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THE INTERNATIONAL STOCK POLLUTANT CONTROL: A STOCHASTIC FORMULATION

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Abstract

In this paper we provide a stochastic dynamic game formulation of the economics of international environmental agreements on the transnational pollution control when the environmental damage arises from stock pollutant that accumulates, for accumulating pollutants such as CO_2 in the atmosphere. To improve the cooperative and the non-cooperative equilibrium among countries, we propose the criteria of the minimization of the expected discounted total cost. Moreover, we consider Stochastic Dynamic Games formulated as Stochastic Dynamic Programming and Cooperative versus Non-cooperative Stochastic Dynamic Games. The performance of the proposed schemes is illustrated by a real data based example.

Keywords: Stochastic Optimal Control, Markov Decision Processes, Stochastic Dynamic Programming, Stochastic Dynamic Games, International Pollutant Control, Environmental Economics, Sustainability

JEL Classification: C610, C630, C730, C44, D70, Q20

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Abstract

In this paper we provide a stochastic dynamic game formulation of the economics of international environmental agreements on the transnational pollution control when the environmental damage arises from stock pollutant that accumulates, for accumulating pollutants such as CO_2 in the atmosphere. To improve the cooperative and the non-cooperative equilibrium among countries, we propose the criteria of the minimization of the expected discounted total cost. Moreover, we consider Stochastic Dynamic Games formulated as Stochastic Dynamic Programming and Cooperative versus Non-cooperative Stochastic Dynamic Games. The performance of the proposed schemes is illustrated by a real data based example.

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1 Introduction

In the last years, the theory on international environmental agreements (IEA) and the prospect of climate change has motivated many game theoretic studies, often focused on cooperation and core solutions.

The necessity of cooperation amongst the countries involved, if a social optimum is to be achieved, has already been addressed in the literature in terms of Game Theory concepts; see e.g. Barrett (2003), Finus (2001), Flam (2006) and references therein for a review on these topics. With a few exceptions this literature works with simple static models of pollution despite the fact that many of the important environmental problems, as climate change, the depletion of the ozone layer or the acid rain problem, are caused by a stock pollutant. However, the stock of pollution may change in the course of the game, as a result of a positive rate of natural decay and emissions of the countries. Thus, the presence of a stock pollutant leads to a dynamic game that is not strictly repeated.

In the framework of a deterministic cooperative game with a dynamic, multi-regional integrated assessment model, Eyckmans and Tulkens (2003) calculated the optimal path of abatement and aggregated discounted welfare for each region. They apply the transfer scheme advocated by Chander and Tulkens (1997) for the Climate Negotiation (CLIMNEG) World Simulation Model (abbreviated as CWSM) with six regions or countries. The idea of surplus sharing is used for determining the transfer scheme, and they compute all possible partial agreement Nash equilibria. They found that allocation in the full cooperation lies in the core of the emission abatement game under this specific transfer scheme. Their CWSM derived from the seminal multi-region economy-climate RICE (Regional Integrated model of Climate and the Economy)model of Nordhaus and Yang (1996).

Germain, Toint, Tulkens, and de Zeeuw (2003) have addressed the issue of how many countries will be interested in signing an IEA with stock pollutant, adopting a cooperative game-theory approach. They extend the result established by Chander and Tulkens (1995) and (1997) for flow pollutants to the larger context of closed-loop (feedback) dynamic games with a stock pollutant. In this context, cooperation is negotiated at each period but financial transfers provide incentives to the countries that ensures the implementation of the grand coalition at each period. Their model, thus yields a sequence of full cooperative international agreements, so that full cooperation is also achieved in a dynamic setting with a stock pollutant.

Another paper related with this issue using a cooperative game-theory approach is Petrosjan and Zaccour (2003). However, in this paper the authors assume that all the countries decide to cooperate at the initial time-consistent decomposition of each player's total cost, as given by Shapley value, so that the countries stick at each moment to the full cooperative solution agreed at initial time, supposing that the global allocation problem has been solved. Nevertheless, there are only a few attempts in the stock pollutant control literature modelling that issue in a stochastic control framework.

Stochastic Programming is considered by Dechert and O'Donnell (2006) in a particular application that explore some fundamental issues of the optimal level of pollution in a lake with competing uses, they show how the model can be interpreted as an open loop dynamic game, where the control variables are the levels of phosphorus discharged into the watershed of the lake, the state of the system is the accumulated level of phosphorus in the lake and the random shock (a multiplicative noise factor on the control variables of the players) is the rainfall that washes the phosphorus in the lake.

The use of stochastic control models to develop climate-economy models has been advocated by Haurie and Viguier (2003) to represent the possible competition between Russia and China on the international market of carbon emissions permits, their model includes a representation of the uncertainty concerning the date of entry of developing countries on this market in the form of an event tree. Also by Bahn, Haurie, and Malhamé (2008), they show how a piecewise deterministic stochastic control model, over an infinite time horizon, can be used as a paradigm for the design of efficient climate policy, their model recognizes the existing uncertainty concerning the true sensitivity of climate, and the fact that the solution to the climate change issue may reside in the introduction of new carbon-free technologies. Keller, Bolker, and Bradford (2004) have already explored the combined effects of uncertainty and learning about a climate threshold (an uncertain ocean thermohaline circulation collapse) in an economic optimal growth model.

The stability of an International Environmental Agreement among n countries that emit pollutant are studied using differential games, defined in continuous time, by Jorgensen, Martín-Herrán, and Zaccour (2003) and (2004), Rubio and Casino (2005), among others.

As far as we know, none stochastic formulation for the finite horizon dynamic analysis of international agreements on transnational pollution control has been introduce as an extension of the issues presented in Germain, Toint, Tulkens, and de Zeeuw (2003). We adopt this point of view because to consider randomness on the factors in the model is closer to reality (see Casas and Romera (2005)).

The main purpose of this work is to suggest a stochastic dynamic game formulation for the Stock Pollutant Control, for both cooperative and non cooperative models. These models proposed are directly linked with the Kyoto or post-Kyoto agreement mechanisms.

The stochastic formulation for this Stock Pollutant Control Model involves the use of Stochastic Dynamic Programming with discrete and finite planing horizon, for searching both cooperatives and non cooperatives equilibria. Stochastic optimization problems should be solved by Stochastic Dynamic Programming Techniques (see Bertsekas (2000)).

The paper is organized as follows: In Section 2 we present the international stock pollutant model with its components, the cost functional components and their elements, the underlaying Markov Decision Process (MDP), and the description of the modes of countries behaviour. In Section 3, we describe the international stock pollutant control cooperative model and we solve the problem of minimize the expected discounted total cost for each period of time and for all the countries jointly. An analysis of particular expected damages functions is included. In Section 4, we describe what happen if the countries do not sign a voluntary international agreement and we solve the non cooperative model. In Section 5, we present a numerical example based on real scenarios borrowed from the work by Eyckmans and Tulkens (2003). In Section 6, we present some conclusions and extensions of our work.

2 Stock Pollutant Control Model

We adopt the point of view of the issues presented in Germain, Toint, Tulkens, and de Zeeuw (2003).

In our model, we introduce a stochastic dynamic game formulation, with finite and discrete planing horizon analysis of IEA on transnational pollution control, as an extension of these issues.

Model Components

We consider a Markovian Game described by a tuple

$$G = \{J, S, (E_i, r_i)_{i \in J}, p, \mathcal{T}\}$$

with the following elements

- There are n players and $J = \{1, 2, ..., n\}$ denotes the set of countries or regions which we simplify refer to as countries in the sequel.
- S is a Borel subset of some Polish (i.e., complete, separable, metric) countable and non empty space; is the state space of the game, with typical element s. The state transition dynamics is a function of the current state of the system and an additive noise factor on each period of time. The state of the system is the accumulated level of pollution in the atmosphere, given by s_t as stock of pollutant at each period t, $s_t \in S$, according to the state equation

$$s_t = (1-\delta)s_{t-1} + \sum_{i=1}^{n} e_{it} + \xi_t \quad , \qquad 1 \leq t \leq T \tag{1}$$

  Where

  - s_0 is the initial stock of pollutant or preindustrial level, given.
  - δ is the pollutant's natural rate of atmospheric absorption of CO_2 between two periods of time, such that $0 < \delta < 1$.

- p specifies the law of motion (or transition probabilities) for the game by associating with each $(s, a) \in S \times E$ a probability $p(\cdot|s, e)$ over the Borel sets of S.

- A finite planing horizon with discrete-time periods t, such that

$$t \in \mathcal{T} = \{1, 2, ..., T\} \subset \mathbb{Z}^+.$$

- The control variables are $e_t = (e_{1t}; e_{2t}; ...; e_{nt})'$ vector of the different countries emissions of pollutant at each period t, entailed by economic activity, where $e_{it} \in E$ and E is the countable and non empty overall action space, and

$$E = \bigcup_{s \in S} E(s),$$

  where $E(s)$ is the set of *admissible actions* (emissions), when the system is in each state (pollutant level) s. For each $s \in S$ the set $E(s)$ is finite.

- The random disturbance ξ_t is a noise process: a sequence of i.i.d. random variables and independent of the initial state s_0, with

$$\mathbb{E}[\xi_t] = 0, \qquad \sigma^2 = \mathbb{E}\left[\xi_t^2\right] < \infty, \qquad \forall t = 1, 2, ..., T-1. \tag{2}$$

We consider stock of pollutant in a wide sense, not restricted to the carbon dioxide (CO_2) stock level. Inclusion of manifold pollutants is important. To wit, the 1997 Kyoto Protocol to the Framework Convention on Climate Change limits aggregate emissions of six direct greenhouse gases, such as: carbon dioxide (CO_2), methane (CH_4), nitrous oxide (N_2O), hydrofluorocarbons ($HFCs$), perfluorocarbons ($PFCs$), sulphur hexafluoride (SF_6)), as well for the indirect greenhouse gases such as SO_2, NOx, CO and/or micro particles of industrial pollution (between 0.1 y 2.5 μ-meters). The emissions are aggregated and considered as CO_2 equivalents.

Functional Cost Components

Following Jorgensen and Zaccour (2001) among many others, we assume that the emissions are proportional to production. Additionally we consider

- Future costs are discounted by the constant and positive *discount factor* β with $0 < \beta \leq 1$.

- $c_i(e_{it})$: function that measures in monetary terms the total cost incurred by country $i \in J$ at period $t \in \mathcal{T}$ from limiting its own industrial emissions to e_{it}; is a differentiable, decreasing ($c_i' < 0$) and strictly convex function ($c_i'' > 0$).

- $d_i(s_t)$: function that measures in monetary terms the damages caused by the stock of pollutant s_t during the time period t for the i-th country; is a differentiable, increasing ($d_i' > 0$) and convex function ($d_i'' \geq 0$).

- $r_i(e_{it}, s_t) = c_i(e_{it}) + d_i(s_t)$: function that measures in monetary terms the total cost incurred by country $i \in J$ from limiting its emissions to e_{it}, and the damages caused by the stock of pollutant s_t during the time period t for the i-th country; $r_{it} \in R$, where R is the cost set and R is a subset of $\mathbb{R}$.

We consider that the only way to control the stock of pollution is through the control of emissions, that is reducing pollution is done through the reduction of emissions, and not through the cleaning of the environment. The marginal cost c_i of reducing emissions is higher for lower levels of emissions.

The decreasing character of the cost functions c_i show the evident phenomenon of the increasing costs related to the emissions reduction, i.e. The increasing cost to decrease the emissions could be associated with filter installations or the use of other techniques.

The underlaying Markov Decision Process

We considere by MDP a Markov Decision Process together with an optimality criteria. The problems considered in this work are discrete-time, finite-horizon and stationary MDP with expected total reward. Then, we can express the elements of our random scenarios through the following MDP

$$\Gamma = (S, E, R, P, \beta),$$

where the *state space* S and the overall *action space* $E = \bigcup_{s \in S} E(s)$ are both countable and nonempty, $E(s)$ is the set of *admissible actions* (emissions), when the system is in each state (pollutant level) s. For each $s \in S$ the set $E(s)$ is finite. The *cost set* R is a bounded countable subset of $\mathbb{R}$. For each $t \geq 1$, let s_t, e_t and r_t denote the state (pollutant level) of the system, the action (emissions) taken by the decision maker (pays), and the cost incurred at period of time t, respectively.

The stationary, single-stage, conditional *transition probabilities* are defined by

$$p^e_{i,j,r} := Prob\left(s_{t+1} = j, r_t = r / s_t = i, e_t = e\right),$$

$$\forall i, j \in S \quad , \quad e \in E(i) \quad , \quad r \in R \quad , \quad t \geq 1,$$

$$\sum_{j \in S, r \in R} p^e_{i,j,r} = 1 \quad , i \in S \quad , \quad e \in E(i).$$

Modes of countries behavior

The damages in each country's environment depend on the emissions of pollutant of all different countries at each time-period t that contribute to a stock s_t.

In cooperative form the countries jointly choose at each period its emissions levels in order to minimize the expected total discount costs, then the resulting trajectories of emissions and stock constitute the international optimum.

In non-cooperative form, each country considers only the damages of the stock of pollutant over itself. In the sense of a Nash equilibrium, the countries minimize, at each period, only its own expected discounted costs, with knowledge of the emissions vector e_{jt}, with $j \neq i$, of the other countries.

3 Cooperative Model

In this case, one assumes that the countries behave in an internationally optimal way, i.e. that each of them takes account of the impact of its own industrial pollution not only on itself but on all other countries as well. It is clear that the damages to the environment of country i will depend on the emissions of all countries. We solve the problem of minimize the expected discounted total cost for each period $t \in \mathcal{T}$, where $\mathcal{T}$ is a discrete and finite set, and for all the countries jointly $(P1)$

$$\begin{aligned} (P1) \qquad & \min_{\{e_{it}\}} \quad \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} \beta^t \left[c_i(e_{it}) + d_i(s_t)\right]\right] \\ & \text{s.t.} \quad s_t = (1-\delta)s_{t-1} + \sum_{i=1}^{n} e_{it} + \xi_t \\ & \qquad e_{it} \geq 0, \qquad \forall i = 1, \cdots, n; \quad \forall t = 1, \cdots, T \\ & \qquad s_0 > 0 \end{aligned}$$

Remark The resulting family of trajectories of emissions (policies) e^*_{it} for all players $i \in J$ determined together with the resulting stock s^*_t, constitute the international optimum for all periods $t \in \mathcal{T}$ or a cooperative equilibrium (see Dutta and Sundaram (1998)).

Note that the objective function in the model $(P1)$ is equivalent to

$$\min_{\{e_{it}\}} \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n}\beta^t\left[c_i(e_{it})+d_i(s_t)\right]\right] \Leftrightarrow \min_{\{e_{it}\}}\sum_{t=1}^{T}\sum_{i=1}^{n}\beta^t\left(c_i(e_{it})+\mathbb{E}\left[d_i(s_t)\right]\right) \quad (3)$$

Proposition 1. *Problem $(P1)$ has an equilibrium $\{e^*_{it}\}$.*

Proof. The convexity of the functions $c_i(e_{it})$ and $d_i(s_t)$, for all $i \in J$ and for all periods $t \in \mathcal{T}$, suffices to guarantee that the minimum exists and is unique (see for instance, Puterman (2005) or Hernández-Lerma (1999)). $\square$

This problem $(P1)$ can be solved by using Stochastic Dynamic Programming tools. The *expected value function* W, according to Bellman's principle of optimality, satisfies the Dynamic Programming equations for $(P1)$

$$(P1.1) \qquad W(T, s_{T-1}) = \min_{e_{iT}} \mathbb{E}\left[\sum_{i=1}^{n}\left(c_i(e_{iT})+d_i(s_T)\right)\right],$$

$$(P1.2) \qquad W(t, s_{t-1}) = \min_{e_{it}} \mathbb{E}\left[\sum_{i=1}^{n}\left[c_i(e_{it})+d_i(s_t)\right]+\beta W(t+1, s_t)\right],$$

$$\forall t = 1, 2, ..., T-1$$

$$\begin{aligned} \text{s.t.} \qquad s_t &= (1-\delta)s_{t-1} + \sum_{i=1}^{n} e_{it} + \xi_t \\ e_{it} &\geq 0 \qquad \forall t \in \mathcal{T} \quad , \quad \forall i \in J \\ s_0 &> 0 \end{aligned}$$

The stochastic Dynamic Programming equations $(P1.1)$ and $(P1.2)$ are equivalent, respectively, to

$$(P1.1) \quad \Leftrightarrow \quad W(T, s_{T-1}) = \min_{e_{iT}}\left\{\sum_{i=1}^{n}\left(c_i(e_{iT})+\mathbb{E}\left[d_i(s_T)\right]\right)\right\}$$

$$(P1.2) \quad \Leftrightarrow \quad W(t, s_{t-1}) = \min_{e_{it}}\left\{\sum_{i=1}^{n}\left(c_i(e_{it})+\mathbb{E}\left[d_i(s_t)\right]\right)+\beta W(t+1, s_t)\right\}$$

If countries cooperate, they jointly solve $(P1.1)$ at final period of time T, the country i's expected total cost is

$$W_i(T, s) = c_i(e^*_{iT}) + d_i(s^*_T),$$

where $e^*_{iT} = \{e^*_{1T}, e^*_{2T}, \ldots, e^*_{nT}\}$ is the vector of optimal emission levels or policy, and s^*_T denotes the resulting stock of pollutant at final period T, given

$$s^*_T = [1 - \delta]s + \sum_{i=1}^{n} e^*_{iT}$$

where s is the inherited stock of pollutant at the begin of period T.

In earlier periods, if countries cooperate they solve the problem $(P1.2)$ for $1 \leq t \leq T - 1$. Optimal levels of emissions and resulting stock of pollutant are denoted by e^*_{it} and s^*_t respectively.

Then let denotes the country i's expected discounted equilibrium cost by

$$W_i(t, s) = c_i(e^*_{it}) + d_i(s^*_t) + \beta W_i(t + 1, s^*_t), \qquad \forall t = 1, \cdots, T - 1$$

with

$$s^*_t = [1 - \delta]s + \sum_{i=1}^{n} e^*_{it}$$

where s is the inherited stock of pollutant at the begin of period t.

Let define as τ-expected discounted total cost by

$$W_i^\tau \equiv \sum_{t=1}^{\tau} W_i(t, s^*_{t-1}), \quad 1 \leq \tau \leq T - 1$$

$$\text{and total cost} \qquad W_i \equiv \sum_{t=1}^{T} W_i(t, s^*_{t-1}).$$

3.1 Cooperative Alternative Problem

We present an equivalent cooperative problem which can be solved by using Linear Programming tools.

The recurrence equation (1), of the contamination stock s_t, gives a dynamic character to the cooperative model also to the non cooperative model, but, using the recurrence expression, considering the state variables s_t and the control variables e_{it} as decision variables, and the state equations as equality restrictions, besides having as an objective function a differentiable convex function, we may write an associated model, writing s_t as a function of the known initial stock s_0, of the emissions e_{it} from each country $i \in J$, and of the random disturbance vector ξ_t in each period of time $t \in \mathcal{T}$.

From the recurrence equation (1) we obtain

$$s_1 = (1-\delta)s_0 + \sum_{i=1}^{n} e_{i1} + \xi_1.$$

$$\begin{aligned} s_2 &= (1-\delta)^2 s_0 + (1-\delta)\sum_{i=1}^{n} e_{i1} + (1-\delta)\xi_1 + \sum_{i=1}^{n} e_{i2} + \xi_2. \\ s_3 &= (1-\delta)^3 s_0 + (1-\delta)^2\sum_{i=1}^{n} e_{i1} + (1-\delta)^2\xi_1 + \\ &\quad +(1-\delta)\sum_{i=1}^{n} e_{i2} + (1-\delta)\xi_2 + \sum_{i=1}^{n} e_{i3} + \xi_3. \end{aligned}$$

By induction we get

$$\begin{aligned} s_t &= (1-\delta)^t s_0 + (1-\delta)^{t-1}\sum_{i=1}^{n} e_{i1} + (1-\delta)^{t-1}\xi_1 + \cdots + \\ &\quad + \cdots + (1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau} + (1-\delta)^{t-\tau}\xi_\tau + \cdots + \sum_{i=1}^{n} e_{it} + \xi_t. \end{aligned}$$

Recursively we obtain the general form

$$s_t = (1-\delta)^t s_0 + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau} + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau.$$

Explicitly developing the previous recurrence equation, we obtain the following system of restrictions

$$
\begin{aligned}
s_1 &= (1-\delta)s_0 + e_{11} + e_{21} + \cdots + e_{n1} + \xi_1. \\
s_2 &= (1-\delta)^2 s_0 + (1-\delta)e_{11} + (1-\delta)e_{21} + \cdots + \\
&\qquad +(1-\delta)e_{n1} + (1-\delta)\xi_1 + e_{12} + \cdots + e_{n2} + \xi_2. \\
s_3 &= (1-\delta)^3 s_0 + (1-\delta)^2 e_{11} + \cdots + (1-\delta)^2 e_{n1} + (1-\delta)^2 \xi_1 + \\
&\quad +(1-\delta)e_{12} + \cdots + (1-\delta)e_{n2} + (1-\delta)\xi_2 + e_{13} + \cdots + e_{n3} + \xi_3. \\
&\vdots \\
s_t &= (1-\delta)^t s_0 + (1-\delta)^{t-1} e_{11} + \cdots + (1-\delta)^{t-1} e_{n1} + (1-\delta)^{t-1}\xi_1 + \cdots + \\
&\quad +(1-\delta)^{t-2} e_{12} + \cdots + (1-\delta)^{t-2} e_{n2} + (1-\delta)^{t-2}\xi_2 + \cdots + (1-\delta)e_{1t-1} \\
&\quad + \cdots + (1-\delta)e_{nt-1} + (1-\delta)\xi_{t-1} + e_{1t} + \cdots + e_{nt} + \xi_t.
\end{aligned}
$$

By using the **Markov's condition or the Property of causality**, $\forall j, r \in \{0, 1, \ldots, N-1\}$ with $j < r$, it is shown that the state x_r only depends on the state x_j and the intermediate controls $\{u_j, u_{j+1}, \ldots, u_{r-1}\}$. Then, we conclude that the actual contamination stock depends on the initial stock s_0 and the set of controls or emission vector $e_1, e_2, ..., e_T$ for each period of time $t \in \mathcal{T}$.

Note that, by definition $e_{it} \geq 0$ for all $t \in \mathcal{T}$, and $s_t \geq 0$ for all $t \in \mathcal{T}$ provided that $0 < \delta < 1$.

Then, we can consider equivalently the following problem with convex objective function and $T+1$ linear constraints

$$
\begin{aligned}
\min_{\{e_{it}\}_{t\in\mathcal{T}}} \quad & \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} \beta^t \left[c_i(e_{it}) + \tilde{d}_i(s_0, e_{it}, \xi_t)\right]\right] \\
\text{s.t.} \quad Ae &= b - \xi \\
e &\geq 0 \\
s_0 &> 0
\end{aligned}
$$

where

$$
\begin{aligned}
e' &= (e_{11}; e_{21}; \cdots; e_{n1}; e_{12}; e_{22}; \cdots; e_{n2}; \cdots; e_{1T}; e_{2T}; \cdots; e_{nT}) \\
b' &= \left(-(1-\delta)s_0; -(1-\delta)^2 s_0; \cdots; -(1-\delta)^T s_0\right) \\
\xi' &= \left(\xi_1; (1-\delta)\xi_1 + \xi_2; \ldots; \ldots; (1-\delta)^{T-1}\xi_1 + (1-\delta)^{T-2}\xi_2 + \cdots + \xi_T\right)
\end{aligned}
$$

The independent vector b and random disturbance vector ξ are of order T.

The matrix A is a $T \times Tn$, lower triangular matrix, with the following structure

$$A = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ (1-\delta) & \cdots & (1-\delta) & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ (1-\delta)^2 & \cdots & (1-\delta)^2 & (1-\delta) & \cdots & (1-\delta) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-\delta)^{T-1} & \cdots & (1-\delta)^{T-1} & (1-\delta)^{T-2} & \cdots & (1-\delta)^{T-2} & \cdots & 1 & \cdots & 1 \end{pmatrix}$$

By using the development presented in this section one can find the solutions e^*_{it} of optimal emissions for each country $i \in J$, and one obtains the stock levels of contamination s^*_t in each period of time $t = 1, 2, ..., T$.

3.2 Analysis of particular Damage Functions

Note that although the cost function c_i, depends only on the emissions e_{it} of each country $i \in J$ at each period of time $t \in \mathcal{T}$, the damages function d_i depends on the initial stock s_0, the emissions of the each one others countries e_{it} with $i \neq j$, the emissions e_{it} and the random disturbance ξ_t, for each period of time t. This fact determines the stochastic structure of the objective function to be considered in the optimization problem $(P1)$, as it is shown in (3).

We analyze useful cases of damage functions that appear in the economic literature, and we present the particular programming problems to be solved in each case. This analysis remains valid for both models, cooperative and non cooperative with some slight modification.

3.2.1 Linear Case

One assume

$$d_i(s_t) = as_t + b, \qquad a, b \in \mathbb{R}.$$

Following (3) the objective function of the cooperative model $(P1)$ has the following form

$$\min_{\{e_{it}\}} \quad \left\{ \sum_{t=1}^{T} \sum_{i=1}^{n} \beta^t \left(c_i(e_{it}) + a\mathbb{E}\left[d_i(s_t) \right] \right) \right\}$$

because

$$\begin{aligned}
\mathbb{E}\left[d_i(s_t)\right] &= \mathbb{E}\left[as_t + b\right], \\
&= a\mathbb{E}\left[s_t\right] + b, \\
&= a\mathbb{E}\left[(1-\delta)^t s_0 + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau} + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau\right] + b, \\
&= a\left[(1-\delta)^t s_0 + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau} + \mathbb{E}[\sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau]\right] + b, \\
&= a\left[(1-\delta)^t s_0 + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau}\right] + b.
\end{aligned}$$

Then the objective function of model $(P1)$ is equal to the objective function of the following linear programming

$$\min_{\{e_{it}\}} \quad \left\{\sum_{t=1}^{T}\sum_{i=1}^{n}\beta^t\left(c_i(e_{it}) + a(1-\delta)^t s_0 + a\sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau} + b\right)\right\}.$$

3.2.2 Quadratic Case

One assume that

$$d_i(s_t) = (s_t)^2.$$

The objective function of the cooperative model has the following form

$$\min_{\{e_{it}\}} \quad \left\{\sum_{t=1}^{T}\sum_{i=1}^{n}\beta^t\left(c_i(e_{it}) + \mathbb{E}\left[(s_t)^2\right]\right)\right\}.$$

then

$$\begin{aligned}\mathbb{E}\left[(s_t)^2\right] &= \mathbb{E}\left[\left((1-\delta)^t s_0 + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau} + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau\right)^2\right],\\ &= \mathbb{E}\left[\varphi^2 + \left(\sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau\right)^2 + 2\varphi\left(\sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau\right)\right],\\ &= \varphi^2 + \mathbb{E}\left[\left(\sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau\right)^2\right] + 2\varphi\mathbb{E}\left[\sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau\right].\end{aligned}$$

where

$$\varphi = (1-\delta)^t s_0 + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau}.$$

Provided that $\{\xi_t\}$ are iid and condition (2), we have

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau\right)^2\right] &= \mathbb{E}\left[\sum_{\tau=1}^{t}(1-\delta)^{2(t-\tau)}\xi_\tau^2 + 2\sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau\sum_{j=1}^{t}(1-\delta)^{t-j}\xi_j\right],\\ &= \sum_{\tau=1}^{t}(1-\delta)^{2(t-\tau)}\mathbb{E}\left[\xi_\tau^2\right] + 2\sum_{\tau=1}^{t}\sum_{j=1}^{t}(1-\delta)^{t-\tau}(1-\delta)^{t-j}\mathbb{E}\left[\xi_\tau\xi_j\right],\\ &= \sum_{\tau=1}^{t}(1-\delta)^{2(t-\tau)}\sigma_t^2,\end{aligned}$$

then $$\mathbb{E}\left[(s_t)^2\right] = \varphi^2 + \sum_{\tau=1}^{t}(1-\delta)^{2(t-\tau)}\sigma_t^2.$$

Then in this case, our problem is transformed in an quadratic programming problem.

3.2.3 Exponential Case

Finally, one assume

$$d_i(s_t) = \exp(s_t).$$

The objective function of the cooperative model has the following form

$$\min_{\{e_{it}\}} \quad \left\{ \sum_{t=1}^{T} \sum_{i=1}^{n} \beta^t \left(c_i(e_{it}) + \mathbb{E}\left[\exp(s_t)\right] \right) \right\}.$$

Now

$$\begin{aligned}
\mathbb{E}\left[\exp(s_t)\right] &= \mathbb{E}\left[\exp\left((1-\delta)^t s_0\right) \exp\left(\sum_{\tau=1}^{t} (1-\delta)^{t-\tau} \sum_{i=1}^{n} e_{i\tau} \right) \exp\left(\sum_{\tau=1}^{t} (1-\delta)^{t-\tau} \xi_\tau \right) \right] \\
&= \exp\left((1-\delta)^t s_0\right) \exp\left(\sum_{\tau=1}^{t} (1-\delta)^{t-\tau} \sum_{i=1}^{n} e_{i\tau} \right) \mathbb{E}\left[\exp\left(\sum_{\tau=1}^{t} (1-\delta)^{t-\tau} \xi_\tau \right) \right]
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\sum_{\tau=1}^{t} (1-\delta)^{t-\tau} \xi_\tau \right) \right] &= \prod_{\tau=1}^{t} \mathbb{E}\left[\exp(1-\delta)^{t-\tau} \xi_\tau \right], \\
&= \prod_{\tau=1}^{t} \varphi_\xi \left[(1-\delta)^{t-\tau} \right].
\end{aligned}$$

We recognize φ_ξ as the z-transformed function if ξ follows a discrete random variable.

Depending on the expression of this φ_ξ function, we get different types of objective functions, and therefore different types of mathematic programming problems, usually they will be non-linear optimization problems.

4 Non-Cooperative Model

In an alternative mode of behaviour, we describe what would happen if the countries do not sign a voluntary international environmental agreement. One may assume that countries behave non cooperatively in the sense of Nash equilibrium, where each of them minimizes at each period only its own discounted costs, taking given the emissions of the other countries. A Nash equilibrium is a family of strategies, one for each player, that minimize every country i's cost, given the strategies of all other players $j \neq i$. In such an equilibrium, no individual country has an incentive to deviate as long as the other countries stick to their equilibrium strategies.

The considered problem is a dynamic game in discrete time and finite horizon with only one player or country. We can adopt the perspective of an Optimal Control Problem (OCP), where the dynamic model is a system in discrete time $s_{t+1} = \phi(s_t, e_t, \xi_t)$ for all $t \in \mathcal{T}$ with initial condition s_0 and finite horizon $T < \infty$.

Formally, there are n problems to solve. Actually, at each period of time $t \in \mathcal{T}$, each country $i \in J$ solves the following problem $(P2)$

$$
\begin{aligned}
(P2) \qquad & \min_{\{e_{i\tau}\}_{\tau\in\{t,\ldots,T\}}} && \mathbb{E}\left[\sum_{\tau=t}^{T} \beta^{\tau}\left[c_i(e_{i\tau}) + d_i(s_\tau)\right]\right] \\
& \text{s.t.} && s_t = (1-\delta)s_{t-1} + \sum_{i=1}^{n} e_{it} + \xi_t \\
& && e_{it} \geq 0 \quad \forall t \in \mathcal{T}; \quad \forall i \in J \\
& && s_0 > 0
\end{aligned}
$$

Note that the objective function in the model $(P2)$ is equivalent to

$$
\min_{\{e_{i\tau}\}_{\tau\in\{t,\ldots,T\}}} \mathbb{E}\left[\sum_{\tau=t}^{T} \beta^{\tau}\left[c_i(e_{i\tau}) + d_i(s_\tau)\right]\right] \Leftrightarrow \min_{\{e_{i\tau}\}_{\tau\in\{t,\ldots,T\}}} \sum_{\tau=t}^{T} \beta^{\tau}\left(c_i(e_{i\tau}) + \mathbb{E}\left[d_i(s_\tau)\right]\right)
$$

Proposition 2. *Problem $(P2)$ has an equilibrium $\{e_{it}^{N}\}$.*

Proof. A particular case the convexity of the functions $c_i(e_{it})$ and $d_i(st)$, for all $i \in J$ and for all periods $t \in \mathcal{T}$, suffices to guarantee that the Nash equilibrium exists and is unique (see for instance, Puterman (2005) and Hernández-Lerma (1999)). □

The *expected value functions* N_i, according to Bellman's principle of optimality, can be found by solving the Stochastic Dynamic Programming equations for $(P2)$

$$
\begin{aligned}
(P2.1) \qquad & N_i(T, s_{T-1}) = \min_{e_{iT}} \mathbb{E}\left[c_i(e_{iT}) + d_i(s_T)\right] \\
(P2.2) \qquad & N_i(t, s_{t-1}) = \min_{e_{it}} \mathbb{E}\left[c_i(e_{it}) + d_i(s_t) + \beta N_i(t+1, s_t)\right]
\end{aligned}
$$

$$
\forall t = 1, 2, \ldots, T-1
$$

$$
\begin{aligned}
\text{s.t.} \qquad s_t &= (1-\delta)s_{t-1} + \sum_{i=1}^{n} e_{it} + \xi_t \\
e_{it} &\geq 0 \qquad \forall t \in \mathcal{T} \quad , \quad \forall i \in J \\
s_0 &> 0
\end{aligned}
$$

Remark The resulting family of trajectories of emissions (policies) e_{it}^{N} determined for each country $i \in J$, together with the resulting stock s_t^{N}, constitute a non-cooperative Nash equilibrium for all periods $t \in \mathcal{T}$ (see Dutta and Sundaram (1998)).

The Stochastic Dynamic Programming equations $(P2.1)$ and $(P2.2)$ are equivalent, respectively, to

$$\begin{aligned}(P2.1) \quad &\Leftrightarrow \quad N_i(T, s_{T-1}) = \min_{e_{iT}} c_i(e_{iT}) + \mathbb{E}\left[d_i(s_T)\right], \\ (P2.2) \quad &\Leftrightarrow \quad N_i(t, s_{t-1}) = \min_{e_{it}} c_i(e_{it}) + \mathbb{E}\left[d_i(s_t)\right] + \beta N_i(t+1, s_t).\end{aligned}$$

In the non cooperative equilibrium the country i's expected total cost at period final T is

$$N_i(T, s) = c_i(e_{iT}^N) + d_i(s_T^N) \qquad ; \qquad s_T^N = [1-\delta]s + \sum_{i=1}^{n} e_{iT}^N.$$

where $e_{iT}^N = \{e_{1T}^N, e_{2T}^N, \ldots, e_{nT}^N\}$ is the vector that denotes the emissions equilibrium level and s_T^N denotes the resulting stock of pollutant at final period of time T, where s is the inherited stock of pollutant at the begin of period T.

Let define as τ-expected discounted total cost by

$$N_i^\tau \equiv \sum_{t=1}^{\tau} N_i(t, s_{t-1}^N), \qquad 1 \leq \tau \leq T-1$$

$$\text{and total cost} \qquad N_i \equiv \sum_{t=1}^{T} N_i(t, s_{t-1}^N).$$

4.1 Non-Cooperative Alternative Problem

By the recurrence equation (1) and considering that each country minimizes its own costs, given the emissions vector e_{jt}, with $j \neq i$, of the all others countries and considering the random disturbance vector ξ_t in each period of time $t \in \mathcal{T}$, for each country $i \in J$ we obtain that

$$\begin{aligned} s_1 &= (1-\delta)s_0 + e_{i1} + \sum_{j \neq i}^{n} e_{j1} + \xi_1. \\ s_2 &= (1-\delta)^2 s_0 + (1-\delta)e_{i1} + (1-\delta)\sum_{j \neq i}^{n} e_{j1} + (1-\delta)\xi_1 + e_{i2} + \sum_{j \neq i}^{n} e_{j2} + \xi_2. \\ s_3 &= (1-\delta)^3 s_0 + (1-\delta)^2 e_{i1} + (1-\delta)^2 \sum_{j \neq i}^{n} e_{j1} + (1-\delta)^2 \xi_1 + (1-\delta)e_{i2} + \\ &\quad +(1-\delta)\sum_{j \neq i}^{n} e_{j2} + (1-\delta)\xi_2 + e_{i3} + \sum_{j \neq i}^{n} e_{j3} + \xi_3. \end{aligned}$$

Proceeding in a similar way by induction till the moment t, we get to the following expression

$$\begin{aligned} s_t &= (1-\delta)^t s_0 + (1-\delta)^{t-1} e_{i1} + (1-\delta)^{t-1}\sum_{j\neq i}^{n} e_{j1} + (1-\delta)^{t-1}\xi_1 + \cdots + \\ &+ (1-\delta)^{t-\tau} e_{it} + (1-\delta)^{t-\tau}\sum_{j\neq i}^{n} e_{j\tau} + (1-\delta)^{t-\tau}\xi_\tau + \cdots + e_{it} + \sum_{j\neq i}^{n} e_{jt} + \xi_t, \end{aligned}$$

in general form

$$s_t = (1-\delta)^t s_0 + \sum_{\tau=1}^{t}\sum_{j\neq i}^{n}(1-\delta)^{t-\tau} e_{j\tau} + \sum_{\tau=1}^{t} e_{i\tau} + \sum_{i=1}^{t}(1-\delta)^{t-\tau}\xi_i.$$

Explicitly developing the previous recurrence equation (1), the constraints system is transformed obtaining

$$\begin{aligned} s_1 &= (1-\delta)s_0 + e_{11} + e_{21} + \cdots + e_{n1} + \xi_1, \\ s_2 &= (1-\delta)^2 s_0 + (1-\delta)e_{11} + (1-\delta)e_{21} + \cdots \\ &\quad +(1-\delta)e_{n1} + (1-\delta)\xi_1 + e_{12} + \cdots + e_{n2} + \xi_2, \\ s_3 &= (1-\delta)^3 s_0 + (1-\delta)^2 e_{11} + \cdots + (1-\delta)^2 e_{n1} + (1-\delta)^2\xi_1 + (1-\delta)e_{12} + \\ &+ \cdots + (1-\delta)e_{n2} + (1-\delta)\xi_2 + e_{13} + \cdots + e_{n3} + \xi_3, \\ &\vdots \\ s_t &= (1-\delta)^t s_0 + (1-\delta)^{t-1} e_{11} + \cdots + (1-\delta)^{t-1} e_{n1} + (1-\delta)^{t-1}\xi_1 + \cdots \\ &+ (1-\delta)^{t-2} e_{12} + \cdots + (1-\delta)^{t-2} eE_{n2} + (1-\delta)^{t-2}\xi_2 + \cdots \\ &+ (1-\delta)e_{1t-1} + \cdots + (1-\delta)e_{nt-1} + (1-\delta)\xi_{t-1} + e_{1t} + \cdots + e_{nt} + \xi_t. \end{aligned}$$

In the non cooperative case we have n problems to solve, one for each country $i \in \{1, 2, ..., n\}$. Let i fixed, then

$$\begin{aligned} \min_{\{e_t\}_{t\in\{1,\ldots,T\}}} \quad & \mathbb{E}\left[\sum_{t=1}^{T}\beta^t\left[c_i(e_t) + \tilde{d}_i(s_0, e_t, \xi_t)\right]\right] \\ \text{s.a.} \quad B_i e &= b_i + \xi \\ e &\geq 0 \qquad \forall t \in \mathcal{T} \\ s_0 &> 0 \end{aligned}$$

where

$$\begin{aligned}
e_i' &= (e_{i1}; e_{i2}; \cdots ; e_{iT}) \\
b_i' &= (b_{i1}; b_{i2}; \cdots ; b_{iT}) \\
b_{it} &= -(1-\delta)^t s_0 - \sum_{\tau=1}^{t} \sum_{j \neq i}^{n} (1-\delta)^{t-\tau} e_{j\tau} \\
\xi' &= \big(\xi_1; (1-\delta)\xi_1 + \xi_2; \ldots ; \ldots ; (1-\delta)^{T-1}\xi_1 + (1-\delta)^{T-2}\xi_2 + \cdots + \xi_T\big)
\end{aligned}$$

The matrix B_i is a square matrix, lower triangular, of order T, with ones in the principal diagonal. The vector b and the random disturbance ξ have order T.

The structure of the matrix B_i is as follows

$$B_i = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
(1-\delta) & 1 & 0 & 0 & 0 & \cdots & 0 \\
(1-\delta)^2 & (1-\delta) & 1 & 0 & 0 & \cdots & 0 \\
(1-\delta)^3 & (1-\delta)^2 & (1-\delta) & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
(1-\delta)^{T-1} & (1-\delta)^{T-2} & (1-\delta)^{T-3} & \cdots & \cdots & (1-\delta) & 1
\end{pmatrix}$$

then we can may obtain the inverse matrix of the matrix B_i, which is quasi diagonal

$$B_i^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-(1-\delta) & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & -(1-\delta) & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & -(1-\delta) & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & \cdots & -(1-\delta) & 1
\end{pmatrix}$$

As in the cooperative model solution, by using the development presented in this section one can find the parameters e_{it}^N of optimal emissions for each country $i \in J$, and one obtains the stock levels of contamination s_t^N in each period of time $t = 1, 2, ..., T$.

Note that the particular analysis for linear, quadratic and exponential damage functions developed in section 3.2.1, holds for the non cooperative case with little change in the objective function.

5 A Numerical Example

In the following, we show some numerical results obtained by application of the algorithms developed in the preceding sections of cooperative ($P1$) and non cooperative ($P2$) problems to a real scenario considering six regions or countries. The six regions or countries considered are USA, Japan, European Union (EU), China, Former Soviet Union (FSU) and Rest of the World (ROW). Periods of time are years, the initial period 0 refers to year 1990, following the Kyoto Protocol.

The model and the values of the parameters used are based on the paper by Eyckmans and Tulkens (2003). In that paper the model named the Climate Negotiation (CLIMNEG) World Simulation Model, is considered as well as a deterministic dynamic analysis about how many countries will be interesting in signing an international environmental agreement (IEA) with accumulating pollutant in discrete time. All computations were made by use of the software Matlab 7.3.0 (R2006b).

5.1 Model and parameters

The temperature change equation is taken from the climate economy model RICE (Regional Integrated model of Climate and the Economy) by Nordhaus and Yang (1996) and Nordhaus and Boyer (2000), as well as most of the parameter values and all basic data on GDP, population, capital stock, carbon emissions and concentration and global mean temperature. A complete overview of the equations and parameter values of the Climate Negotiation (CLIMNEG) World Simulation Model (abbreviated as CWSM) can be found in Eyckmans and Tulkens (2003).

The division of the world is the same as in the RICE model. There are 6 countries or regions: USA, Japan, European Union (EU), China, Former Soviet Union (FSU) and Rest of the World (ROW). The time is divided in years, the initial period (period $t = 0$) refers to year 1990. To take account on the long term impacts of stock pollutant, we take a long planning horizon of 100 years, but we will only consider results until 2030 in order to avoid boundary problems.

The CO_2 emissions in each region or country $i \in J$ at period of time $t \in \mathcal{T}$ are denoted by e_{it}, with $e_{it} \geq 0$ for all $i \in J$ and for all $t \in \mathcal{T}$, and $e_t = (e_{1t}; e_{2t}; ...; e_{nt})$ is the corresponding vector of emissions of CO_2 in each of n regions or countries i at period of time t. Emissions of region i at time t are considered due to economic activity and proportional to the potential GDP named Y_{it}, according to expression

$$e_{it} = \sigma_{it}(1 - \eta_{it})Y_{it} \tag{4}$$

The optimal abatement rate of control of emissions, in each country or region i and in every period of time t, is the endogenous vector $\eta_t = (\eta_{1t}; \eta_{2t}; ...; \eta_{nt})$ with $0 \leq \eta_{it} \leq 1$, for all $i \in J$ and for all $t \in \mathcal{T}$. Note that $\eta_t = 0$ for all t determines the "business-as-usual" (BAU) scenario in this model, i.e. a trajectory in which the emissions are not reduced with respect to their maximum values.

The emissions of CO_2 to output ratio σ_{it}, of each country or region i at each period of time t, declines exogenously over time t due to an assumed autonomous energy efficiency increase. Given e_{it} and Y_{it}, and the BAU scenario, one may obtains

$$\sigma_{i,t} = \frac{e_{it}}{Y_{it}}.$$

The potential GDP denoted by Y_{it} is the output(exogenous) of country or region i at period of time t, in billion 1990 USA dollars, and g_{it} is the annual growth rates of each country or region i at each period of time t.

$$Y_{i,t+1} = (1 + g_{it})Y_{it}. \tag{5}$$

The next equation modelizes the stock pollutant part of the model.

The emissions contribute to the stock of CO_2 in the atmosphere, in billion tons of carbon CO_2, according to equation (1)

$$s_t \quad = \quad (1-\delta)s_{t-1} + \sum_{i=1}^{n} e_{it} + \xi_t, \qquad \forall t = 1, ...T.$$

or equivalently

$$s_t = (1-\delta)^t s_0 + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\sum_{i=1}^{n} e_{i\tau} + \sum_{\tau=1}^{t}(1-\delta)^{t-\tau}\xi_\tau.$$

where the initial stock or preindustrial level of the CO_2 atmospheric stock, is taken as 590 billion tons of carbon equivalent.

The parameter δ, such that $0 < \delta < 1$, the rate of decay or absorption of CO_2 in the atmosphere between two periods of time t and $t-1$, is assumed as $\delta = 0.0833$ per decade or $\delta = 0.0909512$ per year.

The random disturbance ξ_t is a noise process as in (2), i.e. sequence of i.i.d. random variables and independent of the initial state s_0, with normal distribution and

$$\mathbb{E}\left[\xi_t\right] = 0, \qquad \sigma^2 = \mathbb{E}\left[\xi_t^2\right] = 1, \qquad \forall t = 1, 2, ..., T-1.$$

In our simulations we have estimate the expectation of the damages functions over 100 runs carried out after the corresponding 100 values of the standard normal disturbance ξ_t.

The stock s influences in turn the variation of atmospheric temperature w.r.t. its preindustrial or initial level s_0, according to the following equations

$$\Delta T_t = \gamma \ln\left(\frac{s_t}{s_0}\right),$$

where the annual discount rate γ is an exogenous positive parameter. This parameter is calibrated such that a doubling of CO_2 atmospheric concentration results in an increase of temperature of 2.5 degrees with respect to its preindustrial level, and we take its value as

$$\gamma = \frac{2.5}{\ln(2)}.$$

The next two equations describe the economic part of the model, i.e. the costs c_{it} of reducing the emissions of CO_2 on the one hand, and the costs of the damages d_{it} due to stock pollutant and climate change on the other.

The abatement cost function c_{it} of country i at each period of time t, measured in billion 1990 USA dollars, is given by

$$c_{it}(e_{it}) = a_{i1}\eta_{it}^{a_{i2}} Y_{it} = a_{i1}\left[1 - \frac{e_{it}}{\sigma_{it} Y_{it}}\right]^{a_{i2}} Y_{it},$$

where the functions c_{it} are decreasing ($c'_{it} < 0$) and strictly convex ($c''_{it} > 0$), as is assumed in Section 2.

Damages due to stock pollutant and climate change are assumed to follow from the increase of the atmospheric temperature, in billion 1990 USA dollars, according to

$$d_{it}(s_t) = b_{i1}\Delta T_t^{b_{i2}} Y_{it} = b_{i1}\left[\gamma \ln\left(\frac{s_t}{s_0}\right)\right]^{b_{i2}} Y_{it}, \tag{6}$$

where the functions d_{it} are increasing ($d'_{it} > 0$) and convex ($d''_{it} > 0$), according to the hypotheses of the model in Section 2.

The regional parameter values a_{i1}, a_{i2}, b_{i1} and b_{i2} for all countries $i \in J$ are exogenous and positive. These regional parameter values, characterizing damage functions d_{it} and abatement cost functions c_{it}, are derived from Eyckmans and Tulkens (2003), and are given in Table 1.

We now describe the exogenous parameters appearing in the problems $(P1)$ and $(P2)$. The initial output Y_{it}, i.e. 1990 potential GDP, of the different region or countries are given by the vector

$$Y_{1990} = [5464.796, \quad 2932.055, \quad 6828.042, \quad 370.024, \quad 855.207, \quad 4628.621],$$

expressed in billion 1990 USA dollars and the total of the world, at this year, is 21078.750 billions USA dollars.

Table 1: Regional parameter values per country

i	USA	JAP	EU	CHI	FSU	ROW
a_{i1}	0.07	0.05	0.05	0.15	0.15	0.1
a_{i2}	2.887	2.887	2.887	2.887	2.887	2.887
b_{i1}	0.01102	0.01174	0.01174	0.015523	0.00857	0.02093
b_{i2}	2.0	2.0	2.0	2.0	2.0	2.0

The average annual output growth rates g_{it} in per cent for each country at each period of time t, given in Table 2, are calculated from Kverndokk (1994). After (5) it is possible to evaluate Y_{it} for all $i \in J$ and for all $t \in \mathcal{T}$, the cumulative output of region or country i during the period of time t.

Table 2: Average annual output growth rates g_{it} in %, per country for each period of time t (per decade)

period t	USA	JAP	EU	CHI	FSU	ROW
1990-2000	2.60	2.20	2.20	4.60	2.60	3.70
2000-2020	2.20	1.70	1.70	4.40	2.10	3.40
2020-2050	1.60	1.30	1.30	3.40	1.60	2.70
2050-2080	1.00	1.00	1.00	2.50	1.00	1.50
2080-2110	1.00	1.00	1.00	2.00	1.00	1.00

We face now the calculation of the initial value σ_{1990} for the optimization problem.

The initial CO_2 vector of emissions e_{1990}, in absence of any control are taken from the RICE model and these emissions are measured in billion tons of carbon.

$$e_{1990} = [1.37, \quad 0.29, \quad 0.872, \quad 0.805, \quad 1.066, \quad 3.43]$$

Given e_{1990} and the annual GDP Y_{1990} value, following (4) we obtain the initial emissions of CO_2 to output ratio σ_{1990}

$$\sigma_{1990} = [0.0002506, 0.0000989, 0.0001277, 0.0021755, 0.0012464, 0.000741]$$

Given e_{1990} and the annual emissions growth rates g_{it}, following (5) it is easy to calculate the output ratio σ_{it} for all country $i \in J$ and for all period of time $t \in \mathcal{T}$, that is the CO_2 emission/output ratio of region or country i during the period of time t.

In this example we borrow the output Y_{it} and CO_2 emission/output ratio time series from different versions of the RICE model, developed by Nordhaus and Yang (1996) and Nordhaus and Boyer (2000).

Finally the discount factor per year, that appears in the objective functions of problems $(P1)$ and $(P2)$ is taken as

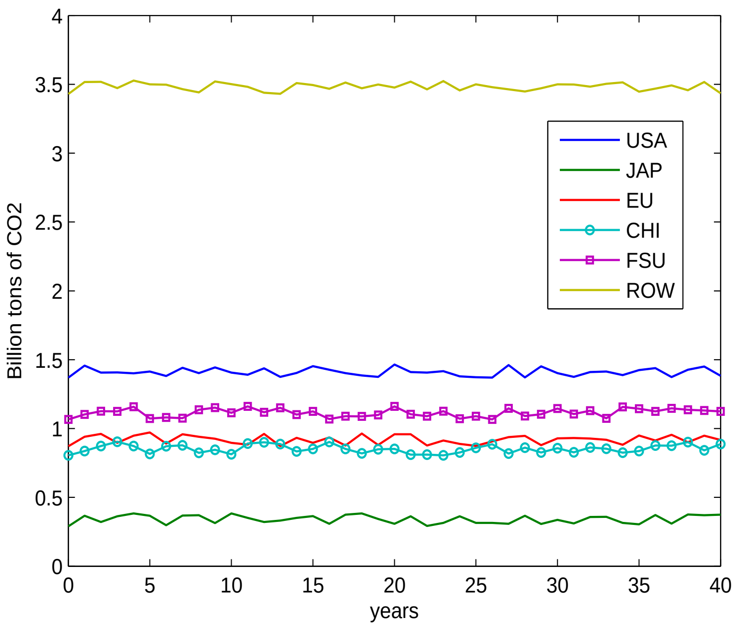
$$\beta = \frac{1}{(1+\rho)^1} = 0.98$$

where the annual discount rate is chosen as $\rho = 0.02$.

5.2 The Numerical Results

In this subsection we present the reference scenario which corresponds to the values of the parameters given in the last subsection. The simulations are made for a time horizon of 100 years, but we give the results only up to 2030, i.e. for the first 40 years, in order to avoid boundary problems.

Figure 1: Optimal cooperative emissions e^*_{it} for each country at each period of time t in billion tons of carbon equivalent.



We have implemented the equivalent formulation of problems $(P1)$ and $(P2)$ given in Subsections 3.1 and 4.1, respectively. The damages function (6) considered in our example, is more complex than the particular cases analyzed in Section 3.2.

Thus, we have developed specific code for our example. All the tables are included in the Appendix.

Note that the optimal abatement rates for each country can be directly obtained after the optimal emissions by applying (4). This is in fact one of the outputs more frequently analyzed by the economic literature concerning stock pollutant control.

Table 3 gives the optimal cooperative emissions e^*_{it} in billion tons of CO_2 equivalent for each country during each period of time t. These results are related with problem $(P1)$. The last row gives the cumulated emissions per country until the end of the horizon T in billion tons of carbon. Figure 1 shows the optimal cooperative emissions e^*_{it} for each country i and per each period of time t.

Figure 2: Optimal Cooperative Value Function W_{it} per country i for each period of time t in billions of 1990 USA dollars.

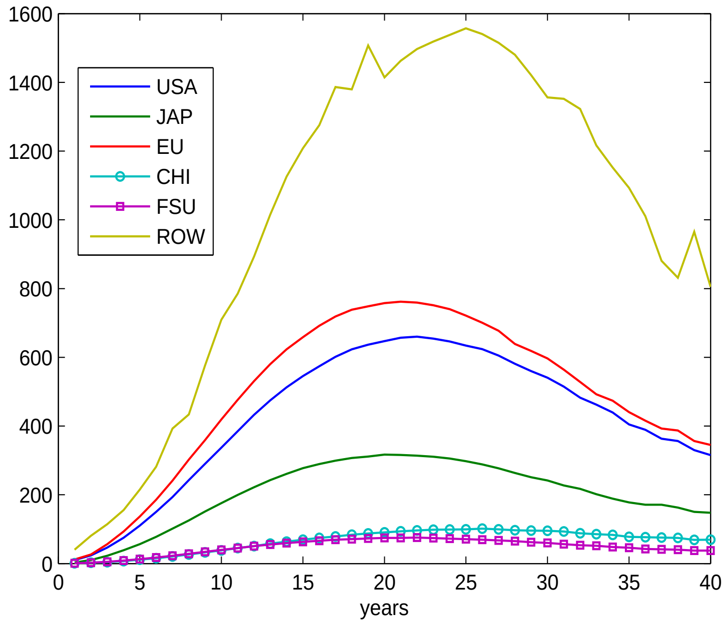


Table 4 gives the optimal cooperative value function W_{it} for each country during each period of time t in billions of 1990 USA dollars. These results are related with problem $(P1)$. The last row gives the cumulated value function per country and the total of the world at the end of the final period T, measured in billions of 1990 USA dollars. Figure 2 shows the optimal cooperative value function W_{it} for each country i and per each period of time t in billions of 1990 USA dollars.

Figure 3: Optimal non cooperative emissions e_{it}^N per each country for each period of time t in billion tons of carbon equivalent.

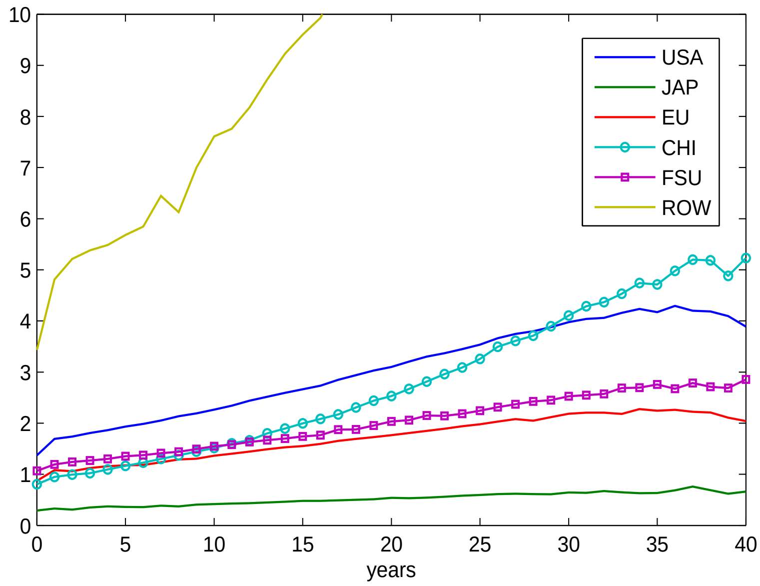
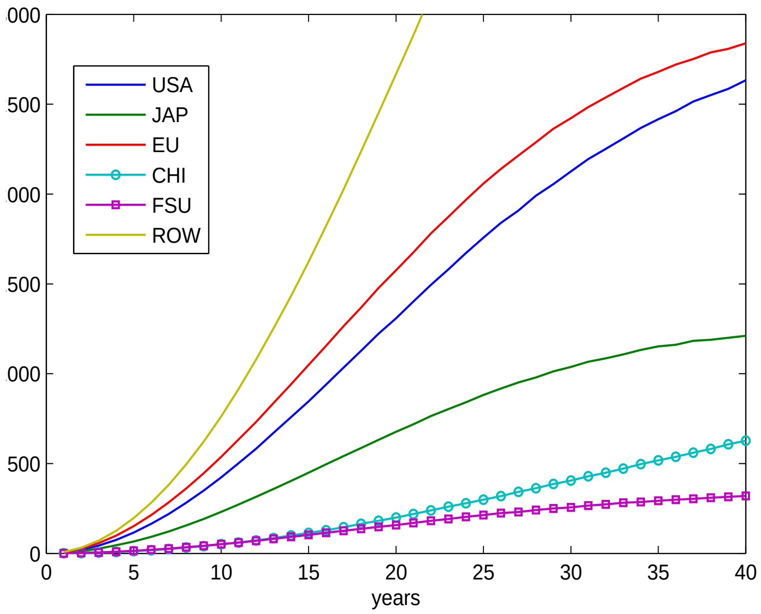


Table 5 gives the optimal non cooperative emissions e_{it}^N per country i during the period of time t. These results are related with problem $(P2)$. The last row gives the cumulated emissions per country i until the end of the period of time T in billion tons of carbon. Figure 3 shows the optimal non cooperative emissions e_{it}^N for each country i and for each period of time t.

Table 6 gives the optimal non cooperative value function N_{it} for each country during each period of time t in billions of 1990 USA dollars. These results are related with problem $(P2)$. The last row gives the cumulated value function for each country and the total of the world until the end of the horizon T, measured in billions of 1990 USA dollars. Note that the Figure 4 shows the optimal non cooperative value function N_{it} for each country i and per each period of time t in billions of 1990 USA dollars.

Although optimal emissions increase with time for both cases, in the cooperative case, see Figure 1, it is not a remarkable issue. Nevertheless, we discover an increasing trend of the optimal emissions in the non cooperative case, as is shown

Figure 4: Optimal Non Cooperative Value Function N_{it} per country i for each period of time t in billions of 1990 USA dollars.



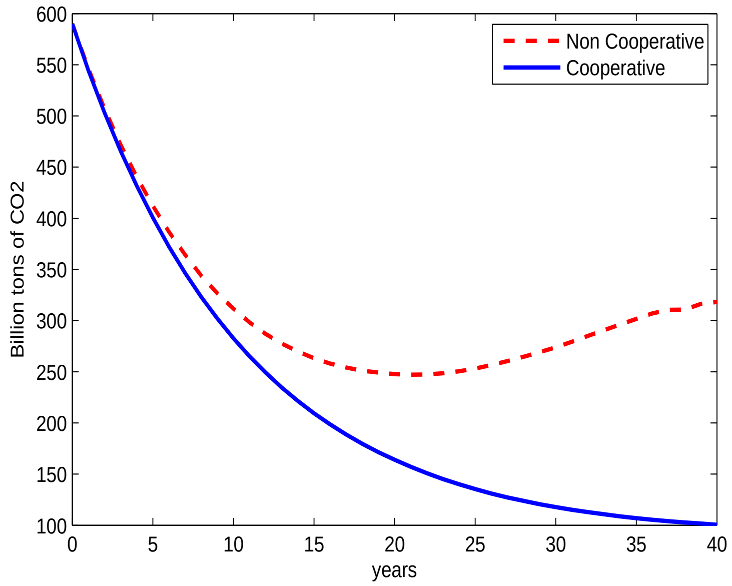
in Figure 3. As it is expected, the total of the optimal non cooperative emissions for each country are bigger that the total of the optimal cooperative emissions as is shown in Tables 3 and 5.

The total Optimal Cooperative Value Function is smaller than the total Optimal Non Cooperative Value Function, as it is shown in Tables 4 and 6, and Figures 2 and 4. We observe that this result is consistent with what is obtained in the seminal paper for the deterministic model provided by Germain, Toint, Tulkens, and de Zeeuw (2003). In fact this result was expected after the definition of the optimum.

We are now to compare the optimal stocks of pollutant. Table 7 gives the cooperative optimal stock of pollutant, s_t^*, the non cooperative optimal stock of pollutant s_t^N and the differences between them at each period of time t in billion tons of carbon. We observe a great improvement of the cooperative behavior with respect to the non cooperative one over the time.

Figure 5 depicts the optimal cooperative and non-cooperative stocks of pollutant, s_t^* and s_t^N respectively for each period of time t in billion tons of carbon equivalent.

Figure 5: Optimal cooperative and non-cooperative stocks



We observe in Figure 5 that the optimal stock cooperative s_t^* decreases faster than the non-cooperative stocks s_t^N. This result is consistent with the expected behavior of the solutions of problems $(P1)$ and $(P2)$.

We have checked our model in different scenarios by changing the values of the noise process parameters including the deterministic case, (i.e. $E[\xi_t] = 0$, $Var[\xi_t] = 0$). All the results we have found were consistent, and for the deterministic case we have obtained optimal stationary strategies for both problems $P1$ and $P2$, as we expected.

6 Conclusions and Extensions

We have developed a useful stochastic formulation which extends the stock pollutant control model developed by Germain, Toint, Tulkens, and de Zeeuw (2003). Our model lets to include through the random disturbance term, random elements not considered in the deterministic model. Moreover, our proposal lets to evaluate the magnitud of this effects by estimating, for example, the variance of the additive noise process. In principle we have assume independence for this process but we can also extend our work by considering some time series structures for the noise process.

Additionally, our example shows that the stochastic formulation produce consistent results in comparison to the deterministic model of reference, but simultaneously provides more flexibility than the former one. Note that the example proposed to illustrate our formulation is very close to the CLIMNEG model, which has been in fact analyzed from the deterministic point of view. So, in somehow we also extend this model to a stochastic setting. On the other hand, we want to remark that our real data based example is strongly driven by the original values taken at 1990 according to Kyoto Protocol.

Summarizing our results, for each country $i \in J$ and each period $t \in \mathcal{T}$ we obtain the following stocks pollution, emissions and values functions for each model

Cooperative Model ($P1$): Pareto equilibrium

$$\{s_t^*\}, \qquad \{e_{it}^*\}, \qquad \{W_i(t, s_{t-1}^*)\}.$$

Non-Cooperative Model ($P2$): Nash equilibrium

$$\{s_t^N\}, \qquad \{e_{it}^N\}, \qquad \{N_i(t, s_{t-1}^N)\}.$$

One might think an extension of our stochastic model by considering monetary transfers to induce cooperation, having in mind the significative differences between the optimal cooperative and non cooperative stock pollutant pointed out in our example, see for instance Figure 5.

Stochastic performance criteria based on bounds of probability could be also considered, as an extension of this work.

Finally, further research could be done if we consider uncertainty about the random perturbation, say the variance of the i.i.d. sequence. We propose to estimate the parameter recursively and to include the estimation in the stochastic control problem.

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A Appendix-Tables

Table 3: Optimal cooperative emissions e^*_{it} for each country i at each period of time t in billion tons of carbon equivalent.

t	USA	Japan	EU	China	FSU	ROW	Total
0	1.3700	0.2920	0.8720	0.8050	1.0660	3.4300	7.8350
1	1.4573	0.3661	0.9403	0.8369	1.1023	3.5168	8.2197
2	1.4059	0.3216	0.9617	0.8730	1.1256	3.5182	8.2061
3	1.4072	0.3623	0.8965	0.9046	1.1240	3.4727	8.1673
4	1.4006	0.3836	0.9480	0.8734	1.1582	3.5268	8.2906
5	1.4143	0.3664	0.9715	0.8160	1.0721	3.5000	8.1402
6	1.3814	0.2980	0.8910	0.8709	1.0808	3.4965	8.0187
7	1.4415	0.3677	0.9579	0.8782	1.0752	3.4648	8.1851
8	1.4019	0.3702	0.9398	0.8229	1.1360	3.4410	8.1117
9	1.4435	0.3138	0.9263	0.8455	1.1521	3.5200	8.2012
10	1.4069	0.3841	0.8962	0.8128	1.1143	3.5015	8.1157
11	1.3907	0.3505	0.8837	0.8911	1.1608	3.4809	8.1577
12	1.4373	0.3208	0.9604	0.9003	1.1176	3.4383	8.1748
13	1.3754	0.3321	0.8722	0.8874	1.1500	3.4314	8.0485
14	1.4040	0.3510	0.9323	0.8341	1.1012	3.5088	8.1315
15	1.4534	0.3642	0.8962	0.8516	1.1240	3.4941	8.1835
16	1.4266	0.3078	0.9349	0.9028	1.0691	3.4667	8.1079
17	1.4019	0.3747	0.8767	0.8520	1.0897	3.5128	8.1077
18	1.3850	0.3839	0.9637	0.8201	1.0878	3.4709	8.1114
19	1.3748	0.3430	0.8802	0.8494	1.0983	3.4988	8.0446
20	1.4651	0.3080	0.9586	0.8524	1.1605	3.4766	8.2213
21	1.4096	0.3621	0.9591	0.8101	1.1035	3.5196	8.1640
22	1.4058	0.2926	0.8770	0.8111	1.0901	3.4636	7.9401
23	1.4165	0.3153	0.9127	0.8053	1.1259	3.5227	8.0984
24	1.3790	0.3626	0.8883	0.8259	1.0720	3.4557	7.9836
25	1.3729	0.3153	0.8741	0.8596	1.0896	3.4996	8.0111
26	1.3705	0.3150	0.9062	0.8864	1.0667	3.4784	8.0233
27	1.4606	0.3087	0.9377	0.8184	1.1468	3.4630	8.1353
28	1.3706	0.3667	0.9465	0.8606	1.0907	3.4478	8.0829
29	1.4518	0.3075	0.8794	0.8262	1.1036	3.4711	8.0396
30	1.4024	0.3371	0.9284	0.8578	1.1453	3.4994	8.1706
31	1.3755	0.3113	0.9312	0.8280	1.1051	3.4988	8.0499
32	1.4095	0.3581	0.9278	0.8626	1.1300	3.4822	8.1702
33	1.4138	0.3585	0.9186	0.8529	1.0737	3.5033	8.1209
34	1.3879	0.3144	0.8818	0.8241	1.1575	3.5139	8.0797
35	1.4239	0.3040	0.9488	0.8358	1.1447	3.4466	8.1038
36	1.4388	0.3716	0.9129	0.8765	1.1241	3.4689	8.1928
37	1.3743	0.3092	0.9545	0.8749	1.1468	3.4918	8.1516
38	1.4264	0.3758	0.8998	0.9008	1.1369	3.4564	8.1960
39	1.4501	0.3712	0.9478	0.8423	1.1314	3.5167	8.2595
40	1.3826	0.3748	0.9176	0.8873	1.1243	3.4344	8.1210
Total	56.3971	13.7018	36.8385	34.1218	44.6082	139.3719	325.0393

Table 4: Optimal Cooperative Value Function W_{it} per country i for each period of time t in billions of 1990 USA dollars.

t	USA	Japan	EU	China	FSU	ROW	Total
1	10.369	3.045	11.474	0.791	0.822	40.105	66.607
2	23.705	9.996	26.217	2.116	2.566	80.578	145.178
3	46.273	22.435	56.221	4.111	5.107	114.704	248.851
4	74.977	38.791	93.267	7.221	8.481	155.732	378.469
5	110.474	56.526	136.585	11.066	12.790	215.051	542.492
6	150.045	77.489	185.374	15.534	17.407	281.586	727.435
7	193.445	101.335	241.070	20.622	22.536	392.502	971.510
8	242.665	125.296	302.217	26.312	28.053	433.630	1158.172
9	289.747	151.391	358.967	32.281	33.853	575.998	1442.238
10	337.331	175.750	419.299	38.329	39.610	709.607	1719.925
11	385.025	199.606	476.437	44.817	44.978	785.260	1936.124
12	432.906	222.136	530.729	50.773	50.390	892.969	2179.902
13	474.958	242.581	580.072	57.794	55.083	1015.229	2425.717
14	512.754	260.732	623.311	63.756	59.215	1126.005	2645.772
15	545.188	277.428	658.670	69.500	62.971	1208.345	2822.102
16	573.908	289.570	691.630	74.551	65.994	1275.034	2970.687
17	601.616	299.142	718.567	78.833	69.498	1386.343	3153.998
18	623.185	307.191	738.315	83.789	70.964	1379.837	3203.282
19	636.706	311.093	748.382	87.706	72.697	1507.285	3363.869
20	646.949	316.818	757.791	90.567	74.612	1414.530	3301.268
21	656.947	315.800	761.817	93.821	74.647	1462.772	3365.805
22	660.043	313.585	759.360	96.347	75.480	1496.716	3401.530
23	654.250	310.591	751.377	98.160	73.763	1518.743	3406.884
24	645.977	305.770	739.954	98.828	72.478	1537.814	3400.821
25	633.975	297.621	721.166	99.575	71.054	1557.018	3380.409
26	623.530	288.232	700.424	101.497	69.405	1540.371	3323.459
27	605.082	277.149	677.445	99.436	67.152	1515.212	3241.477
28	581.221	263.688	638.926	96.741	64.914	1480.543	3126.033
29	560.102	250.809	618.510	95.807	61.931	1421.172	3008.332
30	540.727	241.449	596.758	95.131	59.797	1356.449	2890.311
31	514.518	227.031	564.016	93.019	56.418	1352.034	2807.036
32	482.633	217.063	528.249	88.252	53.187	1322.749	2692.133
33	462.261	201.736	492.236	85.544	51.605	1216.818	2510.199
34	439.454	188.740	473.924	83.498	48.224	1152.185	2386.026
35	404.442	178.001	440.403	77.556	46.016	1092.967	2239.386
36	389.319	171.133	415.472	76.438	42.149	1010.633	2105.144
37	363.086	171.140	392.953	75.772	41.336	880.546	1924.833
38	356.628	162.549	386.854	74.481	40.175	831.407	1852.094
39	329.668	150.447	356.566	68.723	37.583	965.291	1908.279
40	315.541	147.684	345.070	69.529	37.492	804.884	1720.200
Total	17131.631	8168.569	19716.072	2628.625	1942.436	40506.656	90093.990

Table 5: Optimal non cooperative emissions e_{it}^N for each country at each period of time t in billion tons of carbon equivalent.

t	USA	Japan	EU	China	FSU	ROW	Total
0	1.3700	0.2920	0.8720	0.8050	1.0660	3.4300	7.8350
1	1.6936	0.3297	1.0815	0.9466	1.1932	4.8111	10.0556
2	1.7369	0.3093	1.0601	0.9934	1.2411	5.2140	10.5549
3	1.8079	0.3495	1.1240	1.0208	1.2706	5.3799	10.9526
4	1.8647	0.3708	1.1569	1.0952	1.3004	5.4845	11.2725
5	1.9336	0.3616	1.1757	1.1651	1.3533	5.6815	11.6707
6	1.9872	0.3573	1.1840	1.2277	1.3755	5.8461	11.9778
7	2.0505	0.3855	1.2348	1.3000	1.4118	6.4445	12.8270
8	2.1349	0.3713	1.2890	1.3697	1.4400	6.1295	12.7344
9	2.1930	0.4059	1.3035	1.4440	1.4940	6.9994	13.8398
10	2.2653	0.4169	1.3646	1.5154	1.5496	7.6107	14.7225
11	2.3415	0.4262	1.4038	1.6046	1.5801	7.7593	15.1154
12	2.4388	0.4363	1.4455	1.6626	1.6325	8.1749	15.7907
13	2.5168	0.4483	1.4887	1.8007	1.6683	8.7246	16.6475
14	2.5943	0.4613	1.5284	1.8951	1.7009	9.2275	17.4075
15	2.6619	0.4818	1.5525	1.9965	1.7420	9.6005	18.0353
16	2.7330	0.4817	1.5957	2.0837	1.7663	9.9276	18.5880
17	2.8491	0.4917	1.6536	2.1720	1.8722	10.5902	19.6287
18	2.9394	0.5020	1.6906	2.3074	1.8779	10.6086	19.9261
19	3.0297	0.5131	1.7288	2.4412	1.9538	11.5005	21.1672
20	3.0989	0.5397	1.7645	2.5294	2.0352	11.0784	21.0461
21	3.2049	0.5342	1.8076	2.6697	2.0574	11.6166	21.8905
22	3.3017	0.5416	1.8511	2.8143	2.1482	12.1397	22.7966
23	3.3693	0.5614	1.8921	2.9590	2.1435	12.6529	23.5781
24	3.4505	0.5830	1.9419	3.0895	2.1831	13.2212	24.4691
25	3.5378	0.5954	1.9804	3.2576	2.2443	13.8653	25.4807
26	3.6605	0.6113	2.0297	3.4956	2.3156	14.3480	26.4608
27	3.7451	0.6215	2.0785	3.6108	2.3686	14.8227	27.2472
28	3.7994	0.6142	2.0474	3.7109	2.4270	15.2640	27.8629
29	3.8773	0.6104	2.1191	3.8951	2.4512	15.5554	28.5083
30	3.9754	0.6441	2.1863	4.1071	2.5262	15.8186	29.2578
31	4.0398	0.6378	2.2058	4.2885	2.5466	16.6141	30.3326
32	4.0601	0.6735	2.2061	4.3691	2.5741	17.2457	31.1286
33	4.1586	0.6465	2.1817	4.5336	2.6893	17.2203	31.4300
34	4.2358	0.6312	2.2762	4.7429	2.6952	17.5216	32.1029
35	4.1714	0.6324	2.2456	4.7126	2.7592	17.8291	32.3502
36	4.2945	0.6870	2.2613	4.9778	2.6721	17.7925	32.6852
37	4.2009	0.7584	2.2238	5.1995	2.7869	16.2654	31.4349
38	4.1874	0.6889	2.2102	5.1854	2.7111	13.3914	28.3744
39	4.0943	0.6202	2.1070	4.8804	2.6863	19.5576	33.9458
40	3.8903	0.6632	2.0420	5.2317	2.8560	16.0003	30.6835
Total	124.1258	20.9963	69.7200	114.3020	81.3007	465.5356	875.9806

Table 6: Optimal Non Cooperative Value Function N_{it} for each country i for each period of time t in billions of 1990 USA dollars.

t	USA	Japan	EU	China	FSU	ROW	Total
1	5.060	5.840	6.771	0.475	0.607	7.933	26.685
2	19.428	11.189	26.000	1.898	2.370	31.620	92.504
3	43.100	25.930	56.806	4.292	5.235	71.063	206.425
4	75.349	45.255	98.798	7.665	9.140	125.683	361.891
5	115.923	65.951	151.237	12.265	14.113	196.021	555.510
6	164.689	91.649	213.453	17.588	20.000	281.700	789.078
7	219.679	122.344	283.644	24.085	26.765	381.763	1058.279
8	282.161	155.795	361.848	32.230	34.084	496.270	1362.387
9	349.473	191.702	446.515	40.716	42.213	621.860	1692.479
10	423.470	231.095	537.756	51.094	51.246	762.281	2056.942
11	501.907	271.616	633.719	60.638	60.394	914.520	2442.794
12	582.701	314.348	732.073	72.192	70.568	1078.521	2850.404
13	670.695	358.674	837.529	85.196	80.669	1253.131	3285.894
14	758.073	403.696	941.222	100.204	91.882	1432.802	3727.879
15	846.312	449.146	1048.522	114.325	102.467	1623.335	4184.106
16	939.339	496.081	1154.979	128.342	114.587	1822.981	4656.309
17	1033.522	541.159	1264.911	146.118	125.492	2025.461	5136.663
18	1127.862	586.757	1368.316	164.338	136.879	2237.544	5621.696
19	1222.511	632.460	1478.109	181.050	148.047	2451.760	6113.936
20	1308.225	677.369	1576.073	199.063	157.745	2667.923	6586.399
21	1401.983	718.263	1675.835	219.710	169.977	2883.851	7069.619
22	1494.404	764.775	1781.170	239.493	181.545	3114.149	7575.536
23	1581.518	803.590	1873.400	259.723	191.216	3332.866	8042.313
24	1672.016	841.284	1968.219	279.329	203.507	3564.322	8528.677
25	1757.224	881.467	2058.035	298.552	213.239	3786.392	8994.909
26	1839.201	917.556	2139.636	318.380	224.032	4011.689	9450.494
27	1907.954	951.243	2214.529	342.946	230.629	4230.270	9877.570
28	1990.590	979.139	2287.867	362.215	241.658	4449.010	10310.478
29	2055.345	1013.474	2364.149	385.946	250.034	4664.065	10733.013
30	2126.256	1037.725	2422.633	405.758	256.618	4869.331	11118.321
31	2195.015	1066.505	2484.037	429.626	266.211	5080.226	11521.619
32	2251.363	1085.618	2538.383	449.499	272.596	5282.732	11880.192
33	2308.942	1107.414	2590.563	472.219	282.141	5481.463	12242.741
34	2367.816	1132.143	2642.893	496.390	285.541	5680.213	12604.995
35	2416.267	1152.267	2679.950	518.215	292.525	5881.186	12940.409
36	2461.812	1161.568	2720.958	538.015	299.275	6065.006	13246.633
37	2514.893	1182.822	2752.021	560.599	303.991	6248.382	13562.708
38	2550.685	1189.119	2788.440	581.490	309.679	6431.110	13850.523
39	2585.639	1200.261	2808.867	607.007	314.701	6597.569	14114.044
40	2633.175	1211.492	2839.462	627.551	320.025	6785.054	14416.758
Total	52801.575	26075.778	60849.327	9836.434	6403.642	118923.057	274889.812

Table 7: Optimal stocks of pollutant cooperative s_t^* and non cooperative s_t^N and their differences for each period t in billion tons of carbon equivalent.

t	s_t^*	s_t^N	Difference
0	590.0000	590.0000	0.0000
1	544.5887	546.4247	1.83592
2	503.2917	507.3095	4.01784
3	465.7098	472.1477	6.43787
4	431.6674	440.5019	8.83448
5	400.5691	412.1310	11.56194
6	372.1760	386.6461	14.47010
7	346.5303	364.3270	17.79666
8	323.1425	343.9441	20.80159
9	301.9700	326.5193	24.54933
10	282.6366	311.5612	28.92458
11	265.1027	298.3557	33.25308
12	249.1796	287.0259	37.84635
13	234.5776	277.5828	43.00516
14	221.3860	269.7580	48.37202
15	209.4455	263.2723	53.82673
16	198.5148	257.9288	59.41401
17	188.5776	254.1118	65.53429
18	179.5473	250.9391	71.39189
19	171.2710	249.2960	78.02505
20	163.9237	247.6811	83.75741
21	157.1870	247.0574	89.87035
22	150.8389	247.3965	96.55765
23	145.2260	248.4863	103.26028
24	140.0085	250.3680	110.35947
25	135.2929	253.0903	117.79739
26	131.0180	256.5451	125.52711
27	127.2437	260.4724	133.22862
28	123.7602	264.6583	140.89812
29	120.5500	269.1092	148.55920
30	117.7626	273.9050	156.14239
31	115.1078	279.3397	164.23183
32	112.8148	285.0763	172.26153
33	110.6808	290.5929	179.91211
34	108.6996	296.2809	187.58129
35	106.9226	301.6991	194.77650
36	105.3961	306.9598	201.56373
37	103.9672	310.4921	206.52493
38	102.7126	310.6428	207.93016
39	101.6355	316.3512	214.71562
40	100.5178	318.2784	217.76052