affiliation

3.1 Why dependent types?

Dependence of types may arise through many channels, such as culture, education, common sources of information, evolution, etc. It is even possible to give a more structured economic model for this, as follows.

Assume, for instance, that the object has an intrinsic value v, which is unknown and modeled as a random variable. The bidders' valuation is given by this intrinsic value plus an idiosyncratic valuation ε_i, assumed to be independent across bidders. Thus, bidder i's type (value) is $t_i = v + \varepsilon_i$. This obviously makes the types dependent (although is not yet sufficient to imply that types are affiliated — see subsection 3.6).

A generalization of this economic model is to assume that there is a random variable v and that the idiosyncratic values (types) are conditionally independent, given v (we also consider this model in more detail in subsection 3.6, where we show that it does not imply affiliation).

This discussion shows that there are meaningful cases of dependence out of affiliation, even for particular economic examples. Nevertheless, affiliation is widely used in auction theory. The main reason for this seems to be its very appealing intuition, based on positive dependence, as we discuss next.

3.2 Definition and intuition for affiliation

In auction theory, it is usual to assume that f is affiliated. The introduction of the affiliation concept was made through an appealing economic intuition: "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely." Milgrom and Weber (1982a), p. 1096.

The formal definition is illustrated by Figure 2.[6]

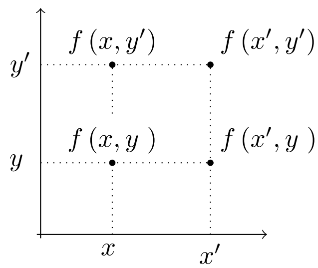


Figure 2: The p.d.f. f is affiliated if $x \leq x'$ and $y \leq y'$ imply $f(x, y') f(x', y) \leq f(x', y') f(x, y)$.

Affiliation requires that the product of the weights at the points (x', y') and (x, y) (where both values are high or both are low) be greater than the product of weights at (x, y') and (x', y) (where they are high and low, alternatively). It is clear from this definition that affiliation captures very well the notion of positive dependence.

In fact, there is a predominant view in auction theory that understands affiliation as a suitable synonym of positive dependence. This can be seen by the intuitions normally given to affiliation, along the same lines as the above quote. One can say that the literature seems to mix two different ideas that, for expositional ease, we would like to state separately: (1) positive dependence is a sensible assumption (an idea that we call *positive dependence intuition*); and (2) affiliation is a suitable mathematical definition of positive dependence (an idea that, for easier future reference, we call *rough identification*).

The positive dependence intuition seems very reasonable because, as we said, many mechanisms may lead to correlated assessments of values. Nevertheless, we argue that the rough identification is misleading because affiliation is too strong to be a suitable definition of positive dependence. In the following subsection, we present some theoretical concepts that also correspond to positive dependence and are strictly weaker than affiliation.

In subsections 3.4 and 3.5 we show that affiliation is also restrictive in topological and measure-theoretic senses.

Since these results do not confirm the usual understanding (rough identification), it is useful to reassess other arguments and models that lead to affiliation. Subsection 3.6 considers the conditional independence model. Subsection 3.7 discusses the use of affiliation in other sciences.

[6]For n players, we say that the density function $f : [\underline{t}, \overline{t}]^n \rightarrow \mathbb{R}_+$ is affiliated if $f(t) f(t') \leqslant f(t \wedge t') f(t \vee t')$, where $t \wedge t' = \left(\min\{t_i, t'_i\}\right)_{i=1}^n$ and $t \vee t' = \left(\max\{t_i, t'_i\}\right)_{i=1}^n$. It is possible to give a definition of affiliation for distributions without density functions. See Milgrom and Weber (1982a). We will assume the existence of a density function and use only this definition.

3.3 The relation between affiliation and positive dependence

In the statistical literature, various concepts were proposed to correspond to the notion of positive dependence. Let us consider the bivariate case, and assume that the two real random variables X and Y have joint distribution F and strictly positive density function f. The following concepts are formalizations of the notion of positive dependence:[7]

Property I - X and Y are positively correlated (PC) if $cov(X,Y) \geqslant 0$.

Property II - X and Y are said to be positively quadrant dependent (PQD) if $cov(g(X), h(Y)) \geqslant 0$, for all non-decreasing functions g and h.

Property III - The real random variables X and Y are said to be associated (As) if $cov(g(X,Y), h(X,Y)) \geqslant 0$, for all non-decreasing functions g and h.

Property IV - Y is said to be left-tail decreasing in X (denoted LTD($Y|X$)) if $\Pr[Y \leqslant y|X \leqslant x]$ is non-increasing in x for all y. X and Y satisfy property IV if LTD($Y|X$) and LTD($X|Y$).

Property V - Y is said to be positively regression dependent on X (denoted PRD($Y|X$)) if $\Pr[Y \leqslant y|X = x] = F(y|x)$ is non-increasing in x for all y. X and Y satisfy property V if PRD($Y|X$) and PRD($X|Y$).

Property VI - Y is said to be Inverse Hazard Rate Decreasing in X (denoted IHRD($Y|X$)) if $\frac{F(y|x)}{f(y|x)}$ is non-increasing in x for all y, where $f(y|x)$ is the p.d.f. of Y conditional to X. X and Y satisfy property VI if IHRD($Y|X$) and IHRD($X|Y$).

We have the following:

Theorem 1 *Let affiliation be Property VII. Then, the above properties are successively stronger, that is,*

$$(VII) \Rightarrow (VI) \Rightarrow (V) \Rightarrow (IV) \Rightarrow (III) \Rightarrow (II) \Rightarrow (I)$$

and all implications are strict.
Proof. *See the appendix.* ■

This theorem illustrates how strong affiliation is.[8] Some implications of Theorem 1 are trivial and most of them were previously established. Our contribution regards

[7] Most of the concepts can be properly generalized to multivariate distributions. See, e.g., Lehmann (1966) and Esary, Proschan and Walkup (1967). The hypothesis of strictly positive density function is made only for simplicity.

[8] We defined only seven concepts for simplicity. Yanagimoto (1972) defines more than thirty concepts of positive dependence and, again, affiliation is the most restrictive of all, but one.

Property VI, that we use later to prove convenient generalizations of equilibrium existence and revenue rank results. We prove that Property VI is strictly weaker than affiliation and is sufficient for, but not equivalent to Property V.

Although Theorem 1 says that affiliation is mathematically restrictive, it would be possible for affiliation to be satisfied in most of the cases with positive correlation. That is, although there are counterexamples for each of the implications above, such counterexamples could be pathologies and affiliation could be true in many cases where positive correlation (property I) holds. Thus, one should evaluate how typical affiliation is.

There are two ways to assess whether a set is typical or not: topological and measure-theoretic. We consider both in the sequel.

3.4 Affiliation is restrictive in the topological sense

We adopt the following notation. Let $\mathcal{C}$ denote the set of continuous density functions $f : [0,1]^2 \to \mathbb{R}_+$ and let $\mathcal{A}$ be the set of affiliated densities. For convenience and consistency with the notation in next sections, we are including in $\mathcal{A}$ all affiliated densities and not only the continuous one, which creates no problem.

Endow $\mathcal{C}$ with the topology of the uniform convergence, that is, the topology defined by the norm of the sup:

$$\|f\| = \sup_{x\in[0,1]^2} |f(x)| .$$

The following theorem shows that the set of continuous affiliated densities is small in the topological sense.

Theorem 2 *The set of continuous affiliated density function $\mathcal{C} \cap \mathcal{A}$ is meager. More precisely, the set $\mathcal{C}\backslash\mathcal{A}$ is open and dense in $\mathcal{C}$.*
Proof. *See the appendix.* ■

In fact, the theorem says more than the set of continuous affiliated density functions is a meager set. A meager set (or set of first category) is the union of countably many nowhere dense sets, which are sets whose closure has empty interior. $\mathcal{C} \cap \mathcal{A}$ is itself a nowhere dense set, by the second claim in the theorem.

The proof of this theorem is given in the appendix, but is not difficult to understand. To prove that $\mathcal{C}\backslash\mathcal{A}$ is open, we take a p.d.f. $f \in \mathcal{C}\backslash\mathcal{A}$ which does not satisfy the affiliated inequality for some points $t, t' \in [0,1]^2$, that is, $f(t) f(t') > f(t \wedge t') f(t \vee t') + \eta$, for some $\eta > 0$. By using such η, we can show that for a function g sufficiently close to f, the above inequality is still valid, that is, $g(t) g(t') > g(t \wedge t') g(t \vee t')$ and, thus, is not affiliated. To prove that $\mathcal{C}\backslash\mathcal{A}$ is dense, we choose a small neighborhood V of a point $\hat{t} \in [0,1]^2$, such that for all $t \in V$, $f(t)$ is sufficiently close to $f(\hat{t})$ — this can be done because f is continuous. Then, we perturb the function in this neighborhood to get a failure of the affiliation inequality.

Maybe more instructive than the proof is to understand why the result is true: simply, affiliation requires an inequality to be satisfied everywhere (or almost everywhere). This is a strong requirement and it is the source of its restrictiveness.

3.5 Affiliation is restrictive in a measure-theoretic sense

Affiliation is also restrictive in the measure-theoretic sense, that is, in an informal way, it is of "zero measure". Obviously, we need to be careful with the formalization of this, since we are now dealing with measures over infinite-dimensional sets (the set of distributions or densities). As is well known, there are no "natural" measures for infinite dimension sets, that is, measures with all of the properties of the Lebesgue measure — see Yasamaki (1985), Theorem 5.3, p. 139.

Thus, before we formalize our results, we informally explain what we mean by "measure-theoretic". Let $\mathcal{D}$ be the set of probabilistic density functions (p.d.f.'s) $f : [0,1]^2 \to \mathbb{R}_+$ and assume that there is a measure μ over it. We define below a sequence $\mathcal{D}^k$ of finite-dimensional subspaces of $\mathcal{D}$ and take the measures μ^k over $\mathcal{D}^k$ induced by the projection of $\mathcal{D}$ over $\mathcal{D}^k$. The result is as follows: if μ^k is absolutely continuous with respect to the Lebesgue measure λ^k over $\mathcal{D}^k$ — as seems reasonable — then μ puts zero measure on the set $\mathcal{A}$ of affiliated p.d.f.'s.

Remark. There is an alternative method of characterizing smallness in the measure-theoretic sense: to show that the set is *shy*, as defined by Anderson and Zame (2001), generalizing a definition of Christensen (1974) and Hunt, Sauer and Yorke (1992). We discuss it in the supplement to this paper.

Now, we formalize our method. Endow $\mathcal{D}$ with the L^1-norm, that is, $\|f\|_1 = \int |f(t)|\, dt$. When there is no peril of confusion with the sup norm previously defined, we write $\|f\|$ for $\|f\|_1$.

For $k \geq 2$, define the transformation $T^k : \mathcal{D} \to \mathcal{D}$ by

$$T^k(f)(x,y) = k^2 \int_{\frac{p-1}{k}}^{\frac{p}{k}} \int_{\frac{m-1}{k}}^{\frac{m}{k}} f(\alpha,\beta)\, d\alpha d\beta,$$

whenever $(x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for $m,p \in \{1,2,...,k\}$. Observe that $T^k(f)$ is constant over each square $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Let $\mathcal{D}^k$ be the image of $\mathcal{D}$ by T^k, that is, $\mathcal{D}^k \equiv T^k(\mathcal{D})$. Thus, T^k is a projection.

Observe that $\mathcal{D}^k$ is a finite dimensional set. In fact, a density function $f \in \mathcal{D}^k$ can be described by a matrix $A = (a_{ij})_{k\times k}$, as follows:

$$f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right], \tag{1}$$

for $m,p \in \{1,2,...,k\}$. The definition of f at the zero measure set of points $\{(x,y) = \left(\frac{m}{k}, \frac{p}{k}\right) : m = 0 \text{ or } p = 0\}$ is arbitrary.

The following result is important to our method:

Proposition 3 *f is affiliated if and only if for all k, $T^k(f)$ also is. In mathematical notation: $f \in \mathcal{A} \Leftrightarrow T^k(f) \in \mathcal{A}$, $\forall k \in \mathbb{N}$, or yet: $\mathcal{A} = \cap_{k\in\mathbb{N}} T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)$.*

Proof. *See the supplement to this paper.* ■

The set of affiliated distributions $\mathcal{A}$ is the countable intersection of the sets $T^{-k}\left(\mathcal{A}\cap\mathcal{D}^k\right)$, and these sets themselves are small. $T^{-k}\left(\mathcal{A}\cap\mathcal{D}^k\right)$ is small in $\mathcal{D}$ because $\mathcal{A}\cap\mathcal{D}^k$ is small in $\mathcal{D}^k$ (by definition, T^k is surjective). In fact, we have the following:

Proposition 4 *If λ^k denotes the Lebesgue measure over $\mathcal{D}^k$, then $\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right)\downarrow 0$ as $k\to\infty$.*
Proof. *See the supplement to this paper.* ■

The convergence is extremely fast, as shown in the following table, obtained by numerical simulations, with 10^7 cases (see the supplement to this paper for the description of the numerical simulation method and other results).

	$k=3$	$k=4$	$k=5$	$k=6$
$\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right)$	1.1%	~0.01%	$\sim 10^{-6}$	$<10^{-7}$

Table 1 - Proportion of affiliated distribution
in the sets $\mathcal{D}^k$.

Now, define the measure μ^k over $\mathcal{D}^k$ as follows: if $E\subset\mathcal{D}^k$ is a measurable subset, put $\mu^k(E)=\mu\left(T^{-k}(E)\right)$. Now, it is easy to obtain the main result of this subsection:

Theorem 5 *If $\mu^k\leq M\lambda^k$ for some $M>0$ then, $\mu(\mathcal{A})=0$.*[9]
Proof. *By Proposition 3, $\mathcal{A}\subset T^{-k}\left(\mathcal{A}\cap\mathcal{D}^k\right)$ for every k. Thus,*

$$\mu(\mathcal{A})\leq\mu\left(T^{-k}\left(\mathcal{A}\cap\mathcal{D}^k\right)\right)=\mu^k\left(\mathcal{A}\cap\mathcal{D}^k\right)\leq M\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right).$$

Since $\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right)\downarrow 0$ as $k\to\infty$, by Proposition 4, we have the conclusion. ■

As the reader may note from the above proof, it is possible to change the condition $\mu^k\leq M\lambda^k$ for some $M>0$ by $\mu^k\leq M^k\lambda^k$ for a sequence M^k, as long as M^k does not go to infinity as fast as $\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right)$ goes to zero. Since the convergence $\lambda^k\left(\mathcal{A}\cap\mathcal{D}^k\right)\downarrow 0$ is extremely fast, as we noted above, this assumption seems mild.

It is useful to observe that Theorem 5 is not empty, that is, there are many measures over $\mathcal{D}$ that satisfy it. A way to see this is to recall that a measure over $\mathcal{D}$ can be constructed from the measures over the finite-dimensional sets $\mathcal{D}^k$ by appealing to

[9] The reader may note that the assumption is slightly stronger than absolute continuity of μ^k with respect to λ^k. In fact, absolute continuity requires only that $\lambda^k(A)=0$ implies $\mu^k(A)=0$. Nevertheless, by the Radon-Nikodym Theorem, absolute continuity implies the existence of a measurable function m^k such that $\mu^k(A)=\int_A m^k d\lambda^k$. Thus, the above assumption is really requiring this function m^k to be bounded: $m^k\leq M$. As we discuss in the paragraph after the Theorem, this bound does not need to be uniform in k.

the Kolmogorov Extension Theorem (see Aliprantis and Border 1991, p. 491). The interested reader will find more comments about this in the supplement to this paper.

The findings presented in this and previous subsections seem to contradict the common understanding that affiliation is reasonable. For instance, models with conditional independence are considered very natural and it is sometimes (wrongly) believed that they imply affiliation. We analyze them in the next subsection.

3.6 Conditional independence

Conditional independence models assume that the signals of the bidders are conditionally independent, given a variable v (which can be the intrinsic value of the object, see Wilson 1969, 1977). Assume that the p.d.f. of the signals conditional to v, $f(t_1, ..., t_n|v)$, is C^2 (twice continuously differentiable) and has full support. It can be proven that the signals are affiliated if and only if

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial t_j} \geqslant 0,$$

and

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial v} \geqslant 0, \tag{2}$$

for all i, j.[10] Conditional independence implies only that

$$\frac{\partial^2 \log f(t_1, ..., t_n|v)}{\partial t_i \partial t_j} = 0.$$

Thus, conditional independence is not sufficient for affiliation. To obtain the latter, one needs to assume (2) or that t_i and v are affiliated. In other words, to obtain affiliation from conditional independence, one has to assume affiliation itself. Thus, the justification of affiliation through conditional independence is meaningless.

A particular case, also described in subsection 3.1, is used as a method of obtaining affiliated signals: to assume that the signals t_i are a common value plus an individual error, that is, $t_i = v + \varepsilon_i$, where the ε_i are independent and identically distributed. This is not yet sufficient for the affiliation of t_1, ..., t_n. Indeed, let g be the p.d.f. of the ε_i, $i = 1$, ..., n. Then, t_1, ..., t_n are affiliated if and only if g is a strongly unimodal function.[11,12]

[10] See Topkis (1978), p. 310.

[11] The term is borrowed from Lehmann (1959). A function is strongly unimodal if $\log g$ is concave. A proof of the affirmation can be found in Lehmann (1959), p. 509, or obtained directly from the previous discussion.

[12] Even if g is strongly unimodal, so that $t_1, ..., t_n$ are affiliated, it is not true in general that t_1, ..., t_n, ε_1, ..., ε_n, v are affiliated.

3.7 Affiliation in other sciences

The above discussion suggests that affiliation is, indeed, a narrow condition and probably not a good description of the world. Nevertheless, we know that affiliation is widely used in Statistics, reliability theory and in many areas of social sciences and economics (possibly under other names). Why is this so if affiliation is restrictive?

In Statistics, affiliation is known as Positive Likelihood Ratio Dependence (PLRD), the name given by Lehmann (1966) when he introduced the concept. PLRD is widely known by statisticians to be a strong property and many papers in the field do use weaker concepts (such as those given by properties V, IV or III).

In Reliability Theory, affiliation is generally referred to as Total Positivity of order two (TP_2), after Karlin (1968). Historical notes in Barlow and Proschan (1965) suggest why TP_2 is convenient for the theory. It is generally assumed that the failure rates of components or systems follow specific probabilistic distributions (exponentials, for instance) and such special distributions usually have the TP_2 property. Thus, it is natural to study its consequences.

In contrast, in auction theory, the types represent information gathered by the bidders and there is no reason for assuming that they have a specific distribution. Indeed, this is rarely assumed (at least in theoretical papers). Thus, the reason for the use of TP_2 (or affiliation) in reliability theory does not apply to auction theory.

Finally, we stress that the type of results from previous subsections are insufficient to regard a hypothesis as inadequate or not useful. This judgement has to be made in the context of the other assumptions of the theory. For instance, it is possible that the hypothesis is not so restrictive *given* the setting where it is assumed. Moreover, the judgment must take into account the most important of all criteria: whether the resulting theory "yields sufficiently accurate predictions"(Friedman (1953), p. 14). Because of this, we study the main consequences of affiliation – namely, equilibrium existence and the revenue rank – in sections 4 and 5. Thus, this paper addresses only the use of affiliation in auction theory. It is a task for the specialists in other fields to analyze whether affiliation is appropriate for their applications.

4 Equilibrium existence

We divide the results of this section into three subsections. In subsection 4.1, we show that pure strategy equilibrium existence has little relation with positive dependence. More precisely, pure strategy equilibrium existence can be generalized from affiliation (Property VII) to Property VI, but not beyond, that is, we have a counterexample satisfying the (strong) Property V, which does not have symmetric increasing pure strategy equilibrium (PSE).

This negative result shows the need for another approach. In subsection 4.2 we argue that the auction phenomena related to dependence can be modeled and analyzed by considering a simpler but sufficiently rich class of distributions, which we introduce there.

Working in this class, we are able to completely characterize the PSE existence question in subsection 4.3. In this subsection, we also show that the proportion of distributions with PSE is small in the set of all densities considered.

4.1 Generalization of equilibrium existence

Before we present our main equilibrium existence result, we call the attention to the fact that the same proof from Milgrom and Weber (1982a) can be used to prove equilibrium existence for Property VI. Indeed, the following property is sufficient:[13]

Property VI′ - The joint (symmetric) distribution of X and Y satisfy property VI′ if for all x, x' and y in $[0,1]$, $x \geq y \geq x'$ imply

$$\frac{F(y|x')}{f(y|x')} \geq \frac{F(y|y)}{f(y|y)} \geq \frac{F(y|x)}{f(y|x)}.$$

It is easy to see that Property VI implies Property VI′ (under symmetry and full support). Unfortunately, however, it is impossible to generalize further the existence of equilibrium for the properties defined in subsection 3.5. Indeed, in the appendix, we give an example of a distribution which satisfies Property V, but does not have equilibrium. These facts are summarized in the following:

Theorem 6 *If $f : [0,1]^2 \to \mathbb{R}$ satisfies property VI′, there is a symmetric pure strategy monotonic equilibrium. Moreover, property V is not sufficient for equilibrium existence.*
Proof. *See the appendix.* ■

The most important message of Theorem 6 is the negative one: that it is impossible to generalize the equilibrium existence for the other still restrictive definitions of positive dependence. This mainly negative result leads us to consider another route to prove equilibrium existence. For this, we begin by considering the set of distributions which we shall work with.

4.2 The class of distributions

Modeling types as continuous real variables is a widespread practice in auction theory. The reason for that is clear: continuous variables allow the use of the convenient tools of calculus, such as derivatives and integrals, to obtain precise characterizations and uniqueness results. The problem with this approach, as long as we try to consider dependence, is that to establish pure strategy equilibrium existence with traditional tools seems difficult, as the result in the previous subsection suggests.

Here we offer an alternative solution. Observe that the value of the single object in the auction is expressed up to cents and is obviously bounded. Thus, the number of actual possible values is finite. Nevertheless, instead of sticking to the (actual) case of discrete values, we allow them to be continuous, but impose, on the other hand, that the density functions are simple. In fact, it is sufficient to consider the particular set of simple symmetric functions $\mathcal{D}^k$, as defined in subsection 3.5. This mathematical

[13]Recently, Monteiro and Moreira (2006) obtained further generalizations of equilibrium existence for non-affiliated variables. Their results are not directly related to positive dependence properties.

restriction implies no economical restriction to the problem we are studying. In sum, we propose considering the set $\mathcal{D}^{\infty} = \cup_{k \in \mathbb{N}} \mathcal{D}^{k}$ of simple p.d.f.'s. Now we describe how the equilibrium existence problem can be completely solved in this set.

First, recall the standard result of auction theory on PSE in private value auctions: if there is a differentiable symmetric increasing equilibrium, it satisfies the differential equation (see Krishna 2002 or Menezes and Monteiro 2005):

$$b^{'}(t) = \frac{t - b(t)}{F(t|t)} f(t|t).$$

If f is Lipschitz continuous, one can use Picard's theorem to show that this equation has a unique solution and, under some assumptions (basically, Property VI' of the previous subsection), it is possible to ensure that this solution is, in fact, equilibrium. Now, for $f \in \mathcal{D}^{\infty}$, the right hand side of the above equation is not continuous and one cannot directly apply Picard's theorem. We proceed as follows.

First, we show that if there is a symmetric increasing equilibrium b, under mild conditions (satisfied by $f \in \mathcal{D}^{\infty}$), b is continuous. We also prove that b is differentiable at the points where f is continuous. Thus, for $f \in \mathcal{D}^{\infty}$, b can be non-differentiable only at the points of the form $\frac{m}{k}$, but it is continuous. See figure 3.

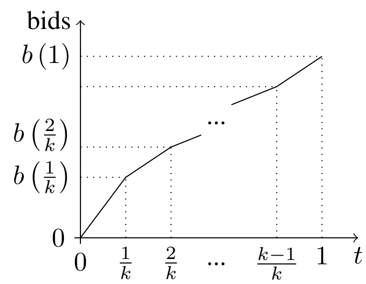


Figure 3: Bidding function for $f \in \mathcal{D}^{k}$.

With the initial condition $b(0) = 0$ and the above differential equation being valid for the first interval $\left(0, \frac{1}{k}\right)$, we have uniqueness of the solution on this interval and, thus, a unique value of $b\left(\frac{1}{k}\right)$. Since b is continuous, this value is the initial condition for the interval $\left(\frac{1}{k}, \frac{2}{k}\right)$, where we again obtain a unique solution and the uniqueness of the value $b\left(\frac{2}{k}\right)$. Proceeding in this way, we find that there is a unique b which can be a symmetric increasing equilibrium for an auction with $f \in \mathcal{D}^{\infty}$. In the supplement to this paper we prove the following:

Theorem 7 *Assume that u is twice continuously differentiable, $u' > 0$, $f \in \mathcal{D}^{k}$, f is symmetric and positive ($f > 0$). If $b : [0,1] \to \mathbb{R}$ is a symmetric increasing equilibrium, then b is continuous in $(0,1)$ and is differentiable almost everywhere in $(0,1)$ (it is may be non-differentiable only in the points $\frac{m}{k}$, for $m = 1, ..., k$). Moreover, b is the unique symmetric increasing equilibrium. If $u(x) = x^{1-c}$, for $c \in [0,1)$, b is given by*

$$b(x) = x - \int_{0}^{x} \exp\left[-\frac{1}{1-c}\int_{\alpha}^{x} \frac{f(s \mid s)}{F(s \mid s)} ds\right] d\alpha. \tag{3}$$

Having established the uniqueness of the candidate for equilibrium, our task is reduced to verifying whether this candidate is, indeed, an equilibrium. We complete this task in the next subsection.

Even if the reader insists on considering the more general set of p.d.f.'s $\mathcal{D}$ — being aware that this is a matter of mathematical generality, but not of economic generality — our set $\mathcal{D}^{\infty}$ is still dense in $\mathcal{D}$ and, thus, may arbitrarily approximate any conceivable p.d.f. in $\mathcal{D}$. In fact, the following result shows that equilibrium existence in the set $\mathcal{D}^{\infty}$ is sufficient for equilibrium existence in $\mathcal{D}$. This provides an additional justification of the method.

Proposition 8 *Let $f \in \mathcal{D}$ be continuous and symmetric. If $T^k(f)$ has a differentiable symmetric pure strategy equilibrium for all $k \geq k_0$, then so does f, and it is the limit of the equilibria of $T^k(f)$ as k goes to infinity.*[14]

4.3 Equilibrium existence results

In the previous subsection, we established the uniqueness of the candidate for symmetric increasing equilibrium for $f \in \mathcal{D}^{\infty}$. Let $b(\cdot)$, given by (3) with $c = 0$, denote such a candidate. Let $\Pi(y, b(x)) = (y - b(x)) F(x|y)$ be the interim payoff of a player with type y who bids as type x, when the opponent follows $b(\cdot)$. Let $\Delta(x, y)$ represent $\Pi(y, b(x)) - \Pi(y, b(y))$. It is easy to see that $b(\cdot)$ is equilibrium if and only if $\Delta(x, y) \leq 0$ for all x and $y \in [0, 1]^2$. Thus, the content of the next theorem is that it is possible to prove equilibrium existence by checking this condition only for a finite set of points:

Theorem 9 *Let $f \in \mathcal{D}^{\infty}$ be symmetric and strictly positive. There exists a finite set $P \subset [0, 1]^2$ (precisely characterized in the supplement to this paper) such that f has a PSE if and only if $\Delta(x, y) \leq 0$ for all $(x, y) \in P$.*

It is useful to say that the theorem is not trivial, since $\Delta(x, y)$ is not monotonic in the squares $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Indeed, the main part of the proof is the analysis of the non-monotonic function $\Delta(x, y)$ in the sets $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$ and the determination of its maxima for each of these sets. It turns out that we need to check a different number of points (between 1 and 5) for some of these squares.

Using Theorem 9, we can classify whether or not there is equilibrium, and, through numerical simulations, obtain the proportion of cases with pure strategy equilibrium. That is, for each trial $f \in \mathcal{D}^k$, we test whether the auction with bidders' types distributed according to f has a symmetric increasing pure strategy equilibrium. The results are shown in the Table 2 below.

[14] See the definition of T^k in subsection 3.5.

For each k, 100%=distributions with equilibrium.

Distribution satisfying	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
Prop. VII (affiliation)	7.7%	0.07%	$< 10^{-7}$	–	–
Prop. VI	1.4%	0.08%	$\sim 10^{-6}$	$< 10^{-7}$	–
Prop. V	5.5%	0.75%	0.01%	$< 10^{-6}$	$< 10^{-7}$
Prop. IV	8.8%	4.1%	0.8%	0.1%	$\sim 10^{-5}$
$\sim$ Prop. IV	76.6%	95%	99.2%	99.9%	100%

Table 2 - Proportion of $f \in \mathcal{D}^k$ with PSE, satisfying properties IV-VII.

Table 2 shows that affiliation is restrictive even in the set of p.d.f.'s with symmetric increasing equilibrium. It is useful to record this fact separately. For this, let us introduce some useful notations. Let μ and μ^k denote the natural measures defined over $\mathcal{D}^\infty = \cup_{k=1}^{\infty} \mathcal{D}^k$ and $\mathcal{D}^k$, respectively, as constructed in the supplement to this paper and let $\mathcal{P}$ and $\mathcal{P}^k$ denote the set of p.d.f.'s in $\mathcal{D}^\infty$ and $\mathcal{D}^k$, respectively, which have a symmetric increasing pure strategy equilibrium. From the above table, we extract the following:

Observation 10 *Let $\mu^k \left(\cdot|\mathcal{P}^k\right)$ denote the measure induced by μ^kin the set $\mathcal{D}^k \cap \mathcal{P}^k$. Then, we have $\mu^k \left(\mathcal{A} \cap \mathcal{D}^k|\mathcal{P}^k\right) \downarrow 0$.*

Another way to say this is: there are many more cases with pure strategy equilibrium than affiliation allows us to prove.

Unfortunately, however, the set of p.d.f.'s with symmetric increasing equilibrium is also small. This result is proved formally in the following:

Theorem 11 *The measure of the set of densities $f \in \mathcal{D}^k$ which has PSE goes to zero as k increases, that is $\mu^k \left(\mathcal{P}^k\right) \downarrow 0$. Consequently, the measure of the set of densities $f \in \mathcal{D}^\infty$ with PSE is zero, that is, $\mu\left(\mathcal{P}\right) = 0$.*
Proof. *See the supplement to this paper.* ■

The proof of this theorem follows a simple idea: the equilibrium existence depends on a series of inequalities, the number of which increases with k. Although some care is needed for rigorously establishing the result, this simple observation is the heart of the argument.

The following table provides the numbers that come from numerical simulations and show that the convergence of $\mu^k \left(\mathcal{P}^k\right)$ to zero is also very fast.

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
With PSE	43.3%	22.2%	11.4%	5.6%	2.7%	1.3%	0.6%
Without PSE	56.7%	77.8%	88.7%	94.4%	97.3%	98.7%	99.4%

Table 3 - Proportion of $f \in \mathcal{D}^k$ with and without PSE.

The result summarized in Table 3 is negative in the sense that it suggests that the focus on symmetric increasing equilibrium may be too narrow. Nevertheless, this is not yet sufficient to conclude that most of the equilibria are in mixed strategies. In fact, while we know that mixed strategy equilibria always exist (Jackson and Swinkels 2005), there is the possibility — not considered in our results — that there are equilibria in asymmetric or non-monotonic pure strategies.

5 The Revenue Ranking of Auctions

This section is also divided in three parts. In the first subsection, we present negative results regarding the generalization of affiliation to other notions of positive dependence. In subsection 5.2 we propose a method for dealing with the problem of revenue ranking. In subsection 5.3 we describe the results obtained with the proposed method.

5.1 Generalization of affiliation's revenue ranking

In this subsection we show that, for private values auctions, it is possible to generalize the existence for Property VI but not for Property V. Moreover, we provide an expression for the difference in revenue from second and first price symmetric auctions that will be useful later.[15] This is the content of the following:

Theorem 12 *If f satisfies Property VI' (see subsection 4.1), then the second price auction gives greater revenue than the first price auction. Specifically, the revenue difference is given by*

$$\int_0^1 \int_0^x b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

where $b(\cdot)$ is the first price equilibrium bidding function, or by

$$\int_0^1 \int_0^x \left[\int_0^y L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \qquad (4)$$

where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$. Moreover, Property V is not sufficient for this revenue rank.
Proof. *See the appendix.* ■

From the expression of expected revenue difference provided by Theorem 12, we see that the revenue superiority of the English (second price) auction over the first price auction seems to be strongly dependent on the condition required for Property VI′ (see subsection 4.1). Moreover, the revenue rank is not valid even for a positive dependence concept as strong as Property V.

[15]The expression is not particularly difficult to obtain, but we were not able to find a reference for it. Milgrom and Weber (1982a)'s argument for affiliation's revenue ranking does not make use of this expression, nor do Krishna (2002) or Menezes and Monteiro (2005).

5.2 How to solve the problem of revenue ranking?

In an article about design economics, Roth (2002) compares the relationship between economic theory and market design to the relationship between physics and engineering and between biology and medicine and surgery. For instance, while physics offers simple and elegant models with clear indications, the actual problems in engineering require dealing with details that are not (and shall not be) considered in the basic physical model. But, as long as the specific environment is sufficiently determined, engineers can detail the basic model to reach more precise conclusions.

Roth (2002)'s analogy suggests an answer to the question in the title of this subsection. Basically, we need a two-level answer. In the first (theoretical) level without specific information, we need a clear prediction, obtained with a simple and basic model. Then, in the applied level, we need a methodology that allows the use of more detailed information (available in the specific case considered), to reach more precise conclusions.

We describe these two levels of the methodology in the sequel.

5.2.1 The basic model, without specific information

Not only the results in subsection 5.1, but also many results in auction theory (e.g. Maskin and Riley 1984, 2000), suggest that it is not possible to give simple predictions about the revenue ranking of auctions. The reason is that the expected difference of revenues can have any sign and the description of the set of distributions which has one sign or another is very difficult.

Fortunately, a simple characterization is available without further assumptions — if we work in the correct level of abstraction. To describe what we mean by "correct level of abstraction", consider the following figure.

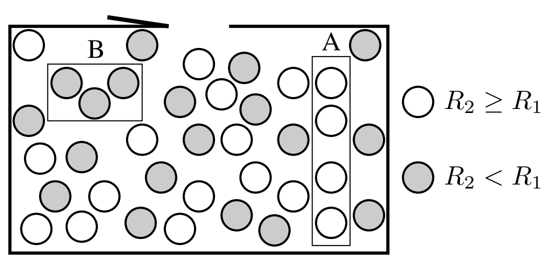


Figure 4: Illustrative representation of the set of distributions with any kind of dependence.

The standard approach to the problem of revenue ranking of auctions is to give conditions (the sets A and B above) under which the revenue ranking is defined. Such an approach is illustrated by Milgrom and Weber (1982)'s affiliation assumption and by Maskin and Riley (2000)'s three different assumptions for asymmetrical independent distributions.[16] Under each of the these assumptions, the authors are able to say pre-

[16]The reader should not be confused with the citation of Maskin and Riley (2000), which do not exactly deal with our problem — the revenue ranking with dependent types — but rather with the problem of revenue ranking with asymmetries. We cite them only to illustrate what we are calling the "standard approach" to the revenue ranking problem (in general).

cisely what the revenue ranking (the color of balls in the above figure) is. Now, if we think of the set of distributions as a black box or an urn, and we want a "prevision" of the color of the ball that we are going to extract from it (the expected revenue ranking), the "correct" type of answer to our problem is the characterization of the *probability* of obtaining one ranking or another.

Since we have the expression of the expected revenue difference, given by (4), we can obtain $\Delta_R^f = R_2^f - R_1^f$ and $r = \frac{R_2^f - R_1^f}{R_2^f}$, for each f. Generating a uniform sample of $f \in \mathcal{D}^k$, we can obtain the probabilistic distribution of Δ_R^f or of r. The procedure to generate $f \in \mathcal{D}^k$ uniformly is described in the supplement to this paper. The results are shown in subsection 5.3 below.

Moreover, we can also obtain theoretical results about what happens for $\mathcal{D}^N$ for a large N and even for $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$. Nevertheless, for the last case, one has to be careful with the meaning of the "uniform" distribution. In the supplement to this paper we show that a natural measure can be defined for $\mathcal{D}^\infty$, which is analogous to Lebesgue measure, although it cannot have all the properties of the finite dimensional Lebesgue measure.

As such, this context-free approach, without specific information, allows one to obtain theoretical results and previsions based on simulations. One possible objection to this approach is that it considers too equally the p.d.f.'s in the sets $\mathcal{D}^k$. But this is exactly what we mean by "context-free". If one has information on the environment where the auction runs, so that one can restrict the set of suitable p.d.f.'s, then this context-free approach should be substituted by the applied approach which follows.

5.2.2 The applied approach, with specific information

Econometricians working in a specific auction environment may be able to restrict or characterize the set of distributions (dependence) that one finds in such an environment.

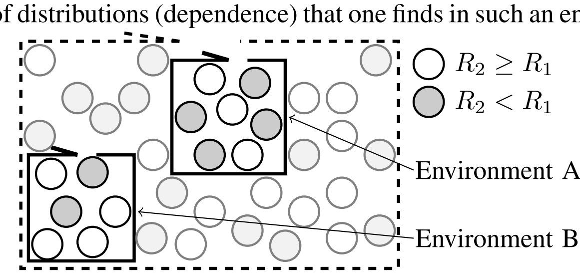


Figure 5: In specific auction environments — e.g. auctions of energy contracts, art objects, timber, etc. — we may *experimentally* find different sets of typical distributions.

If we have the characterization of the typical distributions in an auction environment, we can use the model presented in this paper and make simulations. In this fashion, we will be able to obtain more precise, context-specific indications about the revenue ranking of auctions in such environments.

A least comment is useful. The comparison of figures 4 and 5 may wrongly suggest that this applied approach means to go back to what we called the "standard approach", that is, to consider only special cases of the distributions. It is important to highlight the difference between the two. With the standard approach, econometricians need to begin the analysis believing that the data obey such or such assumption.[17] In our method, no assumption is required on the dependence of information. The restriction that one finds (and which is illustrated in figure 5) comes from the data, not from theoretical restrictions. Methodologically, this is a big difference.

5.3 Results on Revenue Ranking

In the supplement to this paper, we develop the expression of the revenue differences from the second price auction to the first price auction for $f \in \mathcal{D}^k$. Let us denote by R_2^f the expected revenue (with respect to $f \in \mathcal{D}^k$) of the second price auction. Similarly, R_1^f denotes the expected revenue (with respect to $f \in \mathcal{D}^k$) of the first price auction. When there is no need to emphasize the p.d.f. $f \in \mathcal{D}^k$, we write R_1 and R_2 instead of R_1^f and R_2^f. Below, μ refers to the natural measure defined over $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$, as further explained in the supplement to this paper. We observe the following fact in the simulations made:

Observation 13 *The expectation of the (expected) revenue differences, $R_2 - R_1$, is non-negative, that is, $E_\mu \left[R_2^f - R_1^f | f \in \mathcal{P}^k \right] \geq 0$, where $\mathcal{P}^k$ denotes the set of those $f \in \mathcal{D}^k$ for which there is a PSE in the first price auction.*[18]

The simulations were made as follows. We generated the distributions $f \in \mathcal{D}^k$ as described in the supplement to this paper. (It is the same process used in subsection 3.5 and section 4). We evaluate the revenue difference percentage, given by:

$$r = \frac{R_2^f - R_1^f}{R_2^f} \cdot 100\%,$$

that is, we carried out the following:

Numerical experiments

In what follows, we will treat the numerical simulations as giving an "experimental distribution" of r. No confusion should arise between the "experimental distribution" of r and the distributions generated by each $f \in \mathcal{D}^k$. We generated 10^7 distributions $f \in \mathcal{D}^k$, for $k = 3, ..., 9$ and obtained r for each such f. The "experimental distribution" of r is characterized by the table below. It is worth saying that the results are already stable for 10^6 trials.

[17] One may see this in almost all papers about econometrics of auctions.

[18] This was verified for $k \leq 10$, but seems to be valid for larger k's.

Distribution:	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
Expectation	4.5%	8.0%	10.3%	12.1%	13.4%	14.6%	15.5%
Variance	5.3%	6.9%	7.3%	7.2%	7.0%	6.8%	6.6%
5% quantile	-4%	-3%	-2%	0%	1%	3%	4%
10% quantile	-2%	-1%	0%	2%	3%	4%	6%
25% quantile	0%	2%	4%	6%	6%	8%	8%
50% quantile	2%	6%	8%	10%	10%	12.5%	12.5%
75% quantile	6%	10%	12.5%	15%	15%	17.5%	17.5%
90% quantile	10%	15%	17.5%	17.5%	19%	19%	19%
96% quantile	12.5%	17.5%	20%	20%	20%	20%	20%
99% quantile	15%	20%	25%	25%	25%	25%	25%

Table 4 - Expectation of the relative revenue differences (r) for $f \in \mathcal{D}^k$ with PSE.

Figure 6 shows the "experimental density" (histogram) of r for $k = 4$.

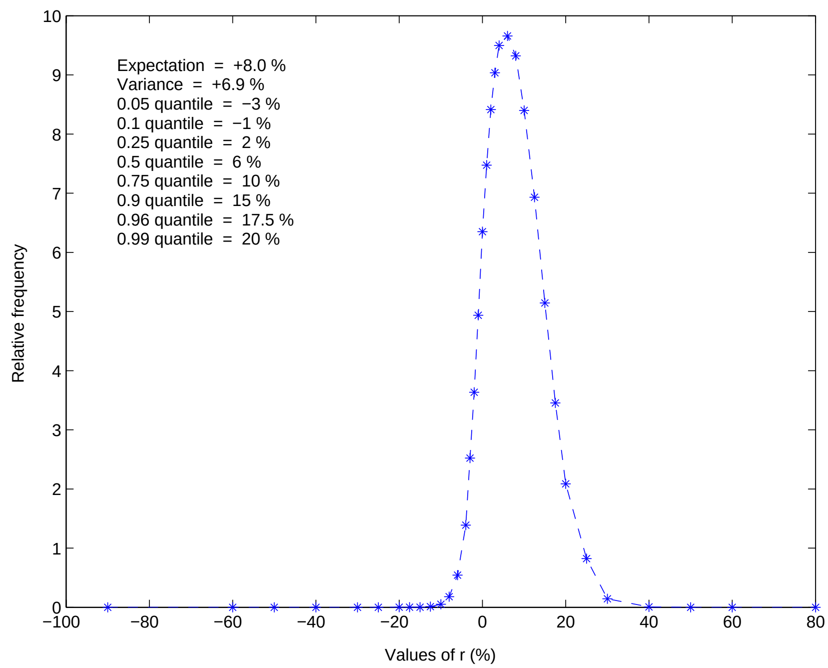


Figure 6: Histogram of r for $k = 4, c = 0$ — for those $f \in \mathcal{D}^k$ with PSE.

In Table 4, we displayed the results only for those f with PSE. If we consider all distributions, with and without PSE, we obtain the results in Table 5 below. This shows that the restriction of PSE existence matters for the distribution of r.

Distribution	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
Expectation	-0.08%	0.13%	0.28%	0.38%	0.46%	0.53%	0.57
Variance	10.5%	10.9%	10.5%	9.9%	9.4%	9.0%	8.5%
5% quantile	-25%	-20%	-20%	-17.5%	-17.5%	-15%	-15%
10% quantile	-15%	-15%	-15%	-15%	-12.5%	-12.5%	-12.5%
25% quantile	-8%	-8%	-8%	-8%	-8%	-8%	-8%
50% quantile	-1%	-1%	-1%	-1%	-1%	-1%	-1%
75% quantile	4%	6%	6%	6%	4%	4%	4%
90% quantile	10%	12.5%	12.5%	10%	10%	10%	10%
96% quantile	15%	15%	15%	15%	15%	15%	12.5%
99% quantile	25%	25%	25%	25%	25%	25%	25%

Table 5 - Expectation of the relative revenue differences (r) for all cases
(with and without PSE).

The following table allows one to compare the effects of dependence and risk aversion to the expected revenue differences. For this, we restrict ourselves to the case of CRRA bidders, that is, bidders with utility function $u(x) = x^{1-c}$, where $c \in [0, 1)$.[19]

Expect.	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$c=0$	4.6%	8.0%	10.3%	12.1%	13.4%	14.6%	15.4%
$c=0.05$	3.7%	6.9%	8.8%	10.0%	10.8%	11.3%	11.6%
$c=0.1$	3.0%	5.7%	7.2%	8.0%	8.4%	8.6%	8.6%
$c=0.15$	2.2%	4.5%	5.7%	6.2%	6.3%	6.3%	6.3%
$c=0.2$	1.4%	3.4%	4.1%	4.4%	4.4%	4.4%	4.2%
$c=0.3$	0.0%	1.0%	1.2%	1.2%	1.1%	1.1%	1.0%
$c=0.4$	-1.4%	-1.5%	-1.8%	-1.8%	-1.8%	-1.7%	-1.5%
$c=0.52$	-2.9%	-4.7%	-5.3%	-5.2%	-4.9%	-4.6%	-4.1%
$c=0.65$	-3.7%	-8.8%	-9.6%	-9.3%	-8.6%	-7.9%	-7.1%
$c=0.8$	-6.3%	-14.7%	-15.9%	-15.2%	-14.1%	-13.1%	-12.1%

Table 6 - Expectation of the relative revenue differences (r) for bidders
with CRRA function $u(x) = x^{1-c}$, where $c \in [0, 1)$.

6 Related literature, the contribution and future work

A few papers have pointed out restrictions or limitations to the implications of affiliation. Perry and Reny (1999) presented an example of a multi-unit auction where the linkage principle fails and the revenue ranking is reversed, even under affiliation. Thus, their criticism seems to be restricted to the generalization of the consequences

[19]In Table 6, we restrict our study to the cases where PSE exists for $c = 0$. We do not have a generalization of the PSE existence result (Theorem 9) — and, thus, we do not have a procedure to test for PSE existence — for $c > 0$. The results in Table 6 should be considered with this in mind.

of affiliation to multi-unit auctions. In contrast, we considered single-unit auctions and non-affiliated distributions.

Klemperer (2003) argues that, in real auctions, affiliation is not as important as asymmetry and collusion. He illustrates his arguments with examples of the 3G auctions conducted in Europe in 2000-2001.

Nevertheless, much more was written in accordance with the conclusions of affiliation. McMillan (1994, p.152) says that the auction theorists working as consultants to the FCC in spectrum auctions, advocated the adoption of an open auction using the linkage principle (Milgrom and Weber 1982a) as an argument: "Theory says, then, that the government can increase its revenue by publicizing any available information that affects the licensee's assessed value". The disadvantages of the open format in the presence of risk aversion and collusion were judged "to be outweighed by the bidders' ability to learn from other bids in the auction" (p. 152). Milgrom (1989, p. 13) emphasizes affiliation as the explanation of the predominance of the English auction over the first price auction.

This paper presents evidence that affiliation is a restrictive assumption. After developing an approach to test the existence of symmetric increasing pure strategy equilibrium (PSE) for simple density functions, we are able to verify that many cases with PSE do not satisfy affiliation. Also, the superiority of the English auction is not maintained even for distributions satisfying strong requirements of positive dependence. Nevertheless, we show that the original conclusion of Milgrom and Weber (1982a) (that positive dependence implies that English auctions gives higher revenue than first price auction) is true for a much larger set of cases , but in a weaker sense — "on average".

We would also like to highlight two conceptual contributions of this paper that may go beyond the actual applications made here.

One of these is the restriction to a simple space of distributions. The proposed space makes possible the complete characterization of the symmetric increasing pure strategy equilibrium existence problem, because it requires only elementary (albeit lengthy) calculations. The set of distributions considered is as general as necessary for economic applications and seems suitable for modeling the asymmetric bidders case as well.

The second conceptual contribution is a way of looking at the problem of revenue ranking. We propose two levels of the answer: an abstract, theoretical level and a simulation-driven applied level.

The first level allows general theoretical conclusions that may be useful as general, context-free guidance. Using the second level, applied economists can reach case-specific conclusions which may be more accurate and valuable.

These ideas are applicable for more general setups than those pursued here. Not only equilibrium existence but also the revenue ranking can be investigated in contexts of n asymmetric bidders, interdependent values, risk aversion and multi-units. Although these generalizations seem feasible, they are by no means trivial.[20]

Nevertheless, the main complement to this work seems to pertain to the fields of

[20]The generalization from $n = 2$ to general n can be pursued, at least for the symmetric case, using the expressions developed in the supplement of this paper. Only, instead of considering the expressions of $f(x|y)$ and $F(x|y)$ as coming from a bivariate distribution, we could write the expressions as coming from n variables but expressing them, as we did, by a matrix. Some expressions would change, but the main results would remain valid. Thus, the most difficult generalization seems to be to the asymmetric case.

econometrics and experimentation. This is to develop a method to characterize the dependence (sets of distributions) typical in each specific situation. With such a method, applied works could characterize what happens in specific auctions. For instance, it is likely that the kind of dependence that appears in mineral rights auctions, or in electricity market auctions, is different from that observed in art auctions.

If such a method is developed and if the correspondent specific characterizations are done — these are big if's — we could find a way to explain why some kinds of markets almost always use a specific auction format, as observed by Maskin and Riley (2000): "rarely is any given kind of commodity sold through more than one sort of auction. Thus, for example, art is nearly always auctioned off according to the English rules, whereas job contracts are normally awarded through sealed bids" (p. 413). Maskin and Riley (2000) comment that the Revenue Equivalence Theorem is not able to explain such case-specific uniformity. The same argument also applies to affiliation, which would predict the use of English auctions for every situation.

An auction theory model capable of general conclusions and context-specific calibrations, using simulations and theoretical analysis, could explain this. If not, at least it would be more realistic and, thus, more useful.

Appendix

Proof of Theorem 1.

It is obvious that $(III) \Rightarrow (II) \Rightarrow (I)$. The implication $(IV) \Rightarrow (III)$ is Theorem 4.3. of Esary, Proschan and Walkup (1967). The implication $(V) \Rightarrow (IV)$ is proved by Tong (1980), chap. 5, p. 80. Thus, we have only to prove that $(VI) \Rightarrow (V)$, since the implication $(VII) \Rightarrow (VI)$ is Lemma 1 of Milgrom and Weber (1982a). Assume that $H(y|x) \equiv \frac{f(y|x)}{F(y|x)}$ is non-decreasing in x for all y. Then, $H(y|x) = \partial_y [\ln F(y|x)]$ and we have

$$1 - \ln[F(y|x)] = \int_y^\infty H(s|x)\, ds \geqslant \int_y^\infty H(s|x')\, ds = 1 - \ln[F(y|x')],$$

if $x \geqslant x'$. Then, $\ln[F(y|x)] \leqslant \ln[F(y|x')]$, which implies that $F(y|x)$ is non-increasing in x for all y, as required by property V.

The counterexamples for each passage are given by Tong (1980), chap. 5, except those involving property (VI): $(V) \nRightarrow (VI)$, $(VI) \nRightarrow (VII)$. For the first counter example, consider the following symmetric and continuous p.d.f. defined on $[0,1]^2$:

$$f(x,y) = \frac{d}{1 + 4(y-x)^2}$$

where $d = [\arctan(2) - \ln(5)/4]^{-1}$ is the suitable constant for f to be a p.d.f. We have the marginal given by

$$f(y) = \frac{k}{2}\left[\arctan 2(1-y) + \arctan 2(y)\right]$$

so that we have, for $(x,y) \in [0,1]^2$:

$$f(x|y) = 2\left[1 + 4(y-x)^2\right]^{-1}\left[\arctan 2(1-y) + \arctan 2(y)\right]^{-1},$$

$$F(x|y) = \frac{\left[\arctan 2(x-y) + \arctan 2(y)\right]}{\arctan 2(1-y) + \arctan 2(y)}$$

and

$$\frac{F(x|y)}{f(x|y)} = 2\left[1 + 4(y-x)^2\right]\left[\arctan(2x-2y) + \arctan(2y)\right].$$

Observe that for $y' = 0.91 > y = 0.9$ and $x = 0.1$,

$$\frac{F(x|y')}{f(x|y')} = 0.366863 > 0.366686 = \frac{F(x|y)}{f(x|y)},$$

which violates property (VI). On the other hand,

$$\begin{aligned} \partial_y\left[F(x|y)\right] &= \frac{\frac{2}{1+4y^2} - \frac{2}{1+4(x-y)^2}}{\arctan(2-2y) + \arctan(2y)} \\ &\quad - \frac{\left[\arctan(2x-2y) + \arctan(2y)\right]\left[\frac{2}{1+4y^2} - \frac{2}{1+4(1-y)^2}\right]}{\left[\arctan(2-2y) + \arctan(2y)\right]^2} \end{aligned}$$

In the considered range, the above expression is non-positive, so that property (V) is satisfied. Then, $(V) \nRightarrow (VI)$.

Now, fix an $\varepsilon < 1/2$ and consider the symmetric density function over $[0,1]^2$:

$$f(x,y) = \begin{cases} k_1, & \text{if } x + y \leqslant 2 - \varepsilon \\ k_2, & \text{otherwise} \end{cases}$$

where $k_1 > 1 > k_2 = 2\left[1 - k_1\left(1 - \varepsilon^2/2\right)\right]/\varepsilon^2 > 0$ and $\varepsilon \in (0, 1/2)$. For instance, we could choose $\varepsilon = 1/3$, $k_1 = 19/18$ and $k_2 = 1/18$. The conditional density function is given by

$$f(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and the conditional c.d.f. is given by:

$$F(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1 y}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2(y+x+\varepsilon-2)+k_1(2-\varepsilon-x)}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and

$$\frac{F(y|x)}{f(y|x)} = \begin{cases} 1, & \text{if } x \leqslant 1-\varepsilon \\ y, & \text{if } x > 1-\varepsilon \text{ and if } y \leqslant 2-\varepsilon-x \\ y+x+\varepsilon-2+k_1/k_2\,(2-\varepsilon-x)\,, & \text{otherwise} \end{cases}$$

Since $1 - k_1/k_2 < 0$, the above expression is non-increasing in x for all y, so that property (VI) is satisfied. On the other hand, it is obvious that property (VII) does not hold:

$$f(0.5, 0.5)\, f\left(1-\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}\right) = k_2 k_1 < k_1^2 = f\left(0.5, 1-\frac{\varepsilon}{2}\right) f\left(0.5, 1-\frac{\varepsilon}{2}\right).$$

This shows that $(VI) \nRightarrow (VII)$.■

Proof of Theorem 2

First, we prove that $C \backslash A$ is open. If $f \in C \backslash A$, then $f(x)\, f(x') > f(x \wedge x')\, f(x \vee x')$, for some $x, x' \in [0,1]^n$. Fix such x and x' and define $K = f(x) + f(x') + f(x \wedge x') + f(x \vee x') > 0$. Choose $\varepsilon > 0$ such that $2\varepsilon K < f(x)\, f(x') - f(x \wedge x')\, f(x \vee x')$ and let $B_\varepsilon(f)$ be the open ball with radius ε and centre in f. Thus, if $g \in B_\varepsilon(f)$, $\|f - g\| < \varepsilon$, which implies $g(x) > f(x) - \varepsilon$, $g(x') > f(x') - \varepsilon$, $g(x \wedge x') < f(x \wedge x') + \varepsilon$, $g(x \vee x') < f(x \vee x') + \varepsilon$, so that

$$\begin{aligned} & g(x)\, g(x') - g(x \wedge x')\, g(x \vee x') \\ > \; & [f(x) - \varepsilon]\,[f(x') - \varepsilon] - [f(x \wedge x') + \varepsilon]\,[f(x \vee x') + \varepsilon] \\ = \; & f(x)\, f(x') - f(x \wedge x')\, f(x \vee x') - \varepsilon\,[f(x) + f(x') + f(x \wedge x') + f(x \vee x')] \\ = \; & f(x)\, f(x') - f(x \wedge x')\, f(x \vee x') - \varepsilon K \\ > \; & \varepsilon K > 0, \end{aligned}$$

which implies that $B_\varepsilon(f) \subset C \backslash A$, as we wanted to show.

Now, let us show that $C \backslash A$ is dense, that is, given $f \in C$ and $\varepsilon > 0$, there exists $g \in B_\varepsilon(f) \cap C \backslash A$. Since $f \in C$, it is uniformly continuous (because $[0,1]^n$ is compact), that is, given $\eta > 0$, there exists $\delta > 0$ such that $\|x - x'\|_{\mathbb{R}^n} < 2\delta$ implies $|f(x) - f(x')| < \eta$. Take $\eta = \varepsilon/4$ and the corresponding δ.

Choose $a \in (4\delta, 1 - 4\delta)$ and define x (x') by specifying that their first $\left\lfloor \frac{n}{2} \right\rfloor$ coordinates are equal to $a - \delta$ ($a + \delta$) and the last ones to be equal to $a + \delta$ ($a - \delta$). Thus, $x \wedge x' = (a - \delta, ..., a - \delta)$ and $x \vee x' = (a + \delta, ..., a + \delta)$. Let x_0 denote the vector $(a, ..., a)$. For $y = x$, x', $x \wedge x'$ or $x \vee x'$, we have: $|f(y) - f(x_0)| < \eta$. Let $\xi : (-1, 1)^n \to \mathbb{R}$ be a smooth function that vanishes outside $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)^n$ and equals 1 in $\left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^n$. Define the function g by

$$\begin{aligned} g(y) \;=\; & f(y) + 2\eta\xi(y - x) + 2\eta\xi(y - x') \\ & -2\eta\xi(y - x \wedge x') - 2\eta\xi(y - x \vee x'). \end{aligned}$$

Observe that $\|g - f\| = 2\eta = \varepsilon/2$, that is, $g \in B_\varepsilon(f)$. In fact, $g \in B_\varepsilon(f) \cap C \backslash A$, because

$$\begin{aligned} g(x) &= f(x) + 2\eta > f(x_0) + \eta; \\ g(x') &= f(x) + 2\eta > f(x_0) + \eta; \\ g(x \wedge x') &= f(x \wedge x') - 2\eta < f(x_0) - \eta; \\ g(x \vee x') &= f(x \vee x') - 2\eta < f(x_0) - \eta, \end{aligned}$$

which implies

$$\begin{aligned} & g(x)\, g(x') - g(x \wedge x')\, g(x \vee x') \\ > \;& [f(x_0) + \eta]^2 - [f(x_0) - \eta]^2 \\ = \;& 4\eta > 0. \blacksquare \end{aligned}$$

Proof of Theorem 6

The equilibrium existence follows from Milgrom and Weber (1982a)'s proof. For the counterexample, consider the p.d.f. defined in the proof of Theorem 1:

$$f(x, y) = \frac{d}{1 + 4(y - x)^2},$$

where $d = [\arctan(2) - \ln(5)/4]^{-1}$. In the proof of Theorem 1, we established that this p.d.f. satisfies Property V but not Property VI and that:

$$F(x|y) = \frac{[\arctan 2(x - y) + \arctan 2(y)]}{\arctan 2(1 - y) + \arctan 2(y)}.$$

From Theorem 7, it is sufficient to prove that

$$b(y) = y - \int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w} dw\right] dz$$

cannot be an equilibrium, that is, to verify the existence of x and y such that

$$(y - b(y))\, F(y|y) < (y - b(x))\, F(x|y).$$

This simplifies to the condition:

$$\frac{\int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w} dw\right] dz}{y - x + \int_0^x \exp\left[-\frac{1}{2}\int_z^x \frac{1}{\arctan 2w} dw\right] dz} < \frac{\arctan 2(x - y)}{\arctan 2y} + 1.$$

Let $y = 0.5$ and $x = 1$. Mathematica gives $\int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w} dw\right] dz = 0.391128$ and $\int_0^x \exp\left[-\frac{1}{2}\int_z^x \frac{1}{\arctan 2w} dw\right] dz = 0.745072$. Thus, we have:

$$\frac{0.391128}{-0.5 + 0.745072} = 1.59597 < 2 = \frac{\arctan 2(x - y)}{\arctan 2y} + 1,$$

which concludes the verification for the counterexample of PSE existence.

Proof of Theorem 12

The dominant strategy for each bidder in the second price auction is to bid his value: $b^2(t) = t$. Then, the expected payment by a bidder in the second price auction, P^2, is given by:

$$\begin{aligned} P^2 &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} y f(y|x)\,dy \cdot f(x)\,dx = \\ &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} [y - b(y)]\, f(y|x)\,dy \cdot f(x)\,dx + \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y)\, f(y|x)\,dy \cdot f(x)\,dx, \end{aligned}$$

where $b(\cdot)$ gives the equilibrium strategy for symmetric first price auctions. Thus, the first integral can be substituted by $\int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \frac{F(y|y)}{f(y|y)} f(y|x)\,dy \cdot f(x)\,dx$, from the first order condition: $b^{'}(y) = [y - b(y)] \frac{f(y|y)}{F(y|y)}$. The last integral can be integrated by parts, to:

$$\begin{aligned} & \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y)\, f(y|x)\,dy \cdot f(x)\,dx \\ = & \int_{[\underline{t},\bar{t}]} \left[b(x)\, F(x|x) - \int_{[\underline{t},x]} b^{'}(y)\, F(y|x)\,dy \right] \cdot f(x)\,dx \\ = & \int_{[\underline{t},\bar{t}]} b(x)\, F(x|x) \cdot f(x)\,dx - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y)\, F(y|x)\,dy \cdot f(x)\,dx \end{aligned}$$

In the last line, the first integral is just the expected payment for the first price auction, P^1. Thus, we have

$$\begin{aligned} D &= P^2 - P^1 \\ &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \frac{F(y|y)}{f(y|y)} f(y|x)\,dy \cdot f(x)\,dx \\ &\quad - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y)\, F(y|x)\,dy \cdot f(x)\,dx \\ &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} f(y|x) - F(y|x) \right] dy \cdot f(x)\,dx \\ &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\,dy \cdot f(x)\,dx \end{aligned}$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\,d\alpha$, where $L(\alpha|t) = \exp\left[-\int_{\alpha}^{t} \frac{f(s|s)}{F(s|s)} ds \right]$. So, we have

$$
\begin{aligned}
b^{'}(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y)\, d\alpha \\
&= \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha .
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
D &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \left[\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

This is the desired expression. For the counterexample, consider the matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1.6938 & 0.3812 & 0.4140 \\ 0.3812 & 2.1318 & 0.5817 \\ 0.4140 & 0.5817 & 2.4206 \end{bmatrix},
$$

and define the p.d.f. as follows:

$$
f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k} \right] \times \left(\frac{p-1}{k}, \frac{p}{k} \right],
$$

for $m, p \in \{1,2,3\}$ and $k = 3$. This distribution satisfies property V (but not property VI) and has a pure strategy equilibrium. The expected revenue from a second price auction is 0.4295, while the expected revenue of a first price auction is 0.4608, which is nearly 7% above.

■

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