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Rewarding cooperation in social dilemmas

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Abstract

One of the most direct human mechanisms of promoting cooperation is rewarding it. We study the effect of sharing a reward among cooperators in the most stringent form of social dilemma. Thus, individuals confront a new dilemma: on the one hand, they may be inclined to choose the shared reward despite the possibility of being exploited by defectors; on the other hand, if too many players do that, cooperators will obtain a poor reward and defectors will outperform them. By appropriately tuning the amount to be shared we can cast a vast variety of scenarios, including traditional ones in the study of cooperation as well as more complex situations where unexpected behavior can occur. We provide a complete classification of the equilibria of the n-player game as well as of the evolutionary dynamics. Beyond, we extend our analysis to a general class of public good games where competition among individuals with the same strategy exists.

Keywords: Reward; Social dilemma; Prisoner's Dilemma; *n*-player game

1 Introduction

Despite the abundance of altruism in nature, the most widely used models of game theory for the study of cooperative behavior, such as the Prisoner's Dilemma and the game of Public Goods, provide scenarios of evolutionary dynamics where defectors dominate cooperators. These models embody the social dilemma of cooperation in which individuals can benefit from mutual cooperation but they can do better by exploiting the cooperation of others. Hence the importance of supporting mechanisms for cooperation, such as iterated interactions and spatial structure formation, which have been explored by many authors since the seminal contributions of Axelrod (1984) and Nowak and May (1992); see Doebeli and Hauert (2005) for a comprehensive review.

The social dilemma is relaxed in some pairwise interaction games for which, depending on the partner's action, cooperating can be the best option. In the famous Snowdrift game (also known as Chicken or Hawk-Dove game), introduced by Maynard-Smith and G. Price (1973) and Sugden (1986), given a partner's decision, the best response is to do the opposite. On the contrary, in the stag-hunt game, another important metaphor for the study of cooperation proposed by Rousseau (1754) as an example of social contract (Skyrms, 2003), the social dilemma presents itself as a coordination problem, where the best response, given a partner's decision, is to do the same. These models can be endowed with an evolutionary dynamics in which cooperators and defectors can coexist at a stable equilibrium, or in which cooperation either evolves or vanishes depending on the initial configuration of the system (Hofbauer and Sigmund, 1998; Nowak, 2006).

Even though the emergence and maintenance of cooperation, from a significant initial proportion of cooperators, can be explained with these evolutionary models or by implementing mechanisms based on iterated or local interactions in spatial structures or in networks, the emergence of cooperation from a single mutation is still a conundrum. The only game model which provides an evolutionary scenario where a small proportion of cooperators invades defectors is the Harmony game (Licht, 1999), where the social dilemma has completely disappeared and it is better to cooperate regardless the opponent's decision.

In this work we present a non trivial solution of this conundrum which captures the essence of the dilemma. The main idea is inspired by one of the most direct human mechanisms of promoting cooperation: to reward. Sigmund *et al.* (2001) analyze a two-stage binary game in which each player can reward her cooperative opponent at the expense of her own payoff. We consider an interaction group of n individuals sharing among cooperators a reward coming from an external source. A context where this approach may apply is in team formation of animal societies (Anderson and Franks, 2001), e.g. in cooperative hunting (Packer and Ruttan, 1988), and in mutualistic situations in which selection imposed by hosts rewards cooperative behavior (see Kiers *et al.* (2003) and references therein). If the reward is large enough for the interaction group it can remove any social dilemma, but if it is not defectors can still exploiting cooperators despite the reward. In this way, by appropriately tuning the reward, we can cast a vast variety of evolutionary scenarios, including traditional ones in the study of cooperation as well as more complex situations where unexpected and counterintuitive behavior can occur.

In order to consider shared reward in the most stringent form of social dilemma, we set our framework on the Prisoner's Dilemma (PD) game. To this aim we introduce a game in which payoffs can be obtained from two sources: first, all players collect payoffs by playing a PD game with their partners, and second, players who have chosen to cooperate share an extra payoff coming from a pool. In the next section we analyze in detail the static game. Closed formulae for the Nash equilibria are obtained and a parametric characterization in terms of the reward is discussed. Situations in which multiple interior equilibria occur are complete determined, as well as the parametric settings in which equilibria increase, decrease or jump discontinuously with the reward. In section 3 we analyze the evolutionary stability of the equilibria and provide the different asymptotic scenarios of cooperation according to the replicator dynamics. A general framework for the study of cooperation based on n-player games is discussed in section 4. There, we extend some results obtained in previous sections to determine the possible different dynamical scenarios that such general games can have. In particular, we prove that there are no more than two interior equilibria for a wide class of social dilemmas introduced by Hauert *et al.* (2006). Section 5 is of conclusions. Proofs of the main results are in the appendix.

2 The shared reward dilemma

In order to model systems that reward cooperation to raise reciprocal beneficial relationships, we present the following game called the shared reward dilemma.

Consider an assembly of n players, each of whom can choose one out of two actions: cooperate (C) or defect (D) with the rest of the $n-1$ players in an one-shot game (i.e. all player's actions are simultaneously performed). From their pairwise matches, players recollect payoffs according to a PD game. In addition, players who have chosen to cooperate obtain an extra payoff coming from a fixed reward ρ that is evenly distributed among all cooperators. We will be referring to this amount as the *shared reward.*

To provide the strategic form of this game, we introduce some notation. For $1 \leq i \leq n$, X_i denotes the strategy of player i: $X_i = 1$ if she cooperates and $X_i = 0$ if she defects. Let $C_i = \sum_{j \neq i} X_j$ be the number of cooperators among the opponents of player i. Payoffs of pairwise interactions are denoted by the standard parameters of the PD game: a defector that exploits a cooperator obtains the temptation T, but when she faces up another defector gets the punishment P; instead, a cooperator meeting another cooperator receives the reward R (not to be confused with the shared reward that we propose in this work!), but obtains the sucker's payoff S when she confronts a defector. Here $T > R > P > S$ and $S + T < 2R$. By using this payoff notation, the total payoff of player i is given by

$$U_i = \begin{cases} C_i R + (n-1-C_i)S + \dfrac{\rho}{C_i+1}, & \text{if } i \text{ cooperates,} \\ C_i T + (n-1-C_i)P, & \text{if } i \text{ defects.} \end{cases} \quad (1)$$

Since the game is symmetric, in the sense that the payoff to a particular player is independent of her label and only depends on the players' actions, a Nash equilibrium is a common strategy for all players. The space of mixed strategies for player i consists of all $0 \leq q_i \leq 1$ such that $P(X_i = 1) = q_i$ and $P(X_i = 0) = 1 - q_i$. In equilibrium, the random variables $X_1, \ldots, X_n$ are i.i.d. according to a Bernoulli distribution and C_i is a binomial random variable independent of X_i. Then, the expected total payoffs of an arbitrary cooperator and of an arbitrary defector when the rest of the players play an equilibrium, are given by

$$f_C(q) = \mathbb{E}[U_i | X_i = 1] = (n-1)qR + (n-1)(1-q)S + \rho\mu_{n-1}(q), \quad (2)$$

$$f_D(q) = \mathbb{E}[U_i | X_i = 0] = (n-1)qT + (n-1)(1-q)P, \quad (3)$$

where $\mu_m(q) = \mathbb{E}[(S_m+1)^{-1}]$, S_m being a binomial random variable which is the sum of m i.i.d. Bernoulli's random variables with mean q. As it was early observed by Chao and Strawderman

(1972), $\mu_m(q)$ has the expression

$$\mu_m(q) = \begin{cases} 1, & \text{for } q = 0, \\ \dfrac{1-(1-q)^{m+1}}{(m+1)q}, & \text{for } 0 < q \leq 1. \end{cases} \tag{4}$$

2.1 The binary case

Let us begin by analyzing the game when there are just two players involved. This case is particularly simple because it reproduces the major binary games used in the study of cooperation. The payoff matrix of this binary game (Gintis, 2000) can be easily obtained from (1) by making $n = 2$, and it is shown in Table 1. Thus, depending on ρ, the game becomes a:

(i) Prisoner's dilemma, if $T > R + \rho/2$ and $P > S + \rho$;

(ii) Snowdrift, if $T > R + \rho/2$ and $P < S + \rho$;

(iii) Stag-hunt, if $T < R + \rho/2$ and $P > S + \rho$;

(iv) Harmony, if $T < R + \rho/2$ and $P < S + \rho$.

	C	D
C	$R+\rho/2$	$S+\rho$
D	T	P

Table 1: Payoff matrix of the binary case.

In order to reduce the number of parameters, it is convenient to introduce the *scaled shared reward* $\delta = \rho/n(n-1)(T-R)$, which in the binary case becomes $\delta = \rho/2(T-R)$, and the *defection ratio* $\zeta = (T-R)/(P-S)$, which compares the excess of payoff of a defector when she confronts a cooperator with the excess of payoff when she faces up a defector. In terms of these two parameters, the conditions above can be rephrased as

(i) Prisoner's dilemma if $\delta < \min(1/2\zeta, 1)$;

(ii) Snowdrift if $1/2\zeta < \delta < 1$;

(iii) Stag-hunt if $1 < \delta < 1/2\zeta$;

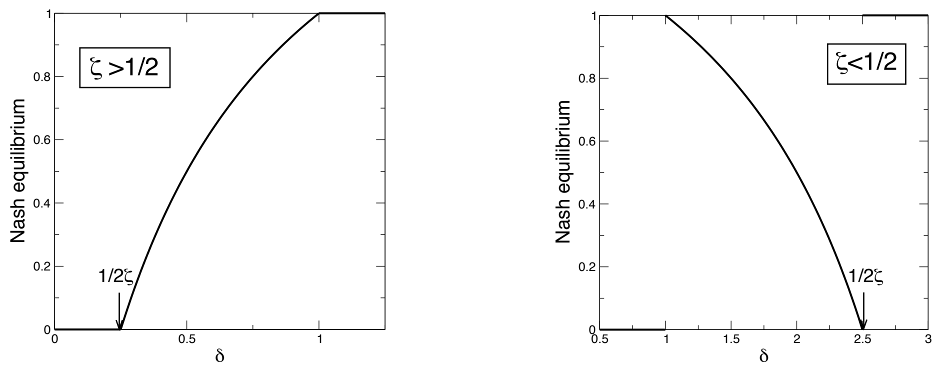

Figure 1: The mixed-strategy Nash equilibrium of the binary game as function of δ. The illustration shows the qualitative behavior of the equilibria for the two scenarios $\zeta > 1/2$ and $\zeta < 1/2$. The graphics correspond to $\zeta = 2$ and $\zeta = 0.2$.

(iv) Harmony if $\delta > \max(1, 1/2\zeta)$.

Therefore, upon increasing δ the game changes from Prisoner's dilemma, where the Nash equilibrium is both players defecting, to Harmony, where it is both players cooperating. But depending on whether $\zeta > 1/2$ or $\zeta < 1/2$, the changes occurs via Snowdrift or via Stag-hunt, respectively. These two games have a Nash equilibrium in mixed strategies, which can be obtained by equating the expected payoff of a cooperator and a defector that confront a mixed strategy in which cooperation is chosen with probability q. These payoffs can be obtained from (2) and (3) by taking $n = 2$ (hence $\mu_1(q) = 1 - q/2$). Accordingly, the Nash equilibrium is

$$q = \frac{1 - 2\delta\zeta}{1 - (1+\delta)\zeta}. \tag{5}$$

If $\zeta > 1/2$, q is a continuous increasing function of δ. If $\zeta < 1/2$, q is a continuous decreasing function on $(1, 1/2\zeta)$, with discontinuity points at $\delta = 1$ and $\delta = 1/2\zeta$. Figure 1 illustrates these two scenarios. It is interesting to notice that, while for $\zeta > 1/2$ there is a continuous monotonic behavior (although with threshold and saturation) of cooperation with the shared reward, for $\zeta < 1/2$ the behavior is discontinuous and counterintuitive (cooperation decreases with increasing shared reward). In section 3 we provide a more sensible discussion of this phenomenon in the context of evolutionary dynamics.

2.2 Nash equilibria for more than two players

Next, we provide a complete characterization of the Nash equilibria that can be reached for different shared rewards when there are more than two players.

Theorem 1. *Let $\delta = \rho/n(n-1)(T-R)$ be the scaled shared reward of the game and $\zeta = (T-R)/(P-S)$ the defection ratio. Then, the following three scenarios can be found for the shared reward dilemma with a number of players $n \geq 3$:*

A) For $\zeta \geq 1/2$,

- *(i) if $\delta \leq 1/n\zeta$, the unique Nash equilibrium is full defection ($q = 0$);*
- *(ii) if $1/n\zeta < \delta \leq 1$, the Nash equilibrium is a continuous function of δ which increases from 0^+ to 1, corresponding to the unique solution on $(0,1]$ of*

$$(\zeta - 1)x + 1 - \delta\zeta \frac{1-(1-x)^n}{x} = 0; \tag{6}$$

- *(iii) if $\delta > 1$ the unique Nash equilibrium is full cooperation ($q = 1$).*

B) For $1/n \leq \zeta < 1/2$,

- *(i) if $\delta \leq 1/n\zeta$ the only Nash equilibrium is full defection;*
- *(ii) if $1/n\zeta < \delta < 1$ the Nash equilibrium is a continuous function of δ which increases from 0^+ to some limit smaller than 1, corresponding to the unique solution on $(0,1)$ of (6);*
- *(iii) if $\delta \geq 1$ there exists $\delta_c > 1$ such that if $\delta > \delta_c$ the unique Nash equilibrium is $q = 1$, whereas if $1 \leq \delta \leq \delta_c$ there are two Nash equilibria corresponding to the solutions $0 < q_1 \leq q_2 \leq 1$ of (6) (equality, $q_1 = q_2$ holds only for $\delta = \delta_c$). The equilibria q_1 and q_2 are continuous monotone functions of δ (increasing and decreasing respectively) and $q_2 = 1$ when $\delta = 1$.*

C) For $\zeta < 1/n$,

- *(i) if $\delta < 1$ the only Nash equilibrium is full defection;*
- *(ii) if $1 \leq \delta < 1/n\zeta$ the Nash equilibrium is a continuous function of δ which decreases from 1 to some limit greater than 0, corresponding to the unique solution on $(0,1]$ of (6);*

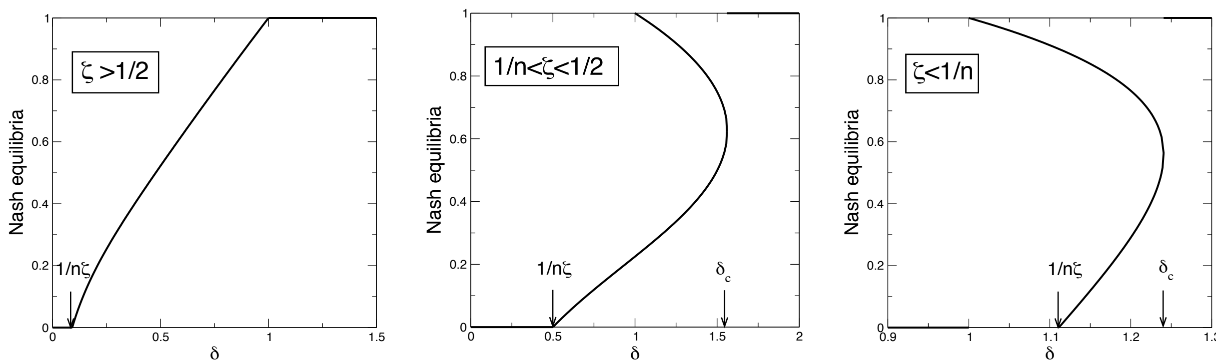

Figure 2: Nash equilibria of the n player game as a function of δ. The illustration shows the qualitative behavior of the equilibria for (a) $\zeta > 1/2$, (b) $1/2 > \zeta > 1/n$ and (c) $\zeta < 1/n$. The graphics correspond to (a) $n = 10$ and $\zeta = 1.1$, (b) $n = 10$ and $\zeta = 0.2$, and (c) $n = 3$ and $\zeta = 0.3$.

(iii) if $\delta \geq 1/n\zeta$ there exists $\delta_c > 1/n\zeta$ such that if $\delta > \delta_c$ the unique Nash equilibrium is $q = 1$, whereas if $1 \leq \delta \leq \delta_c$ there are two Nash equilibria corresponding to the solutions $0 \leq q_1 \leq q_2 < 1$ of (6) (equality, $q_1 = q_2$ holds only for $\delta = \delta_c$). The equilibria q_1 and q_2 are continuous monotone functions of δ (increasing and decreasing respectively) and $q_1 = 0$ when $\delta = 1/n\zeta$.

An upper bound for δ_c is given by

$$\delta_c \leq \frac{1}{4\zeta} \frac{n}{(n-1)} \frac{\left(1 + \dfrac{2\zeta}{n-1}\right)^2}{\left(1 - \dfrac{n-2}{n-1}\zeta\right)}. \tag{7}$$

Proof. See the Appendix.

As an image is worth a thousand words, we outline in Figure 2 the different scenarios of Nash equilibria that the theorem describes.

When the number of players $n \to \infty$ we obtain a simplified asymptotic version of Theorem 1 in which only two of the three cases above remain:

Corollary 1. *Let ρ_n be the shared reward of a shared reward dilemma for n players; let $\delta \geq 0$ be defined as*

$$\delta = \lim_{n \to \infty} \frac{\rho_n}{n(n-1)(T-R)}, \tag{8}$$

and define $\delta_\zeta = 1/4\zeta(1-\zeta)$*. Then, in the limit* $n \to \infty$ *we have*

(i) full defection if $\delta = 0$*;*

(ii) a unique mixed-strategy Nash equilibrium

$$q = \frac{1-\sqrt{1-\delta/\delta_\zeta}}{2(1-\zeta)} \tag{9}$$

if $0 < \delta < 1$*;*

(iii) two mixed-strategy Nash equilibria, $0 < q_1 \leq q_2 < 1$*, where* q_1 *is given by (9) and*

$$q_2 = \frac{1+\sqrt{1-\delta/\delta_\zeta}}{2(1-\zeta)}, \tag{10}$$

if $1 < \delta \leq \delta_\zeta$ *and* $\zeta < 1/2$ *(equality* $q_1 = q_2 = 1/2(1-\zeta)$ *only holds if* $\delta = \delta_\zeta$*), and*

(iv) full cooperation otherwise.

Proof. See the Appendix.

The case $\zeta = 1/2$ is singular because the Nash equilibrium jumps discontinuously from full defection to full cooperation when the shared reward crosses $\delta = 1$. The case $\zeta = 1$ is of particular importance because it reproduces the cost/benefit parametrization of the PD game, by making $T = b$, $R = b - c$, $P = 0$ and $S = -c$, with $b > c > 0$. For this popular framework, suitable for biological applications, our result shows that the equilibrium of the shared reward dilemma only depends on the fixed amount ρ to be shared by the cooperators and of the cost c to cooperate, but it is independent of the benefit b. An analogous result is observed in a spatial evolutionary version of the shared reward dilemma (Jiménez *et al.*, 2007). The limit case $\zeta \to +\infty$ (equivalent to $P \to S^+$) has also received special attention in the analysis of PD games on complex networks (Nowak and Sigmund, 2000; Eguíluz *et al.*, 2005). Using Corollary 1, we can obtain an estimate for the equilibrium when $P \to S^+$, namely the smallest value between $\sqrt{\rho/n(n-1)(T-R)}$ and 1.

3 Evolutionary stability

In population dynamics, the evolution of cooperation can be modeled in several ways. According to the replicator dynamics (Hofbauer and Sigmund, 1998), the cooperative behavior in infinitely large populations is described by

$$\frac{dx}{dt} = x(1-x)\left[f_C(x) - f_D(x)\right], \tag{11}$$

$x(t)$ being the fraction of cooperators at time t and $f_C(x)$ and $f_D(x)$ the average fitness of cooperators and defectors in the population, respectively. In this paper we consider the approach presented by Hauert *et al.* (2006) to study replicator dynamics based on interaction groups of individuals. The standard setup to obtain the replicator equation is to assume a large population of individuals who randomly select partners to play a two-person game. In this alternative approach, players select groups of $n-1$ individuals and play an n-person game instead. This is a sensible approach to study the evolutionary behavior of populations interacting through public goods games (Hauert *et al.*, 2006), and it is also a suitable approach to study the evolutionary behavior of the shared reward dilemma. Let us denote $P_C(k)$ and $P_D(k)$ the payoffs of a cooperator and a defector, respectively, in an interaction group of n players with k cooperators, according to the description of the shared reward dilemma provided in section 2,

$$P_C(k) = (k-1)R + (n-k)S + \frac{\rho}{k}, \quad \text{for } 1 \leq k \leq n, \tag{12}$$

and

$$P_D(k) = kT + (n-1-k)P, \quad \text{for } 0 \leq k \leq n-1. \tag{13}$$

If the population is well-mixed, the number of cooperators at time t in an interaction group of n individuals is a binomial random variable with mean $nx(t)$. Let S_m be a binomial random variable which is the sum of m i.i.d. Bernoulli's random variables with mean $x(t)$. Therefore, the average fitness at time t are given by

$$f_C(x(t)) = \mathbb{E}P_C(S_{n-1}+1) = (n-1)x(t)R + (n-1)[1-x(t)]S + \rho\mu_{n-1}(x(t)) \tag{14}$$

and

$$f_D(x(t)) = \mathbb{E}P_D(S_{n-1}) = (n-1)x(t)T + (n-1)(1-x(t))P, \tag{15}$$

corresponding to formulae (2) and (3) with $q = x(t)$. Substituting (14) and (15) in (11), we model the evolution of cooperation when a shared reward ρ is available for each interaction group.

It is clear that $x = 0$ and $x = 1$ are always zeros of the replicator equation (11), but there will be further zeros at the solutions of $f_C(x^*) = f_D(x^*)$ in the open interval $(0,1)$. By the *folk* *theorem* of evolutionary game theory (Cressman, 2003), the asymptotic stability of these zeros will depend on the sign of $f_C(x) - f_D(x)$. For example, if it is positive, $x = 0$ is unstable whereas $x = 1$ is stable, and if it is negative it is the other way around. But things change if $f_C(x) - f_D(x)$ changes sign in the interval $(0,1)$. By Theorem 1, we can determine how many zeros (none, one or two) has $f_C(x) - f_D(x)$ in the open interval $(0,1)$. On the other hand, since $f_C(0) - f_D(0) = n(n-1)(T-R)(\delta - 1/n\zeta)$, then $x = 0$ is stable if $\delta > 1/n\zeta$ and it is unstable otherwise. Thus, we will find the following stability patterns, depending on the number of zeros of (6) in the interval $(0,1)$:

(I) if $\delta < 1/n\zeta$ (in this case there is either none or just one zero),

  (a) if there are no zeros, $x = 0$ is stable and $x = 1$ unstable;

  (b) if there is one zero $0 < x_1 < 1$, then $x = 0$ is stable, x_1 is unstable and $x = 1$ is stable, with x_1 separating the basins of attraction of $x = 0$ and $x = 1$;

(II) if $\delta > 1/n\zeta$,

  (a) if there are no zeros, $x = 0$ is unstable and $x = 1$ stable;

  (b) if there is one zero $0 < x_1 < 1$, then $x = 0$ is unstable, x_1 is stable and $x = 1$ is unstable;

  (c) if there are two zeros $0 < x_1 < x_2 < 1$, then $x = 0$ is unstable, x_1 is stable, x_2 is unstable and $x = 1$ is stable, and x_2 separates the basins of attraction of x_1 and $x = 1$.

All these situations are illustrated in Figure 3. There are some features worth noticing in this figure. We can see that in the case $\zeta > 1/2$, the outcome of the evolutionary game is as expected: there is a threshold shared reward above which cooperation increases monotonically up to reaching saturation. However, the two cases for which $\zeta < 1/2$ have unexpected outcomes. One feature common to both of them is that there is a critical value of the shared reward, δ_c, at which reached asymptotic cooperation jumps discontinuously from a value $q < 1$ to full cooperation. In both of them there is also a region of δ in which, depending on the initial fraction of cooperators, the outcome may be full cooperation or a smaller fraction of cooperators. This smaller fraction outcome may even be 0 in the case in which $\zeta < 1/n$. We would like to remark here that, being $x = 0$ unstable for any $\delta > 1/n\zeta$, a single mutant in an interaction group of defectors spreads cooperation in the population if the shared reward is sufficiently large.

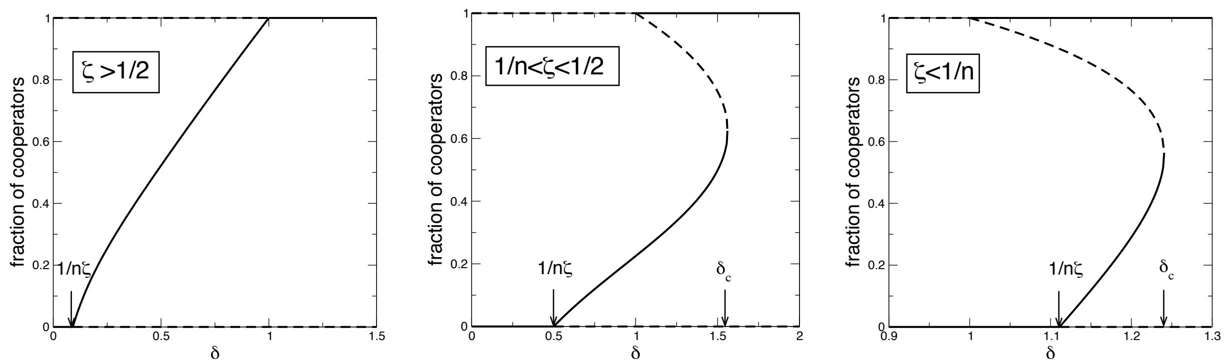

Figure 3: Equilibria of the replicator equation (11). Solid lines represent the asymptotically stable solutions, while dashed lines represent the unstable ones.

4 A general framework for *n*-player games

We can define a more general setup for n-player games with two strategies (cooperate and defect), like the one we have presented in this work, and extract some general conclusions. In this type of n-player games, payoffs are functions of the total number of cooperators, namely k. A rather general representation of the payoffs, which we will be discussing in a moment, amounts to write a cooperator's payoff as

$$P_C(k) = \beta_{\mathrm{C}}(k) - \chi_{\mathrm{C}}(k), \qquad 1 \leq k \leq n \tag{16}$$

and a defector's payoff as

$$P_D(k) = \beta_{\mathrm{D}}(k) - \chi_{\mathrm{D}}(n-k), \qquad 0 \leq k \leq n-1, \tag{17}$$

with β_{C}, β_{D}, χ_{C} and χ_{D} nondecreasing functions.

If $\chi_{\mathrm{C}} = \chi_{\mathrm{D}} \equiv 0$ we recover the usual setup where the payoffs increase with the number of cooperators. For instance, we recover the standard public goods game by taking $\beta_D(k) = bk/n$ and $\beta_C(k) = \beta_D(k) - c$. With a more general increasing function β_C we can model feasible situations where a fixed cost is distributed among cooperators (e.g. $\beta_C(k) = \beta_D(k) - c/k$). In these examples, the difference between payoffs of a cooperator and a defector only enters through the cost of cooperation; however, the general setup also covers situations in which cooperators and defectors are not equally efficient in taking advantage of a common resource (Hauert *et al.*, 2006).

On the other hand, the functions χ_{C} and χ_{D} are suitable to model biological and economical scenarios where competition among individuals with the same strategy exists. A model with competition among defectors has already been considered by Hauert *et al.* (2006), who describe situations in which the strength of competition increases with the number of defectors in the group. Similarly, competition between cooperators may increase with k. The function $\chi_C(k)$ can represent the loss in a cooperator's payoff due to competition with other cooperators (e.g. for their share of a common resource, such as a reward). Likewise, $\chi_D(n-k)$ describes the loss in a defector's payoff due to competition with the other defectors. Under the assumptions we have made, $\chi_D(n-k)$ is a decreasing function of k (i.e. an increasing function of the number of defectors $n-k$). For this reason, $P_D(k)$ is always an increasing function of k and therefore the term $-\chi_D(n-k)$ can be absorbed into $\beta_D(k)$. Thus we can assume, without loss of generality, $\chi_D = 0$ and redefine the defector's payoff as

$$P_D(k) = \beta_D(k), \qquad 0 \leq k \leq n-1. \tag{18}$$

Both, the games introduced by Hauert *et al.* (2006) and the present game are two particular cases of this general framework:

1. The most general model based on synergy and discounting of cooperation in social dilemmas considered by Hauert *et al.* (2006) can be described by the payoff functions

$$P_C(k) = \frac{b}{n}\frac{1-w^k}{1-w} - c, \qquad 1 \leq k \leq n, \tag{19}$$

$$P_D(k) = \frac{b}{n}\frac{1-v^k}{1-v}u^{n-1-k}, \qquad 0 \leq k \leq n-1, \tag{20}$$

   where $0 < u \leq 1$ measures the strength of competition among defectors and $w, v > 0$ are discounting ($0 < w, v \leq 1$) or synergy ($w, v > 1$) factors. Since the payoffs (19) and (20) are increasing functions of k, we can just take $\chi_C \equiv 0$ to express the model in the normal form (16) and (18).

2. The normal form for the shared reward dilemma can be obtained by setting

$$\beta_C(k) = k(R-S+\rho) + nS, \qquad \chi_C(k) = R + \rho\frac{k^2-1}{k}, \qquad 1 \leq k \leq n, \tag{21}$$

$$\beta_D(k) = k(T-P) + (n-1)P, \qquad 0 \leq k \leq n-1. \tag{22}$$

Repeating the arguments of section 3, the average fitnesses in a well-mixed population with a fraction of cooperators equal to x are

$$f_{\mathrm{C}}(x) = \mathbb{E}\beta_{\mathrm{C}}(Z+1) - \mathbb{E}\chi_{\mathrm{C}}(Z+1), \qquad f_{\mathrm{D}}(x) = \mathbb{E}\beta_{\mathrm{D}}(Z), \tag{23}$$

Z being a binomial random variable with mean $(n-1)x$. As we have discussed in section 3, the dynamics can be fully characterized by knowing the number of interior rest points of the replicator equation, i.e. the solutions of $f_{\mathrm{C}}(x) - f_{\mathrm{D}}(x) = 0$ in the open interval $(0,1)$. The difference $f_C(x) - f_D(x)$ is the Bernstein polynomial of degree $n-1$ (Devore and Lorentz, 1993) of the function $\phi : [0,1] \to \mathbb{R}$ defined by

$$\phi(x) = \beta_{\mathrm{C}}\big(x(n-1)+1\big) - \chi_{\mathrm{C}}\big(x(n-1)+1\big) - \beta_{\mathrm{D}}\big(x(n-1)\big). \tag{24}$$

The number of zeros of a Bernstein polynomial on $(0,1)$ is bounded above by the number of sign changes of the corresponding function (Devore and Lorentz, 1993), $\phi(x)$ in our case. This permits to obtain an upper bound to the number of interior rest points straight away from the analysis of the function $\phi(x)$. Specifically, in standard biological and economical settings, where functions β_{C}, χ_{C} and β_{D} have a defined convexity/concavity, such analysis may be straightforward. Incidentally, Bernstein polynomials (the fitness functions that we are considering) are shape preserving, they conserve the convexity of the function that generates them. For the synergy/discounting model considered by Hauert *et al.* (2006), a straightforward analysis of its ϕ function permits to prove that there can be no more than two interior rest points.

5 Conclusions

In this paper we have studied the effect of rewarding cooperation in a strict social dilemma. The shared reward mechanism is based on the distribution of a fixed amount among all cooperative individuals. By adding this payment to the standard payoffs of the Prisoner's Dilemma, cooperators and defectors in an interaction group confront a dilemma: on the one hand, individuals may be inclined to choose for shared reward despite the possibility of being exploited by defectors; on the other hand, if too many players do that, cooperators will obtain a poor reward and defectors will outperform them. In the simplest case with only two players, we recover the traditional binary games for the study of cooperation where the social dilemma is relaxed: stag hunt and snowdrift.

Although intuition suggests that in this game there should be a threshold shared reward above which cooperation increases monotonically up to reaching saturation, the game exhibits more complex situations. Scenarios with multiple interior equilibrium points can be obtained where there are critical values of the shared reward at which cooperation jumps discontinuously. Also, counterintuitive behavior where cooperation decreases as the shared reward increases can be obtained. All arising scenarios have been characterized for the static game as well as for an evolutionary version of the game based on the replicator dynamics. All in all we can conclude that the effects of rewarding cooperation are not as straightforward as one might initially think, and demand a more careful analysis. The origin of this complexity lies in the dilemma that the players confront and the impossibility to know *a priori* how much reward a player can get by cooperating.

We have also provided a general setup to understand two-strategy games in which the payoffs only depend on the total number of cooperators. The shared reward dilemma, as well as versions of the public good games endowed with mechanisms that increase (synergy) or decrease (discounting) the marginal contribution of each additional cooperator (Hauert *et al.*, 2006), are all examples of such general games. For them we provide an upper bound to the number of interior rest points that the corresponding evolutionary games may have.

One important issue for the shared reward dilemma is where this shared reward comes from. In the Introduction we have mentioned situations in Biology which can fit the setup of the shared reward dilemma, as well as mechanisms of direct rewarding to foster more social behavior. To name just one, companies have realized the need of searching for mechanisms that motivate, incentive or encourage cooperative behavior among their employees in order to contribute to the effective success of the teamwork. This context leads to another variant that we have not consider here: the case in which the shared reward is detracted from the payoff of all players. This case is particularly interesting for two reasons: first of all, for the feedback mechanism that it implies, and secondly, because it models a common scenario of taxation and subsequent subside of only certain people. Given the complexity of the shared reward game as we have analyzed it here, the results of this new scenario are presumed very rich. This tax-subsidy scenario has already been explored by some of us (Lugo and Jiménez, 2006) for a spatial evolutionary version of it. This item will be the subject of a forthcoming work.

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Appendix

Proof of Theorem 1. For any $x \in [0,1]$, let us define the "loss function"

$$\phi(x) = \frac{f_D(x) - f_C(x)}{(n-1)(P-S)} = \phi_1(x) - \delta\zeta\phi_2(x), \tag{25}$$

where $\phi_1(x) = x(\zeta - 1) + 1$ and

$$\phi_2(x) = n\mu_{n-1}(1,x) = \begin{cases} n, & \text{for } x = 0, \\ \dfrac{1-(1-x)^n}{x}, & \text{for } 0 < x \leq 1. \end{cases} \tag{26}$$

(c.f. eq. (2)–(4)). First of all, for $\delta = 0$ the only root of the loss function is at $x = 1/(1-\zeta)$, which, for any $\zeta > 0$, is outside the interval $[0,1]$. Hence $\phi(x) > 0$ for all $x \in [0,1]$ and the only Nash equilibrium if full defection. Let us henceforth assume $\delta > 0$. Function $\phi_2(x)$ decreases monotonically with x and, for any $n > 2$, is strictly convex within the interval $[0,1]$; instead, $\phi_1(x)$ is a straight line with nonnegative or negative slope depending on whether $\zeta \geq 1$ or $\zeta < 1$, respectively. For reasons that will be clear in a while, we need to consider separately the cases $\zeta \geq 1$, $\zeta < 1/n$ and $1/n \leq \zeta < 1$.

(a) Case $\zeta \geq 1$:

As $\phi_1(x)$ is nondecreasing, the loss function $\phi(x)$ monotonically increases with x and the only Nash equilibrium depends on the signs of $\phi(0) = 1 - \delta\zeta n$ and $\phi(1) = (1-\delta)\zeta$.

(i) If $\delta \leq 1/n\zeta$ we have $0 \leq \phi(0) < \phi(1)$ and the Nash equilibrium is full defection. This equilibrium is strict for $\delta < 1/n\zeta$.

(ii) If $1/n\zeta < \delta < 1$ we have $\phi(0) < 0$ and $\phi(1) > 0$, and the Nash equilibrium in mixed strategies is the solution $0 < q < 1$ of (6). Note that $\phi(x)$ decreases with δ, thus q increases with δ.

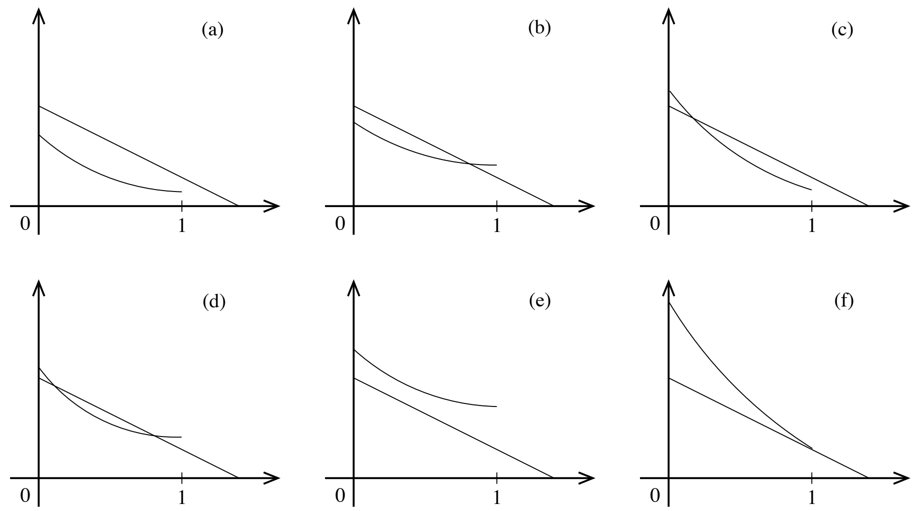

Figure 4: Relative situations of $\phi_1(x)$ and $\delta\zeta\phi_2(x)$ (see text).

(iii) If $\delta \geq 1$ we have $\phi(0) < \phi(1) \leq 0$ and the Nash equilibrium is full cooperation, which is strict for $\delta > 1$.

In the next two cases $\zeta < 1$ and therefore both $\phi_1(x)$ and $\phi_2(x)$ are decreasing functions of x. As $\phi_2(x)$ is convex, the situations that can occur are all sketched in fig. 4.

(b) Case $\zeta < 1/n$:

(i) If $\delta < 1$ then $\phi(0) > 0$ and $\phi(1) > 0$ and we have the situation sketched in fig. 4(a). The only Nash equilibrium is full defection.

(ii) If $1 \leq \delta < 1/n\zeta$ we have $\phi(0) > 0$ and $\phi(1) \leq 0$, so the situation is as sketched in fig. 4(b) and therefore there will be a unique equilibrium $0 < q \leq 1$. Note that $q = 1$ for $\delta = 1$ and decreases as δ goes to $1/n\zeta$.

(iii) If $1/n\zeta \leq \delta$ then $\phi(0) \leq 0$ and $\phi(1) < 0$. Thus we will have one of the two situations plotted in figs. 4(d) and 4(e) depending on the slopes of $\phi_1(x)$ and $\phi_2(x)$ at $x = 0$ at the crossover $\delta = 1/n\zeta$, where $\phi(0)$ changes sign. If $\phi_1'(0) > \phi_2'(0)/n$ the situation will be as illustrated in fig. 4(d), and if $\phi_1'(0) \leq \phi_2'(0)/n$ it will be as in fig. 4(e). In the former case there will be two Nash equilibria, $0 < q_1 < q_2 < 1$, and in the latter the only Nash equilibrium will be $q = 1$. As $\phi_1'(x) = \zeta - 1$ and

$$\phi_2'(x) = \frac{nx(1-x)^{n-1} - 1 + (1-x)^n}{x^2}, \tag{27}$$

we have $\phi_1'(0) = \zeta - 1$ and $\phi_2'(0) = -n(n-1)/2$. The condition $\phi_1'(0) > \phi_2'(0)/n$ reads $\zeta > (3-n)/2$, which holds for any $n \geq 3$. We thus find two equilibria, $0 \leq q_1 < q_2 < 1$, which, upon increasing δ, approach each other (q_1 increases and q_2 decreases) up to δ_c, where they coalesce in a unique Nash equilibrium $q \in (0,1)$. Finally, for $\delta > \delta_c$ the only Nash equilibrium is full cooperation.

(c) Case $1/n \leq \zeta < 1$:

(i) If $\delta < 1/n\zeta$ then $\phi(0) > 0$ and $\phi(1) > 0$ and we have the situation sketched in fig. 4(a). The only Nash equilibrium is again $q = 0$.

(ii) If $1/n\zeta \leq \delta < 1$ (this case is empty if $\zeta = 1/n$) then $\phi(0) \leq 0$ and $\phi(1) > 0$, and we have the situation depicted in fig. 4(c). There is a unique Nash equilibrium $q \in [0,1)$ determined by (6). Also $q = 0$ for $\delta = 1/n\zeta$ and increases as δ goes to 1.

(iii) If $\delta \geq 1$ then $\phi(0) \leq 0$ and $\phi(1) \leq 0$. In this case we may have two equilibria if the situation of fig. 4(d) occurs, or just one if either $\delta > 1$ and we have the situation of fig. 4(e), or $\delta = 1$ and the situation is like in fig. 4(f). The separation between the first case and the last two cases depends on which scenario, fig. 4(d) or fig. 4(f) we have at $\delta = 1$. This, in turn, depends on the slopes of $\phi_1(x)$ and $\phi_2(x)$ at $x = 1$ when $\delta = 1$: if $\phi_1'(1) < \zeta\phi_2'(1)$ then we will have fig. 4(d), and if $\phi_1'(1) \geq \zeta\phi_2'(1)$ we will have fig. 4(f). The former is equivalent to $\zeta < 1/2$, the latter to $\zeta \geq 1/2$. So if $\zeta \geq 1/2$ the only Nash equilibrium is $q = 1$, whereas if $\zeta < 1/2$ there will be, for $1 \leq \delta < \delta_c$, two equilibria, $0 < q_1 < q_2 \leq 1$, which coalesce in a single one at $\delta = \delta_c$. For $\delta > \delta_c$ the only Nash equilibrium is $q = 1$.

The limiting value δ_c can be determined as the value of δ at which the curve $\phi_1(x)$ is tangent to $\delta_c\zeta\phi_2(x)$ at a point $x_c \in (0,1)$. At this point the two equations

$$\phi_1(x_c) = \delta_c\zeta\phi_2(x_c), \qquad \phi_1'(x_c) = \delta_c\zeta\phi_2'(x_c), \tag{28}$$

hold simultaneously. These two equations can be combined to yield

$$\delta_c\zeta(1-x)^n = x_c^2(1-\zeta) - x_c + \delta_c\zeta, \tag{29}$$

$$[(n-1)-(n-2)\zeta]x_c^2 - (n-1+2\zeta)x_c + \delta_c\zeta n = 0. \tag{30}$$

For x_c to exist it is necessary that the second equation has a solution. The condition for this to happen is

$$(n-1+2\zeta)^2 - 4[(n-1)-(n-2)\zeta]\delta_c\zeta n \geq 0. \tag{31}$$

Since $\zeta < 1/2$ then $(n-1)-(n-2)\zeta > 0$, so the above equation holds provided

$$\delta_c \leq \frac{(n-1+2\zeta)^2}{4[(n-1)-(n-2)\zeta]\zeta n} = \frac{\left(1+\frac{2\zeta}{n-1}\right)^2}{4\zeta\left(1-\frac{n-2}{n-1}\zeta\right)}\left(\frac{n-1}{n}\right). \tag{32}$$

This expresses an upper bound for δ_c. ■

Proof of Corollary 1. As $n \to \infty$ only two of the three cases of theorem 1 remain, corresponding now to $\zeta \geq 1/2$ and $0 \leq \zeta < 1/2$. Besides, eq. (6) becomes the quadratic equation

$$(\zeta-1)x^2 + x - \delta\zeta = 0, \tag{33}$$

whose two solutions are

$$q_1 = \frac{1-\sqrt{1-4\zeta(1-\zeta)\delta}}{2(1-\zeta)}, \qquad q_2 = \frac{1+\sqrt{1-4\zeta(1-\zeta)\delta}}{2(1-\zeta)}. \tag{34}$$

Both are real whenever $0 \leq \delta \leq \delta_\zeta = 1/4\zeta(1-\zeta)$. On the other hand, q_1 monotonically increases with δ. If $\zeta \geq 1/2$, q_1 runs from 0 to 1 as δ moves from 0 to 1; if $\zeta < 1/2$, q_1 goes from 0 to $1/2(1-\zeta)$ as δ goes from 0 to δ_ζ. As for q_2, the condition for it to be within the interval $[0,1]$ is $\zeta \leq 1/2$ and $1 \leq \delta \leq \delta_\zeta$. When $\zeta = 1/2$ and $\delta = 1$ then $q_2 = q_1 = 1$. When $\zeta < 1/2$ then q_2 provides a second solution, monotonically decreasing from 1 down to $1/2(1-\zeta)$ as δ runs from 1 to δ_ζ, where it coalesces with q_1.

Finally, for $\delta > \delta_\zeta$ we have

$$(\zeta-1)x^2 + x - \delta\zeta > 0, \tag{35}$$

so the only Nash equilibrium is full cooperation. ■

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