UNIVERSIDAD CARLOS III DE MADRID

working papers

Working Paper 07-55
Economic Series 30
June 2007

Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 916249875

ON THE IMPOSSIBILITY OF REPRESENTING INFINITE UTILITY STREAMS

JUAN A. CRESPO, CARMELO NÚÑEZ, AND JUAN PABLO RINCÓN–ZAPATERO

Abstract

We show that, independently of the topology chosen on the set of all infinity utility streams, there is no Social Welfare Function preserving the von Weizsäcker's overtaking criterion. With our proof we extend the impossibility result of Basu and Mitra.

Keywords: Intergenerational Equity, Pareto, Social Welfare functions, Overtaking Criterion.

JEL Classification: D63, D71, D90.

The first author is partially supported by FEDER-MEC grant MTM2004-06686. The second author is partially supported by MEC grant SEJ2004-00968/ECON. The third author is partially supported by the MEC-FEDER grant MTM2005-06534.

Introduction

Infinity utility streams are a useful tool to understand economic problems with an infinite time horizon. Introduced by Ramsey in [9], they have been used by Koopmans [7], von Weizsäcker [11] and Gale [6] among others to study optimal growth models.

A standard problem in this framework is the choice of an order in the set X of all infinite sequences of consumptions, reflecting the fact that *more is better*. One of the most intensively studied orders is the *overtaking criterion*, introduced by von Weizsächer [11]. With this order, a sequence of consumptions is preferred to another one if, from a certain moment in time, the accumulated consumption of the former is bigger than the accumulated consumption of the latter. This criterion is *ethically acceptable*, in the spirit of Ramsey, because it is not in conflict with an equal treatment for all generations.

In the study of infinite utility streams, a challenging problem is the existence of utility functions representing the order given on the set of these streams, the so called Social Welfare Functions (SWFs). A SWF is a rule that aggregates the consumptions of all generations into a real number preserving the order imposed on X. The existence of a SWF permits not only to decide if an infinite utility stream is better than other, but to say *how much better* are two comparable streams.

In Koopmans [7], Diamond [5] and Basu and Mitra [4] it is shown that a SWF preserving the Pareto order is incompatible with an egalitarian treatment of all generations. More precisely, Koopmans shows that some form of *impatience* arises when the utility function satisfies recursive properties. Diamond proves that a Paretian and egalitarian continuous utility function never exists under certain restrictions on the metric chosen on the set X of infinite utility streams. Finally, Basu and Mitra prove that, independently on the topology chosen on X, it is impossible to construct an utility function that satisfies the *strong Pareto*[1] axiom and which is egalitarian.

Our starting point is the following observation. The strong Pareto axiom demands that, for a given consumption stream, an increment in consumption of just one generation leads to valuate this new stream better than the original. This, in combination

[1] This seems to be the standard name for the axiom, while Basu and Mitra call it just Pareto.

with the egalitarian condition, is mathematically non–realizable, but also seems to be too strict economically because, intuitively, one unit of consumption should not influence in the utility of infinitely many generations. In fact, in Basu and Mitra [3] it is proved that there exist SWFs satisfying the so called *partial Pareto* axiom. This axiom imposes that increments in either finitely many or in all components imply more utility. Nevertheless, nothing is imposed on intermediate cases, that is, where infinitely many components are increased and, at the same time, infinitely many remain the same.

It seems to us that, in an economy with infinitely many generations, the idea of Ramsey of not assigning to a particular generation more importance than to others is preserved if it is considered that the welfare of any generation, in comparison with the welfare of the whole economy, is negligible. In consequence, we propose a criterion where only infinitely many increments affect the value of the SWF. More precisely, a consumption stream is preferred to another one if infinitely many components of the first are greater than the respective ones of the second and *at most* a finite number of them are lower.

Observe that a hypothetical SWF preserving the strong Pareto condition and being egalitarian should also preserve this new order. The converse is obviously not true, so one can wonder if there exists a SWF preserving the order introduced above. We show that this is impossible if the set of consumptions contains at least two possibilities. In particular, we show that a SWF preserving the overtaking criterion does not exist. We remark that, as in Basu and Mitra [4], our result is independent of the topology chosen on the space of all utility streams.

1. Ordering Infinite Utility Streams.

An infinity utility stream is an infinite sequence $x = (x_1, \dots, x_n, \dots)$ where $x_i \in A$, a subset of $\mathbb{R}$. In other words, $x \in A^{\mathbb{N}}$, the infinite cartesian product of A. We will denote $X = A^{\mathbb{N}}$. We will assume from now on that A contains at least two points, if not the problem is tautological. In fact we focus our attention on the case where $A = \{0, 1\}$, that is, at every date there are only two possibilities of consumption, 0 or 1, although A might be any discrete subset of $\mathbb{R}$ containing 0.

Definition 1.1. (Strong Pareto principle) *Let x, y in X. We will say that $x \succ y$ if $x_i \geq y_i$ $\forall i \in \mathbb{N}$ and there exists $j \in \mathbb{N}$ such that $x_j > y_j$.*

The strong Pareto principle is sensible to increments in *at least* one component. As discussed in the Introduction, one unit of consumption should not affect the global utility of an economy where infinitely many consumers are assumed[2]. For this reason we introduce the following variations of the Pareto axiom.

Definition 1.2. (Infinite Pareto principle) *Let x, $y \in X$, we say that $x \succ_{\mathcal{I}} y$ if*

(1) *$x_i \geq y_i$ $\forall i \in \mathbb{N}$, and*
(2) *There exists $A \subset \mathbb{N}$ with $\sharp A = \infty$ such that $x_j > y_j$ $\forall j \in A$.*

Remark 1.3. *In the above definition, if we set $A = \mathbb{N}$ for all x and y, then we obtain the weak Pareto principle.*

Note that, if $x \succ_{\mathcal{I}} y$ then $x \succ y$ but, of course, the converse is not true. An alternative way to define this order is to say that $x \succ_{\mathcal{I}} y$ if the following conditions are satisfied:

(1) $x_j - y_j \geq 0$ $\forall j$
(2) There exists $(s_n)_{n \geq 1}$, an strictly increasing sequence in $\mathbb{N}$, such that $x_{s_j} - y_{s_j} = 1$.

The Infinite Pareto axiom is affected by increments in infinitely many components but, in principle, variations in a finite number of them give no information. In consequence, we are going to define a less strict axiom such that, given two comparable consumption streams (in the sense of Definition 1.2), changes in just a finite number of components in any of them maintains the strict preference.

Definition 1.4. (Infinite quasi–Pareto principle[3]) *Given x, $y \in X$, we say that $x \succ_{\mathcal{I}q} y$ if there exists a finite permutation of the components of x, σx, such that $\sigma x \succ_{\mathcal{I}} y$.*

[2]This point of view agrees with Aumann's [2], where an individual in an economy with a continuum of agents is negligible. This fact has also been pointed out by Lauwers [8]

[3]We have called it quasi-Pareto because the principle has the flavor of a Paretian axiom, but notice that it is quite different, because it makes socially preferable an increment of consumption in infinitely many generations although finitely many may be worse in the proposed distribution.

An alternative way to define this strict preference relation is $x \succ_{\mathcal{I}q} y$ if

(1) There exists $(s_n)_{n=1}^{\infty}$, an strictly increasing sequence in $\mathbb{N}$, such that $y_{s_j} = 0 < 1 = x_{s_j}$.
(2) Only for certain $r_1, r_2, \ldots, r_k$ in $\mathbb{N}$, $x_{r_j} = 0 < 1 = y_{r_j}$ where k and $r_1, r_2, \ldots, r_k$ depend on x and y. Note that $k = 0$ means that no such r_j would exist.

Remark 1.5. *Observe that $x \succ_{\mathcal{I}q} y$ if and only if there are infinitely many components of x that are strictly bigger that the respective components of y and* at most *there is a finite number of components of y which are strictly bigger than the respective in x. In particular, $x \succ_{\mathcal{I}} y$ implies $x \succ_{\mathcal{I}q} y$.*

2. Social Welfare functions and orders.

A Social Welfare Function (SWF) is a function $W : X \to \mathbb{R}$. In this section we analyze the relationship between SWFs preserving the different versions of Pareto principles defined in the previous section.

Definition 2.1. *We say that a SWF W is:*

(1) *Strong Paretian if given $x \succ y$ in X, then $W(x) > W(y)$.*
(2) *Infinite Paretian if given $x \succ_{\mathcal{I}} y$ in X, then $W(x) > W(y)$.*
(3) *Infinite quasi–Paretian if given $x \succ_{\mathcal{I}q} y$ in X, then $W(x) > W(y)$.*

Obviously, if W is Strong Paretian, then W is Infinite Paretian. Now we recall the concept of equity among generations introduced by Diamond [5], in order to compare the types of welfare functions just defined.

Definition 2.2. *Let W be a SWF. We say that W is* egalitarian *(or that W satisfies the* equity *condition) if given $x = (x_1, \ldots, x_n, \ldots) \in X$, then for all $i, j \in \mathbb{N}$*

$$W(x_1, \ldots, x_i, \ldots, x_j, \ldots) = W(x_1, \ldots, x_j, \ldots, x_i, \ldots).$$

This is equivalent to say that the value of $W(x)$ is not affected by finite permutations of the components of x. Next, with the help of this definition, we can state the main relationships between the different types of SWF defined above.

Lemma 2.3. *Let W be a SWF. Then,*

(1) *If W is Infinite quasi–Paretian, then W is Infinite Paretian.*
(2) *If W is Infinite Paretian and egalitarian, then W is Infinite quasi–Paretian.*

Proof. The first implication is obvious by Remark 1.5. For the second one, let W be an Infinite Paretian SWF that satisfies equity. If $x \prec_{\mathcal{I}q} y$, then there exists $(s_n)_{n\geq 1}$, an strictly increasing sequence in $\mathbb{N}$ such that $x_{s_j} = 0 < y_{s_j} = 1$ and, only for certain $r_1, r_2, \ldots, r_k$ in $\mathbb{N}$, one has $y_{r_j} = 0 < 1 = x_{r_j}$. For $j = 1, \ldots, k$, rearranging the components x_{r_j} into the components x_{s_j} and viceversa is just a finite permutation of the components of x. Call this new vector $\sigma x = ((\sigma x)_1, \ldots, (\sigma x)_n, \ldots)$. By the equity condition $W(x) = W(\sigma x)$. Moreover, by construction $y_i - (\sigma x)_i \geq 0$ for all i and $y_{s_j} - (\sigma x)_{s_j} = 1$ for $j = k+1, \ldots$, hence $\sigma x \prec_{\mathcal{I}} y$ and $W(x) = W(\sigma x) < W(y)$ as we wanted to prove. ∎

3. The Impossibility Theorem

In this section we will prove that it is impossible to construct a SWF on X preserving the infinite quasi–Paretian order. Lemma 2.3 shows that an impossibility result for such a welfare function implies that neither Strong Paretian nor Infinite Paretian welfare functions can exist if we impose equity among generations.

Let us recall the immersion of the interval $(0,1)$ in $\{0,1\}^{\mathbb{N}}$, given by Sierpinski in [10], which was used in Basu and Mitra [4] to prove their impossibility theorem. Set an enumeration in $\mathbb{Q} \cap (0,1)$ $q_1, \ldots, q_n, \ldots$, and for $r \in (0,1)$ let the (infinite) sequence $i(r) \in \{0,1\}^{\mathbb{N}}$ be defined as follows

$$i(r)_n = \begin{cases} 1 & \text{if } q_n < r \\ 0 & \text{if } q_n \geq r \end{cases} .$$

Remark 3.1.

(1) *$i(r)$ contains infinitely many ones and infinitely many zeros. This is because there are infinitely many rationals q_n in the interval $(0,r)$ (so $i(r)_n = 1$) as well as infinitely many q_m in $[r,1)$ (and hence $i(r)_m = 1$).*
(2) *If $r < s$ are two numbers in $(0,1)$ then $i(r) \prec_{\mathcal{I}q} i(s)$ in $\{0,1\}^{\mathbb{N}}$. Indeed, if $i(r)_n = 1$ one has $q_n < r < s$ and hence $i(s)_n = 1$. On the other hand,*

the interval (r, s) contains infinitely many rationals q_m, so in the components associated to those rationals $i(s)_m = 1$ while $i(r)_m = 0$.

Proposition 3.2. *Let W be a SWF on X which is Infinite quasi–Paretian. Then, the composite $f = W \circ i : (0, 1) \to (0, 1)$ has, at most, a countable set of discontinuities.*

Proof. The composite $f = W \circ i$ is an strictly increasing function from $(0, 1)$ to $(0, 1)$, because i preserves the strict preferences, by Remark 3.1 (2), and so does W, by hypothesis. Then, f could have at most a countable set of discontinuities. ∎

Our main result extends to our framework the Impossibility Theorem of Basu and Mitra [4]. In fact our proof is inspired by theirs, in the sense that we use Sierpinski's construction to produce an strictly increasing function with a non-countable number of discontinuities.

Theorem 3.3. *There does not exist an Infinite quasi–Paretian SWF on X.*

Proof. We will show that if such a W exists, then the composite $f = W \circ i$ is discontinuous at every $r \in (0, 1)$, contradicting Proposition 3.2.

Fix $r \in (0, 1)$ and consider $i(r)$. We are going to construct an element of X, $i(r)^+$, such that $i(r) \prec_{\mathcal{I}q} i(r)^+ \prec_{\mathcal{I}q} i(s)$ for all $r < s$.

Let $q_1, \ldots, q_k, \ldots$ be the enumeration of the rationals in $(0, 1)$ used to define the immersion i. Consider a decreasing sequence of rational numbers $(z_j)_{j \geq 1}$ in $(0, 1)$, converging to r, and define $i(r)^+$ as follows:

$$i(r)^+_n = \begin{cases} 1 \text{ if } q_n < r \\ 1 \text{ if } q_n = z_j \text{ for some } j \\ 0 \text{ otherwise} \end{cases}$$

So $i(r)^+$ contains a 1 in all the components where $i(r)$ already contained a 1. Moreover, as $(z_j)_{j \geq 1}$ is an infinite sequence, $i(r)^+$ has infinitely many components where $i(r)$ was null and $i(r)^+$ is not. In other words, $i(r) \prec_{\mathcal{I}} i(r)^+$, hence $i(r) \prec_{\mathcal{I}q} i(r)^+$.

On the other hand, let $r < s < 1$ and consider $i(s)$. Then $i(r)^+ \prec_{\mathcal{I}q} i(s)$. Indeed, as $(z_j)_{j \geq 1}$ converges to r, there exists n_0 such that $z_n < s$ for all $n > n_0$, hence there is only a finite number of non zero components in $i(r)^+$ that are null in $i(s)$, those associated to $z_1, \ldots, z_{n_0}$. Moreover, there are infinitely many rational numbers in (r,s) that are not in $(z_j)_{j\geq 1}$, hence there are infinitely many no null components in $i(s)$ that are null in $i(r)^+$.

Observe that the construction of $i(r)^+$ does not depend on s. So, for any given $s \in (0,1)$, with $r < s$, one has $i(r) \prec_{\mathcal{I}q} i(r)^+ \prec_{\mathcal{I}q} i(s)$ as desired and then $W(i(r)) < M < W(i(s))$, where $M = W(i(r)^+)$, so $W \circ i$ is discontinuous at every $r \in (0,1)$. ∎

Corollary 3.4. *There does not exist neither an Infinite nor a Strong Paretian SWF on X satisfying the equity condition.*

Remark 3.5. *Note that in Theorem 3.3 we do not impose the equity condition because, in the Infinite quasi–Paretian criterion, there is an implicit weaker notion of equity. Indeed, under our condition, if a sequence is strictly preferred to another, finite permutations in any of the consumption streams preserve the strict preference, while Diamond's definition of equity implies that a sequence is indifferent with any finite permutation of its components, which is a stronger requirement.*

4. On the non representability of von Weizsächer's order

In this section we study von Weiszächer's overtaking criterion and its extension by Gale known as catching up criterion. Following Asheim and Tungodden [1], we recall those definitions.

Definition 4.1. *Let $x, y \in X$. We say that*

(1) $x \succ_O y \Leftrightarrow \exists n_0$ *such that* $\sum_{k=1}^{n} x_k - y_k > 0, \forall n > n_0$ *(overtaking criterion).*

(2) $x \succ_C y \Leftrightarrow \exists n_0$ *such that* $\sum_{k=1}^{n} x_k - y_k \geq 0, \forall n > n_0$ *and, in addition, for infinitely many n one has* $\sum_{k=1}^{n} x_k - y_k > 0$ *(catching up criterion).*

Of course, $x \succ_O y$ implies $x \succ_C y$, but the reciprocal is not true. For instance, if we consider $x = (1,0,1,0,1,\ldots)$ and $y = (0,1,0,1,0,\ldots)$, we have $x \succ_C y$ but $x \not\succ_O y$.

Remark 4.2. *The reader should note that, by the very definition, $x \succeq_C y$ if and only if there exists n_0 such that $\sum_{k=1}^{n} x_k - y_k \geq 0, \forall n > n_0$. Then $x \sim_C y$ if and only if $\exists n_0$ such that $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k, \forall n > n_0$, and hence $x \succ_C y$ as defined.*

If only two consumption possibilities are allowed in each period, then we find the following relationship between Definition 1.4 (infinite quasi–Pareto principle) and Definition 4.1 (overtaking criterion).

Proposition 4.3. *Let $x, y \in \{0,1\}^{\mathbb{N}}$. Then, $x \succ_{\mathcal{I}q} y \Rightarrow x \succ_O y$.*

Proof. For $x, y \in \{0,1\}^{\mathbb{N}}$, consider the sequence $(z_n)_{n\geq 1}$ with $z_n = \sum_{k=1}^n x_k - y_k$. Then, we claim that $x \succ_{Iq} y$ if and only if there exists n_0 such that the sequence $(z_n)_{n>n_0}$ is positive, monotone increasing and $\lim_{n\to\infty} z_n = \infty$. Indeed, given $x, y \in \{0,1\}^{\mathbb{N}}$, consider the sets $A = \{k : x_k = 0 < 1 = y_k\}$ and $B = \{k : x_k = 1 > 0 = y_k\}$. Clearly, $x \succ_{\mathcal{I}q} y$ implies that A is finite and B infinite. Hence, taking n_0 such that $\#\{k \in B : k \leq n_0\} = \#A$, the sequence $(z_n)_{n>n_0}$ is positive and monotone increasing. Finally, in order to prove that $\lim_{n\to\infty} z_n = \infty$, just consider that $z_{n+1} = 1 + z_n$ for $n + 1 \in B$. Now the result follows. ∎

It is obvious that the above criteria are not equivalent even if one works in $\{0,1\}^{\mathbb{N}}$, as the following simple example shows.

Example 4.4. *Consider*

$$x = (1,1,0,1,1,0,1,1,0,\dots) \quad \textit{and} \quad y = (0,0,1,0,0,1,0,0,1,\dots)$$

It is clear that $x \not\succ_{\mathcal{I}q} y$, but $x \succ_O y$, because $\sum_{k=1}^n x_k - y_k = n/3 + \ell > 0$ where we write $n = 3j + \ell$, $0 \leq \ell < 3$.

In much the same way as in Section 3, we can show the impossibility of representing the overtaking criterion by means of a SWF.

Theorem 4.5. *Let $A \subset \mathbb{R}$ containing at least two points. Consider $X = A^{\mathbb{N}}$ the set of infinity utility streams over A. Then, X does not admit a SWF preserving the order given by the overtaking criterion.*

Proof. We argue by contradiction, supposing that W represents the overtaking criterion, that is, $x \succ_O y \Rightarrow W(x) > W(y)$. Then we would have $x \succ_{\mathcal{I}q} y \Rightarrow x \succ_O y$, by Proposition 4.3, and hence $W(x) > W(y)$, by hypothesis. So W is quasi–Paretian, in contradiction with Theorem 3.3. ∎

5. Concluding Remarks

It is proved in Basu and Mitra [4] that there does not exist any social welfare function which satisfies the strong Pareto and intergenerational equity axioms. They also address the following question: since the proof does not use the full strength of the strong Pareto axiom, could other weaker versions of the Pareto axiom be sufficient to prove their result? Inspired in Aumann's model of competitive markets with a continuum of agents we propose a modification of the Pareto axiom reflecting the negligibility of a single agent. In the same way, Lauwers [8] criticizes the relevance of a single generation, which is the key point of the strong Pareto axiom.

In consequence, our idea is to introduce what we call the Infinite quasi–Pareto principle, in which the welfare of a single agent (and for extension, of any finite set of agents) is negligible, but not the welfare of any infinite set of agents. We have shown that, even under this weaker principle, the impossibility result of Basu and Mitra [4] remains. So, we rise a general question: could it be introduced a weaker version of our Infinite quasi–Pareto principle and still to maintain the impossibility result, or is the Infinite quasi–Pareto principle the weakest version that we can have, in order to satisfy the impossibility result?

In fact one can address a more specific question: can we have (and construct) a social welfare function, defined on either a discrete or a continuum domain, being infinite Paretian and satisfying a weaker version of equity?

References

[1] Asheim, G. and Tungodden, B.: *Resolving distributional conflicts between generations*, Econom. Theory **24** (2004), no. 1, 221–230.

[2] Aumann, R. J.: *Markets with a continuum of traders*, Econometrica **32** (1964), 39–50.

[3] Basu, K. and Mitra, T.: *On the existence of paretian social welfare relations for infinite utility streams with extended anonymity*, Intergenerational Equity and Sustainability, John Roemer and Kotaro Suzumura (eds.). Palgrave. (Forthcoming)

[4] ______: *Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian*, Econometrica **71** (2003), no. 5, 1557–1563.

[5] Diamond, P. A.: *The evaluation of infinite utility streams*, Econometrica **33** (1965), 170–177.

[6] Gale, D.: *On optimal development in a multi-sector economy*, The Review of Economic Studies **34** (1967), no. 1, 1–18.

[7] Koopmans, T. C.: *Stationary ordinal utility and impatience*, Econometrica **28** (1960), 287–309.

[8] Lauwers, L.: *Intertemporal objective functions: strong Pareto versus anonymity*, Math. Social Sci. **35** (1998), no. 1, 37–55.

[9] Ramsey, F. P.: *A mathematical theory of saving*, The Economic Journal **38** (1928), no. 52, 543–559.

[10] Sierpiński, W.: *Cardinal and ordinal numbers*, Second revised edition. Monografie Matematyczne, Vol. 34, PańStowe Wydawnictwo Naukowe, Warsaw, 1965.

[11] von Weizsächer, C. C.: *Existence of optimal programs of accumulation for an infinite time horizon*, The Review of Economic Studies **32** (1965), no. 2, 85–104.

Juan A. Crespo, Carmelo Núñez, and Juan Pablo Rincón–Zapatero

Departamento de Economía

Universidad Carlos III de Madrid,

E-28903 Getafe, Spain

E-mail: jacrespo@eco.uc3m.es, cnunez@eco.uc3m.es, jrincon@eco.uc3m.es,