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CREDIT RISK WITH SEMIMARTINGALES AND RISK-NEUTRALITY

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A no-arbitrage framework to model interest rates with credit risk, based on the LIBOR additive process, and an approach to price corporate bonds in incomplete markets, is presented in this paper. We derive the no-arbitrage conditions under different conditions of recovery, and we obtain new expressions in order to estimate the probabilities of default under risk-neutral measure.

Keywords: Credit-risk, semimartingales, interest-rate modelling.

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CREDIT RISK WITH SEMIMARTINGALES AND RISK-NEUTRALITY

JESÚS PÉREZ COLINO AND WINFRIED STUTE

(WORKING PAPER)

Abstract. A no-arbitrage framework to model interest rates with credit risk, based on the LIBOR additive process, and an approach to price corporate bonds in incomplete markets, is presented in this paper. We derive the no-arbitrage conditions under different conditions of recovery, and we obtain new expressions in order to estimate the probabilities of default under risk-neutral measure.

1. Preliminaries

1.1. **Introduction.** Notoriously, works in mathematical finance should reflect the market reality, and they have to be comprehensible for practitioners. Unfortunately, the ones which are realistic are not necessarily comprehensible and those comprehensible are not necessarily realistic.

But both are needed. Usually the trade-off between reality and simplicity in modelling is not easy to break, and unfortunately, the mathematics of finance are not effortless, and much market practice is based on soft or partial use of these tools, working and pricing with models that do not reflect in a complete manner what is actually going on.

Basically the main goal of this paper is to develop a sufficiently wide model for corporate bonds with credit risk, and develop a set of mathematical tools and results that would allow the practitioner to simplify this framework and conditions in order to implement these models according to the specific needs of the market (with or without continuity and with or without jumps, with or without credit migration or under different types of default).

This paper is organized as follows:

- In *Section 1* we introduce the basics such as definitions and technical notation that will be used during the whole paper. Additionally, we expose here the different assumptions about the dynamics of forward rate models under a semimartingale framework.
- *Section 2* is devoted to develop the basic expressions for corporate bonds under different recovery frameworks, basically extending the results of Heath, Jarrow and Morton (1992) to our framework.
- In *Section 3* we obtain the no-arbitrage expressions for each model and we derive a risk-neutral form for the probability of default.

1.2. **Basic Assumptions for the risk-free Interest Rates model.** We will consider processes on a complete stochastic basis $(\Omega, \mathcal{G}, \mathbb{P})$. Let $G = \{G_t; t \geq 0\}$ be the **LIBOR additive process** (piecewise stationary process) with a given tenor structure $0 = T_0 < T_1 < ... < T_n = T^\star$ with $T^\star$ fixed. The **LIBOR additive process** G is introduced here as a source of uncertainty in our model. Notice that the trajectories of this process belong to the **Skorohod space** $\mathcal{D}$. We can associate with G_t a random

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measure of its jumps, denoted by $\mu_{\eta(t)}$ for any $t \in [0, T^\star]$, $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \in \left[T_{\eta(t)}, T_{\eta(t)+1}\right)$. Actually, set

$$\mu_{\eta(t)}([0,t], A) = \sum_{0<s\leq t} 1_A(\triangle G(s))$$

and let us introduce the measure $\upsilon_{\eta(t)}$ as

$$\upsilon_{\eta(t)}(A) = \mathbb{E}\left(\mu_{\eta(t)}([0,t], A)\right)$$

is called the **Lévy measure** of the process G.

The **Lévy-Khintchine formula**, have shown that the characteristic function has the form:

$$\begin{aligned}\hat{\mu}_t(z) &= \mathbb{E}(\exp[i\langle z, G(t)\rangle]) \\ &= \prod_{j\leq\eta(t)} \mathbb{E}\left(\exp\left[i\left\langle z, \left(G_{t\wedge T_{j+1}} - G_{T_j}\right)\right\rangle\right]\right) \\ &= \exp\left[\sum_{j\leq\eta(t)} (t\wedge T_{j+1} - T_j)\,\psi_j(z)\right]\end{aligned}$$

with $z \in \mathbb{R}^d$ and

$$\psi_j(z) = i\left\langle\gamma_j, z\right\rangle - \frac{1}{2}\langle z, A_j z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,g\rangle} - 1 - i\langle z, g\rangle 1_{\{|g|\leq 1\}}\right)\upsilon_j(dg), \quad j = 0, 1, ..., n$$

and where A_j is a **symmetric nonnegative-definite** $d\times d$ **matrix**, $\gamma_j \in \mathbb{R}^d$, υ_j is the mentioned **Lévy measure** on $\mathbb{R}^d\backslash\{0\}$ and $g \in \mathbb{R}^d$ satisfying

$$\upsilon_j(\{0\}) = 0 \quad \text{and} \quad \textstyle\int_{\mathbb{R}^d}\left(|g|^2 \wedge 1\right)\upsilon_j(dg) < \infty$$

for any $j = 1, ..., n$.

Under these two conditions, and using **Jacod and Shiryaev** $(2003, II.2.29)$, G is a **special semi-martingale** and G has a well known **Lévy-Itô decomposition** or **canonical representation**:

$$G(t) = \int_0^t \alpha_{\eta(s)}ds + \int_0^t \sigma_{\eta(s)}W(ds) + \int_0^t\int_{|g|\leq 1} g\left(\upsilon_{\eta(s)}(ds, dy) - ds\mu_{\eta(s)}(dy)\right)$$

where $\alpha_{\eta(s)} \in \mathbb{R}^d$, W is a standard d-dimensional **Wiener process** with values in $\mathbb{R}^d$ and with **covariance** operator $A_{\eta(t)} = \left(\sigma_{\eta(t)}(i,j)\right)_{i,j\leq d}$.

Let $r(t)$, $t \geq 0$ be the **short rate process**. If at moment 0 one puts into the bank account 1 unit, then at moment t one has

$$B_t = \exp\left[\int_0^t r(s)ds\right]$$

Let $B(t,T)$ be the market price at moment t of a **bond** paying 1 unit at maturity time T. The **forward rate** $f(t,T)$ **curve** is a function defined for $t \leq T < T^\star$ and such that

$$B(t, T_i) = \exp\left[-\int_t^{T_i} f(t,s)ds\right]$$

We postulate here the following dynamic for the **forward rates**

$$df(t,T) = \alpha_{\eta(t)}(t,T)\,dt + \sigma_{\eta(t)}(t,T)\,dW_t + \int_E h(t,T,x)\,\mathbf{1}_{\{|x|\leq 1\}}\left(\mu - \upsilon_{\eta(t)}\right)(dt, dx)$$

Notice that the usual **short rate** is defined as $r(t) = f(t,t)$.

1.3. **Basic Assumptions for the Credit Risk Model.** In this section, we mainly focus on **corporate (defaultable) bond** featuring two different issues,

- *first*, the dynamic of **defaultable instantaneous forward rates in incomplete markets**, which are specified through the **Heath, Jarrow and Morton** (1992) model, driven by a **LIBOR additive processes**,
- and *second*, we additionally assume that the **credit migration** is modelled by a SDE driven by a **multivariate marked point process**.

In order to achieve this aim, we have to establish some assumptions that will be applied during the whole work.

1.3.1. *Assumptions related with the Credit Risk dynamic.*

(1) Given a fixed horizon date $T^{\star} \in \mathbb{R}_{+}$, let us assume that our continuous-time financial economy 'lives' on a "*sufficiently rich*" **stochastic basis** $(\Omega, \mathbb{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T^{\star}]}$. Notice that in our case, a "*sufficiently rich*" stochastic basis is one such that the filtration is generated by two stochastic processes G and C

$$\mathcal{F}_t = \sigma \{G_s, C_s; 0 \leq s \leq t\} \tag{1.1}$$

which satisfies the "*usual conditions*". Notice that we can define the embedded filtration $\mathcal{G}_t \subseteq \mathcal{F}_t$ such that

$$\mathcal{G}_t = \sigma \left\{G_s, \upsilon ([0,s] \times E), W_s; 0 \leq s \leq t, E \in \mathcal{B}\left(\mathbb{R}^d\right)\right\}$$

and additionally, we can define as well a second embedded filtration $\mathcal{C}_t \subseteq \mathcal{F}_t$ such that

$$\mathcal{C}_t = \sigma \{C_s; 0 \leq s \leq t\}$$

Henceforth, we can define $\mathcal{F}_t$ as the **original full filtration** such that

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{C}_t = \sigma \{\mathcal{G}_s, \mathcal{C}_s; 0 \leq s \leq t\}$$

with respect to which all processes are adapted. In following subsections, assumptions about the nature of G and C are detailed.

Additionally to these three filtrations, we also have three smaller filtrations $\mathbb{F}'$, $\mathbb{G}'$ and $\mathbb{C}'$ that will be called **observed filtrations** such that $\mathcal{F}'_t = \mathcal{G}'_t \vee \mathcal{C}'_t$ and $\mathcal{F}'_t \subseteq \mathcal{F}_t$. They are originated directly from the observed time series of market prices and notice that implicitly, we are assuming different notions of equivalent martingale measures, according to which filtration we are interested in.

(2) We assume that the process G is a **LIBOR additive process**. This process is basically a piecewise stationary Lévy process, and according to the results obtained in Colino and Stute (2008), this process has an **infinitely divisible and self-decomposable distribution** and it admits the **Lévy-Khintchine formula** and the Lévy-Itô **decomposition**.

(3) On the other hand, we are assuming also that the **credit quality** of corporate debt is represented by the random variable C categorized into a finite number of (mutually disjoint) **credit rating classes** (credit classes, for short). Each **credit class** is represented by one of $m+1 \in \mathbb{N}^{+}$ elements of a finite state space, say $\mathcal{K} = \left\{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, 1\right\}$ (state space). By convention, the state 1 is always assumed to correspond to the **default event**. In addition, the states are ordered so that the state 0 represents the highest ranking, whereas the state $\frac{m-1}{m}$ represents the lowest ranking.

Let us define the **credit migration process** by C_t for any $0 \leq t \leq T^{\star}$ as a random variable on $(\Omega, \mathcal{C}, \mathbb{P})$ adapted to the filtration $\mathcal{C}{=}(\mathcal{C}_t)_{t \in [0,T^{\star}]}$. Let us assume that the dynamic of this process can be defined by the following stochastic differential equation with values in $[0, T^{\star}] \times \mathcal{K}$

$$dC_t = \sum_{a,b \in \mathcal{K}} (b-a)\, 1_{\{C_{t-}=a\}} dN_{ab}(t)\,,\ C_0 \in \mathcal{K} \backslash \{1\} \tag{1.2}$$

where both the $m+1$-vector point process $N_t = \left(N_{a,0}(t), ..., N_{a,\frac{m-1}{m}}(t), N_{a,1}(t)\right)$, such that

$$N_{ab}(t) \text{ has } (\mathbb{P}, \mathcal{C})\text{-intensity } \lambda_{ab}(t) \text{ for } a, b \in \mathcal{K}$$

where $\lambda_{ab} : [0, T^{\star}] \times \mathbb{R}^d \to [0, \infty]$ are bounded functions with bounded gradients.

(4) The double sequence $(\tau_k, C_{\tau_k})_{k \geq 1}$ is called a $\mathcal{C}$-adapted **multivariate marked point process**[1]. Notice that the τ_k's form a sequence of **stopping times** that define the moments of time that the credit rate C changes. More explicitly, for any $k \in \mathbb{N}_+$ the random variable (random stopping time) τ_k will be defined as

$$\begin{cases} \tau_0 := 0 \\ \tau_k := \inf \left\{ t > \tau_{k-1} / \, C_t \neq C_{\tau_{k-1}} \right\} \wedge T^{\star} \\ \tau_k^{\star} := \inf \left\{ t > \tau_{k-1} / \, C_t = 1 \right\} \wedge T^{\star} \end{cases} \tag{1.3}$$

and it represents the time of the k^{th} jump or transition for C. Therefore, $\tau_k : \Omega \to \mathbb{R}_+$ is a non-negative random variable defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For convenience, we assume for any $k \in \mathbb{N}_+$ that $\mathbb{P}\{\tau_k = 0\} = 0$ and hence $\mathbb{P}\{\tau_k > 0\} = 1$.

Notice that the **default time** $\tau_k^{\star}$ is the first moment when the rating process hits the state 1 or reaches the state of default. Default is, by definition, an **absorbing state** and sometimes, for the sake of simplicity, it will appear as $\tau^{\star}$.

(5) The usual approach to **continuous-time Markov** chains is based on **transition semigroups**, and the principal mathematical object is then the **infinitesimal generator**. The transition semigroup is the continuous-time analogue of the iterates of the transition matrix in discrete time.

Given an initial rating C_0 of a defaultable bond, as in the discrete case, the future changes in its ratings are described by a $\mathcal{K}$valued stochastic process C_t referred to as the **migration process** under the real-world probability $\mathbb{P}$ that follows a continuous-time homogeneous $\mathbb{C}$-Markov chain, with the **transition semigroup** P of the following form:

$$\mathbb{P}(t) = [p_{ab}(t)]_{a,b \in \mathcal{K}} \;\; , \text{ with } 0 < t \leq T^{\star} \tag{1.4}$$

where

$$p_{ab}(t) := \mathbb{P}\left(C_{t+s} = b \,| C_s = a \right) \text{ for every } s, t \in [0, T^{\star}] \tag{1.5}$$

In a credit risk framework, we shall postulate that the **default state** $C(t) = 1$ is absorbing, i.e. $p_{1,1}(t) = 1$ or equivalently $p_{1,b}(t) = 0$ for any $b \in \mathcal{K} \backslash \{1\}$.

On the other hand, it is also very well-known that the right-hand side continuity at time $t = 0$ of $P(\cdot)$ implies the right-hand side differentiability at $t = 0$. More specifically, the following finite limit exists for every $a, b \in \mathcal{K}$ and equals

$$\lambda_{ab} := \lim_{t \downarrow 0} \frac{p_{ab}(t) - p_{ab}(0)}{t} = \lim_{t \downarrow 0} \frac{p_{ab}(t) - \delta_{ab}}{t}. \tag{1.6}$$

Observe that for every $a \neq b$ we have $\lambda_{ab} \geq 0$, and $\lambda_{aa} = -\sum_{a=0, a \neq b}^{1} \lambda_{ab}$. The matrix $\Lambda := [\lambda_{ab}]_{0 \leq a,b \leq 1}$ is called the **infinitesimal generator matrix** for a Markov chain associated with $P(\cdot)$. Since each entry of λ_{ab} of the matrix Λ can be shown to represent the **intensity** of transition from the state a to the state b, the infinitesimal generator matrix Λ is also commonly known as the **intensity matrix**.

1.3.2. *Assumptions related with the stochastic process.*

(1) Let us define the **LIBOR additive process with the credit rating** G_t^C as a $\mathcal{F}_t$-adapted **LIBOR additive process** that is also a function of the credit state such that

$$G_t^C = G(t, C(t))$$

Notice that $(G_t)_{t \geq 0}$ is a $\mathcal{G}_t$-adapted **LIBOR additive process**[2] on $\mathbb{R}^d$, and $(C_t)_{t \geq 0}$ is a $\mathcal{C}_t$-adapted **multivariate point process** on $\mathcal{K}$. We have in mind a map from $\mathcal{D}^d \times \mathcal{D}$ into $\mathcal{D}^d$, where $\mathcal{D}^d$=$\mathcal{D}\left(\mathbb{R}^d, I\right)$, with $I = [0, T^{\star}] \subset \mathbb{R}^+$, is a **d-dimensional Skorohod space** and $\mathcal{D}$=$\mathcal{D}(\mathcal{K}, I)$ is also a **Skorohod space**, with $\mathcal{K} = \left\{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, 1\right\}$ as a finite space in $[0,1]$, and $t \in [0, T^{\star}]$. For the sake of clarity, in the future the final **d-dimensional Skorohod space** will be denoted

[1] See **Liptser and Shiryaev** (1989) 3.4 p.168 or **Brémaud** (1981) 2.1 p.19

[2] Piecewise stationary Lévy process (process with independet increments, stochastic continuity and piecewise stationary) See **Colino** (2008) for complete definitions and properties.

as $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$. Several proofs and developments in this context will be provided in sections (2.4) and (2.5) in this paper.

(2) We will take for granted the structure of **infinitely divisible distributions** on $\mathbb{R}^d$, and in particular the **Lévy-Khintchine formula**. We have seen that $(G_t^c)_{t\geq 0}$ is a **LIBOR additive process** on $\mathbb{R}^d$, and for every $c\in\mathcal{K}\backslash\{1\}$ given, and for every t, G_t^c has an **infinitely divisible** distribution, and the driving process G_t^c has a **triplet characteristic** $\left(\gamma_j^c(t),A_j^c,\nu_j^c\right)_{j\geq 0}$, that is connected with the mentioned **canonical (Lévy-Itô) decomposition** of G^c for any $c\in\mathcal{K}\backslash\{1\}$ in the following manner:

$$\begin{aligned} G\left(t,c\right) &= G_t^c \\ &= G_0^c+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\alpha_j(u,c)du+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\sigma_j(u,c)dW_u \\ &\quad+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta\left(u,x\right)\left(\mu_j^c-\upsilon_j^c\right)\left(du,dx\right) \end{aligned} \tag{1.7}$$

where $\sigma_{\eta(t)}(u,c)dW_t$ is the **continuous martingale part** of G_t^c and, on the other hand, $\upsilon_{\eta(t)}^c$ is the **random measure** associated with the jumps of G_t^c, and $\mu_{\eta(t)}^c\left(du,dx\right)=du\ F_{\eta(t)}^c\left(dx\right)$ is its (non-random) **compensator**.

Notice that W_t is the usual **standard d-dimensional Brownian motion** and $\sigma_{\eta(t)}(u,c)$ is a d-dimensional vector that is the diagonal of the square-root of the **symmetric nonnegative-definite matrix** $A_{\eta(t)}^c$;

Also we had assumed that the process G^c has jumps **bounded** by a constant $h=1$, however this **truncation function** can be any $h\in\mathbb{R}^+$ by replacing G^c by G^c/h (which has jumps bounded by 1) and $\gamma_{\eta(t)}^c(t)$ by $h\gamma_{\eta(t)}^c(t)$, so the $\left(\mathcal{G}_t\right)^c$ and the rates are unchanged. It is clear that any martingale solution will depend on the choice of the truncation function. In the sequel we fix one truncation function and sometimes do not mention the dependence of the characteristics on this truncation function.

1.3.3. *Assumptions related with the forward rates with credit risk.*

(1) Let us define $f\left(t,T,c\right)$ as the **instantaneous defaultable forward rates** at time $t\in[0,T]$ for any $T<T^\star$ and for every $c\in\mathcal{K}\backslash\{1\}$. It corresponds to the rate that one can contract for a time t, on a loan with credit risk c that begins at date T and is returned an instant later. It is usually defined by

$$f\left(t,T,c\right)=-\frac{\partial\log B(t,T,c)}{\partial T} \tag{1.8}$$

where $B(t,T,c)$ is the value in t of a **zero-coupon bond conditional to the credit rate** c until maturity T, or in other words

$$B(t,T,c):=\left.B(t,T,C_t)\right|_{C_t=c}\ \text{for every } c\in\mathcal{K}\backslash\{1\}$$

and therefore, the conditional zero-coupon bond with maturity T and the credit rate c follows

$$B(t,T,c)=\exp\left\{-\int_t^T f\left(t,s,c\right)ds\right\} \tag{1.9}$$

(2) We assume that the evolution of this forward rate is driven by a d-dimensional **LIBOR additive process** for a given credit rate $c\in\mathcal{K}\backslash\{1\}$ that admits the **Lévy-Itô decomposition,** such that the dynamics of the **instantaneous forward rate** $f\left(t,T,c\right)$ given the credit rating $c\in\mathcal{K}\backslash\{1\}$

in $t \leq T \in I$, under the real-world probability $\mathbb{P}$, which we assume as follows[3]:

$$df(t,T,c) = \alpha_{\eta(t)}(t,T,c)\,dt + \sigma_{\eta(t)}(t,T,c)\,dW_t + \int_E \delta(t,T,x)\left(\mu^c_{\eta(t)} - \upsilon^c_{\eta(t)}\right)(dt,dx) \tag{1.10}$$

when $t \leq T \in I$ where $\eta(t) = \sup\{j \geq 0 : T_j \leq t\}$ with $j = 0, 1, ..., n$ (for the sake of clarity, we will denote this by the generic index j), W_t is a d-dimensional standard **Wiener process** in $\mathbb{R}^d$ and it is identical for any $c \in \mathcal{K}\setminus\{1\}$, $\mu^c_{\eta(t)}$ is a **random measure** for a given credit rating $c \in \mathcal{K}\setminus\{1\}$ such that $m \in \mathbb{N}^+$ with the **compensator** $\upsilon^c_{\eta(t)}(dt,dx)$.

Notice that a forward rate is not a financial asset issued by a company that has the probabilities of default, therefore $c := C_t|\,\mathcal{F}_t \in \mathcal{K}\setminus\{1\}$ for every $t \in [0,T]$, $T \in I$. This means that modelling these forward rates, we are not considering the probabilities of credit migration that usually appear in a specific corporate bond valuation.

On the other hand, we are implicitly assuming that $(f(t,T,c))_{c\in\mathcal{K}\setminus\{1\}}$ is a sequence of **semimartingales**, because G^c is a semimartingale, and also the risk-free forward rate $f(t,T,0) := f(t,T)$ is another semimartingale.

(3) Let us make some assumptions on the coefficients. Basically the functions $\alpha_j : \Omega \times [0,T^\star] \times [0,T^\star] \to \mathbb{R}$ and $\sigma_j : \Omega \times [0,T^\star] \times [0,T^\star] \to \mathbb{R}_+$ for any $j = 0, 1, ..., n$ and both are $\mathcal{R}_+ * \mathcal{B}([0,T^\star])$-measurable. The coefficient $\delta : \Omega \times \mathbb{R}^r \times [0,T^\star] \times [0,T^\star]$ is $\mathcal{R}_+ * \mathcal{B}([\mathbb{R}^r]) * \mathcal{B}([0,T^\star])$ measurable as well, and all the coefficients cited previously are finite for all times t, and fixed $T \geq t$, or in other words

$$\sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\leq j\leq\eta(T)} \int_{t\vee T_j}^{T_{j+1}\wedge T} |\alpha_j(u,s,c)|\,ds \right) du < \infty \tag{1.11}$$

$$\sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\leq j\leq\eta(T)} \int_{t\vee T_j}^{T_{j+1}\wedge T} |\sigma_j(u,s,c)|^2\,ds \right) du < \infty \tag{1.12}$$

and

$$\sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \int_E \left(\sum_{\eta(u)\leq j\leq\eta(T)} \int_{t\vee T_j}^{T_{j+1}\wedge T} |\delta(u,s,x)|^2\,ds \right) \upsilon^c_j(du,dx) < \infty \tag{1.13}$$

Notice that all coefficients are equal to zero for $T < t$, and we are assuming that $E = \mathbb{R}^d$. Also to abbreviate the formulae we will use $\tilde{\mu}^c_j := \upsilon^c_j - \mu^c_j$ where $c \in \mathcal{K}\setminus\{1\}$, at this moment, is a fixed credit rate.

(4) Additionally, for every $s, t, T \in [0,T^\star]$ and $s, t \leq T$, and $c \in \mathcal{K}\setminus\{1\}$ such that there is a constant $\tilde{C} < \infty$ and

$$\begin{aligned} |\alpha_j(s,T,c) - \alpha_j(t,T,c)| &\leq \tilde{C}\,|s-t| \\ |\sigma_j(s,T,c) - \sigma_j(t,T,c)| &\leq \tilde{C}\,|s-t| \\ |\delta(u,s,x) - \delta(u,t,x)| &\leq \tilde{C}\,|s-t| \end{aligned}$$

then the Equation (2.10) will admit a unique (strong) solution (see **Fujiwara and Kunita** (1989), **Tang and Li** (1994) or Theorem V.38 in **Protter** (2004)).

(5) By definition $r^c_t = f(t,t,c)$ is the **instantaneous spot rate** or simply the **spot rate** given a credit rating (called in the literature also as short-rate). Also let us define the concept of **instantaneous spread rate**, as

$$s(t,T_j,c) := f(t,T_j,c) - f(t,T_j,0) \tag{1.14}$$

[3]Notice that if $c = 0$ we are considering the risk-free or default-free case.

1.3.4. *Assumptions related with the Corporate Bond dynamic.*

(1) We consider a **continuous-time trade economy** for every t inside the trading interval $[0, T^{\star}]$ with a fixed $T^{\star} > 0$. Assume the existence of a (frictionless) **continuous-time bond market** where a set of assets $B(t, T, C_{t-})$ stand for the price at time $t \leq T \leq T^{\star}$ of a **zero-coupon bond with rating** $C_{t-} \in \mathcal{K} \backslash \{1\}$, maturity at time $T \leq T^{\star}$ and **recovery-rate** q in case of default.

(2) For the sake of simplicity, let us select a subset of n-corporate bonds with maturity T_i with $i = 0, 1, ..., n$ and $T_j \leq T^{\star}$ for every $i = 0, 1, ..., n$. Notice that a (frictionless) market for T_i-**corporate bonds** with rating $C_{t-} \in \mathcal{K} \backslash \{1\}$ generate a **family of bond prices** for $i = 0, 1, ..., n$ with the same rating C_{t-}. It basically means a finite family of strictly positive real-valued adapted processes $B(t, T_j, C_{t-})$, with $t \in [0, T_i]$, and the terminal (par) value at maturity $B(T_i, T_i, C_{t-}) = 1$ for every $T_i \in [0, T^{\star}]$ given a $C_{t-} \in \mathcal{K} \backslash \{1\}$. Let us assume that the **price process of a defaultable bond with credit migrations and fractional recovery** should satisfy

$$B(t, T_i, C_t) = \mathbb{E}_{\mathbb{Q}_{T_i}} \left(B(t, T_i, 0) \, 1_{\{\tau^{\star} > T_i\}} + q B^{\star} 1_{\{\tau^{\star} \leq T_i\}} \middle| \mathcal{G}_t \right) \tag{1.15}$$

where $\mathbb{Q}_{T_i}$ is the forward martingale measure for the date T_i, for every $i = 0, 1, ..., n$, with $0 \leq t \leq T_i$, and q can be defined as the **recovery rate** or the fractional part of $B^{\star}$ that the investor will recover in case of default, such that $q \in [0, 1]$. Notice that this structure of bond maturities is the time structure that mark the tenor structure in the LIBOR additive process.

(3) Additionally, the value of **the bond in case of default** can be defined as

$$B^{\star} := \begin{cases} = B\left(\tau^{\star}, T_i, C_{\tau_{k-1}}\right) & \rightarrow \text{ market value} \\ = B\left(\tau^{\star}, T_i, 0\right) & \rightarrow \text{ treasury value} \\ = 1 & \rightarrow \text{ par value} \end{cases} \tag{1.16}$$

(4) On the other hand notice that if $C_{t-} \in \mathcal{K} \backslash \{1\}$, we shall interpret $B(t, T_i, C_{t-})$ as the **pre-default value** of a T_i-maturity zero-coupon corporate bond, or more formally

$$\begin{aligned} B(t, T_i, C_{t-}) &= B(t, T_i, 0) \cdot \exp\left(- \int_t^{T_j} s(t, u, C_{t-}) du \right) \\ &= B(t, T_i) \cdot S(t, T_i, C_{t-}) \end{aligned} \tag{1.17}$$

where $s(t, u, C_{t-})$ is the instantaneous spread rate (see expression 2.14).

2. The building-blocks for Interest-Rate modelling

Given the mentioned conditions in the previous section, here we characterize the functional forms of the dynamic of corporate bonds, when the LIBOR additive process is driving the forward rates dynamic, and more specifically, when we consider different frameworks in credit risk. Basically the term structure model is based on an exogenous specification of the dynamics of instantaneous, continuously compounded forward rates $f(t,T,c)$. Our aim in this section is to recover the functional form of corporate bonds from (2.10) for different frameworks of credit risk.

This section is organized as follows:

- *Subsection 1* is devoted to develop the simplest case, the risk-free case, following closely **Bjork et al**. (1997) and **Eberlein et al**. (2006) for the **Heath, Jarrow and Morton** approach (1992), but now, introducing the **LIBOR additive process**.
- Basically the next subsections are extensions of the first one, in the sense that we include different credit risk frameworks for the corporate bond. Therefore, *subsection 2* includes the **credit risk** but without the possibility to have credit migration between different rates.
- And it is in *subsection 3* where we introduce the **credit migration** and we obtain specific functional forms for corporate bonds with these characteristics.

2.1. **Risk-free Bond Market Structure.** In this subsection, we introduce some well-known results due to **Bjork, Di Masi, Kabanov and Runggaldier** (1997) for risk-free bonds, that will be extended later for different credit-risk frameworks. Basically, here, we present the functional expression for the discounted default-free bond when the forward rates are driven by a LIBOR additive process.

According to the assumptions shown in section 2.1.4, it is easy to conclude that we have to consider a model of the **dynamics of the risk-free forward curve** with the following SDE:

$$\begin{aligned} df(t,T,0) &= df(t,T) \\ &= \alpha_{\eta(t)}(t,T)\,dt + \sigma_{\eta(t)}(t,T)\,dW_t + \int_{\mathbb{R}^r} \delta_{\eta(t)}(t,x,T)\left(\upsilon_{\eta(t)} - \mu_{\eta(t)}\right)(dt,dx) \end{aligned}$$

Basically, this model for instantaneous forward rates with credit risk, is an extension of the discretized **Heath, Jarrow and Morton** (1992) model as in **Bjork, Kabanov and Runggaldier** (1997) and **Bjork, Di Masi, Kabanov and Runggaldier** (1997).

Henceforth, we can define the price of a **discounted default-free zero-coupon bond** as

$$\begin{aligned} Z(t,T_i) &= \frac{B(t,T_i)}{B_t} \\ &= \exp\left\{-\int_0^t r(s)\,ds - \int_t^{T_i} f(t,s)\,ds\right\} \end{aligned} \tag{2.1}$$

for any $0 \leq t \leq T_i$, with $T_i \in [0,T^\star]$, and $i = 0,1,...,n$.

Proposition 1. *The* ***discounted bond price process*** $Z(t,T_i)$ *has the form*

$$\begin{aligned} Z(t,T_i) = Z(0,T_i)\exp \quad & \left\{ \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \tilde{a}_j(u,T_i)du + \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} b_j(u,T_i)dW_u \right. \\ & \left. + \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \int_{\mathbb{R}^d} h(u,x,T_i)\,\tilde{\mu}_j(du,dx) \right\} \end{aligned} \tag{2.2}$$

and satisfies the linear stochastic differential equation

$$\frac{dZ\left(t,T_{i}\right)}{Z\left(t-,T_{i}\right)} = \tilde{a}_{\eta(t)}(t,T_{i})dt + b_{\eta(t)}(t,T_{i})dW_{t} + \int_{\mathbb{R}^{d}} h\left(t,x,T_{i}\right)\tilde{\mu}_{\eta(t)}\left(dt,dx\right) + \int_{\mathbb{R}^{d}}\left(e^{h(t,x,T_{i})} - 1 - h\left(t,x,T_{i}\right)\right)\upsilon_{\eta(t)}\left(dt,dx\right) \quad (2.3)$$

with

$$\tilde{a}_{\eta(t)}(t,T_{i}) = a_{\eta(t)}(t,T_{i}) + \frac{1}{2}\left|b_{\eta(t)}(t,T_{i})\right|^{2}$$

and

$$a_{\eta(t)}(t,T_{i}) = -\sum_{\eta(t)\leq j\leq i-1}\int_{T_{j}\vee t}^{T_{j+1}}\alpha_{j}\left(t,s\right)ds \quad (2.4)$$

$$b_{\eta(t)}(t,T_{i}) = -\sum_{\eta(t)\leq j\leq i-1}\int_{T_{j}\vee t}^{T_{j+1}}\sigma_{j}\left(t,s\right)ds \quad (2.5)$$

$$h\left(t,x,T_{i}\right) = -\int_{t}^{T_{j}}\delta\left(t,x,s\right)ds \quad (2.6)$$

Proof. This proof follows the same ideas as in **Heath, Jarrow and Morton** (1992), **Brace, Gatarek and Musiela** (1997), **Bjork et al**. (1997), **Glasserman and Kou** (1997) or **Musiela and Rutkowski** (2004).

Notice that according to the assumptions mentioned in Section 1.2 we have

$$\begin{aligned} B(t,T_{i}) &= \exp\left\{-\int_{t}^{T_{i}} f\left(t,s\right)ds\right\} \\ &= \exp\left\{-\int_{t}^{T_{i}} f\left(0,s\right)ds\right. \\ &\quad -\sum_{\eta(u)\leq j\leq i-1}\int_{T_{j}\vee t}^{T_{j+1}}\left(\sum_{j\leq\eta(t)}\int_{T_{j}}^{t\wedge T_{j+1}}\alpha_{j}\left(u,s\right)du\right)ds \\ &\quad -\sum_{\eta(u)\leq j\leq i-1}\int_{T_{j}\vee t}^{T_{j+1}}\left(\sum_{j\leq\eta(t)}\int_{T_{j}}^{t\wedge T_{j+1}}\sigma_{j}\left(u,s\right)dW_{u}\right)ds \\ &\quad \left.-\sum_{\eta(u)\leq j\leq i-1}\int_{T_{j}\vee t}^{T_{j+1}}\left(\sum_{j\leq\eta(t)}\int_{T_{j}}^{t\wedge T_{j+1}}\int_{\mathbb{R}^{d}}\delta_{j}\left(u,x,s\right)1_{\{|x|\leq 1\}}\tilde{\mu}_{j}\left(du,dx\right)\right)ds\right\} \end{aligned}$$

whence using the stochastic version of *Fubini's theorem* (see **Protter** (1995) Theorem $IV.4.45$) we have

$$\begin{aligned}
\ln B(t,T_i) &= -\int_t^{T_i} f(t,s)\,ds \\
&= -\int_t^{T_i} f(0,s)\,ds \\
&\quad - \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\le j\le i-1} \int_{T_j\vee t}^{T_{j+1}} \alpha_j(u,s)\,ds \right) du \\
&\quad - \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\le j\le i-1} \int_{T_j\vee t}^{T_{j+1}} \sigma_j(u,s)\,ds \right) dW_u \\
&\quad - \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \int_{\mathbb{R}^d} \left(\sum_{\eta(u)\le j\le i-1} \int_{T_j\vee t}^{T_{j+1}} \delta_j(u,x,s)\,1_{\{|x|\le 1\}} ds \right) \tilde{\mu}_j(du,dx)
\end{aligned}$$

Splitting the integrals, we obtain

$$\begin{aligned}
= &\int_0^t f(0,s)\,ds - \int_0^{T_i} f(0,s)\,ds \\
&+ \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\le j\le\eta(t)} \int_{T_j\vee u}^{T_{j+1}\wedge t} \alpha_j(u,s)\,ds - \sum_{\eta(u)\le j\le i-1} \int_{T_j\vee u}^{T_{j+1}} \alpha_j(u,s)\,ds \right) du \\
&+ \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\le j\le\eta(t)} \int_{T_j\vee u}^{T_{j+1}\wedge t} \sigma_j(u,s)\,ds - \sum_{\eta(u)\le j\le i-1} \int_{T_j\vee u}^{T_{j+1}} \sigma_j(u,s)\,ds \right) dW_u \\
&+ \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \int_{\mathbb{R}^d} \left(\sum_{\eta(u)\le j\le\eta(t)} \int_{T_j\vee u}^{T_{j+1}\wedge t} \delta_j(u,x,s)\,1_{\{|x|\le 1\}} ds - \sum_{\eta(u)\le j\le i-1} \int_{T_j\vee u}^{T_{j+1}} \delta_j(u,x,s)\,1_{\{|x|\le 1\}} ds \right) \tilde{\mu}_j(du,dx)
\end{aligned}$$

For the sake of simplicity, let us rename

$$\begin{aligned}
a_j(t,T_i) &:= -\sum_{\eta(t)\le j\le i-1} \int_{T_j\vee t}^{T_{j+1}} \alpha_j(t,s)\,ds \\
b_j(t,T_i) &:= -\sum_{\eta(t)\le j\le i-1} \int_{T_j\vee t}^{T_{j+1}} \sigma_j(t,s)\,ds \\
h(t,x,T_i) &:= -\int_t^{T_j} \delta(t,x,s)\,ds
\end{aligned}$$

and notice that the sum of the four integrals in the left-hand side of the last equality coincides with the expression for the integrated short rate

$$\begin{aligned}\textstyle\int_0^t r_s ds = & \int_0^t f(0,s)ds \\ & + \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\le j\le i-1} \int_{T_j\vee u}^{T_{j+1}} \alpha_j(u,s)\,du \right) ds \\ & + \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\le j\le i-1} \int_{T_j\vee u}^{T_{j+1}} \sigma_j(u,s)\,dW_u \right) ds \\ & + \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \left(\sum_{\eta(u)\le j\le i-1} \int_{T_j\vee u}^{T_{j+1}} \int_X \delta(u,x,s)\,1_{\{|x|\le 1\}}\,\tilde{\mu}_j(du,dx) \right) ds\end{aligned}$$

Hence we obtain

$$\begin{aligned}\ln B(t,T_i) &= \ln B(0,T_i) \\ &+ \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} a_j(u,T_i)du \\ &+ \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} b_j(t,T_i)dW_u \\ &+ \sum_{j\le\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \int_{\mathbb{R}^d} h(u,x,T_i)\,\tilde{\mu}_j(du,dx) \\ &+ \int_0^t r_s ds\end{aligned}$$

and this proves (2.2). By the **Itô formula for semimartingales**[4], we have that

$$\begin{aligned}dZ(t,T_i) &= Z(t-,T_i)\Big[a_{\eta(t)}(u,T_i)dt + \frac{1}{2}\left|b_{\eta(t)}(t,T_i)\right|^2 dt + b_{\eta(t)}(t,T_i)dW_t \\ &+ \int_{\mathbb{R}^d} h(u,x,T_i)\,1_{\{|x|\le 1\}}\,\tilde{\mu}_{\eta(t)}(dt,dx) + \int_{\mathbb{R}^d}\left(e^{h(u,x,T_i)} - 1 - h(u,x,T_i)\right)\mu_{\eta(t)}(dt,dx)\Big]\end{aligned}$$

whence if we define $\tilde{a}_{\eta(t)}(u,T_i) = a_{\eta(t)}(u,T_i) + \frac{1}{2}\left|b_{\eta(t)}(t,T_i)\right|^2$, (2.3) follows. □

2.2. **Conditional Corporate-Bond market structure.** In this subsection we introduce the **credit risk** for corporate bonds. Let us go one step further, including the **credit rating** $c \in \mathcal{K}\backslash\{1\}$ in the model of the dynamics of the **instantaneous forward rate** $f(t,T,c)$ in $t \le T \in I$, using the form (1.10) under $\mathbb{P}$,

$$df(t,T,c) = \alpha_{\eta(t)}(t,T,c)\,dt + \sigma_{\eta(t)}(t,T,c)\,dW_t + \int_{\mathbb{R}^r} \delta(t,T,x)\left(\mu^c_{\eta(t)} - \upsilon^c_{\eta(t)}\right)(dt,dx)$$

Additionally, assume that the price of a **defaultable bond zero coupon bond** with credit rate $C_{t_-} \in \mathcal{K}\backslash\{1\}$ with $c = C_{t_-}$, can be expressed as

$$B(t,T_i,c) = \exp\left\{-\int_t^{T_i} f(t,s,c)\,ds\right\}$$

for any $0 \le t \le T_i$, with $T_i \in [t,T^\star]$, and $i = 0,1,...,n$. Notice that this value is the price of a corporate bond *conditional* that between t and T_i there is no possibility of credit migration. This is theoretically

[4]See **Jacod and Shiryaev** (1987) Ch.1 (4.57), or **Cont and Tankov** (2004) Ch. 8

possible to define but impossible to find in the real world. However, it is worthy to develop this definition as a basic tool for the next section.

Theorem 1. *For any $C_{t_-} \in \mathcal{K} \backslash \{1\}$, the* ***discounted defaultable zero coupon bond price process*** $Z\left(t, T_i, C_{t_-}\right)$ *with* $0 \leq t \leq T_i \leq T^{\star}$, *has the form*

$$Z\left(t, T_i, C_{t_-}\right) = Z\left(0, T_i, C_{t_-}\right) \exp \left\{ \sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} \tilde{a}_j\left(u, T_i, C_{t_-}\right) du + \sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} b_j\left(u, T_i, C_{t_-}\right) dW_u + \sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} \int_{\mathbb{R}^d} h\left(u, x, T_i\right) \tilde{\mu}_j^{C_{t_-}}\left(du, dx\right) \right\} \tag{2.7}$$

and satisfies the following linear stochastic differential equation

$$\begin{aligned} d\left(Z(t, T_i, C_{t_-})\right) &= Z(t_-, T_i, C_{t_-}) \left[\tilde{a}_{\eta(t)}\left(t, T_i, C_{t_-}\right) dt + b_{\eta(t)}(t, T_i, C_{t_-}) dW_t \right. \\ &\quad + \int_{\mathbb{R}^d} h\left(t, x, T_i\right) 1_{\{|x| \leq 1\}} \tilde{\mu}_{\eta(t)}^{C_{t_-}}\left(dt, dx\right) \\ &\quad \left. + \int_{\mathbb{R}^d} \left(e^{h(t,x,T_i)} - 1 - h\left(t, x, T_i\right)\right) \mu_{\eta(t)}^{C_{t_-}}\left(dt, dx\right) \right] \end{aligned} \tag{2.8}$$

with

$$\tilde{a}_{\eta(t)}(t, T_i, C_{t_-}) := a_{\eta(t)}(t, T_i, C_{t_-}) + \frac{1}{2}\left|b_{\eta(t)}(t, T_i, C_{t_-})\right|^2 + s(t, T_i, C_{t_-})$$

and

$$\begin{aligned} a_{\eta(t)}(t, T_i, C_{t_-}) &: = - \sum_{\eta(t) \leq j \leq i-1} \int_{T_j \vee t}^{T_{j+1}} \alpha_j\left(t, s, C_{t_-}\right) ds \\ b_{\eta(t)}(t, T_i, C_{t_-}) &: = - \sum_{\eta(t) \leq j \leq i-1} \int_{T_j \vee t}^{T_{j+1}} \sigma_j\left(t, s, C_{t_-}\right) ds \\ h\left(t, x, T_i\right) &: = - \int_u^{T_i} \delta\left(t, x, s\right) ds \end{aligned}$$

Proof. Following the similar procedure as in the proof of Proposition 1 we obtain

$$\begin{aligned} \ln B\left(t, T_i, C_{t_-}\right) &= - \int_0^{T_i} f\left(0, s, C_{t_-}\right) ds \\ &\quad + \sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} a_j(u, T_i, C_{t_-}) du \\ &\quad + \sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} b_j(u, T_i, C_{t_-}) dW_u \\ &\quad + \sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} \int_{\mathbb{R}^d} h\left(u, x, T_i\right) \tilde{\mu}_j^{C_{t_-}}\left(du, dx\right) \\ &\quad + \sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} r_s^{C_{t_-}} ds \end{aligned}$$

Additionally if we decompose the defaultable short rate into the risk-free short-rate and short term credit spread such that

$$\int_0^t r_t^{C_{t_-}} dt = \int_0^t r_t^0 dt + \int_0^t s\left(t, t, C_{t_-}\right) dt$$

then this proves (2.7). By the **Itô formula for semimartingales**[5], we get from that

$$\begin{aligned} dZ(t,T_i,C_{t\_}) &= Z(t,T_i,C_{t\_})\Bigg[\left(a_{\eta(t)}\left(t,T_i,C_{t\_}\right)+s\left(t,t,C_{t\_}\right)+\frac{1}{2}\left|b_{\eta(t)}(t,T_i,C_{t\_})\right|^2\right)dt \\ &+b_{\eta(t)}(t,T_i,C_{t\_})dW_t \\ &+\int_{\mathbb{R}^d} h\left(t,x,T_i\right)1_{\{|x|\le 1\}}\ \tilde{\mu}_{\eta(t)}^{C_{t-}}\left(dt,dx\right) \\ &+\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h\left(t,x,T_i\right)\right)\mu_{\eta(t)}^{C_{t-}}\left(dt,dx\right)\Bigg] \end{aligned}$$

If we define $\tilde{a}_{\eta(t)}\left(t,T_i,C_{t\_}\right):=a_{\eta(t)}\left(t,T_i,C_{t\_}\right)+s\left(t,t,C_{t\_}\right)+\frac{1}{2}\left|b_{\eta(t)}(t,T_i,C_{t\_})\right|^2$ then (2.8) holds. □

2.3. **Corporate-Bond Market Structure with Credit Migration and Default.** This subsection is devoted to expose the dynamics of the corporate-bond prices with **credit migration**, and **different structures of recovery** in case of default (see assumptions for the credit risk model). Basically here we consider that the **price process of a defaultable bond with credit migrations and fractional recovery** should satisfy

$$B\left(t,T_i,C_t\right)=\mathbb{E}_{\mathbb{Q}_{T_i}}\left(B\left(t,T_i,0\right)1_{\{\tau^\star>T_i\}}+qB^\star 1_{\{\tau^\star\le T_i\}}\middle|\,\mathcal{G}_t\right) \tag{2.9}$$

where $\mathbb{Q}_{T_j}$ is the forward martingale measure for the date T_i for every $i=0,1,...,n$, with $0\le t\le T_i$, where $q\in[0,1]$ represents the fractional part of $B^\star$ that the investor will recover in case of default (recovery rate). More specifically, here we will consider three possible cases

$$B^\star=\begin{cases} =B\left(\tau^\star,T_i,C_{\tau_{k-1}}\right) & \rightarrow \text{ market value} \\ =B\left(\tau^\star,T_i,0\right) & \rightarrow \text{ treasury value} \\ =1 & \rightarrow \text{ par value}\end{cases}$$

In the first case, the investor only can recover a fraction of the **market value** of the bond just quoted in the moment prior to default. In the second case, we consider the recovery in case of a default, of a fractional part of a different bond, usual a **risk-free or treasury bond**, and finally, a fractional recovery of the **par value** of the bond.

2.3.1. *Corporate-bond dynamics with fractional recovery of market value.* Consider the price of a **defaultable zero-coupon bond** with a given recovery rate[6], i.e.,

$$B\left(t,T_i,C_t\right)=\mathbb{E}_{\mathbb{Q}_{T_i}}\left(B\left(t,T_i,0\right)1_{\{\tau^\star>T_i\}}+qB\left(\tau^\star,T_i,C_{\tau_{k-1}}\right)1_{\{\tau^\star\le T_i\}}\middle|\,\mathcal{G}_t\right) \tag{2.10}$$

where $\mathbb{Q}_{T_i}$ is the forward martingale measure for the date T_i for every $i=0,1,...,n$, with $0\le t\le T_i$, $k\in\mathbb{N}_+$, and $C_t\in\mathcal{K}\backslash\{1\}$ with

$$C_t=C_0+\sum_{a,b\in\mathcal{K}}\int_0^t\left(b-a\right)1_{\{C_{s-}=a\}}dN_{ab}\left(s\right)$$

where $C_{t-}\in\mathcal{K}\backslash\{1\}$ is the credit rate in the prior to the moment of jump in t (see assumptions in section 2.1.3).

We define the price of a **discounted defaultable zero coupon bond** as

$$\begin{aligned} Z\left(t,T_i,C_t\right) &= \frac{B\left(t,T_i,C_t\right)}{B_t} \\ &= \mathbb{E}_{\mathbb{Q}_{T_i}}\left(Z\left(t,T_i,0\right)1_{\{\tau^\star>T_i\}}+qZ\left(\tau^\star,T_i,C_{\tau_{k-1}}\right)1_{\{\tau^\star\le T_i\}}\middle|\,\mathcal{G}_t\right)\end{aligned}$$

[5]See **Jacod and Shiryaev** (1987) Ch.1 (4.57), **Goll and Kallsen** (2000) Lemma A.5, or **Cont and Tankov** (2004) Ch. 8

[6]See details in **Schönbucher** (2003)

Proposition 2. *For any* $C_{t-}\in\mathcal{K}\backslash\{1\}$, *the* ***discounted defaultable zero coupon bond price process*** $Z(t,T_i,C_t)$ ***with fractional recovery*** q ***of market value*** *on* $[0,T_i]$ *satisfies the following linear stochastic differential equation*

$$dZ(t,T_i,C_t)=dZ(t,T_i,C_{t-})-l\cdot Z(t,T_i,C_{t-})\cdot d\left(1_{\{C_t=1\}}\right) \tag{2.11}$$

where l *is the loss rate* $l=1-q$.

Proof. Under the fractional-recovery of market value hypothesis[7], since $1_{\{C_t=h\}}$ is a process of finite variation, for any $h=0,..,1-\frac{1}{m},1$, with $m\in\mathbb{N}^+$, therefore, an application of Itô's rule yields

$$dZ(t,T_i,C_t)=\sum_{h=0}^{1-\frac{1}{m}}\left[dZ(t,T_i,C_{t-})1_{\{C_{t-}=h\}}+Z(t,T_i,C_{t-})d\left(1_{\{C_t=h\}}\right)\right]+Z(t,T_i,C_{t-})\cdot q\cdot d\left(1_{\{C_t=1\}}\right)$$

For the sake of clarity, let us define $c=C_{t-}\in\mathcal{K}\backslash\{1\}$. Notice that C_t is a $\mathcal{C}_t$-adapted process. Therefore

$$\sum_{h=0}^{1-\frac{1}{m}}dZ(t,T_i,C_{t-})1_{\{C_{t-}=h\}}=dZ(t,T_i,c)$$

and

$$\begin{aligned}\sum_{h=0}^{1-\frac{1}{m}}Z(t,T_i,C_{t-})d\left(1_{\{C_t=h\}}\right) &= Z(t,T_i,c)\sum_{h=0}^{1-\frac{1}{m}}d\left(1_{\{C_t=h\}}\right)\\ &= -Z(t,T_i,c)d\left(1_{\{C_t=1\}}\right)\end{aligned}$$

using the fact that $\sum_{i=0}^{1-\frac{1}{m}}1_{\{C_t=i\}}=1-1_{\{C_t=1\}}$ whence $\sum_{i=0}^{1-\frac{1}{m}}d\left(1_{\{C_t=i\}}\right)=-d\left(1_{\{C_t=1\}}\right)$.

As a direct result, we have the expression

$$dZ(t,T_i,C_t)=dZ(t,T_i,c)+(q-1)\,Z(t,T_i,c)d\left(1_{\{C_t=1\}}\right)$$

and taking into account that $l=1-q$, we proved (2.11). □

Theorem 2. *For any* $C_{t-}\in\mathcal{K}\backslash\{1\}$, *the* ***discounted defaultable zero coupon bond price process*** $Z(t,T_i,c)$ *on* $[0,T_i]$ *follows*

$$\begin{aligned}\frac{dZ(t,T_i,C_t)}{Z(t_-,T_i,C_{t-})} &= \left(a_j(t,T_i,c)+s(t,t,c)+\frac{1}{2}\left|b_j(t,T_i,c)\right|^2\right)dt\\ &\quad +b_j(t,T_i,c)dW_t\\ &\quad +\int_{\mathbb{R}^d}h(t,x,T_i)\,1_{\{|x|\le 1\}}\,\tilde{\mu}_j^c(dt,dx)\\ &\quad +\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h(t,x,T_i)\right)\mu_j^c(dt,dx)\\ &\quad -l\cdot d\left(1_{\left\{C_{\tau_k}=1\right\}}\right)\end{aligned}$$

Proof. Directly, using Theorem 1 and Proposition 2. □

[7]See **Duffie and Singleton** (1999) for an extensive mathematical work of valuation under a "recovery of market value" framework.

2.3.2. *Corporate-bond dynamics with fractional recovery of treasury.* In this second case, we are assuming that under this model, the issuer of the corporate bond, in the default case, will pay a fractional part of a risk-free bond with identical maturity, such that

$$B(t,T_i,C_t) = \mathbb{E}_{\mathbb{Q}_{T_i}}\left(B(t,T_i,0)\,1_{\{\tau^\star>T_i\}} + qB(\tau^\star,T_i,0)\,1_{\{\tau^\star\le T_i\}}\,\middle|\,\mathcal{G}_t\right) \tag{2.12}$$

Proposition 3. *For any $c = C_{t-}\in\mathcal{K}\backslash\{1\}$, the* ***discounted defaultable zero coupon bond price process*** *$Z(t,T_i,C_t)$* ***with fractional recovery*** *q* ***of treasury-bond value*** *on $[0,T_i]$ satisfies the following linear stochastic differential equation*

$$\begin{aligned}
\frac{dZ(t,T_i,C_t)}{Z(t_-,T_i,C_{t-})} &= \left(a_j(t,T_i,c)+s(t,t,c)+\frac{1}{2}\left|b_j(t,T_i,c)\right|^2\right)dt\\
&\quad +b_j(t,T_i,c)dW_t\\
&\quad +\int_{\mathbb{R}^d} h(t,x,T_i)\,1_{\{|x|\le 1\}}\;\tilde{\mu}^c_j(dt,dx)\\
&\quad +\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h(t,x,T_i)\right)\mu^c_j(dt,dx)\\
&\quad -\left(1-\frac{q}{S(t,T_i,c)}\right)\cdot d\left(1_{\{C_{\tau_k}=1\}}\right)
\end{aligned}$$

where $S(t,T_i,c)=\exp\left\{-\int_t^{T_j} s(t,s,c)\,ds\right\}$ and $j=\eta(t)$.

Proof. In this case, under the fractional recovery of treasury hypothesis, we have

$$\begin{aligned}
dZ(t,T_i,C_t) &= \sum_{h=0}^{1-\frac{1}{m}}\left[dZ(t,T_i,C_{t-})1_{\{C_{t-}=h\}}+Z(t,T_i,C_{t-})d\left(1_{\{C_t=h\}}\right)\right]\\
&\quad +B(t,T_i,0)\cdot q\cdot d\left(1_{\{C_t=1\}}\right)
\end{aligned}$$

where if we set $c=C_{t-}\in\mathcal{K}\backslash\{1\}$, and using the following expressions (see Proposition 2)

$$\sum_{h=0}^{1-\frac{1}{m}} dZ(t,T_i,C_{t-})1_{\{C_{t-}=h\}} = dZ(t,T_i,c)$$

and

$$\begin{aligned}
\sum_{h=0}^{1-\frac{1}{m}} Z(t,T_i,C_{t-})d\left(1_{\{C_t=h\}}\right) &= Z(t,T_i,c)\sum_{h=0}^{1-\frac{1}{m}} d\left(1_{\{C_t=h\}}\right)\\
&= -Z(t,T_i,c)d\left(1_{\{C_t=1\}}\right)
\end{aligned}$$

We obtain the assertion upon using the fact that $\sum_{i=0}^{1-\frac{1}{m}} 1_{\{C_t=i\}} = 1-1_{\{C_t=1\}}$: $\sum_{i=0}^{1-\frac{1}{m}} d\left(1_{\{C_t=i\}}\right) = -d\left(1_{\{C_t=1\}}\right)$.

Therefore we obtain the following expression

$$\begin{aligned}
dZ(t,T_i,C_t) &= dZ(t,T_i,c)-Z(t,T_i,c)d\left(1_{\{C_t=1\}}\right)+\frac{Z(t,T_i,c)}{S(t,T_i,c)}\cdot q\cdot d\left(1_{\{C_t=1\}}\right)\\
&= Z(t,T_i,c)\left[\left(a_j(t,T_i,c)+s(t,t,c)+\frac{1}{2}\left|b_j(t,T_i,c)\right|^2\right)dt\right.\\
&\quad +b_j(t,T_i,c)dW_t+\int_{\mathbb{R}^d} h(t,x,T_i)\,1_{\{|x|\le 1\}}\;\tilde{\mu}^c_j(dt,dx)\\
&\quad \left.+\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h(t,x,T_i)\right)\mu^c_j(dt,dx)\right]\\
&\quad -Z(t,T_i,c)\left(1-\frac{q}{S(t,T_i,c)}\right)d\left(1_{\{C_t=1\}}\right)
\end{aligned}$$

□

2.3.3. *Corporate-bond dynamics with fractional recovery of par value.* Finally, we assume that the issuer of the corporate bond, in the default case, will pay a fractional part of the par value, such that

$$B\left(t,T_j,C_t\right)=\mathbb{E}_{\mathbb{Q}_{T_j}}\left(B\left(t,T_j,0\right)1_{\{\tau^\star>T_j\}}+q1_{\{\tau^\star\leq T_j\}}\middle|\,\mathcal{G}_t\right) \tag{2.13}$$

Proposition 4. *For any* $c=C_{t-}\in\mathcal{K}\backslash\{1\}$, *the* ***discounted defaultable zero coupon bond price process*** $Z\left(t,T_i,C_t\right)$ ***with fractional recovery*** q ***of par value*** *on* $[0,T_i]$ *satisfies the following linear stochastic differential equation*

$$\begin{aligned}\frac{dZ(t,T_i,C_t)}{Z\left(t_-,T_i,C_{t-}\right)} &= \left(a_j\left(t,T_i,c\right)+s\left(t,t,c\right)+\frac{1}{2}\left|b_j(t,T_i,c)\right|^2\right)dt\\ &\quad +b_j(t,T_i,c)dW_t+\int_{\mathbb{R}^d}h\left(t,x,T_i\right)1_{\{|x|\leq1\}}\;\tilde{\mu}_j^c\left(dt,dx\right)\\ &\quad +\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h\left(t,x,T_i\right)\right)\mu_j^c\left(dt,dx\right)\\ &\quad -\left(1-\frac{q}{Z\left(t,T_i,c\right)}\right)\cdot d\left(1_{\{C_{\tau_k}=1\}}\right)\end{aligned}$$

Proof. Similar to the previous Proposition 3. □

3. Absence of arbitrage and dynamics under a martingale measure

Let us recall that we are given a stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$ where $\mathbb{P}$ can be interpreted as the **real-world probability measure** and the **original filtration** $\mathbb{F} = (\mathcal{F}_t)$ with respect to which all processes are adapted; it is the filtration generated by G and C.

Let us define $\mathcal{Q}_{\mathcal{F}}$ as the **set of all probability measures** $\tilde{\mathbb{P}}$ with $\tilde{\mathbb{P}}_{\mathcal{F}_t} \backsim \mathbb{P}_{\mathcal{F}_t}$ for all finite t such that all the discounted zero-coupon bond prices process, $Z(t,T,C_t)$, are local $\tilde{\mathbb{P}}-$ **martingales** for every $T_i \in J$ and relative to $(\mathcal{G}_t)$, or in other words

$$\begin{aligned}\mathcal{Q}_{\mathcal{F}} \quad : \quad &= \Big\{ \tilde{\mathbb{P}} \in \mathcal{M}^1(\Omega, \mathcal{G}) : \tilde{\mathbb{P}}_{\mathcal{G}_t} \backsim \mathbb{P}_{\mathcal{G}_t} \text{ and } (Z(t,T,C_t))_{0 \le t \le T} \\ &\text{for } C_t \in \mathcal{K} = \left\{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, 1\right\} \text{ is a local } \tilde{\mathbb{P}}\text{-martingale for any } 0 \le t \le T \in [0, T^\star] \Big\}\end{aligned}$$

where $\mathcal{M}^1(\Omega, \mathcal{G})$ denotes the set of all probability measures on the measurable space $(\Omega, \mathcal{G})$.

We say that a model admits the existence of an **equivalent martingale measure property** (EMM) if the set $\mathcal{Q}_{\mathcal{F}}$ is *non empty*, and the economy represented by this model is **complete** if this martingale measure is *unique*. Then two questions naturally arise:

(1) *Can our model be an equilibrium or no-arbitrage model?* Or equivalently: *Can we find the martingale measure using our model?* The answer is "no" unless we have a very special structure for the coefficients of our model. The present section is devoted to show these conditions under different credit risk frameworks.
(2) Assuming that there exists an equivalent martingale measure, *Is our model complete?* or in other words, *Is this martingale measure unique?* The answer is "no", even if the dimension of the LIBOR market process is one (see **Eberlein et al.** (2006)) due to the introduction of the credit migration.

This section is basically focused to derive the necessary and sufficient conditions on the forward rate process with credit risk, such that there exists an equivalent martingale measure according to the well-known theorems of asset-pricing that appear in **Harrison and Kreps** (1979) and **Harrison and Pliska** (1981). Basically we generalize the corresponding results of **Heath, Jarrow and Morton** (1992) and **Björk et al.** (1997) and we obtain the no-arbitrage expressions for different frameworks of credit risk and we derive a new risk-neutral form for the probability of default. An outline of this section is as follows:

- In *subsection 1* we mainly focus to obtain the no-arbitrage conditions when we assume corporate bonds with credit migration and fractional recovery of market value. Under this framework we obtain the necessary conditions to have a discounted corporate bond martingale, and we derive some relevant results and expressions for the forward rate process and probability of default.
- In *subsection 2* we derive similar results but when we assume corporate bonds with credit migration and fractional recovery of treasury.
- And identically, in *subsection 3* we study how to obtain the equivalent martingale measure in the case of corporate bonds with credit migration and fractional recovery of par value.

In order to construct this set $\mathcal{Q}_{\mathcal{F}}$ we will follow **Jacod and Shiryaev** (1989), **Björk et al.** (1997) and **Eberlein et al.** (2006). Let us consider the sequence of pairs (β_j, Y_j) such that

- $\beta_j = \left(\beta_j^{i'}(t)\right)_{i' \le d}$ is a predictable $\mathbb{R}^d$-valued process such that

$$\sum_{j \le \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} \left(\beta_j^{'} A_j \beta_j\right) ds < \infty : \text{ for any } t \in [0, T^\star] \text{ a.s.} \tag{3.1}$$

- $Y_j = (Y_j(\omega, t, x))$ is a sequence of $\mathcal{R}^d$-measurable $(0, \infty)$-valued function such that

$$\sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} \int_{\mathbb{R}^d} (Y_j(s, x) - 1)\, d\upsilon_{\eta(t)}(dx, ds) < \infty \ : \ \text{for } t \in [0, T^\star] \ \text{a.s.} \tag{3.2}$$

Using these definitions, let us formulate a modified 'short' version of **Girsanov's Theorem for semimartingales**[8],

Theorem 3. *Let the sequence of pairs $(\beta_j, Y_j)_{j=0,1,\ldots,n}$ be defined as above, and let us define the density process M by*

$$dM_t = M_t \beta_{\eta(t)} dW_t + M_{t-} \int_{\mathbb{R}^d} \left(Y_{\eta(t)}(t, x) - 1\right) \left(\mu_{\eta(t)} - \upsilon_{\eta(t)}\right)(dt, dx)$$

with $M_0 = 1$ and suppose that for all finite t

$$\mathbb{E}^{\mathbb{P}}[M_t] = 1$$

Then there exists a probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}$ locally equivalent to $\mathbb{P}$ with

$$d\tilde{\mathbb{P}}_t = M_t d\mathbb{P}_t$$

such that:

(i): $\tilde{W}_t := W_t - \sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} \beta_j(s)\, ds$ *is a $\tilde{\mathbb{P}}$-Wiener process, and*
(ii): $\upsilon^{\tilde{\mathbb{P}}}_{\eta(t)}(t, dx) = Y_{\eta(t)}(t, x) \cdot \upsilon_{\eta(t)}(t, dx)$ *is the $\tilde{\mathbb{P}}$-compensator of $\mu_{\eta(t)}$.*

Remark 1. *Notice that the real-world probability measure $\mathbb{P}$ itself belongs to $\mathcal{Q}_{\mathcal{F}}$ if we use directly as Girsanov's quantities $\left(\beta_j = 0, Y_j = 1\right)$ for any $j = 0, 1, ..., n$.*

In the following three subsections, for sake of clarity, we assume directly that $\mathbb{P} \in \mathcal{Q}_{\mathcal{F}}$ or equivalently, using the **Girsanov's quantities** $\left(\beta_j = 0, Y_j = 1\right)$ for any $j = 0, 1, ..., n$.

3.1. **Absence of arbitrage condition in a corporate-bond market with fractional recovery of market value.** Basically this subsection is devoted to show the main results concerning the existence of an equivalent martingale measure when we assume credit migration with fractional recovery of market value. They generalize the corresponding results of **Heath, Jarrow and Morton** (1992) and **Björk et al.** (1997). Let us recall that a model has the **equivalent martingale measure property** (EMM) if the set $\mathcal{Q}_{\mathcal{F}}$ is not empty.

Proposition 5. *The initial probability measure $\mathbb{P}$ itself belongs to $\mathcal{Q}_{\mathcal{F}}$ if and only if the following two conditions hold, for every $T_i \in J$:*

$$\sum_{j \leq \eta(t)} \int_{T_j}^{t \wedge T_{j+1}} \int_{\mathbb{R}^d} \left(e^{h(t,x,T_i)} - 1 - h(t, x, T_i)\right) \mu_j^{C_{t-}}(ds, dx) < \infty \tag{3.3}$$

and

$$\tilde{a}_{\eta(t)}(t, T_i, C_{t-}) - l \cdot \lambda_{c_{t_-},1} + \textstyle\int_{\mathbb{R}^d} \left(e^{h(t,x,T_i)} - 1 - h(t, x, T_i)\right) \mu_{\eta(t)}^{C_{t-}}(dt, dx) = 0 \tag{3.4}$$

for any $t \in [0, T_i]$, and any $C_{t_-} \in \mathcal{K} \backslash \{1\}$ where [9]

$$\tilde{a}_{\eta(t)}(t, T_i, C_{t-}) := a_{\eta(t)}(t, T_i, C_{t-}) + s(t, t, C_{t-}) + \frac{1}{2}\left|b_{\eta(t)}(t, T_i, C_{t-})\right|^2 \tag{3.5}$$

[8]The reader can find an extended and complete version of this Girsanov Theorem for Semimartingales in the Chapter 1 of this thesis (Theorem 48).

[9]Notice that if $C_{\tau_k} = 0$ it means default-free asset and $s_t^0(T_i) = 0$. It is the risk-free bond.

Proof. [$\Leftarrow$] According to Theorem 2 and assuming, for the sake of simplicity, that $j=\eta(t)$

$$\begin{aligned}\frac{dZ(t,T_i,C_t)}{Z(t_-,T_i,C_{t-})} &= \left(a_j(t,T_i,c)+s(t,t,c)+\frac{1}{2}\left|b_j(t,T_i,c)\right|^2\right)dt\\ &+b_j(t,T_i,c)dW_t\\ &+\int_{\mathbb{R}^d} h(t,x,T_i)\,1_{\{|x|\le 1\}}\,\tilde{\mu}_j^c(dt,dx)\\ &+\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h(t,x,T_i)\right)\mu_j^c(dt,dx)\\ &-l\cdot d\left(1_{\{C_{\tau_k}=1\}}\right)\end{aligned}$$

Notice that using the **Doob-Meyer** expression

$$d\left(1_{\{C_t=1\}}\right)=dM_1(t)+\lambda_{C_{t_-},1}dt$$

we get

$$\begin{aligned}\frac{dZ(t,T_i,C_t)}{Z(t_-,T_i,C_{t-})} &= \left(\tilde{a}_j(t,T_i,c)-l\cdot\lambda_{C_{t_-},1}\right)dt\\ &+b_j(t,T_i,c)dW_t\\ &+\int_{\mathbb{R}^d} h(t,x,T_i)\,1_{\{|x|\le 1\}}\,\tilde{\mu}_j^c(dt,dx)\\ &+\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h(t,x,T_i)\right)\mu_j^c(dt,dx)\\ &-l\cdot dM_1(t)\end{aligned}$$

which has a local martingale solution if

$$\left(\tilde{a}_j(t,T_i,c)-l\cdot\lambda_{C_{t_-},1}\right)dt+\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h(t,x,T_i)\right)\mu_j^c(dt,dx)=0$$

and

$$\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h(t,x,T_i)\right)\mu_j^c(dt,dx)<\infty$$

[$\Rightarrow$] Let us define the process $M:=\left[Z(t_-,T_i,C_{\tau_k})\right]^{-1}Z(t,T_i,C_t)$ that is a local martingale. Let μ^M be the jump measure of M, and υ^M be its compensator. According to **Jacod and Shiryaev** (1989) $(II.2.29)$ we have that $\int_{\mathbb{R}^d}|x|\wedge|x|^2\,\upsilon_i^M(dx)<\infty$ for finite t. Hence, for every $j=0,1,...,n$,

$$\int_{\mathbb{R}^d}\left(\left|e^{h(t,x,T_i)}-1\right|\wedge\left|e^{h(t,x,T_i)}-1\right|^2\right)\upsilon_j^M(dx)=\int_{\mathbb{R}^d}\left(|x|\wedge|x|^2\right)\upsilon_j^M(dx)<\infty$$

Since $\int_{\mathbb{R}^d}|h(t,x,T_i)|^2\,\upsilon_t(dx)<\infty$ the first condition holds, by virtue of the following inequality

$$e^{h(t,x,T_i)}-1-h(t,x,T_i)\le C\left(\left|e^{h(t,x,T_i)}-1\right|\wedge\left|e^{h(t,x,T_i)}-1\right|^2+h(t,x,T_i)^2\right)$$

where C is a constant. Using the dynamic of $\left[Z(t_-,T_i,C_{t_-})\right]^{-1}Z(t,T_i,C_t)$, we infer that M is a local martingale only if the process given by the left hand side is equal to zero. □

Remark 2. *This is a generalization of the* ***Heath, Jarrow and Morton*** *(1992) drift condition when the credit migration and default are possible. It reveals that in a simple remarkable way, this model can be specified under a (local) martingale measure.*

Remark 3. *Notice that under this framework, the risk-neutral condition has a direct relationship with the intensity matrix or with the default probabilities in the following sense:*

$$\lambda_{c_{t_-},1}=\frac{1}{(1-q)}\left[\tilde{a}_j(t,T_i,C_{t-})+\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)}-1-h(t,x,T_i)\right)\mu_j^{C_{t-}}(dt,dx)\right]$$

Additionally, we can obtain, almost directly, the following results:

- The *first* one (Proposition 6) is related with the **risk-neutral dynamics of instantaneous forward rates**, and this result will be invariant with respect to the recovery framework we use.
- The *second* proposition (Proposition 7) basically exposes the **dynamics of corporate-bonds with fractional recovery of market value** when we impose the risk-neutrality using conditions (2.34) and (2.35).

Proposition 6. *Assume that we specify the* ***forward rate dynamics*** *under a martingale measure* $\mathbb{P}$ *by*

$$df\left(t,T,c\right)=\alpha_{\eta(t)}\left(t,T,c\right)dt+\sigma_{\eta(t)}\left(t,T,c\right)dW_{t}+\int_{\mathbb{R}^{r}}\delta_{\eta(t)}\left(t,x,T\right)\left(\mu_{\eta(t)}^{c}-\upsilon_{\eta(t)}^{c}\right)\left(dt,dx\right). \tag{3.6}$$

Then the following relation holds

$$\begin{aligned}\alpha_{\eta(t)}\left(t,T,c\right) &= -\sigma_{\eta(t)}\left(t,T,c\right)^{\intercal}b_{j}(t,T_{i},c)+s\left(t,t,c\right)\\ &\quad+\int_{\mathbb{R}^{d}}\left(e^{h(t,x,T_{i})}-1-h\left(t,x,T_{i}\right)\right)\mu_{j}^{c}\left(dt,dx\right)\end{aligned} \tag{3.7}$$

Proof. Since we are working under a martingale measure $\mathbb{P}$ we have by Proposition 5 that

$$\tilde{a}_{j}\left(t,T_{i},C_{t-}\right)-l\cdot\lambda_{c_{t_{-}},1}+\int_{\mathbb{R}^{d}}\left(e^{h(t,x,T_{i})}-1-h\left(t,x,T_{i}\right)\right)\mu_{j}^{C_{t-}}\left(dt,dx\right)=0$$

and differentiating this equation with respect T_i gives us the equation (3.6). □

Proposition 7. *The* ***corporate-bond price dynamics*** *under a* ***martingale measure*** $\mathbb{P}$*, and under the* ***fractional recovery of market value*** *hypothesis, will follow the stochastic differential equation*

$$\frac{dB\left(t,T_{i},C_{t}\right)}{B\left(t_{-},T_{i},C_{t_{-}}\right)}=r_{t}dt+b_{j}(t,T_{i},C_{t_{-}})dW_{t}+\int_{\mathbb{R}^{d}}h\left(t,x,T_{i}\right)1_{\{|x|\leq1\}}\,\tilde{\mu}_{j}^{c}\left(dt,dx\right)-l\cdot dM_{1}\left(t\right) \tag{3.8}$$

where $\tilde{\mu}_{j}^{c}$ *is the* $\mathbb{P}$*-compensated Lévy measure,* $l=1-q\in(0,1)$ *is the portion of the market value that the investor will lose in the default case.*

Proof. We know by definition that

$$B\left(t,T_{i},C_{\tau_{k}}\right)=B_{t}Z\left(t,T_{i},C_{t}\right).$$

Then, under the risk-neutral measure,

$$\begin{aligned}dB\left(t,T_{j},C_{\tau_{k}}\right) &= Z\left(t,T_{i},C_{t}\right)dB_{t}+B_{t}dZ\left(t,T_{i},C_{t}\right)\\ &= B\left(t_{-},T_{i},C_{t_{-}}\right)\Big[r_{t}dt+b_{j}(t,T_{i},C_{t_{-}})dW_{t}\\ &\quad+\int_{\mathbb{R}^{d}}h\left(t,x,T_{i}\right)1_{\{|x|\leq1\}}\,\tilde{\mu}_{j}^{c}\left(dt,dx\right)-l\cdot dM_{1}(t)\Big]\end{aligned}$$

□

3.2. **Absence of arbitrage in a corporate-bond market with fractional recovery of treasury.** In this second subsection, we basically reproduce the results given in the last subsection, but under fractional recovery of treasury framework.

Proposition 8. *The initial probability measure* $\mathbb{P}$ *itself belongs to* $\mathcal{Q}_{\mathcal{F}}$ *if and only if the following two conditions hold, for every* $T_i\in J$*:*

$$\sum_{j\leq\eta(t)}\int_{T_{j}}^{t\wedge T_{j+1}}\int_{\mathbb{R}^{d}}\left(e^{h(t,x,T_{i})}-1-h\left(t,x,T_{i}\right)\right)\mu_{j}^{C_{t-}}\left(dt,dx\right)<\infty \tag{3.9}$$

and

$$\tilde{a}_{j}\left(t,T_{i},C_{t-}\right)-\left(1-\tfrac{q}{S(t,T_{i},c)}\right)\cdot\lambda_{c_{t_{-}},1}+\int_{\mathbb{R}^{d}}\left(e^{h(t,x,T_{i})}-1-h\left(t,x,T_{i}\right)\right)\mu_{j}^{C_{t-}}\left(dt,dx\right)=0 \tag{3.10}$$

for any $t \in [0, T_i]$, *and any* $C_{t\_} \in \mathcal{K} \backslash \{1\}$ *where* [10]

$$\tilde{a}_j(t, T_i, C_{t-}) := a_j(t, T_i, C_{t-}) + s(t, t, C_{t-}) + \frac{1}{2}|b_j(t, T_i, C_{t-})|^2 \tag{3.11}$$

and

$$S(t, T_i, c) = \exp\left\{-\int_t^{T_j} s(t, s, c)\, ds\right\}$$

Proof. Basically the proof is the same as for Proposition 5. □

Remark 4. *Notice how in this case we obtain the following expression for the default intensity, under risk-neutral measure*

$$\lambda_{c_{t\_},1} = \frac{S(t, T_i, c)}{S(t, T_i, c) - q}\left[\tilde{a}_j(t, T_i, C_{t-}) + \int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)} - 1 - h(t, x, T_i)\right)\mu_j^{C_{t-}}(dt, dx)\right]$$

Proposition 9. *The* ***corporate-bond price dynamics*** *under a martingale measure* $\mathbb{P}$*, and under the fractional recovery of market value hypothesis, will follow*

$$\begin{aligned} \frac{dB(t, T_i, C_t)}{B(t_-, T_i, C_{t\_})} &= r_t dt - \left(1 - \frac{q}{S(t, T_i, c)}\right) \cdot dM_1(t) \\ &\quad + b_j(t, T_i, C_{t\_})dW_t + \int_{\mathbb{R}^d} h(t, x, T_i)\, 1_{\{|x|\le 1\}}\, \tilde{\mu}_j^c(dt, dx) \end{aligned} \tag{3.12}$$

where r_t *is the usual short-rate,* $M_1(t)$ *is the martingale from the Doob-Meyer decomposition of the default indicator and* $\tilde{\mu}_j^c$ *is the* $\mathbb{P}$*-compensated Lévy measure.*

Proof. As in Proposition 7. □

3.3. **Absence of arbitrage in a corporate-bond market with fractional recovery of par value.** And finally, in this third subsection, we give the results under the hypothesis that in the case of a default the investor will recover a fractional part of par value .

Proposition 10. *The initial probability measure* $\mathbb{P}$ *itself belongs to* $\mathcal{Q}_{\mathcal{F}}$ *if and only if the following two conditions hold, for every* $T_i \in J$*:*

$$\sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)} - 1 - h(t, x, T_i)\right)\upsilon_j^{C_{t-}}(dt, dx) < \infty \tag{3.13}$$

and

$$\tilde{a}_j(t, T_i, C_{t-}) - \left(1 - \tfrac{q}{Z\left(t, T_i, C_{t\_}\right)}\right)\cdot\lambda_{c_{t\_},1} + \textstyle\int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)} - 1 - h(t, x, T_i)\right)\upsilon_j^{C_{t-}}(dt, dx) = 0 \tag{3.14}$$

for any $t \in [0, T_i]$, *and any* $C_{t\_} \in \mathcal{K} \backslash \{1\}$ *where* [11]

$$\tilde{a}_j(t, T_i, C_{t-}) := a_j(t, T_i, C_{t-}) + s(t, t, C_{t-}) + \frac{1}{2}|b_j(t, T_i, C_{t-})|^2 \tag{3.15}$$

Proof. Basically the proof is the same as for Proposition 60. □

Remark 5. *Notice how in this case we obtain the following expression for the default intensity, under risk-neutral measure*

$$\lambda_{c_{t\_},1} = \frac{Z(t, T_i, c)}{Z(t, T_i, c) - q}\left[\tilde{a}_j(t, T_i, C_{t-}) + \int_{\mathbb{R}^d}\left(e^{h(t,x,T_i)} - 1 - h(t, x, T_i)\right)\upsilon_j^{C_{t-}}(dt, dx)\right]$$

---

[10]Notice that if $C_{\tau_k} = 0$ it means default-free asset and $s_t^0(T_i) = 0$. It is the risk-free bond.

[11]Notice that $C_t = 0$ means default-free bond and consequently the spread $s_t^0(T_i) = 0$.

Proposition 11. *The* ***corporate-bond price dynamics*** *under a martingale measure* $\mathbb{P}$*, and under the fractional recovery of market value hypothesis, will follow*

$$\begin{aligned}\frac{dB\left(t,T_i,C_t\right)}{B\left(t_-,T_i,C_{t_-}\right)} &= r_t dt-\left(1-\frac{q}{Z(t,T_i,c)}\right)\cdot dM_1\left(t\right)\\ &\quad +b_j(t,T_i,C_{t_-})dW_t+\int_{\mathbb{R}^d} h\left(t,x,T_i\right)1_{\{|x|\le 1\}}\ \tilde{\mu}_j^c\left(dt,dx\right) \end{aligned} \tag{3.16}$$

where r_t *is the usual short-rate,* $M_1\left(t\right)$ *is the martingale from the Doob-Meyer decomposition of the default indicator and* $\tilde{\mu}_j^c$ *is the* $\mathbb{P}$*-compensated Lévy measure.*

Proof. As in Proposition 5. □

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