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# WEAK CONVERGENCE IN CREDIT RISK

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**Abstract**

In the present paper, we study both the approximation of a continuous-time model by a sequence of discrete-time price models driven by semimargingales with credit risk, and the convergence of these price processes (in terms of the triplets) under a framework that allows the practitioner a multiple set of models (semimartingale) and credit conditions (migration and default).

**Keywords:** Weak convergence, semimartingales, incomplete markets, corporate bonds.

**2000 MSC:** Primary 60B10, 60F05; Secondary 46N30.

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# WEAK CONVERGENCE IN CREDIT RISK

JESÚS PÉREZ COLINO

*(WORKING PAPER)*

ABSTRACT. In the present paper, we study both the approximation of a continuous-time model by a sequence of discrete-time price models driven by semimartingales with credit risk, and the convergence of these price processes (in terms of the triplets) under a framework that allows the practitioner a multiple set of models (semimartingale) and credit conditions (migration and default).

## 1. INTRODUCTION

In order to implement certain models and pricing rules, continuous-time models are often too complex to handle. Therefore it is convenient to both discretize time and space and show that the discretization is 'good' in the sense that the discretized models and pricing rules converge to the continuous-time model as the discretization steps tend to zero. The main subject of this paper treats this aspect of mathematical finance when credit risk and incompleteness appears. We study both the approximation of a continuous-time model by a sequence of discrete-time models and the convergence of price processes for corporate bonds with credit migration.

Some authors have presented different discrete-time solutions to model defaultable or corporate bonds (**Jarrow and Turnbull** (1995), **Jarrow, Turnbull and Lando** (1997) or **Schönbucher** (2002) among others) as a mixture of trees, one for the risk-free interest rates, and a second one for the credit risk factor (untradeable). However, though different authors propose a useful methodology to price credit derivatives, they do not talk about the conditions of weak convergence when both factors are mixed using discrete frameworks. In fact, there is no guarantee that these compositions between random factors converge in a measure in continuous time, and all the usual tools from stochastic analysis in continuous time can be applied.

On the other hand, at least in the case of the interest-rates derivatives market, the market quotes caps/floors and swaptions (plain vanilla interest-rates derivatives) using the well-known **Black-76** model. It basically implies that any alternative model written or developed to price exotic options has to guarantee that, at least, it will recover the prices of the plain-vanilla options priced by **Black-76**. In other words, our model has to converge in distribution to the implied distribution given by the market through the quoted volatilities. Therefore, any new interest rate model has to prove that when it goes to continuous time, it is able to recover the implied distribution in the **Black-76** model, which is driven by the usual Brownian motion (continuous process). Therefore we have to prove that our LIBOR additive process (semimartingale) is able to converge weakly to the implied probability given by the market. However, our problem is more complex if we add the reactive or credit risk part to our usual stochastic differential equation.

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In this paper, we turn our attention to **the Reaction-Additive system** which can be defined as a system of stochastic differential equations (SDEs) where the first of these SDEs is based on the **bond price process** $B$ as a **tradeable risky asset** which is modelled using semimartingales (**Colino** 2008). On the other hand, the second SDE models the non-tradeable **factor** of uncertainty (**credit migration**) which is represented by a finite-state process $C$ driven by a **multivariate point process**. The present paper has as main goal to study the weak convergence conditions in this kind of stochastic systems. Basically we present a set of new limit theorems for semimartingales with stochastic triplet, that specifically are well suited to explain the behaviour of corporate bonds with random evolution of their credit risk.

In order to achive the mentioned goal, we have structured this paper in the following two sections:

- The next section in this paper (*section 2*), is devoted to present the mentioned system of stochastic equations. Basically, we review some basics stochastic features for the mentioned set to equations, such as the markovian property or the belonging to Skorohod space.
- The *section 3* is focused in the weak convergence conditions of this financial corporate bond model in incomplete markets. Here, we show the conditions of relative compacteness, weak convergence and we translate this conditions in terms of moduli of continuity to conditions in terms of the semimartinagale triplet.

## 2. Corporate-Bond valuation in a Reaction-Additive system

The present section is mainly devoted to show how due to this system of stochastic equations we can obtain a realistic sample-path for corporate bond prices, driven by the composition of these two processes. In order to achieve this goal, we proceed in three different steps:

- *First*, we introduce some ideas and references where the reader can find the proof of **existence** and **uniqueness** of solutions for such SDEs systems.
- In the *second* subsection, we face the question of **markovianity** and **uniqueness** of the sample paths generated by the composition of these two processes.
- Finally, in the *third* subsection, we prove that these sample paths are '*càdlàg*' or equivalently, they 'live' in a **Skorohod space**.

In order to clarify the framework, let us briefly summarize some assumptions and results that we have developed in **Colino** (2008). Let us define $B(t, T_i, C_t)$ as the price of a **zero-coupon corporate bond**, valued in $t \in [0, T_i]$ for any fixed maturity $T_i \leq T^\star$, with credit rating $C_t \in \mathcal{K}$ for every $t \in [0, T_i]$. Additionally we assume that the **corporate bond** is modelled with a **fractional recovery of market value** in case of default. Notice that $B(t, T_i, C_t)$ is a strictly positive and $\mathcal{F}$-adapted process, defined on a '*sufficiently rich*' **stochastic basis** $(\Omega, \mathbb{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^\star]}$ generated by a $d$-dimensional **LIBOR additive process** $G$ and the **credit migration process** $C$, or in other words $\mathcal{F}_t = \sigma\{G_s, C_s; 0 \leq s \leq t\}$.

Let us fix the following elements: *first*, fix $m \in \mathbb{N}_+$ such that $\mathcal{K} = \left\{0, \frac{1}{m}, ..., 1 - \frac{1}{m}, 1\right\}$, *second* fix the time horizon $T_i \in [0, T^\star]$, as the maturity of the corporate bond; and *finally*, a domain, that in our case will be $\mathbb{R}^d$. Given any starting point $(t_0, B_{t_0}, C_{t_0}) \in [0, T_i] \times \mathbb{R}^d \times \mathcal{K}$ we have the following system of SDEs, where $(B, C)$ is a solution, under **risk-neutral probability**, with values in $\mathbb{R}^d \times \mathcal{K}$ :

$$\begin{cases} \frac{dB(t,T_i,C_t)}{B\left(t_-,T_i,C_{t\_}\right)} = \left(r_t + l \cdot \lambda_{C_{t-},1}\right) dt + b_{\eta_{(t)}}(t, T_i, C_{t\_})dW_t + \int_{\mathbb{R}^d} h\left(t, x, T_i\right) 1_{\{|x|\leq 1\}}\ \tilde{\mu}^{C_{t-}}_{\eta_{(t)}}\left(dt, dx\right) \\ dC_t = \sum\limits_{a,b\in K} (b - a)\, 1_{\left\{C_{t\_}=a\right\}} \left(C_{t\_}\right) dN_{ab}\left(t\right) \end{cases} \tag{2.1}$$

where $r_t = f(t,t,0)$ is the usual risk-free 'short-rate', $l$ is the rate of losses in case of a default, such that $l \in [0,1]$, $b_{\eta_{(t)}}$ is the coefficient of diffusion such that $b_{\eta_{(t)}} : [0,T_i] \times \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}^d$ is $C^1$ with respect to $(t,B) \in [0,T_i] \times \mathbb{R}^d$ for any $t \leq T_i$ with $t, T_i \in [0,T^\star]$.

Notice that we have three sources of randomness,

- *first*, $W = (W^i)_{i=1...d}$ is the usual $\mathbb{R}^d$-valued $(\mathbb{P},\mathcal{G})$-**Brownian motion;**
- *second*, $\tilde{\mu}^{C_{t-}}_{\eta_{(t)}}$ is the **compensated random measure**, for a given credit rate $C_{t-}$, that satisfy the usual integrability conditions for any $t \leq T_i$ with $t, T_i \in [0,T^\star]$
- and *finally*, $N = (N_{ab})_{a,b\in\mathcal{K}}$ is a $\mathcal{F}$-adapted **multivariate point process** such that $(N_{ab}(t))$ has $(\mathbb{P},\mathcal{G})$-intensity $\lambda^{ab}(t,B)$ for $a,b \in \mathcal{K} = \left\{0, \frac{1}{m}, ..., 1-\frac{1}{m}, 1\right\}$.

This model is a non-standard SDE system because of its dependence structure. The coefficients in the SDE (2.1) for corporate bond dynamic contain the credit risk rating and, on the other hand, the intensities in the multivariate point process $N$ that drive the credit migration process, depend in turn on the credit rate of the corporate bond.

2.1. **Classical and Viscosity solutions for reaction-additive systems under local regularity.** In this section we mention different methodologies to derive existence and uniqueness results for classical and viscosity solutions of interacting systems of partial integro-differential equations (PIDEs). Such systems will be called **reaction-additive equations** and play a key role in subsequent sections. These methodologies and results have been studied previously by different authors, such that **Bensoussan and Lions** (1982), **Crandall and Lions** (1983), **Barles, Buckdahn and Pardoux** (1997), or **Pardoux, Pradeilles and Rao** (1997). We mention here , very briefly, some of the main results, in order to show the existence of solutions to the reaction-additive system.

We consider the **system of integral-partial differential equations** (PIDEs) of parabolic type for $c \in \mathcal{K}\backslash\{1\}$, $j = 0,1,...,n$ and boundary conditions at terminal time $T$. Denote by USC (respectively LSC) the class of upper semicontinuous (respectively, lower semicontinuous) functions $u : (0,T] \times \mathcal{K} \times \mathbb{R}^d \to \mathbb{R}$ and, on the other hand, let us define by $C_p^+\left((0,T] \times \mathcal{K} \times \mathbb{R}^d; \mathbb{R}\right)$ the set of measurable functions on $[0,T] \times \mathcal{K} \times \mathbb{R}^d$ with polynomial growth of the degree $p$ at $+\infty$, Lipschitz and bounded on $[0,T] \times \mathcal{K} \times \mathbb{R}^-$ such that

$$\varphi \in C_p^+((0,T] \times \mathcal{K} \times \mathbb{R}) \Longleftrightarrow \exists K, p > 0, |\varphi(t,x)| \leq K\left(1 + |x|^p 1_{\{x>0\}}\right)$$

Consider the following **system of backward integral-partial differential equation** of parabolic type (initial-boundary value problem) on $(0,T] \times \mathcal{K} \times \mathbb{R}$ for all $j = 0,1,...,n$

$$\begin{cases} -\frac{\partial}{\partial t} u_j(t,x,c) - \mathcal{L}u_j(t,x,c) - f_j\left(t,x,c,u_j,\left(\nabla u_j b_j^c\right)(t,x,c), \mathcal{B}u_j(t,x,c)\right) = 0 \\ u_j(T,x,c) = g_j(x,c) \end{cases} \quad (2.2)$$

where the **second-order integral-differential operator** $\mathcal{L}$ for any $c \in \mathcal{K}\backslash\{1\}$ on sufficiently smooth functions has the form

$$\mathcal{L} = \mathcal{A} + \mathcal{J}$$

with

$$\begin{aligned} \mathcal{A}u_j(t,x,c) &= \sum_{i=1}^{d} a_{\eta(t)}(t,T,c) \frac{\partial u_j}{\partial x^i}(t,x,c) + \frac{1}{2}\sum_{i,j=1}^{d} b_{\eta(t)}(t,T,c) \frac{\partial^2 u_j^c}{\partial x^i \partial x^j}(t,x,c) \\ &\quad + \sum_{c\in\mathcal{K}} \lambda_{b,c}(t)\left[u_j(x,t,c) - u_j(x,t,b)\right] \\ \mathcal{J}u_j(t,x,c) &= \int_{\mathbb{R}^d} \left[u_j(x+y,t,c) - u_j(x,t,c) - y 1_{\{|y|<1\}}\right] \mu_j^c(dy) \end{aligned}$$

and $\mathcal{B}$ is an integral operator defined as

$$\mathcal{B}u_j(x,c) = \int_{\mathbb{R}^d} [u_j(x+y,t,c) - u_j(x,t,c)]\,\mu_j^c(dy)$$

Existence and uniqueness of classical solutions for the PIDEs considered above in Sobolev-Hölder spaces have been studied in **Bensoussan and Lions** (1982) and **Garroni and Menaldi** (2002) in the case where the diffusion component is not degenerated (basically with $\sigma_{\eta(t)}(t,T,c) > 0$ for every $t \in [0,T]$ and $c \in \mathcal{K}\backslash\{1\}$).

However, as we have already mentioned **Colino** (2008), many of the examples of the **LIBOR additive process** can be pure-jump processes (with $\sigma_{\eta(t)}(t,T,c) > 0$ for every $t \in [0,T]$ and $c \in \mathcal{K}\backslash\{1\}$) for which such results are not available. A notion of solution that yields existence and uniqueness for such equations without requiring nondegeneracy of coefficients or a prior knowledge of smoothness of solutions is the notion of **viscosity solution** introduced by **Crandall and Lions** (1983) for PDEs, and extended for integro-differential equations of the type considered here in **Alvarez and Tourin** (1996), **Barles, Buckdahn and Pardoux** (1997), or **Pardoux, Pradeilles and Rao** (1997).

For such a system, we introduce the notion of a viscosity solution

**Definition 1.** *We say that* $u \in C_p^{1,2,+}\left((0,T]\times\mathcal{K}\times\mathbb{R}^d;\mathbb{R}\right)$ *is*

$(i)$ *a* ***viscosity subsolution*** *of* (2.2) *if*

$$u_j(T,x,c) \leq g_j(x,c) \ \text{ with } x \in \mathbb{R}^d$$

*and if for all* $c \in \mathcal{K}$, *all* $j = 0,...,n$, *and* $\varphi_j \in C_p^{1,2,+}\left((0,T]\times\mathcal{K}\times\mathbb{R}^d;\mathbb{R}\right)$, *such that* $(t,x) \in [0,T]\times\mathbb{R}^d$ *is a global minimum point of* $u_j - \varphi_j$, *we have*

$$-\frac{\partial}{\partial t}\varphi_j(t,x,c) - \mathcal{A}\varphi_j(t,x,c) - \mathcal{J}\left(u_j,\varphi_j\right)(t,x,c) - f_j\left(t,x,u_j(t,x,c),\left(\nabla\varphi_j b_j^c\right)(t,x,c),\mathcal{B}_c\varphi_j(t,x,c)\right) \geq 0$$

$(ii)$ *a* ***viscosity supersolution*** *of* (2.2) *if*

$$u_j(T,x,c) \geq g_j(x,c) \ \text{ with } x \in \mathbb{R}^d$$

*and if for all* $c \in \mathcal{K}$, *all* $j = 0,...,n$, *and* $\varphi_j \in C_p^{1,2,+}\left((0,T]\times\mathcal{K}\times\mathbb{R}^d;\mathbb{R}\right)$, *such that* $(t,x) \in [0,T]\times\mathbb{R}^d$ *is a global maximum point of* $u_j - \varphi_j$,

$$-\frac{\partial}{\partial t}\varphi_j(t,x,c) - \mathcal{A}\varphi_j(t,x,c) - \mathcal{J}\left(u_j,\varphi_j\right)(t,x,c) - f_j\left(t,x,u_j(t,x,c),\left(\nabla\varphi_j b_j^c\right)(t,x,c),\mathcal{B}_c\varphi_j(t,x,c)\right) \leq 0$$

$(iii)$ *a* ***viscosity solution*** *of* (2.2) *if it is both a sub and a supersolution of* (2.2)

Note that existence and uniqueness of viscosity solutions for such parabolic integro-differential equations in $\mathbb{R}^d$ are discussed in **Alvarez and Tourin** (1996) in the case where $\upsilon$ is a finite measure, and in **Barles, Buckdahn and Pardoux** (1997) or **Pham** (1998).

**Theorem 1.** *Under the conditions that* $u$ *belongs to the set of measurable functions on* $[0,T]\times\mathcal{K}\times\mathbb{R}^d$ *with polynomial growth of the degree* $p$ *at* $+\infty$, *Lipschitz and bounded on* $[0,T]\times\mathcal{K}\times\mathbb{R}^-$, *then the function* $u$ *is a* ***viscosity solution*** *of the system of backward PIDEs* (2.2).

*Proof.* cf. **Barles, Buckdahn and Pardoux** (1997) Theorem 3.4. or **Pardoux, Pradeilles and Rao** (1997) Theorem 4.1. □

2.2. **Markov property and uniqueness.** A standard way to show the **Markov property** is to prove uniqueness of a corresponding (time-inhomogeneous) **martingale problem**. We expose here a direct argument in the mentioned way.

Consider the mentioned reaction-additive system of SDE under risk neutral probability

$$\begin{cases} \frac{dB(t,T_i,C_t)}{B\left(t_-,T_i,C_-\right)} = \left(r_t + l\cdot\lambda_{C_{t-},1}\right)dt + b_{\eta_{(t)}}(t,T_i,C_{t_-})dW_t + \int_{\mathbb{R}^d} h\left(t,x,T_i\right)1_{\{|x|\le 1\}}\,\tilde{\mu}^{C_{t-}}_{\eta_{(t)}}\left(dt,dx\right) \\ dC_t = \sum_{a,b\in K}\left(b-a\right)1_{\left\{C_{t_-}=a\right\}}\left(C_{t_-}\right)dN_{ab}\left(t\right) \end{cases}$$

Let $h\left(B_t,C_t\right)$ be a Borel-measurable function, where $B_t$ means $B\left(t,T_i,C_t\right)$. Define the function $u(t,B,C)\in C_b\left(\mathbb{R};\mathbb{R}\times\mathcal{K}\right)$, for any $t\in[0,T_i]$, that satisfies the following PIDE system

$$\begin{aligned} 0 \;=\; & \frac{\partial u_{\eta_{(t)}}}{\partial t}(t,B_t,C_t) \\ & + \left(r_t + l\cdot\lambda_{C_{t-},1}\right)B_{t_-}\frac{\partial u_{\eta_{(t)}}}{\partial B}(t,B_t,C_t) \\ & + \frac{b_{\eta_{(t)}}(t,T,C_{t_-})^2}{2}B_{t_-}\frac{\partial^2 u_{\eta_{(t)}}}{\partial B^2}(t,B_t,C_t) \\ & + \int_{\mathbb{R}^d}\left[u_{\eta_{(t)}}\left(t,B_{t_-}e^x,C_{t_-}\right) - u_{\eta_{(t)}}\left(t,B_{t_-},C_{t_-}\right) - B_{t_-}\left(e^x-1\right)\right]\upsilon^{C_{t-}}_{\eta_{(t)}}\left(dx\right) \\ & + \frac{\partial u_{\eta_{(t)}}}{\partial C}(t,B_t,C_t)\sum_{a,b\in K}\left[u_{\eta_{(t)}}(t,B_t,a) - u_{\eta_{(t)}}(t,B_t,b)\right]1_{\left\{C_{t_-}=b\right\}}\left(C_{t_-}\right)\lambda^{ab}_t \end{aligned} \tag{2.3}$$

for $a,b\in\mathcal{K}$, $x\in\mathbb{R}^d$ and terminal condition

$$u_{\eta_{(T)}}(T,B,C) = h(B_T,C_T)$$

**Proposition 1.** *For $u_{\eta_{(t)}}$ given as above, the process $u_{\eta_{(t)}}(t,B_t,C_t)$ with $t\in[0,T_i]$ is a **martingale**.*

*Proof.* Applying **Itô formula** to $u_{\eta_{(t)}}(t,B_t,C_t)$ yields

$$\begin{aligned} du_{\eta_{(t)}}(t,B_t,C_t) \;=\; & \frac{\partial u_{\eta_{(t)}}}{\partial B_t}(t,B_t,C_t)dB_t \\ & + \left(\frac{\partial u_{\eta_{(t)}}}{\partial t}(t,B_t,C_t) + \frac{1}{2}b_j\left(t,T,c\right)^2\frac{\partial^2 u}{\partial B_t^2}(t,B_t,C_t)\right)dt \\ & + \int_{\mathbb{R}^d}\left[u_{\eta_{(t)}}\left(t,B_{t_-}e^x,C_{t_-}\right) - u_{\eta_{(t)}}\left(t,B_{t_-},C_{t_-}\right) - B_{t_-}\left(e^x-1\right)\right]\upsilon^{C_{t-}}_{\eta_{(t)}}\left(dx\right) \\ & + \sum_{a,b\in K}\left[u_{\eta_{(t)}}(t,B_{t_-},a) - u_{\eta_{(t)}}(t,B_{t_-},b)\right]1_{\left\{C_{t_-}=b\right\}}\left(C_{t_-}\right)dN^{ab}_t \end{aligned}$$

where substituting the **Doob-Meyer decomposition**, $dN^{ab}_t = dM^{ab}_t + \lambda^{ab}_t dt$ and using **PIDE** (2.2) yields that the drift term vanishes on $[0,T]$. From this we obtain

$$\begin{aligned} du_{\eta_{(t)}}(t,B_t,C_t) \;=\; & B_{t_-}\frac{\partial u_{\eta_{(t)}}}{\partial B_t}(t,B_t,C_t)b_{\eta_{(t)}}(t,T_i,C_{t_-})dW_t \\ & + B_{t_-}\frac{\partial u_{\eta_{(t)}}}{\partial B_t}(t,B_t,C_t)\int_{\mathbb{R}^d} h\left(t,x,T_i\right)1_{\{|x|\le 1\}}\,\tilde{\mu}^c_j\left(dt,dx\right) \\ & + \sum_{a,b\in K}\left[u_{\eta_{(t)}}(t,B_{t_-},a) - u_{\eta_{(t)}}(t,B_{t_-},b)\right]1_{\left\{C_{t_-}=b\right\}}\left(C_{t_-}\right)dM^{ab}_t \end{aligned}$$

Therefore the process $u_j(t, B_t, C_t)$, $t \in [0, T_j]$ is a martingale. □

Notice that the last theorem is equivalent to the following fact: for any continuous function $f\left(t, G_t^C\right)$ on $[0,T] \times \left[\mathbb{R}^d \times \mathcal{K}\right]$ with compact support that is of class $\mathcal{C}^2$ in $g$, the process

$$f\left(t, G_t^C\right) - f\left(0, G_0^C\right) - \int_0^t \mathcal{L}_s f\left(s, G_s^C\right) ds$$

is a martingale, with the operator $\mathcal{L}_s$ being given by

$$\begin{aligned}\mathcal{L}_s f\left(s, G_s^C\right) \quad &= \quad \sum_{i=1}^d \alpha^i_{\eta_{(s)}}(t, C) \frac{\partial f}{\partial G^{i,C}}(t, G_t^C) + \frac{1}{2} \sum_{i,j=1}^d \beta^{i,j}_{\eta_{(s)}}(t, C) \frac{\partial^2 f}{\partial G^{i,C} \partial G^{j,C}}(t, G_t^C) \\ &+ \sum_{b,c \in \mathcal{K}} \lambda_{b,c}(t) \left[f\left(t, G_t^c\right) - f\left(t, G_t^b\right)\right] + \int_{\mathbb{R}^d} \left[f\left(t, G_t^c + g\right) - f\left(t, G_t^c\right) - g 1_{\{|g|<1\}}\right] \mu_{\eta(s)}(dg)\end{aligned}$$

**Proposition 2.** *$G_t^C$ with $t \in [0, T_j]$ is a (time-inhomogeneous)* ***Markov process*** *with respect to $\mathbb{P}$ and $\mathcal{F}$. Its distribution is uniquely determined by the SDE system.*

*Proof.* For any $h \in C_b\left(\mathbb{R}^d \times \mathcal{K}, \mathbb{R}\right)$ there is a unique viscosity solution $u$ to the **PIDE** (2.2). By **Barles, Buckdahn and Pardoux** (1997) Theorem 3.4 or **Pardoux, Pradeilles and Rao** (1997) Theorem 4.1 we have

$$\mathbb{E}\left[h(G_T, C_T) | \mathcal{F}_t\right] = \mathbb{E}\left[u(T, G_T, C_T) | \mathcal{F}_t\right] = u(t, G_t, C_t)$$

for $0 \le t \le T \le T^\star$, and this establishes the Markov property of $(G, C)$ (by Theorem 38.*ii*).

To show uniqueness of the finite-dimensional distributions by induction, let $h_1, ..., h_{m-1}, h_m$ be arbitrary continuous bounded functions. For any times $t_0 \le ... \le t_{m-1} \le t_m$ conditioning on $\mathcal{F}_{m-1}$

$$\mathbb{E}\left[\prod_{j=0}^m h_j(G_{t_j}, C_{t_j}) \middle| \mathcal{F}_{m-1}\right] = \mathbb{E}\left[\left(\prod_{j=0}^{m-1} h_j(G_{t_j}, C_{t_j}) \middle| \mathcal{F}_{m-1}\right) u_{\eta_{(t_{m-1})}}(G_{t_{m-1}}, C_{t_{m-1}})\right]$$

where $u_{\eta_{(t_j)}}$ denotes the solution to the **PIDE** (2.2) in $t_j$. Since the right-hand side of the last equation is determined by the n-dimensional distributions, the claims follow. □

2.3. **Sample-paths in the Skorohod space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$.** Let us recall that the main "engine" that moves our system of stochastic differential equation is a **LIBOR additive process with credit transitions** $G^C$ such that $G^C : (\omega, t) \to (G_t, C_t) \equiv \left(G_t^C\right)_{C \in \mathcal{K}}$ for every $t \in [0, T^\star]$ and $C \in \mathcal{K}^{(m)} = \left\{0, \frac{1}{m}, ..., 1 - \frac{1}{m}, 1\right\}$ with a fixed $m \in \mathbb{N}_+$. The main goal of this subsection is to specify the necessary and sufficient conditions for $G^C \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$, or in other words, $G^C$ has sample paths in the **Skorohod space** or the space of real functions $G^C(t)$ on $[0, T^\star] \times \mathbb{R}^d \times \mathcal{K}^{(m)}$ that are '*càdlàg*' (right-continuous with left-limits).

Notice that if we consider $C$ fixed with value $c \in \mathcal{K}$ with $\mathcal{K} \in [0,1] \cap \mathbb{Q}$, then $G^c \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ by direct application of Theorem 11.1 in **Sato** (1999), or roughly speaking, if $G^c$ is a stochastically continuous and Markov process then it has a version in the Skorohod space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. Properties of such processes and spaces are very well-known and perfectly reflected in **Lipster and Shiryaev** (1989) or **Jacod and Shiryaev** (1999). Our aim in this subsection is to extend these results when $G^C$ is a semimartingale that depends directly on the process $C$.

There exist different ways to prove that $G^C$ has sample-functions in the Skorohod space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. In our case, we will follow the most generic manner, closely related to **Billingsley** (1999), **Carmona, Kesten and Walsh** (1986), **Jacod and Shiryaev** (1987), **Liptser and Shiryaev** (1989), and **Vostrikova** (1988).

Let us begin with the following definitions:

- Let $\Pi_\delta^{(n)}$ be the partition $\{0 = t_0 < t_1 < ... < t_n \leq T^\star \leq 1\}$ of the time interval $[0, T^\star]$, where $n \in \mathbb{N}_+$, satisfying the condition $\min_{0 \leq j \leq n-1} (t_j - t_{j-1}) > \delta$ on the "normalized" interval $[0, 1]$
- And let $\mathcal{K}^{(m)}$ be the set of credit-ratings $\{0 = C_0 < C_1 < ... < C_m = 1\}$ where $m \in \mathbb{N}_+$.

Let us define the following **moduli of continuity** in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$

$$\mathbb{L}_\delta^{(m)}(G) = \inf_{\substack{\Pi_\delta^n \\ t_{j+1} - t_j > \delta}} \max_{0 \leq j \leq n} \sup_{s,t \in [t_j, t_{j+1})} \sup_{c \in \mathcal{K}^{(m)}} \left| G_s^c - G_t^c \right| \tag{2.4}$$

$$\mathbb{LL}_\delta^{(m)}(G) = \sup_{0 \leq t \leq 1} \sup_{\substack{C', C \in \mathcal{K}^{(m)} \\ \left|C' - C\right| \leq \delta}} \left| G_t^{C'} - G_t^C \right| \tag{2.5}$$

with $n, m \in \mathbb{N}_+, T^\star \in \mathbb{R}_+$ and $\delta \in \mathbb{R}_+$.

Additionally, let us define an additional modulus of continuity in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$

$$\mathbb{L}_\delta^C(G) = \inf_{\substack{\Pi_\delta^n \\ t_{j+1} - t_j > \delta}} \max_{0 \leq j \leq n} \sup_{s,t \in [t_j, t_{j+1})} \left| G_s^C - G_t^C \right| \tag{2.6}$$

**Theorem 2.** *$G^C$ belongs to $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ if and only, if for every $n, m \in \mathbb{N}_+, T^\star \in \mathbb{R}_+$ the following conditions hold:*

$$\lim_{\delta \to 0} \mathbb{L}_\delta^C(G) = 0 \tag{2.7}$$

*and*

$$\lim_{\delta \to 0} \mathbb{LL}_\delta^{(m)}(G) = 0 \tag{2.8}$$

In order to prove this theorem, we need three additional results: The *first* result (Lemma 1) came directly from **Billingsley** (1999), and jointly with the *second* one (Lemma 2), both give us the basic condition or criteria to establish when $G^C$ belongs to $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. The *third* result (Lemma 3) develops a basic tool to be used in the proof of Theorem 75.

**Lemma 1.** *For each $G^C$ in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ with $m \in \mathbb{N}_+$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}_+$ such that*

$$0 = t_0 < t_1 < ... < t_n \leq T^\star \leq 1$$

*and*

$$\max_{0 \leq j \leq n} \sup_{s,t \in [t_j, t_{j+1})} \sup_{C \in \mathcal{K}_\delta^{(m)}} \left| G_s^C - G_t^C \right| < \varepsilon$$

*Proof.* cf. **Billingsley** (1999) Lemma 1 p.122. □

Lemma 1 is equivalent to the following assertion.

**Lemma 2.** *$G^C$ belongs to $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ if and only if for every $n, m \in \mathbb{N}_+, T^\star \in \mathbb{R}_+$ and $\delta \in \mathbb{R}_+$, the following conditions hold:*

$$\lim_{\delta \to 0} \mathbb{L}_\delta^{(m)}(G) = 0$$

*Proof.* Fixed a $T^\star \leq 1$, we defined $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ as the **Skorohod space** of functions in $[0, T^\star]$ with values in $\mathcal{K}^{(m)}$ which are right-continuous at any point in $[0, T^\star]$ and left-limits at any point in $[0, T^\star]$ for every $m \in \mathbb{N}_+$. It is clear that $G^C$ belongs to $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ if it belongs to$D_{[0,1]}\left(\mathbb{R}^d\right)$ for any $T^\star \in [0, 1]$ and for any $m \in \mathbb{N}_+$, and it is not difficult to see that **Billingsley's proof** of Lemma 26 for $D_{[0,1]}\left(\mathbb{R}^d\right)$

can be extended for $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$ changing the absolute value $|\ |$ of $\mathbb{R}$ with the expression $\sup_{C\in\mathcal{K}^{(m)}}|\ |$ for any $m\in\mathbb{N}_+$. □

Before we came to the next lemma, let us introduce the following notation: fix any $m\in\mathbb{N}_+$ and $\bar{\delta}>0$, then $\mathcal{K}^{(m)}$ can be covered by a finite union of open-balls with radius $\bar{\delta}$. Therefore there are $C_1,...,C_{r(\bar{\delta})}$ in $\mathcal{K}^{(m)}$ such that $B\left(C_k,\bar{\delta}\right)$ with $k=1,...,r(\bar{\delta})$

$$\mathcal{K}^{(m)}\subseteq B\left(C_1,\bar{\delta}\right)\cup...\cup B\left(C_{r(\bar{\delta})},\bar{\delta}\right)$$

**Lemma 3.** *For any $n,m\in\mathbb{N}_+$, $\delta>0$ and $\bar{\delta}>0$, the following conditions hold:*

(1)

$$\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G)\leq\mathbb{L}_\delta^{(m)}(G)$$

(2) *There is a $\delta_0$, $0\leq\delta_0\leq\delta$ such that*

$$\mathbb{L}_{\delta_0}^{(m)}(G)\leq 2\max_{1\leq k\leq r(\bar{\delta})}\mathbb{L}_\delta^{C_k}(G)+2\mathbb{LL}_{\bar{\delta}}^{(m)}(G)$$

*Proof.* The *inequality* (1) follows from taking into account that

$$\left|G_s^C-G_t^C\right|\leq\sup_{C\in\mathcal{K}^{(m)}}\left|G_s^C-G_t^C\right|.$$

Therefore

$$\inf_{\substack{\Pi_\delta^{(n)}\\ t_{j+1}-t_j>\delta}}\max_{0\leq j\leq n}\sup_{s,t\in[t_j,t_{j+1})}\left|G_s^C-G_t^C\right|\leq\inf_{\substack{\Pi_\delta^{(n)}\\ t_{j+1}-t_j>\delta}}\max_{0\leq j\leq n}\sup_{s,t\in[t_j,t_{j+1})}\sup_{C\in\mathcal{K}^{(m)}}\left|G_s^C-G_t^C\right|$$

and this proves *inequality* (1).

In order to prove the *inequality* (2), notice that for any $k=1,...,r(\bar{\delta})$, we can write

$$\left|G_s^C-G_t^C\right|\leq\left|G_s^C-G_s^{C_k}\right|+\left|G_s^{C_k}-G_t^{C_k}\right|+\left|G_t^{C_k}-G_t^C\right|.$$

Therefore

$$\sup_{C\in B\left(C_k,\bar{\delta}\right)\cap\mathcal{K}^{(m)}}\left|G_s^C-G_t^C\right|\leq 2\mathbb{LL}_{\bar{\delta}}^{(m)}(G)+\left|G_s^{C_k}-G_t^{C_k}\right|$$

Notice that we can easily establish

$$\sup_{C\in\mathcal{K}^{(m)}}\left|G_s^C-G_t^C\right|=\max_{1\leq k\leq r\left(\bar{\delta}\right)}\sup_{C\in B\left(C_k,\bar{\delta}\right)\cap\mathcal{K}^{(m)}}\left|G_s^C-G_t^C\right|\leq 2\mathbb{LL}_{\bar{\delta}}^{(m)}(G)+\max_{1\leq k\leq r\left(\bar{\delta}\right)}\left|G_s^{C_k}-G_t^{C_k}\right|$$

and for any $\delta'>0$ we have

$$\begin{aligned}\mathbb{L}_{\delta'}^{(m)}(G) &\leq 2\mathbb{LL}_{\bar{\delta}}^{(m)}(G)+\inf_{\substack{\Pi_\delta^{(n)}\\ (t_{j+1}-t_j)>\delta}}\max_{0\leq j\leq n}\sup_{s,t\in[t_j,t_{j+1})}\max_{1\leq k\leq r\left(\bar{\delta}\right)}\left|G_s^{C_k}-G_t^{C_k}\right|\\ &\leq 2\mathbb{LL}_{\bar{\delta}}^{(m)}(G)+\inf_{\substack{\Pi_\delta^{(n)}\\ (t_{j+1}-t_j)>\delta}}\max_{1\leq k\leq r\left(\bar{\delta}\right)}\max_{0\leq j\leq n}\sup_{s,t\in[t_j,t_{j+1})}\left|G_s^{C_k}-G_t^{C_k}\right|\end{aligned}$$

To achieve the *inequality* (2) in our Lemma 3, we just need to prove the existence of a $\delta_0,\ 0 \le \delta_0 \le \delta$ such that

$$\begin{aligned}\inf_{\substack{\Pi^{(n)}_{\delta_0}\\(t_{j+1}-t_j)>\delta_0}} \max_{1\le k\le r(\bar\delta)} \max_{0\le j\le n} \sup_{s,t\in[t_j,t_{j+1})} \left|G^{C_k}_s - G^{C_k}_t\right| &\le 2 \max_{1\le k\le r(\bar\delta)} \inf_{\substack{\Pi^{(n)}_{\delta}\\(t_{j+1}-t_j)>\delta}} \max_{0\le j\le n} \sup_{s,t\in[t_j,t_{j+1})} \left|G^{C_k}_s - G^{C_k}_t\right| \\ &= 2 \max_{1\le k\le r(\bar\delta)} \mathbb{L}^{C_k}_{\delta}(G)\end{aligned}$$

Notice that by definition of infimum, for every $k = 1, ..., r(\bar\delta)$ there exists a subpartition on $[0, T^\star]$ $\{t_{k,j}\}_{0\le j\le n_k+1}$ such that $t_{k,j+1} - t_{k,j} \ge \delta$ for any $j = 0, ..., n_k - 1$ and

$$\max_{0\le j\le n_k} \sup_{s,t\in[t_{k,j},t_{k,j+1})} \left|G^{C_k}_s - G^{C_k}_t\right| \le 2\mathbb{L}^{C_k}_{\delta}(G)$$

Define $\{t_p\}_{0\le p\le l+1}$ as the subpartition of the interval $[0, T^\star]$ built with the set of different $t_{k,j}$ with $j = 0, ..., n_k + 1,\ k = 1, ..., r(\bar\delta)$. Finally let us define as $\delta_0$ the minimum distance between two points of the given partition. Therefore we have $0 < \delta_0 \le \delta$ and

$$\max_{0\le k\le r(\bar\delta)} \max_{0\le p\le l} \sup_{s,t\in[t_p,t_{p+1})} \left|G^{C_k}_s - G^{C_k}_t\right| \le 2 \max_{1\le k\le r(\bar\delta)} \mathbb{L}^{C_k}_{\delta}(G)$$

so that we finally obtain

$$\inf_{\substack{\Pi^{(n)}_{\delta_0}\\t_{j+1}-t_j>\delta_0}} \max_{1\le k\le r(\bar\delta)} \max_{0\le j\le n} \sup_{s,t\in[t_j,t_{j+1})} \left|G^{C_k}_s - G^{C_k}_t\right| \le 2 \max_{1\le k\le r(\bar\delta)} \mathbb{L}^{C_k}_{\delta}(G)$$

□

Finally, we can prove the Theorem 2, using the last three results:

*Proof.* (**Theorem 2**) Here we attempt to show if the condition that appear in Lemma 2 is equivalent to conditions (1) and (2) in Theorem 2.

*First*, let us assume the conditions in Theorem 2 in the following sense: given a $\varepsilon > 0$ there exists a $\delta_1 > 0$ with $\delta \le \delta_1$ such that

$$\mathbb{LL}^{(m)}_{\delta}(G) \le \frac{\varepsilon}{4}$$

and related with condition (1) in Theorem 2, there exist a $\delta_2 > 0$ with $\delta \le \delta_2$ such that, for every $k = 1, ..., r(\delta_1)$

$$\mathbb{L}^{C_k}_{\delta}(G) \le \frac{\varepsilon}{4}$$

Notice that the condition (2) in Lemma 3 guarantees the existence of $\delta_0$ such that $0 \le \delta_0 \le \delta_2$ and

$$\mathbb{L}^{(m)}_{\delta_0}(G) \le 2 \max_{1\le k\le r(\bar\delta)} \mathbb{L}^{C_k}_{\delta}(G) + 2\mathbb{LL}^{(m)}_{\bar\delta}(G).$$

It is easy to see that $\mathbb{L}^{(m)}_{\delta}(G)$ is an increasing function of $\delta$, therefore we have that for every $\delta \le \delta_0$

$$\mathbb{L}^{(m)}_{\delta}(G) \le \varepsilon.$$

This proves the condition of Lemma 2.

*Second*, notice that if we assume the condition that appears in Lemma 2 and using the first assertion in Lemma 3, then the first condition in Theorem 2 is proved for every $C \in \mathcal{K}^{(m)}$. On the other hand, to establish the condition (2) in Theorem 2, let us consider a $\varepsilon > 0$, and assuming the condition that appears in Lemma 3, then there exists $\Delta_0 > 0$ and a partition $0 = t_0 < t_1 < ... < t_n = 1$ on $[0,1]$ such that $t_j - t_{j-1} > \Delta_0$ for any $j = 1, ..., n$ and

$$\max_{0 \leq j \leq n} \sup_{s,t \in [t_j, t_{j+1})} \sup_{C \in \mathcal{K}^{(m)}} \left| G_s^C - G_t^C \right| \leq \frac{\varepsilon}{3}.$$

Notice that for any $j = 1, ..., n$ the mapping $G_{t_j} : C_{t_j} \to G_{t_j}^C$ from $[0,1]$ to $\mathbb{R}^d$ is a continuous function over the compact $\mathcal{K}^{(m)}$. Hence it is uniformly continuous. Additionally for any $j = 1, ..., n$ there is a $\delta_j > 0$ such that for any $\delta \in [0, \delta_j]$

$$\sup_{\substack{C', C \in \mathcal{K}^{(m)} \\ \left|C' - C\right| \leq \delta}} \left| G_{t_j}^{C'} - G_{t_j}^C \right| \leq \frac{\varepsilon}{3}$$

Let us define $\delta^* = \min(\Delta_0, \delta_0, ..., \delta_n)$. Then for any $t \in [0, T^\star]$, and choosing $t_j$ under the condition that $t \in [t_{j-1}, t_j)$, we have for any $\delta > 0$

$$\begin{aligned} \sup_{\substack{C', C \in \mathcal{K}^{(m)} \\ \left|C' - C\right| \leq \delta}} \left| G_t^{C'} - G_t^C \right| \quad &\leq \quad \sup_{C \in \mathcal{K}^{(m)}} \left| G_{t_j}^C - G_t^C \right| + \sup_{C' \in \mathcal{K}^{(m)}} \left| G_{t_j}^{C'} - G_t^{C'} \right| \\ &+ \sup_{\substack{C', C \in \mathcal{K}^{(m)} \\ \left|C' - C\right| \leq \delta}} \left| G_{t_j}^C - G_{t_j}^{C'} \right|. \end{aligned}$$

Using the inequalities shown before, for any $0 < \delta \leq \delta^*$, we conclude the condition (2) of Theorem 2 from

$$\sup_{0 \leq t \leq T^\star} \sup_{\substack{C', C \in \mathcal{K}^{(m)} \\ \left|C' - C\right| \leq \delta}} \left| G_t^{C'} - G_t^C \right| \leq \varepsilon$$

□

## 3. Uniform Weak Convergence in the Reaction-Additive system

This current section 3 is concerned with the approximation of a financial corporate bond model in incomplete markets, where the corporate bond dynamics is driven by a **LIBOR additive process** (tradeable) conditioned to a **multivariate point process (**non-tradeable).

According to the previous section, we have seen that our reaction-additive system has sample-paths in the Skorohod space. Therefore, in order to study the weak convergence of this system in such spaces, we have structured this section in the following parts:

- The *first* subsection is basically devoted to give a quick review of the basics related with weak convergence in the Skorohod space.
- The *second* one is devoted to show the conditions of relative compactness of a subset in the Skorohod space. Basically, in this subsection, we expose an equivalent result to the well-known **Arzelà-Ascoli** Theorem for the Skorohod space.
- In the *third* subsection our efforts are mainly focused to find the basic conditions in order to obtain the mentioned convergence in law, in the space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$.

- However these conditions would be too theoretical to have any practical application in finance, and in the *fourth* subsection, we translate these conditions in terms of the characteristic triplets, which are the basic parameters for any derivative pricing.

3.1. **Introduction to Weak Convergence.** Let us first introduce a preliminary section in order to give a brief introduction to weak convergence of stochastic processes and semimartingales. Inclusion of this material is justified not only because of the complexity of the subject but also because it is necessary to establish some basic framework, notation and theorems that will be used later.

Therefore, this preliminary section is an attempt to gather some basic and typical results to describe several main concepts and theorems that will give us the main directions of this chapter. Again, here it is not intended to give a systematic presentation of the most important results or to explain how to prove them; for this purposes one would need more pages. A more comprehensive picture of the present state of the art can be obtained from **Billingsley** (1968), **Lipster and Shiryaev** (1989) or **Jacod and Shiryaev** (1999).

3.1.1. *Weak Convergence, Continuous Mapping and Skorohod Embedding.* In this preliminary subsection we recall some results concerning **tightness** and **convergence** of sequences of semimartingales. As we are concerned with the **weak convergence** (in distribution), we suppose that for a sequence $(G^n)_{n\in\mathbb{N}}$ of processes, $G^n$ is defined on a stochastic basis $(\Omega^n, \mathbb{G}^n, \mathbb{P}^n)$. Additionally, a process $G$ is defined on some $(\Omega, \mathbb{G}, \mathbb{P})$, and we denote **weak convergence** of $G^n$ to $G$ i.e. $\mu^n = \mathcal{L}(G^n|_{\mathbb{P}^n}) \to \mu = \mathcal{L}(G|_{\mathbb{P}})$, by $G^n \to_{\mathcal{L}} G$ if there is no ambiguity about $\mathbb{P}^n$ and $\mathbb{P}$.

Following **Jacod and Shiryaev** (1999), let us consider a **Polish space** $(E, d)$ (that is a complete and separable metric space) with its Borel $\sigma$-field $\mathcal{E} = \mathcal{B}_E$, and consider the space $\mathcal{P}(E)$ of all probability measures on $(E, \mathcal{E})$. The set $\mathcal{P}(E)$ is endowed with the **weak topology** which is the coarsest topology for which the mapping $\mu \to \mu(f) = \int_E f d\mu$ is continuous, for all bounded continuous functions $f$ on $E$. $\mathcal{P}(E)$ is itself a Polish space for this topology.

**Definition 2.** *The sequence $(\mu^n)_{n\in\mathbb{N}}$ **converges weakly** to $\mu$ if, for every bounded continuous function $f$ on $E$, $(\mu^n(f))_{n\in\mathbb{N}}$ converges to $\mu(f)$.*

The weak convergence of random variables is defined through the weak convergence of probabilities measures: let $G$ be an $E$-valued random variable on some probability space $(\Omega, \mathbb{G}, \mathbb{P})$ The image of $\mathbb{P}$ under $G$ is denoted by $\mu = \mathbb{P}_G$. It is called the **law** or **distribution** of $G$.

**Definition 3.** *$(G^n)_{n\in\mathbb{N}}$ **converges in law** (or in distribution) if $(\mu^n)_{n\in\mathbb{N}}$ **converges weakly** to $\mu$ in $\mathcal{P}(E)$.*

This is equivalent to saying that $\mathbb{E}_{\mathbb{P}_n}[f(G^n)] \to \mathbb{E}_{\mathbb{P}}[f(G)]$ when $n \to \infty$, for all bounded continuous functions $f$ on $E$.

Notice that we can not use other standard modes of convergence on $E$ such as convergence almost surely or convergence in probability because the random variables $G^n$ may be defined on different probability spaces.

Now, let us recall two well-known results concerning weak convergence, namely the **continuous mapping theorem** and the **Skorohod embedding theorem**.

**Theorem 3.** ***(Continuous Mapping Theorem)*** *Let $(E, d)$ and $(E', d')$ be two metric spaces, endowed with the Borel-$\sigma$-algebras $\mathcal{B}_E$ and $\mathcal{B}_{E'}$ respectively, and let $\mu$, $(\mu^n)_{n\in\mathbb{N}}$ be probability measures on $(E, \mathcal{B}_E)$. Let furthermore $\varphi^n, \varphi : E \to E'$ be a sequence of measurable functions and denote by $D$ the set of all*

*$g \in E$ such that there exist a sequence $(g^n)_{n\in\mathbb{N}}$ with $g^n \to g$ but $\varphi^n(g^n) \nrightarrow \varphi(g)$. If $E'$ is separable, then $D \in \mathcal{B}_{E'}$ and in this case the assumptions $\mu^n \to \mu$ and $\mu(D) = 0$ imply $\mu^n(\varphi^n)^{-1} \to \mu(\varphi)^{-1}$.*

*Proof.* cf. **Billingsley** (1968) Theorem 5.5. □

**Remark 1.** *Notice that if $\varphi^n = \varphi$ for all $n \in \mathbb{N}^+$, Theorem 3 reduces to the usual continuous mapping theorem, in the sense that if $\varphi$ is $\mu$-a.e. continuous, then $\mu^n \to \mu$ implies $\mu^n(\varphi)^{-1} \to \mu(\varphi)^{-1}$.*

**Theorem 4.** ***(Skorohod Embedding Theorem)*** *Let $(E, d)$ be a separable metric space endowed with the Borel-$\sigma$-algebra $\mathcal{B}_E$, and let $\mu$, $(\mu^n)_{n\in\mathbb{N}}$ be probability measures on $(E, \mathcal{B}_E)$ with $\mu^n \to \mu$. Then there exist a probability space $(\Omega, \mathbb{G}, \mathbb{P})$ and $E$-valued random variables $G$ and $G^n$, all defined on $(\Omega, \mathbb{G}, \mathbb{P})$ with distributions $\mu$ and $\mu^n$ respectively, and such that $G^n \to G$ $\mathbb{P}$-a.s.*

*Proof.* cf. **Ethier and Kurtz** (1986), Theorem 3.1.8. □

3.1.2. *Tightness of Sequence of càdlàg Processes.* Let us consider the **Polish space** $(E, \mathcal{E})$ with its Borel $\sigma$-field $\mathcal{E}$. Consider the space $\mathcal{P}(E)$ of all probability measures on $(E, \mathcal{E})$ with the weak topology.

**Definition 4.** *A subset $A$ of $\mathcal{P}(E)$ is called uniformly **tight** in $E$ if for every $\varepsilon$ there exists a compact subset $K$ in $E$ such that $\mu(E - K) \leq \varepsilon$ for all $\mu \in A$.*

Then, the **Prohorov Theorem** reads as follows

**Theorem 5.** ***(Prohorov)*** *A subset $A$ of $\mathcal{P}(E)$ is **relatively compact** (for the weak topology) if and only if it is uniformly **tight**.*

*Proof.* cf. **Billingsley** (1968) Theorem 5.1. □

In this subsection we consider only $\mathbb{R}^d$-valued *càdlàg* processes. Let $G$ be such a process, defined on a triple $(\Omega, \mathbb{G}, \mathbb{P})$. Then it may be considered as a random variable taking its values in the Polish space $\mathbb{D}(\mathbb{R}^d)$. Consequently its law $\mu = \mathcal{L}(G)$ is an element of $\mathcal{P}(\mathbb{D}(\mathbb{R}^d))$.

**Definition 5.** *A sequence $(G^n)_{n\in\mathbb{N}}$ is said to be uniformly **tight** if for every $\varepsilon > 0$ there exists a compact set $K$ in $E$ such that $\mathbb{P}[G^n \notin K] \leq \varepsilon$ for all $n \in \mathbb{N}$*

**Remark 2.** *Notice that using **Prohorov's theorem**, we can conclude that the sequence $\{\mathcal{L}(G^n)\}$ is **relatively compact** in $\mathcal{P}(\mathbb{D}(\mathbb{R}^d))$ if and only if the sequence $(G^n)_{n\in\mathbb{N}}$ is uniformly **tight**.*

The next result is concerned with **tightness** of sequences of *càdlàg* processes. Recall that in the **Skorohod space** we have the following **modulus of continuity**. Let us fix $T^\star > 0$ such that $I = [0, T^\star] \subset \mathbb{R}_+$. Let us assume a time partition $0 = t_0 \leq t_1 \leq ... \leq t_n = T^\star$, $\delta > 0$ and $G \in \mathbb{D}(\mathbb{R}^d)$ we define

$$\mathbb{L}_{\delta,t_n}(G) = \inf\left\{\max_{i\leq n} \mathbb{L}(G; [t_{i-1}, t_i)) : n \in \mathbb{N}_+, 0 = t_0 < ... < t_n = T^\star, \inf_{i<n}(t_i - t_{i-1}) \geq \delta\right\}$$

where $\mathbb{L}(G; I) = \sup_{s,t\in I} |G(s) - G(t)|$ for an interval $I = [0, T^\star] \subset \mathbb{R}_+$.

**Theorem 6.** *Let $(G^n)_{n\in\mathbb{N}}$ be a sequence of càdlàg processes. Then $(\mathcal{L}(G^n|\mathbb{P}_n))_{n\in\mathbb{N}}$ is **tight** if and only if the following two conditions hold:*

- **(i):** *for all $T^\star \in \mathbb{R}_+$, $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}_+, K > 0$ such that for all $n \geq n_0$*

$$\mathbb{P}_n\left[\sup_{t\leq t_n} |G^n| > K\right] \leq \varepsilon$$

**(ii):** *for all* $T^\star \in \mathbb{R}_+$, $\varepsilon > 0$, $\delta > 0$ *there exist* $n_0 \in \mathbb{N}_+, \theta > 0$ *such that for all* $n \geq n_0$

$$\mathbb{P}_n \left[\mathbb{L}_{t_n} \left(G^n, \theta\right) \geq \delta\right] \leq \varepsilon$$

*Proof.* cf. **Jacod and Shiryaev** (1987) Theorem VI.3.21. □

3.1.3. *Convergence Results for Sequences of Semimartingale.* Concerning limit theorems for stochastic processes, it is necessary to introduce **characteristics** of semimartingales, a concept heavily used later in the following theorems. The idea is to associate to a **semimartingale** a triplet of predictable processes which describe drift, volatility and jumps, in analogy to the concept of **characteristic triplet** of infinitely divisible distributions, which in turn describes drift, volatility and jumps of the associated **Lévy process**. The reader should notice that any theorem related with weak convergence of stochastic processes will be necessarily related with semimartingales theory. In fact, in the following sections, we show that if the characteristics triplet of a sequence of the LIBOR additive processes (semimartingales) are known, one can show convergence in distribution via convergence of the characteristics.

First of all, let us assume a time partition $t_0 \leq t_1 \leq ... \leq t_n$ of $[0, T^\star]$ and let us denote with $G^n$ a $d$-dimensional semimartingale with independent increments ($PII$). On the other hand, $G$ is a $d$-dimensional $PII$ without fixed time of discontinuity. Then the distribution of the process $G$ is characterized by a triplet of characteristics $(\gamma, A, \upsilon)$ relative to some fixed truncation function $h$, or in other words, let $G$ be a semimartingale and $h$ a truncation function and define the process $G(h)$ by

$$G(h)_t = G_t - \sum_{s \leq t} \left(\Delta G_s - h\left(\Delta G_s\right)\right)$$

Note that $\sum_{s \leq t} \left(\Delta G_s - h\left(\Delta G_s\right)\right) = \int_0^t \left(g - h\left(g\right)\right) \mu^G(ds)$ where $\mu^G$ is the random measure associated with the jumps of $G$, and since $\Delta G_s - h\left(\Delta G_s\right) \neq 0$ only for finitely many $s$, this sum converges. Furthermore $\Delta G_s - h\left(\Delta G_s\right)$ is bounded so $G\left(h\right)$ is a special semimartingale with canonical decomposition

$$G(h) = G_0 + M\left(h\right) + \gamma\left(h\right)$$

where $M\left(h\right)$ is a local $\mathbb{P}$-martingale and $\gamma\left(h\right)$ is a predictable process with finite variation. Therefore, the triplet $(\gamma, A, \upsilon)$ with

| | |
|---|---|
| $\gamma = \gamma\left(h\right)$ | from the canonical decomposition |
| $A = \left(\left\langle \tilde{G}_i, \tilde{G}_j \right\rangle\right)_{1 \leq i,j \leq d}$ | where $\tilde{G}$ is the continuous part of $G$ |
| $\upsilon = \upsilon^P$ | is the $\mathbb{P}$-compensator of $\mu^G$ |

is called the triplet of $\mathbb{P}$-characteristics of $G$ relative to the truncation function $h$ or simply characteristics if there is no ambiguity about the measure and the truncation function involved. Sometimes $\gamma$ is called the first, $A$ the second, and $\upsilon$ the third characteristic of $X$.

Obviously only the first and the modified second characteristic depend on the choice of the truncation function. In the sequel we fix one truncation function and sometimes do not mention the dependence of the characteristics on this truncation function.

Concerning limit theorems for stochastic processes, it is necessary to define the modified second characteristic of $G$, $\tilde{A}^n$ which is *càdlàg* and increasing in the set of all $d \times d$ symmetric nonnegative matrices for their natural order, by

$$\tilde{A}^n_{ij} = \left\langle M\left(h\right)^i, M\left(h\right)^j \right\rangle$$

where $M(h)$ is the **martingale part** in the usual decomposition of a semimartingale[1].

[1]See **Jacod and Shiryaev** (1987) Proposition II.2.17

Recall that we have mentioned that $G$ has no fixed time of discontinuity, it means that $\mu\left(\{t\}\times\mathbb{R}^d\right)=0$ and $\tilde{A},\gamma^n(t)$ and $\varphi\cdot\nu$ are continuous functions, then we can state the following theorem[2]

**Theorem 7.** *Let $G^n, G$ be $\mathbb{R}^d$-valued processes with independent increments and characteristics $(\gamma^n(t), A^n, \nu^n)$ and $(A,\nu,\gamma(t))$ respectively. Let $\tilde{A}^n$ and $\tilde{A}$ be the modified second characteristics of $G^n$ and $G$ respectively, and let $D$ be a dense subset of $\mathbb{R}_+$. Then $G^n \xrightarrow{\mathcal{L}} G$ if and only if the following three conditions hold:*

- **(i):** $\sup_{s\leq t}|\gamma^n(s)-\gamma(s)|\to 0$ *for all* $t\geq 0$
- **(ii):** $\left|\tilde{A}^n_t-\tilde{A}_t\right|\to 0$ *for all* $t\in D$
- **(iii):** $\int_{\mathbb{R}^d}\varphi\cdot\upsilon^n_t dg\to\int_{\mathbb{R}^d}\varphi\cdot\upsilon_t dg$ *for all* $t\in D,\varphi\in C\left(\mathbb{R}^d\right)$

*where*

$$C\left(\mathbb{R}^d\right):=\left\{f\in\mathcal{C}_b\left(\mathbb{R}^d\right);\exists\varepsilon>0\ \forall g\in U_\varepsilon(0)\ \ f(g)=0 \text{ and } \lim_{|g|\to\infty} f(g) \text{ exists}\right\}$$

*and $\mathcal{C}_b\left(\mathbb{R}^d\right)$ is the class of all continuous and bounded functions $f:\mathbb{R}^d\to\mathbb{R}$.*

*Proof.* cf. **Jacod and Shiryaev** (1987) Theorem VII.3.4. □

3.1.4. *Convergence of Stochastic Integrals and Stochastic Differential Equations.* In many cases the semimartingales under consideration are stochastic integrals or solutions of stochastic differential equations driven by a converging sequence of semimartingales. Then one is faced with the question whether the convergence carries over to these new processes. The discussion of this issue dates back to **Wong and Zakai** (1965) and has received growing interest for obvious reasons. **Slominski** (1989), **Jakubowski, Mémin and Pagès** (1989) and **Kurtz and Protter** (1991) established sufficient conditions for the convergence of stochastic integrals and solutions of stochastic differential equations in terms of uniform tightness of the converging processes, which have the drawback that they are not easy to formulate and sometimes hard to verify. **Duffie and Protter** (1992) introduce the notion of goodness of a sequence of semimartingales and state simple (but not very general) sufficient conditions. See **Kurtz and Protter** (1996) for more general results.

For a sequence $(\Omega^n,\mathbb{F}^n,\mathbb{P}^n)$, let $(G^n)_{n\in\mathbb{N}}$ and $(H^n)_{n\in\mathbb{N}}$ be sequences of *càdlàg* processes where each $G^n$ is an $\mathbb{R}^d$-valued $(\mathbb{F}^n,\mathbb{P}^n)$-semimartingale and $H^n$ is $\mathbb{F}^n$-adapted and takes values in $\mathbb{R}^{d'\times d}$. Recall that the total variation process of a process $A$ of finite variation is denoted by $Var(A)=\int|dA|$.

**Definition 6.** *A sequence $(G^n)_{n\in\mathbb{N}}$ of semimartingales is called **good** (with respect to $(\mathbb{P}^n)_{n\in\mathbb{N}}$ and $\mathbb{P}$) if for any sequence $(H^n)_{n\in\mathbb{N}}$ the convergence of $\mathcal{L}\left(G^n,H^n|_{\mathbb{P}^n}\right)\xrightarrow{w}\mathcal{L}\left(G,H|_{\mathbb{P}}\right)$ implies convergence of $\mathcal{L}\left(G^n,H^n,\int H^n_-dG^n\big|_{\mathbb{P}^n}\right)\xrightarrow{w}\mathcal{L}\left(G,H,\int H_-dG\big|_{\mathbb{P}}\right)$.*

**Proposition 3.** *Let $(G^n)_{n\in\mathbb{N}}$ be good and suppose $\mathcal{L}\left(G^n,H^n|_{\mathbb{P}^n}\right)\xrightarrow{w}\mathcal{L}\left(G,H|_{\mathbb{P}}\right)$. Then $\left(\int H^n_-dG^n\right)$ is also good.*

*Proof.* cf. **Duffie and Protter** (1992) Theorem 4.1. □

We next provide a sufficient condition for the convergence of solutions of stochastic differential equations.

**Theorem 8.** *Let $(G^n)_{n\in\mathbb{N}}$ be good, let $G$ be a semimartingale, and let $f:\mathbb{R}_+\times\mathbb{R}^{d'}\to\mathbb{R}^{d'\times d}$ satisfy*

[2]This Theorem can be relaxed to a wider class of functions, however it will be sufficient for our purposes.

**(i):** *$y \mapsto f(t,y)$ is* ***Lipschitz****, uniformly in $t$*
**(ii):** *$t \mapsto f(t,y)$ is* ***left-continuous with right-limits****, for all $y$*

*Furthermore let $Y^n$ and $Y$ be the (unique) solutions of*

$$dY_t^n = f\left(t, Y_{t-}^n\right) dG_t^n, \quad Y_t^n \in \mathbb{R}^{d'}$$
$$dY_t = f\left(t, Y_{t-}\right) dG_t, \quad Y_t \in \mathbb{R}^{d'}$$

*If $G^n \xrightarrow{\mathcal{L}} G$, then $(Y^n, G^n) \xrightarrow{\mathcal{L}} (Y, G)$.*

*Proof.* cf. **Duffie and Protter** (1992) Theorem 4.4. □

3.2. **Relative compactness in the space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$.** This section is devoted to study the conditions for relative compactness of a subset in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. Basically, we look for a result equivalent to Theorem 6 or the **Arzelà-Ascoli Theorem** for the **Skorohod space** $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. In the sequel, we consider for any $n \geq 1$, a semimartingale $G^{n,C}$ that depends on a process $C$,and defined on the stochastic bases $(\Omega^n, \mathbb{F}^n, \mathbb{P}^n)$. The main aim of this subsection is to find conditions such that $\left(G^{n,C}\right)_{n\in\mathbb{N}}$ are **relatively compact** in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$.

Let us begin recalling some definitions:

- Let $\Pi_\delta^{(n)}$ be the partition of time $\{0 = t_0 < t_1 < ... < t_n \leq T^\star \leq 1\}$ where $n \in \mathbb{N}_+$, satisfying the condition $\min_{0\leq j\leq n-1}(t_j - t_{j-1}) > \delta$ on the "normalized" interval $[0,1]$
- And let $\mathcal{K}^{(m)}$ be the set of credit-ratings $\{0 = C_0 < C_1 < ... < C_m = 1\}$ where $m \in \mathbb{N}_+$.

Let us define the following **moduli of continuity** in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$

$$\begin{aligned} \mathbb{L}_\delta^{(m)}(G) &= \inf_{\substack{\Pi_\delta^{(n)} \\ t_{j+1}-t_j>\delta}} \max_{0\leq j\leq n} \sup_{s,t\in[t_j,t_{j+1})} \sup_{c\in\mathcal{K}^{(m)}} \left|G_s^c - G_t^c\right| \\ \mathbb{LL}_\delta^{(m)}(G) &= \sup_{0\leq t\leq 1} \sup_{\substack{C',C\in\mathcal{K}^{(m)} \\ \left|C'-C\right|\leq\delta}} \left|G_t^{C'} - G_t^C\right| \end{aligned}$$

with $n, m \in \mathbb{N}_+, T^\star \in \mathbb{R}_+$ and $\delta \in \mathbb{R}_+$.

Additionally, let us define another modulus of continuity in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$

$$\mathbb{L}_\delta^C(G) = \inf_{\substack{\Pi_\delta^{(n)} \\ t_{j+1}-t_j>\delta}} \max_{0\leq j\leq n} \sup_{s,t\in[t_j,t_{j+1})} \left|G_s^C - G_t^C\right|$$

Basically, for the most part, we are concerned with the **relative compactness** of sequences $\{\mathbb{P}_n\}$; this means that every subsequence $\{\mathbb{P}_{n_i}\}$ contains a further subsequence $\left\{\mathbb{P}_{n_{i(m)}}\right\}$ such that $\mathbb{P}_{n_{i(m)}} \to_m \mathbb{Q}$ for some probability measure.

**Theorem 9.** *Let us assume $C \in \mathcal{K}^{(m)}, n, m \in \mathbb{N}_+$, and $a, \varepsilon > 0$. Additionally, assume that the following conditions hold*

1)

$$\lim_{a\to+\infty} \overline{\lim_{n\to\infty}} \mathbb{P}_n\left(\sup_{0\leq t\leq T^\star} \sup_{C\in\mathcal{K}^{(m)}} \left|G_t^{n,C}\right| \geq a\right) = 0$$

2)
$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\mathbb{P}_n\left(\mathbb{L}^C_\delta\left(G^{n,C}\right)>\varepsilon\right)=0$$

3)
$$\lim_{\delta\to 0}\sup_{n\geq 1}\mathbb{P}_n(\mathbb{LL}^{(m)}_\delta\left(G^n\right)>\varepsilon)=0$$

*Then the processes $G^{n,C}$ with $n\geq 1$ have sample-paths in the* ***Skorohod space*** $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$, *and conditions* 1), 2) *and* 3) *hold if and only if the set of processes* $\left(G^{n,C}\right)_{n\geq 1}$ *are* ***relatively compact*** *in* $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$.

In order to prove the Theorem 9, it will be useful to prove the following result.

**Proposition 4.** *Assuming condition* 3) *in Theorem* 9*, then the condition* 2) *in Theorem* 9 *is satisfied if and only if the following expression*

$$\lim_{\delta\to 0}\overline{\lim_{n\to+\infty}}\mathbb{P}_n\left(\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}^C_\delta\left(G^{n,C}\right)>\varepsilon\right)=0 \tag{3.1}$$

*is true for every $m\in\mathbb{N}_+$.*

*Proof.* It is obvious that (3.1) implies the condition 2) in Theorem 9. Conversely, assuming that conditions 2) and 3) in Theorem 9 are true, let us prove that both implies (3.1).

Fix a $\eta>0$. Using directly condition 3) in Theorem 9, there is a $\delta'>0$ such that

$$\sup_{n\geq 1}\mathbb{P}_n(\mathbb{LL}^{(m)}_{\delta'}\left(G^n\right)>\frac{\varepsilon}{4})\leq\frac{\eta}{2} \tag{3.2}$$

Following the same notation that in Lemma 3, fix any $m\in\mathbb{N}_+$, then $\mathcal{K}^{(m)}$ can be covered by a finite union of open-balls with radius $\delta'$. Therefore there are $C_1,...,C_{r(\delta')}$ in $\mathcal{K}^{(m)}$ such that $B\left(C_k,\delta'\right)$ with $k=1,...,r(\delta')$

$$\mathcal{K}^{(m)}\subseteq B\left(C_1,\delta'\right)\cup...\cup B\left(C_{r(\delta')},\delta'\right)$$

On the other hand, assuming that the condition 2) in Theorem 9 is satisfied, there is a $\delta''>0$ such that for every $k=1,...,r(\delta')$

$$\overline{\lim_{n\to\infty}}\mathbb{P}_n\left(\mathbb{L}^{C_k}_{\delta''}\left(G^n\right)>\frac{\varepsilon}{4}\right)\leq\frac{\eta}{2r\left(\delta'\right)} \tag{3.3}$$

Using directly Lemma 3, there exists a $\delta^0$ such that $0<\delta^0\leq\delta''$ and

$$\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}^C_{\delta^0}\left(G^n\right)\leq 2\max_{1\leq k\leq r(\delta')}\mathbb{L}^{C_k}_{\delta''}\left(G^n\right)+2\mathbb{LL}^{(m)}_{\delta'}\left(G^n\right)$$

notice that $\mathbb{L}^C_\delta\left(G^{n,C}\right)$ is an increasing function of $\delta$, therefore we have for every $\delta\leq\delta^0$

$$\mathbb{P}_n\left(\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}^C_\delta\left(G^n\right)>\varepsilon\right)\leq\sum_{k=1}^{r(\delta')}\mathbb{P}_n\left(\mathbb{L}^{C_k}_{\delta''}\left(G^n\right)>\frac{\varepsilon}{4}\right)+\mathbb{P}_n\left(\mathbb{LL}^{(m)}_{\delta'}\left(G^n\right)>\frac{\varepsilon}{4}\right) \tag{3.4}$$

Finally taking into account the expressions (3.2), (3.3) and (3.4) we obtain for every $\delta<\delta^0$

$$\overline{\lim_{n\to+\infty}}\mathbb{P}_n\left(\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}^C_\delta\left(G^{n,C}\right)>\varepsilon\right)\leq\eta$$

which implies (3.1) and the proof is complete. □

*Proof.* (**Theorem** 9) Let us *first* prove that if conditions 1), 2) and 3) hold then $G^{n,C}$ have sample-paths in the **Skorohod space** $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$.

We know that the process $G^{n,C}$, defined in $(\Omega^n,\mathbb{F}^n,\mathbb{P}^n)$, has sample-paths in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$ if and only if

$$\mathbb{P}_n\left(\left\{\omega\in\Omega^n, G^n(\omega)\in D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)\right\}\right)=1$$

or, equivalently

$$\mathbb{P}_n\left(\left\{\omega\in\Omega^n, G^n(\omega)\notin D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)\right\}\right)=0$$

and, using directly Theorem 2, we have that

$$\begin{aligned}&\left\{\omega\in\Omega^n, G^n(\omega)\notin D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)\right\}\\ =\;&\left\{\omega\in\Omega^n,\exists C\in\mathcal{K},\lim_{\delta\to0}\mathbb{L}_\delta^C(G^n)>0\right\}\cup\left\{\omega\in\Omega^n,\exists m\in\mathbb{N},\lim_{\delta\to0}\mathbb{LL}_\delta^{(m)}(G^n)>0\right\}\\ \subseteq\;&\left\{\omega\in\Omega^n,\exists m\in\mathbb{N},\lim_{\delta\to0}\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G^n)>0\right\}\cup\left\{\omega\in\Omega^n,\exists m\in\mathbb{N},\lim_{\delta\to0}\mathbb{LL}_\delta^{(m)}(G^n)>0\right\}\end{aligned}$$

Therefore, our goal is to prove that

$$\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\lim_{\delta\to0}\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G^n)>0\right\}\right)=0 \tag{3.5}$$

and

$$\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\lim_{\delta\to0}\mathbb{LL}_\delta^{(m)}(G^n)>0\right\}\right)=0 \tag{3.6}$$

Let us begin with (3.5). Notice that (3.5) can be proved using the following equivalence

$$\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\lim_{\delta\to0}\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G^n)>0\right\}\right)=\lim_{k\to+\infty}\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\lim_{\delta\to0}\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G^n)>\frac{1}{k}\right\}\right)$$

and noting that $\mathbb{L}_\delta^C(G^n)$ is an increasing function of $\delta$ therefore we have

$$\lim_{k\to+\infty}\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\lim_{\delta\to0}\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G^n)>\frac{1}{k}\right\}\right)\le\lim_{k\to+\infty}\lim_{\delta\to0}\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G^n)>\frac{1}{k}\right\}\right).$$

Notice that for every $n\ge1$ and for every $C\in\mathcal{K}^{(m)}$, the process $G^{n,C}$ is a **semimartingale** and it has sample-paths in the **Skorohod space** $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$. Therefore we have, for every $k\in\mathbb{N}_+$ (cf. **Liptser and Shiryayev** (1986) Theorem 6.1.6).

$$\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\mathbb{L}_\delta^C(G^n)>\frac{1}{k}\right\}\right)=0 \tag{3.7}$$

Notice that following the same reasoning than in Proposition 4 we can show that the equality (3.7) and the condition 3) in Theorem 9 implies for every $k\in\mathbb{N}_+$

$$\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G^n)>\frac{1}{k}\right\}\right)=0$$

and notice that when $k\to+\infty$ we obtain (3.5).

Similarly, we can prove (3.6) following a similar reasoning, noting also that $\mathbb{LL}_\delta^{(m)}(G^n)$ is an increasing function of $\delta$,

$$\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\lim_{\delta\to 0}\mathbb{LL}_\delta^{(m)}(G^n)>0\right\}\right)\le\lim_{k\to+\infty}\lim_{\delta\to 0}\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\mathbb{LL}_\delta^{(m)}(G^n)>\frac{1}{k}\right\}\right)$$

where

$$\lim_{\delta\to 0}\mathbb{P}_n\left(\left\{\omega\in\Omega^n,\mathbb{LL}_\delta^{(m)}(G^n)>\frac{1}{k}\right\}\right)=0$$

and finally, we conclude that the process $G^n$ has **sample-paths** in the space $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$.

In the *second* part, we have to show that these three conditions in Theorem 9 are necessary and sufficient conditions to have **relative compactness** of $\left(G^{n,C}\right)_{n\ge 1}$ in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$.

According to Theorem $VI.1.14$ in **Jacod and Shiryaev** (1989), a subset $A$ of $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$ is **relatively compact** for the **Skorohod topology** for every $m\in\mathbb{N}_+$ and for a fixed $T^\star$ if and only if

$$1)\quad \sup_{G\in A}\sup_{0\le t\le T^\star}\sup_{C\in\mathcal{K}^{(m)}}\left|G_t^C\right|<+\infty$$

$$2)\quad \lim_{\delta\to 0}\sup_{G\in A}\mathbb{L}_\delta^{(m)}(G)=0$$

Notice that using a similar reasoning as in Theorem 2, condition 2) can be split into these two equivalent conditions

$$1)\quad \lim_{\delta\to 0}\sup_{G\in A}\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G)=0$$

$$2)\quad \lim_{\delta\to 0}\sup_{G\in A}\mathbb{LL}_\delta^{(m)}(G)=0$$

Therefore we can show that a subset $A$ of $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$ is **relatively compact** for the **Skorohod topology** for every $m\in\mathbb{N}_+$ and for a fixed $T^\star$ if and only if

$$\begin{aligned}&1)\quad \sup_{G\in A}\sup_{0\le t\le T^\star}\sup_{C\in\mathcal{K}^{(m)}}\left|G_t^C\right|<+\infty\\&2)\quad \lim_{\delta\to 0}\sup_{G\in A}\sup_{C\in\mathcal{K}^{(m)}}\mathbb{L}_\delta^C(G)=0\\&3)\quad \lim_{\delta\to 0}\sup_{G\in A}\mathbb{LL}_\delta^{(m)}(G)=0\end{aligned}\tag{3.8}$$

Now we prove that conditions 1), 2) and 3) in Theorem 9 are necessary and sufficient to guarantee than $(G^n)_{n\ge 1}$ is **relatively compact** in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$.

Notice that using **Prohorov Theorem** (Theorem 5), if the subset $A$ is **relatively compact**, then it is **tight**. Let us assume that the set $(G^n)_{n\ge 1}$ is relatively compact and let us give a $\varepsilon>0$. Therefore, according to **Prohorov Theorem,** there exist a compact $A_\varepsilon$ in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$ such that if $\bar{A}_\varepsilon$ is the complement of $A_\varepsilon$ then we have for any $n\ge 1$

$$\mathbb{P}_n\left(\bar{A}_\varepsilon\right)\le\varepsilon$$

where $\mathbb{P}_n$ is the law of $G^n$ in the space $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$. Then, using the set of conditions (3.8), and replacing $A$ by $A_\varepsilon$ we have that for any $C\in\mathcal{K}^{(m)}$ with $m\in\mathbb{N}_+$, with a fixed $T^\star$, and any $\eta>0$ there exist $a_{m,T^\star}, \delta_{m,T^\star,\eta} \in \mathbb{R}$ such that

$$1) \quad A_\varepsilon \subseteq \left\{ G^C \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right), \sup_{0\le t\le T^\star} \sup_{C\in\mathcal{K}^{(m)}} \left|G_t^C\right| \le a_{m,T^\star} \right\}$$

$$2) \quad A_\varepsilon \subseteq \left\{ G^C \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right), \sup_{C\in\mathcal{K}^{(m)}} \mathbb{L}^C_{\delta_{m,T^\star,\eta}}(G) \le \eta \right\}$$

$$3) \quad A_\varepsilon \subseteq \left\{ G^C \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right), \mathbb{LL}^{(m)}_{\delta_{m,T^\star,\eta}}(G) \le \eta \right\}$$

and for any $C \in \mathcal{K}^{(m)}$, with a fixed $T^\star$ and any $\eta > 0$, we finally obtain conditions 1), 2) and 3) in Theorem 9

$$1) \quad \sup_{n\ge1} \mathbb{P}_n \left( \sup_{0\le t\le T^\star} \sup_{C\in\mathcal{K}^{(m)}} \left|G_t^{n,C}\right| \le a_{m,T^\star} \right) \le \varepsilon$$

$$2) \quad \sup_{n\ge1} \mathbb{P}_n \left( \sup_{C\in\mathcal{K}^{(m)}} \mathbb{L}^C_{\delta_{m,T^\star,\eta}}(G^n) \le \eta \right) \le \varepsilon$$

$$3) \quad \sup_{n\ge1} \mathbb{P}_n \left( \mathbb{LL}^{(m)}_{\delta_{m,T^\star,\eta}}(G^n) \le \eta \right) \le \varepsilon$$

Conversely, in order to prove the '*only if*' part of this theorem, let us assume the conditions 1), 2) and 3) holds, for any $C \in \mathcal{K}^{(m)}$,with a fixed $T^\star$ and any $\eta > 0$, and for any $n \ge 1$. Let $\mathbb{P}_n$ be the law of $G^n$ in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. Notice that using condition 1) we can obtain some results about tightness, or more specifically, there is an $a_{m,T^\star}$ large enough such that

$$A = \left\{ G^C \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right), \sup_{0\le t\le T^\star} \sup_{C\in\mathcal{K}^{(m)}} \left|G_t^C\right| \le a_{m,T^\star} \right\}$$

therefore, we have

$$\varliminf_{n\to+\infty} \mathbb{P}_n(A) \ge 1 - \frac{\varepsilon}{3}$$

And similarly, using conditions 2) and 3) we can assume that exists a $\delta_{m,T^\star,k}$ small enough such that

$$A_{m,T^\star,k} = \left\{ G^C \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right), \sup_{C\in\mathcal{K}} \mathbb{L}_{\delta_{m,T^\star,k}}(G) \le \frac{1}{k} \right\}$$

and

$$B_{m,T^\star,k} = \left\{ G^C \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right), \mathbb{LL}^{(m)}_{\delta_{m,T^\star,k}}(G) \le \frac{1}{k} \right\}$$

such that

$$\varliminf_{n\to+\infty} \mathbb{P}_n\left(A_{m,T^\star,k}\right) \ge 1 - \frac{\varepsilon}{3}$$

and

$$\varliminf_{n\to+\infty} \mathbb{P}_n\left(B_{m,T^\star,k}\right) \ge 1 - \frac{\varepsilon}{3}$$

Let us define now the set $K$ in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ as the set $K = A \cap A_{m,T^\star,k} \cap B_{m,T^\star,k}$, therefore $K$ is a compact set in the space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$, and verify that

$$\varliminf_{n\to+\infty} \mathbb{P}_n(K) \ge 1 - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \ge 1 - \varepsilon$$

and it shows directly that $(G^n)_{n\ge1}$ is **tight** in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$, and **relatively compact**, according to the **Prohorov theorem.** □

3.3. **Weak Convergence in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$.** As in the last subsection, let us consider, for any $n\geq 1$, the semimartingale $G^{n,C}$ with $C\in\mathcal{K}$, defined on *'sufficiently rich'* stochastic bases $(\Omega^n,\mathbb{G}^n,\mathbb{P}^n)$ endowed with the filtration $\mathbb{G}^n=(\mathcal{G}^n_t)_{t\in[0,T^\star]}$, and additionally let us define the stochastic process $G^C$ with $C\in\mathcal{K}$, also in a *'sufficiently rich'* stochastic basis $(\Omega,\mathbb{G},\mathbb{P})$ endowed with the filtration $\mathbb{G}=(\mathcal{G}_t)_{t\in[0,T^\star]}$. This subsection is mainly devoted to find the basic conditions in order to obtain a $G^{n,C}$ with sample-paths in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$, and convergence in law to $G^C$, in the space $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$.

**Theorem 10.** *We assume that the finite-dimensional distributions of $G^{n,C}$ converge weakly to the finite-dimensional distributions of $G^C$. Let us assume $C\in\mathcal{K}^{(m)}$, $n,m\in\mathbb{N}_+$, and $\varepsilon,a>0$. Additionally, assume that the following conditions hold*

*1)*

$$\lim_{a\to+\infty}\overline{\lim_{n\to\infty}}\,\mathbb{P}_n\left(\sup_{0\leq t\leq T^\star}\sup_{C\in\mathcal{K}^{(m)}_\delta}\left|G^{n,C}_t\right|\geq a\right)=0$$

*2)*

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\,\mathbb{P}_n\left(\mathbb{L}^C_\delta\left(G^n\right)>\varepsilon\right)=0$$

*3)*

$$\lim_{\delta\to 0}\sup_{n\geq 1}\mathbb{P}_n(\mathbb{LL}^{(m)}_\delta\left(G^n\right)>\varepsilon)=0$$

*Then the processes $G^{n,C}$ with $n\geq 1$ have sample-paths in the **Skorohod space** $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$, and the **weak convergence** of $G^{n,C}\overset{w(\mathbb{P}_n)}{\to}G^C$ in the Skorohod space $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$ takes place.*

*Proof. First*, we have to prove that the process $G^{n,C}$ has **sample paths in the Skorohod space** $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$, and that $\left(G^{n,C}\right)_{n\geq 1}$ is relatively compact. In order to show that, it is enough to see that the conditions in Theorem 10 are the same as the conditions in Theorem 2 and Theorem 9.

*Second*, we have to prove the **weak convergence** of the sequence $\left(G^{n,C}\right)_{n\geq 1}$. Let us define $\mathbb{Q}^n$ as the law of $G^n$ in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$. Also let us consider two subsequences $(\mathbb{Q}^{n\prime})$ and $(\mathbb{Q}^{n\prime\prime})$, from the sequence $(\mathbb{Q}^n)_{n\geq 1}$, that converge to the probabilities $\mathbb{Q}'$ and $\mathbb{Q}''$ in the measurable space $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$. According to Theorem 2.8 in **Billingsley** (1968) in order to prove weak convergence, it is enough to prove that $\mathbb{Q}'$ and $\mathbb{Q}''$ are the same.

For $0\leq t_1<...<t_k\leq 1$ with $k\in\mathbb{N}_+$, let us define the natural projection from $D\left(\mathbb{R}^\infty\right)$ to $\mathbb{R}^k$ by $\Pi_{t_1,...,t_k}\left(G\right)=\left(G_{t_1},...,G_{t_k}\right)$, such that for $A\in\mathbb{R}^n$ we can denote

$$\Pi^{-1}_{t_1,...,t_k}\left(A\right)=\left\{\left(G_{t_1},...,G_{t_k}\right)\in A\right\}$$

and notice that $\Pi_{t_1,...,t_k}$ is continuous for all $i=1,...,k$, $t_i\in\Lambda\left(G\right)=\left\{t>0,\Delta G_t=0\right\}\cup\left\{0\right\}$.

Now, let us define

$$\begin{aligned}\Lambda_{\mathbb{Q}'} &= \left\{t>0:\mathbb{Q}'\left(G\in D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right),\Delta G_t=0\right)=1\right\}\cup\left\{0\right\}\\ \Lambda_{\mathbb{Q}''} &= \left\{t>0:\mathbb{Q}''\left(G\in D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right),\Delta G_t=0\right)=1\right\}\cup\left\{0\right\}\end{aligned}$$

Then we can say that, for any $t_1,...,t_k\in\Lambda_{\mathbb{Q}'}\cap\Lambda_{\mathbb{Q}''}$, the sequences $\left(\mathbb{Q}^{n\prime}\circ\Pi^{-1}_{t_1,...,t_k}\right)$ and $\left(\mathbb{Q}^{n\prime\prime}\circ\Pi^{-1}_{t_1,...,t_k}\right)$ converge weakly to $\left(\mathbb{Q}'\circ\Pi^{-1}_{t_1,...,t_k}\right)$ and $\left(\mathbb{Q}''\circ\Pi^{-1}_{t_1,...,t_k}\right)$ respectively.

Now, let us fix $t_1, ..., t_k \in \Lambda_{\mathbb{Q}'} \cap \Lambda_{\mathbb{Q}''}$ and $0 \leq C_1 < ... < C_m \leq 1$ with $C_1, ..., C_m \in \mathbb{R}_+$ and $m \in \mathbb{N}_+$. Additionally let us define the continuous mapping

$$\Pi_{C_1,...,C_m}(G_{t_1}, ..., G_{t_k}) = \left(G_{t_1}^{C_1}, ..., G_{t_1}^{C_m}, ..., G_{t_k}^{C_1}, ..., G_{t_k}^{C_m}\right)$$

from $\mathbb{R}^k$ to $\mathbb{R}^{k \cdot m}$. Notice that according to the **continuous mapping theorem** (Theorem 3), the sequences $\left(\mathbb{Q}^{n\prime} \circ \Pi^{-1}_{t_1,...,t_k} \circ \Pi^{-1}_{C_1,...,C_m}\right)$ and $\left(\mathbb{Q}^{n\prime\prime} \circ \Pi^{-1}_{t_1,...,t_k} \circ \Pi^{-1}_{C_1,...,C_m}\right)$ converge weakly to $\left(\mathbb{Q}' \circ \Pi^{-1}_{t_1,...,t_k} \circ \Pi^{-1}_{C_1,...,C_m}\right)$ and $\left(\mathbb{Q}'' \circ \Pi^{-1}_{t_1,...,t_k} \circ \Pi^{-1}_{C_1,...,C_m}\right)$ respectively.

Notice that, if we assume that the finite-dimensional distributions of $G^{n,C}$ converge weakly to the finite-dimensional distributions of $G^C$, then we have that

$$\mathbb{Q}' \circ \Pi^{-1}_{t_1,...,t_k} \circ \Pi^{-1}_{C_1,...,C_m} = \mathbb{Q}'' \circ \Pi^{-1}_{t_1,...,t_k} \circ \Pi^{-1}_{C_1,...,C_m} \tag{3.9}$$

Let us prove that, for any $C_1, ..., C_m \in \mathbb{R}_+$ with $m \in \mathbb{N}_+$, we have

$$\mathbb{Q}' \circ \Pi^{-1}_{t_1,...,t_k} = \mathbb{Q}'' \circ \Pi^{-1}_{t_1,...,t_k} \tag{3.10}$$

or in other words, if we denote $\mathcal{C}_k$ as the $\sigma$-field from $(\mathcal{K})^k$ , and $\bar{\mathcal{C}}_k$ as the $\sigma$-field from $(\mathcal{K})^k$ generated by $\Pi^{-1}_{C_1,...,C_m}(A)$ , with $C_1, ..., C_m \in \mathbb{R}_+$, $m \in \mathbb{N}_+$ and $A \in \mathcal{B}\left(\mathbb{R}^{k \cdot m}\right)$ , then we have to prove that $\mathcal{C}_k = \bar{\mathcal{C}}_k$.

Notice first that, for any $C_1, ..., C_m \in \mathbb{R}_+$, the application $\Pi_{C_1,...,C_m}$ such that

$$\Pi_{C_1,...,C_m} : G = (G_1, ..., G_k) \to \left(G_1^{C_1}, ..., G_1^{C_m}, ..., G_k^{C_1}, ..., G_k^{C_m}\right)$$

is continuous. Therefore $\bar{\mathcal{C}}_k \subseteq \mathcal{C}_k$.

On the other hand, in order to show the inclusion in the opposite direction, it is enough to show that any ball in $(\mathcal{K})^k$ is $\bar{\mathcal{C}}_k$-measurable, because $\mathcal{K}$ is a complete and separable space. Let us choose $C$ from $C_1, ..., C_{r(\delta)}$ in the compact $\mathcal{K}_i$ such that $\mathcal{K}_i \subset B(C_1, \delta) \cup ... \cup B\left(C_{r(\delta)}, \delta\right)$ and such that $r(\delta)$ is the minimum number of balls with radius $\delta$ that cover $\mathcal{K}_i$. Therefore let us define, the following $\bar{\mathcal{C}}_1$-measurable application with

$$G \to d(G, Y) = \sum_{i=0}^{+\infty} 2^{-i} \frac{\sup_{C \in \mathcal{K}_i} \left|G^C - Y^C\right|}{1 + \sup_{C \in \mathcal{K}_i} \left|G^C - Y^C\right|}$$

Using basic properties of measurable mappings we have that

$$G \to \sup_{C \in \mathcal{K}_i} \left|G^C - Y^C\right|$$

for any $Y \in \mathbb{R}^d$, is also a $\bar{\mathcal{C}}_1$-measurable mappings, for any $i \in \mathbb{N}_+$. Therefore for any $\varepsilon \in \mathbb{R}_+$

$$\left\{G \in \mathbb{R}^d, \sup_{C \in \mathcal{K}_i} \left|G^C - Y^C\right| \leq \varepsilon\right\} = \bigcap_{C \in \mathcal{K}_i} \left\{G \in \mathbb{R}^d, \left|G^C - Y^C\right| \leq \varepsilon\right\}$$

which are $\bar{\mathcal{C}}_1$-measurable. Henceforth, (3.9) implies (3.10) .

Now notice that, according to **Liptser and Shiryaev** (1989) we know that $\Lambda_{\mathbb{Q}'}$ and $\Lambda_{\mathbb{Q}''}$ are dense in $\mathbb{R}_+$. Therefore $\Lambda_{\mathbb{Q}'} \cap \Lambda_{\mathbb{Q}''}$ is also dense in $\mathbb{R}_+$, and we conclude that $\mathbb{Q}' = \mathbb{Q}''$ because a probability in the Borelian $\sigma$-field of $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ is entirely determined by its finite-dimensional distributions. Therefore the sequence $\left(G^{n,C}\right)_{n \geq 1}$ weakly converges in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ towards a certain process $\tilde{G}^C$ with sample paths in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ .

In order to complete the proof, we have to show the link between $\tilde{G}^C$ and $G^C$. According to **Billingsley** (1999), and using the continuity property of the following mappings

$$\Pi_{C_1,...,C_m} : G \to \left(G^{C_1}, ..., G^{C_m}\right)$$

from $D\left(\mathbb{R}^d\right)$ to $D\left(\mathbb{R}^d, \mathcal{K}\right)$, and

$$\Pi_{t_1,...,t_k} : \left(G^{C_1}, ..., G^{C_m}\right) \to \left(G^{C_1}_{t_1}, ..., G^{C_m}_{t_1}, ..., G^{C_1}_{t_k}, ..., G^{C_m}_{t_k}\right)$$

from $D\left(\mathbb{R}^d, \mathcal{K}\right)$ to $\mathbb{R}^{d \cdot n \cdot m}$, for every $t_1, ..., t_n \in \bar{\Lambda}_{\mathbb{Q}'}$ where

$$\bar{\Lambda}_{\mathbb{Q}'} = \left\{t > 0, \mathbb{Q}' \circ \Pi^{-1}_{C_1,...,C_m}\left(G \in D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right), \Delta G_t = 0\right) = 1\right\} \cup \{0\}$$

and we obtain the weak convergence in $\mathbb{R}^{d\cdot k\cdot m}$ from $\mathbb{Q}^{n\prime} \circ \Pi^{-1}_{C_1,...,C_m} \circ \Pi^{-1}_{t_1,...,t_k}$ towards $\mathbb{Q}' \circ \Pi^{-1}_{C_1,...,C_m} \circ \Pi^{-1}_{t_1,...,t_k}$. We know that $\mathbb{Q}^{n\prime} \circ \Pi^{-1}_{C_1,...,C_m} \circ \Pi^{-1}_{t_1,...,t_k}$ is the law in $\mathbb{R}^{d\cdot k\cdot m}$ of a vector $\left(G^{C_1}_{t_1}, ..., G^{C_m}_{t_1}, ..., G^{C_1}_{t_k}, ..., G^{C_m}_{t_k}\right)$. Using the weak convergence of the finite-dimensional distribution we know that, for every $t_1, ..., t_k \in \bar{\Lambda}_{\mathbb{Q}'}$

$$\left(G^{n,C_1}_{t_1}, ..., G^{n,C_m}_{t_1}, ..., G^{n,C_1}_{t_k}, ..., G^{n,C_m}_{t_k}\right) \to \left(G^{C_1}_{t_1}, ..., G^{C_m}_{t_1}, ..., G^{C_1}_{t_k}, ..., G^{C_m}_{t_k}\right)$$

or

$$\mathbb{Q}' \circ \Pi^{-1}_{C_1,...,C_m} \circ \Pi^{-1}_{t_1,...,t_k} = \mathbb{Q}_{C_1,...,C_m} \circ \Pi^{-1}_{t_1,...,t_k}$$

where $\mathbb{Q}_{C_1,...,C_m}$ is the law in $D\left(\mathbb{R}^d, \mathcal{K}\right)$ of the processes $\left(G^{C_1}, ..., G^{C_m}\right)$. Because a probability in $D\left(\mathbb{R}^d, \mathcal{K}\right)$, is entirely determined by its finite-dimensional distributions then

$$\mathbb{Q}' \circ \Pi^{-1}_{C_1,...,C_m} = \mathbb{Q}_{C_1,...,C_m}$$

Hence we have the link between $\tilde{G}^C$ and $G^C$ : for every $m \in \mathbb{N}_+$ and $C_1, ..., C_m \in \mathbb{R}_+$ the law in $D\left(\mathbb{R}^d, \mathcal{K}\right)$ of the processes $\left(\tilde{G}^{C_1}, ..., \tilde{G}^{C_m}\right)$ and $\left(G^{C_1}, ..., G^{C_m}\right)$ coincide. □

3.4. **Previsibility conditions for weak convergence of $G^C$ in the space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$.** This section is devoted to develop the conditions in Theorem 10 in terms of the characteristics or triplet. Roughly speaking, the idea that we have in mind is related with the following question: Under which conditions, in terms of the triplet, our model will converge in distribution to the continuous time model that the market imposes?

Let us first define the setting assuming that we are given a *"sufficiently rich"* stochastic bases $(\Omega^n, \mathbb{F}^n, \mathbb{P}^n)$ and $(\Omega, \mathbb{F}, \mathbb{P})$ endowed with the filtrations $\mathbb{F}^n = (\mathcal{F}^n_t)_{t\in[0,T^\star]}$ and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^\star]}$. Let $G^{n,C} = \left(G^{n,C}(t)\right)_{C\in\mathcal{K},t\geq 0}$ and $G^C = \left(G^C(t)\right)_{C\in\mathcal{K},t\geq 0}$ be semimartingales depending on the process $C = (C_t)_{t\in[0,T^\star]}$ with $C \in \mathcal{K}^{(m)}$, defined on the spaces $(\Omega^n, \mathbb{F}^n, \mathbb{P}^n)$ and $(\Omega, \mathbb{F}, \mathbb{P})$ respectively.

Additionally, we assume that the semimartingale $G^{n,C}$ is a special[3] and locally square-integrable martingale, or in other words, a martingale that admits the canonical decomposition

$$G^{n,C}_t = G^{n,C}_0 + M^{n,C}_t + B^{n,C}_t$$

where $M^{n,C}_t$ is a local $\mathbb{P}$-martingale and $B^{n,C}_t$ is a predictable process with finite variation over each finite interval. Let us assume that $G^C$ is a continuous semimartingale[4], that admits also the canonical decomposition

$$G^C_t = G^C_0 + M^C_t + B^C_t$$

[3] See Liptser and Shiryayev (1986) definition 2.2

[4] It is well-known that interest rates derivatives quotes use the Black-76 model. This model is a continuous-time model, and it generates continuous sample-paths. However, taking into account the tenor structure, we can strip the quoted volatilities into a set of forward volatilities with skews/smile for each tenor. It basically implies that if $n$ = number of tenors, $G^{n,C}$ has to be a LIBOR additive process with jumps and $G^C$ is a LIBOR additive process without discontinuities.

This means that the characteristics of $G^{n,C}$ and $G^{C}$ are $\left(\gamma^{n,C}, A^{n,C}, \upsilon^{n,C}\right)$ and $\left(\gamma^{C}, A^{C}, \upsilon^{C}\right)$, respectively, with $\upsilon^{C}=0$ in the second case.

The aim of this section is to give conditions expressed in terms of the predictable characteristics $\left(\gamma^{n,C}, A^{n,C}, \upsilon^{n,C}\right)$ and $\left(\gamma^{C}, A^{C}, \upsilon^{C}\right)$ of the semimartingales $G^{n,C}$ and $G^{C}$, providing the existence of the modifications processes $G^{n,C}=\left(G^{C,n}(t)\right)_{C\in\mathcal{K},t\geq 0}$ and $G^{C}=\left(G^{C}(t)\right)_{C\in\mathcal{K},t\geq 0}$ with trajectories in $D_{[0,T^{\star}]}\left(\mathbb{R}^{d},\mathcal{K}\right)$ for which a weak convergence in $D_{[0,T^{\star}]}\left(\mathbb{R}^{d},\mathcal{K}\right)$ takes place. During the present subsection, basically we follow **Liptser and Shiryaev** (1989) section 8.3.

In addition to the previous definitions of $G^{n,C}$ and $G^{C}$, and for the sake of simplicity, let us introduce $\tilde{G}^{n,C}$ and $\tilde{G}^{C}$ as the **continuous martingale components** of $G^{n,C}$ and $G^{C}$, respectively, or in other words,

$$\tilde{G}^{n,C}_{t}=\tilde{G}^{n,C}_{0}+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\sigma^{n}_{j}(u,C)dW^{n}_{u}$$

and

$$\tilde{G}^{C}_{t}=\tilde{G}^{C}_{0}+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\sigma_{j}(u,C)dW_{u}$$

with the usual conditions on the coefficients (**Colino** (2008)).

Therefore, using the assumption that $G^{n,C}_{t}$ is a **special semimartingale**, we have

$$G^{n,C}_{t}=G^{n,C}_{0}+\tilde{B}^{n,C}_{t}+\tilde{G}^{n,C}_{t}+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^{d}}\delta(u,x)\left(\mu^{n,C}_{j}-\upsilon^{n,C}_{j}\right)(du,dx)$$

where $\tilde{B}^{n,C}_{t}$ is a process, with locally integrable variation, such that

$$\tilde{B}^{n,C}_{t}=\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\alpha^{n}_{j}(u,C)du<\infty$$

Let us assume that for any $C,C'\in\mathcal{K}^{(m)}\backslash\{1\}$ the previsible quadratic variation of $W^{C}_{t}$ and $W^{C'}_{t}$ is, for every $t\geq 0$,

$$\left\langle W^{C},W^{C'}\right\rangle_{t}=\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}cov_{j}\left(u,W^{C},W^{C'}\right)du$$

Additionally, and for the sake of simplicity in the sequel, we denote with $M^{n,C}=\left(M^{n,C}_{t}\right)_{C\in\mathcal{K},t\geq 0}$ the **martingale part** of the semimartingale $G^{n,C}$, for every $t\geq 0$, such that

$$M^{n,C}_{t}=\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\sigma^{n}_{j}(u,C)dW^{n}_{u}+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^{d}}\delta(u,x)\left(\mu^{n,C}_{j}-\upsilon^{n,C}_{j}\right)(du,dx)$$

and the quadratic variation $\left\langle M^{n,C}\right\rangle_{t}$ is given by (according to **Liptser and Shiryaev** (1989) Theorem 3.5.1)

$$\left\langle M^{n,C}\right\rangle_{t}=\left\langle\tilde{G}^{n,C}\right\rangle_{t}+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^{d}}\delta^{2}(u,x)\upsilon^{n,C}_{j}(du,dx)-\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^{d}}\delta(u,x)\left(\upsilon^{n,C}_{j}(\{u\},dx)\right)^{2}$$

Let us recall that our goal is to translate the conditions of Theorem 10 in terms of the previsible characteristics of the semimartingales. Notice that assuming that $G^{n,C}$ is a special and locally square-integrable martingale, and the semimartingale $G^C$ is continuous with deterministic initial value, is equivalent to condition 1 in Theorem 10. Therefore, in order to obtain the weak convergence, we have to establish the following group of conditions:

**Condition Group (1)**: in order to establish *Condition* (2) in Theorem 10 for every $C \in \mathcal{K}$ we need

$$\begin{cases} (1.1) & \left|G_0^{C,n} - G_0^C\right| & \underset{n\to\infty}{\overset{\mathbb{P}^n}{\to}} & 0 \\ (1.2) & \sum_{j\leq n} \int_{T_j}^{T_{j+1}} \int_{\mathbb{R}\setminus\{0\}} \delta^2(u,x)\, 1_{(|x|>a)} dv_j^{n,C}(du,dx) & \underset{n\to\infty}{\overset{\mathbb{P}^n}{\to}} & 0 \\ (1.3) & \sup_{0\leq t\leq T^\star} \left|\tilde{B}_t^{n,C} - \sum_{j\leq \eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \alpha_j(t,C,G^{n,C})ds\right| & \underset{n\to\infty}{\overset{\mathbb{P}^n}{\to}} & 0 \\ (1.4) & \sup_{0\leq t\leq T^\star} \left|\left\langle M^{n,C,h}\right\rangle_t - \sum_{j\leq \eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \sigma_j(t,C,G^{n,C})ds\right| & \underset{n\to\infty}{\overset{\mathbb{P}^n}{\to}} & 0 \end{cases}$$

**Condition Group (2)**: Additionally, to obtain weak convergence in the 'fidis', jointly with condition group 1, for every $C, C' \in \mathcal{K}^{(m)} \setminus \{1\}$ we need

$$(2.1) \quad \sup_{0<t\leq 1} \left|\left\langle \tilde{G}^{n,C}, \tilde{G}^{n,C'}\right\rangle_t - \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} cov_j\left(u, W^C, W^{C'}\right) du\right| \underset{n\to\infty}{\overset{\mathbb{P}^n}{\to}} 0$$

**Condition Group (3)**: Finally, to obtain the *Condition* (3) in Theorem 10, assume $p \geq 2$ and $\alpha > m$ such that for a bounded stopping times $\tau$ and $C, C' \in \mathcal{K}^{(m)} \setminus \{1\}$, $C \neq C'$ we have

$$\begin{cases} (3.1) & \sup_n \left(\mathbb{E}^n \left|\gamma_\tau^{C,n} - \gamma_\tau^{C',n}\right|^p \Big/ \left|C - C'\right|^\alpha\right) & \leq & c(p) \\ (3.2) & \sup_n \left(\mathbb{E}^n \left|\left\langle \bar{G}_\tau^{C,n} - \bar{G}_\tau^{C',n}\right\rangle\right|^{p/2} \Big/ \left|C - C'\right|^\alpha\right) & \leq & c(p) \\ (3.3) & \sup_n \left(\mathbb{E}^n \int_0^\tau \int_E (g_1 - g_2)^p \, d\mu^{n,C,C'} \Big/ \left|C - C'\right|^\alpha\right) & \leq & c(p) \end{cases}$$

where $\tau = \inf\left\{0 < t \leq 1 : \sup_{|C-C'|\leq\delta} \left|G_t^{C,n} - G_t^{C',n}\right| \geq \epsilon\right\}$ and $c(p)$ is a positive constant.

**Theorem 11.** *We suppose that the condition groups* (1) *to* (3) *are satisfied. Then, the processes* $G^{C,n} = \left(G^{C,n}(t)\right)_{C\in\mathcal{K},t\geq 0}$ *and* $G^C = \left(G^C(t)\right)_{C\in\mathcal{K},t\geq 0}$ *have paths in* $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ *such that weak convergence* $G^{C,n} \overset{w(\mathbb{P}^n)}{\to} G^C$ *in* $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ *takes place.*

Basically, the proof of Theorem 11 requires the verification of the conditions of Theorem 10 according to the following plan:

- *First*, we suppose that $G^{C,n} = \left(G^{C,n}(t)\right)_{C\in\mathcal{K},t\geq 0}$ and $G^C = \left(G^C(t)\right)_{C\in\mathcal{K},t\geq 0}$ are the paths in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. Using the conditions of group 1 we establish a weak convergence of the process $G^{C,n}$ to $G^C$ with $C \in \mathcal{K}^{(m)}$, given $m \in \mathbb{N}_+$, in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. In turn, this convergence implies the condition 2) in Theorem 10.
- *Second*, using the conditions of group 1 and 2 we establish the convergence of the finite-dimensional distributions
$$\left(G^{n,C_1}, G^{n,C_2}, ..., G^{n,C_m}\right) \underset{n\to\infty}{\overset{\mathbb{P}^n}{\to}} \left(G^{C_1}, G^{C_2}, ..., G^{C_m}\right)$$
in the Skorohod space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ for every $m \geq 2$.
- *Third*, using the conditions of group 3, the result from **Valkelia and Dzaparidze** (1990), we prove that for every $C, C' \in \mathcal{K}^{(m)} \setminus \{1\}$ and a bounded stopping time $\tau$ :
$$\sup_n \mathbb{E}^n \left(\left|G_\tau^{C,n} - G_\tau^{C',n}\right|^p \Big/ \left|C - C'\right|^\alpha\right) \leq \bar{C}_i$$

and using specially chosen stopping times, this inequality and the lemma about the estimation of the modulus of continuity allow us to verify the condition 2) of Theorem 10.

In order to prove the Theorem 11, we need the following results.

**Proposition 5.** *Let $G^C$ be a stochastic process defined on the stochastic basis $(\Omega,\mathcal{F},\mathbb{P})$ with sample-paths in the Skorohod space $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$. Then for any $\varepsilon>0$ the function $\tau$ defined in $\Omega$ by*

$$\tau(\omega)=\inf\left(t>0,\ \sup_{\substack{C',C\in\mathcal{K}_\delta^{(m)}\\ \left|C'-C\right|\le\delta}}\left|G_t^{C'}(\omega)-G_t^C(\omega)\right|\ge\varepsilon\right)\wedge T^\star$$

*is a $\mathcal{F}$-stopping time.*

*Proof.* According to **Lipster and Shiryaev** (1989) let us define the process $Z$ as

$$Z_t(\omega)=\sup_{\substack{C',C\in\mathcal{K}_\delta^{(m)}\\ \left|C'-C\right|\le\delta}}\left|G_t^{C'}(\omega)-G_t^C(\omega)\right|$$

and notice that it is enough to prove that $Z$ is a process with sample-paths in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$, $\mathcal{F}_t$-adapted. Let $\omega\in\Omega$. Then we have for every $s,t\ge0$

$$\begin{aligned}\left|Z_s(\omega)-Z_t(\omega)\right| &= \sup_{\substack{C',C\in\mathcal{K}_\delta^{(m)}\\ \left|C'-C\right|\le\delta}}\left|G_s^{C'}(\omega)-G_s^C(\omega)\right|-\sup_{\substack{C',C\in\mathcal{K}_\delta^{(m)}\\ \left|C'-C\right|\le\delta}}\left|G_t^{C'}(\omega)-G_t^C(\omega)\right|\\ &\le \sup_{\substack{C',C\in\mathcal{K}_\delta^{(m)}\\ \left|C'-C\right|\le\delta}}\left|G_s^{C'}(\omega)-G_s^C(\omega)-G_t^{C'}(\omega)+G_t^C(\omega)\right|\\ &\le 2\sup_{C',C\in\mathcal{K}_\delta^{(m)}}\left|G_s^C(\omega)-G_t^C(\omega)\right|\end{aligned}$$

Notice that $G^C$ has sample-paths in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$. Therefore we have that

$$\lim_{\delta\to0}\inf_{\substack{\Pi_\delta^{(n)}\\ t_{j+1}-t_j>\delta}}\max_{0\le j\le n}\sup_{s,t\in[t_j,t_{j+1})}\left|Z_s(\omega)-Z_t(\omega)\right|=0$$

This, according to Lemma 2, proves that $Z$ is a process with sample-paths in $D_{[0,T^\star]}\left(\mathbb{R}^d,\mathcal{K}\right)$.

In order to prove that $Z$ is a $\mathcal{F}_t$-adapted process, notice that for any $\omega\in\Omega$, and any $t\in\mathbb{R}_+$, the mapping $C\to G_t^C(\omega)$ from $\mathcal{K}$ to $\mathbb{R}^d$ is continuous. Therefore we have that, for every $t\in\mathbb{R}_+$ and any $C\in\mathcal{K}^{(m)}$, $Z$ is a $\mathcal{F}_t$-adapted process. □

**Proposition 6.** *The condition* 3) *in Theorem* 4 *is equivalent to*

$$\lim_{\delta\to0}\sup_{n\ge1}\mathbb{P}_n\left(\sup_{\substack{C',C\in\mathcal{K}_\delta^{(m)}\\ \left|C'-C\right|\le\delta}}\left|G_\tau^{n,C'}(\omega)-G_\tau^{n,C}(\omega)\right|\ge\varepsilon\right)=0$$

*where $\tau$ is the $\mathcal{F}$-stopping time defined in Proposition* 5.

*Proof.* Let us recall that according to Proposition 5, the stopping time $\tau(\omega)$ was defined as

$$\tau(\omega) = \inf\left( t > 0, \sup_{\substack{C',C \in \mathcal{K}_\delta^{(m)} \\ \left|C'-C\right| \leq \delta}} \left| G_t^{n,C'}(\omega) - G_t^{n,C}(\omega) \right| \geq \varepsilon \right) \wedge T^\star$$

Hence

$$\mathbb{P}_n(\mathbb{LL}_\delta^{(m)}(G^n) > \varepsilon) = \mathbb{P}_n( \sup_{0 \leq t \leq T^\star} \sup_{\substack{C',C \in \mathcal{K}_\delta^{(m)} \\ \left|C'-C\right| \leq \delta}} \left| G_t^{n,C'} - G_t^{n,C} \right| > \varepsilon) \leq \mathbb{P}_n( \sup_{\substack{C',C \in \mathcal{K}_\delta^{(m)} \\ \left|C'-C\right| \leq \delta}} \left| G_\tau^{n,C'} - G_\tau^{n,C} \right| \geq \varepsilon)$$

Conversely, it is easy to see

$$\mathbb{P}_n( \sup_{\substack{C',C \in \mathcal{K}_\delta^{(m)} \\ \left|C'-C\right| \leq \delta}} \left| G_\tau^{n,C'} - G_\tau^{n,C} \right| \geq \varepsilon) \leq \mathbb{P}_n(\mathbb{LL}_\delta^{(m)}(G^n) \geq \varepsilon)$$

□

Another relevant result that we need to prove Theorem 11, is the next one, given by **Dzhaparidze and Valkeila** (1990)

**Lemma 4.** *Let $M$ be a locally square integrable martingale with $M_0 = 0$. Let $T$ be a stopping time. If $\upsilon$ is the compensator of the jump measure of $M$, then there exist for every $p \geq 2$ constants $k_p$ and $K_p$ such that*

$$\begin{aligned} k_p \mathbb{E}\left( \langle M \rangle_T^{p/2} + \int_0^T \int_{\mathbb{R}\setminus\{0\}} |g|^p \, d\upsilon \right) &\leq \mathbb{E}\left( \sup_{0 \leq t \leq T} |M_t|^p \right) \\ &\leq K_p \mathbb{E}\left( \langle M \rangle_T^{p/2} + \int_0^T \int_{\mathbb{R}\setminus\{0\}} |g|^p \, d\upsilon \right) \end{aligned}$$

*Proof.* cf. **Dzhaparidze and Valkeila** (1990) Lemma 2.1 p.108, based on the **Burkholder-Gundy** inequality for martingales. □

And the last useful result in our proof is the following,

**Proposition 7.** *Let $\tau$ be a finite stopping time and $C_1, ..., C_m \in \mathcal{K}$ and $m \in \mathbb{N}_+$. Additionally let define $\mu^n$ the integer-valued random measure for jumps of a process $\left(G^{n,C_1}, ..., G^{n,C_m}\right)$ and $\upsilon^n$ the compensator, let $f$ be a real function on $\mathbb{R}$ such that the following integrals has sense for any $i = 1, ..., m$*

*1)* $\int_0^t \int_{\mathbb{R}\setminus\{0\}} f(g_i) \, \mu^n(dt, dg_1, ..., dg_i, ..., dg_m) = \int_0^\tau \int_{\mathbb{R}\setminus\{0\}} f(g) \, d\mu^{n,C_i}$

*2)* $\int_0^t \int_{\mathbb{R}\setminus\{0\}} f(g_i) \left(\mu^n - \upsilon^n\right)(dt, dg_1, ..., dg_i, ..., dg_m) = \int_0^\tau \int_{\mathbb{R}\setminus\{0\}} f(g) \, d\left(\mu^{n,C_i} - \upsilon^{n,C_i}\right)$

*Proof.* Both results can be proved using similar ways. Let us define two local martingales $M = (M_t)_{t \geq 0}$ and $M' = (M'_t)_{t \geq 0}$ for any $t \in \mathbb{R}^+$,

$$M_t = \int_0^t \int_{\mathbb{R}\setminus\{0\}} f(g_i) \left(\mu^{n,C_i} - \upsilon^{n,C_i}\right)(dt, dg_1, ..., dg_i, ..., dg_m)$$

and

$$M_t' = \int_0^t \int_{\mathbb{R}\backslash\{0\}} f(g)\, d\left(\mu^{n,C_i} - \upsilon^{n,C_i}\right)$$

Notice that $M$ and $M'$ are purely discontinuous processes and to prove that they are indistinguishable, it is enough to show for any $t \in \mathbb{R}^+$

$$\Delta M_t = \Delta M_t'$$

or

$$\Delta M_t = \int_{\mathbb{R}\backslash\{0\}} f(g_i)\left(\mu^n(\{t\}, dg_1, ..., dg_i, ..., dg_m) - \upsilon^n(\{t\}, dg_1, ..., dg_i, ..., dg_m)\right)$$

and

$$\Delta M_t' = \int_{\mathbb{R}\backslash\{0\}} f(g)\, d\left(\mu^{n,C_i}(\{t\}, dg) - \upsilon^{n,C_i}(\{t\}, dg)\right)$$

Notice that the processes

$$\left(\int_{\mathbb{R}\backslash\{0\}} f(g_i)\left(\upsilon^n(\{t\}, dg_1, ..., dg_i, ..., dg_m)\right)\right)_{t\geq 0} \quad \text{and} \quad \left(\int_{\mathbb{R}\backslash\{0\}} f(g)\left(\upsilon^{n,C_i}(\{t\}, dg)\right)\right)_{t\geq 0}$$

are respectively the compensators of

$$\left(\int_{\mathbb{R}\backslash\{0\}} f(g_i)\left(\mu^n(\{t\}, dg_1, ..., dg_i, ..., dg_m)\right)\right)_{t\geq 0} \quad \text{and} \quad \left(\int_{\mathbb{R}\backslash\{0\}} f(g)\left(\mu^{n,C_i}(\{t\}, dg)\right)\right)_{t\geq 0}$$

and it is enough to establish the following equality for any $t \in \mathbb{R}^+$ :

$$\int_{\mathbb{R}\backslash\{0\}} f(g_i)\left(\mu^n(\{t\}, dg_1, ..., dg_i, ..., dg_m)\right) = \int_{\mathbb{R}\backslash\{0\}} f(g)\left(\mu^{n,C_i}(\{t\}, dg)\right)$$

and in order to complete the proof we only need to see that

$$\int_{\mathbb{R}\backslash\{0\}} f(g_i)\left(\mu^n(\{t\}, dg_1, ..., dg_i, ..., dg_m)\right) = f\left(\Delta G_t^{n,C_i}\right)$$

and

$$\int_{\mathbb{R}\backslash\{0\}} f(g)\left(\mu^{n,C_i}(\{t\}, dg)\right) = f\left(\Delta G_t^{n,C_i}\right)$$

□

Finally, we can proceed with the proof of Theorem 11.

*Proof.* (**Theorem 11**) Let us proceed with the *first* part. To prove weak convergence condition according with Theorem 10 or equivalently, to prove the weak convergence from $G^{C,n}$ to $G^C$ with $C \in \mathcal{K}^{(m)}$, given $m \geq \mathbb{N}_+$, in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$, means, according to **Lipster and Shiryaev** (1989) Theorem 8.3.1 (p.625), the following conditions, for every $C \in \mathcal{K}^{(m)}$, given $m \geq \mathbb{N}_+$, for every $a \in (0,1]$ and $0 < t \leq T^\star \leq 1$:

1′) $\left|G_0^{C,n} - G_0^C\right| \overset{\mathbb{P}^n}{\to} 0$

2′) $\sum_{j\leq n} \int_{T_j}^{T_{j+1}} \int_{\mathbb{R}\backslash\{0\}} 1_{(|x|>a)} d\upsilon_j^{n,C} \overset{\mathbb{P}^n}{\to} 0$

3′) $\sup_{0\leq t\leq T^\star} \left|B_t^{n,C} - \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \alpha_{\eta(s)}(t, C, G^{n,C}) ds\right| \overset{\mathbb{P}^n}{\to} 0$

4′) $\sup_{0\leq t\leq T^\star} \left|\left\langle M^{n,C,h}\right\rangle_t - \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \sigma_{\eta(s)}(t, C, G^{n,C}) ds\right| \overset{\mathbb{P}^n}{\to} 0$

Now, we have to prove that conditions 1.1, 1.2, 1.3 and 1.4 (in the condition group 1) imply $1'$, $2'$, $3'$ and $4'$. It is clear that 1.1 implies $1'$. The condition $2'$ can be proved using 1.2 and the inequality, for any $\varepsilon > 0$,

$$\mathbb{P}^n\left(\sum_{j\le n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}} 1_{(|x|>h)}d\upsilon_j^{n,C} \ge \varepsilon\right) \le \mathbb{P}^n\left(\sum_{j\le n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}}^{2} \delta^2 1_{(|x|>h)}d\upsilon_j^{n,C} \ge \varepsilon a^2\right)$$

Related with the condition $3'$, we use the canonical decomposition for semimartingales and special semimartingales (using directly definitions in **Liptser and Shiryaev** (1989) 4.1.1 and **Jacod and Shiryaev** (1999) $II.2.38$)

$$\begin{aligned} B_t^{n,C} &= G_t^{n,C} - G_0^{n,C} - \tilde{G}_t^{n,C} - \sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta(u,x)\,1_{\{|x|\le 1\}}\left(\mu_j^{n,C} - \upsilon_j^{n,C}\right)(du,dx) \\ &\quad - \sum_{j\le\eta(t)}\int_{T_j}^{t^{(n)}\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta(u,x)\,1_{\{|x|>1\}}\mu_j^{n,C}(du,dx) \end{aligned}$$

and

$$\tilde{B}_t^{n,C} = G_t^{n,C} - G_0^{n,C} - \tilde{G}_t^{n,C} - \sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta(u,x)\left(\mu_j^{n,C} - \upsilon_j^{n,C}\right)(du,dx)$$

Therefore

$$B_t^{n,C} - \tilde{B}_t^{n,C} = -\sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta(u,x)\,1_{\{|x|>1\}}\upsilon_j^{n,C}(du,dx)$$

and we obtain

$$\begin{aligned} \mathbb{P}^n\left(\sup_{0\le t\le T^\star}\left|B_t^{n,C} - \tilde{B}_t^{n,C}\right| \ge \varepsilon\right) &\le \mathbb{P}^n\left(\sum_{j\le n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^d}|\delta(u,x)|\,1_{\{|x|>1\}}\upsilon_j^{n,C}(du,dx) \ge \varepsilon\right) \\ &\le \mathbb{P}^n\left(\sum_{j\le n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^d}(\delta(u,x))^2\,1_{\{|x|>1\}}\upsilon_j^{n,C}(du,dx) \ge \varepsilon\right) \end{aligned}$$

This shows that conditions 1.3 and $3'$ are equivalent, using condition 1.2.

Related with condition $4'$, let us recall

$$\begin{aligned} \left\langle M^{n,C,h}\right\rangle_t &= \left\langle \tilde{G}^{n,C}\right\rangle_t + \sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta^2(u,x)\,1_{\{|x|\le h\}}\upsilon_j^{n,C}(du,dx) \\ &\quad - \sum_{j\le\eta(t)}\sum_{T_j\le u\le t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta(u,x)\,1_{\{|x|\le h\}}\left(\upsilon_j^{n,C}(\{u\},dx)\right)^2 \end{aligned}$$

and

$$\begin{aligned} \left\langle M^{n,C}\right\rangle_t &= \left\langle \tilde{G}^{n,C}\right\rangle_t + \sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta^2(u,x)\,\upsilon_j^{n,C}(du,dx) \\ &\quad - \sum_{j\le\eta(t)}\sum_{T_j\le u\le t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta(u,x)\left(\upsilon_j^{n,C}(\{u\},dx)\right)^2 \end{aligned}$$

Let us simplify the notation redefining $\delta(u,x)=\delta$. Therefore

$$\begin{aligned}
\left|\left\langle M^{n,C,h}\right\rangle_t-\left\langle M^{n,C}\right\rangle_t\right| &\leq \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta^2 1_{\{|x|>h\}}\upsilon_j^{n,C}(du,dx)\\
&\quad+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}\left|\int_{\mathbb{R}^d}\delta 1_{\{|x|\leq h\}}\left(\upsilon_j^{n,C}(\{u\},dx)\right)^2\right.\\
&\quad\left.-\int_{\mathbb{R}^d}\delta\left(\upsilon_j^{n,C}(\{u\},dx)\right)^2\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta^2 1_{\{|x|>h\}}\upsilon_j^{n,C}(du,dx)+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}\int_{\mathbb{R}^d}|\delta|\,1_{\{|x|>h\}}\upsilon_j^{n,C}(\{u\},dx)\\
&\quad\times\left(\int_{\mathbb{R}^d}|\delta|\,1_{\{|x|\leq h\}}\upsilon_j^{n,C}(\{u\},dx)+\int_{\mathbb{R}^d}|\delta|\,\upsilon_j^{n,C}(\{u\},dx)\right)\\
&= \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta^2 1_{\{|x|>h\}}\upsilon_j^{n,C}(du,dx)+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}\int_{\mathbb{R}^d}|\delta|\,1_{\{|x|>h\}}\upsilon_j^{n,C}(\{u\},dx)\\
&\quad\times\left(2\int_{\mathbb{R}^d}|\delta|\,1_{\{|x|\leq h\}}\upsilon_j^{n,C}(\{u\},dx)+\int_{\mathbb{R}^d}|\delta|\,1_{\{|x|>h\}}\upsilon_j^{n,C}(\{u\},dx)\right)
\end{aligned}$$

taking into account that, using **Liptser and Shiryaev** (1989), for any $t\geq 0$ and any $\omega\in\Omega$

$$\upsilon_j^{n,C}(\omega,\{t\}\times\mathbb{R})\leq 1$$

we have

$$\begin{aligned}
\left|\left\langle M^{n,C,h}\right\rangle_t-\left\langle M^{n,C}\right\rangle_t\right| &\leq \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d\setminus\{0\}}\delta^2 1_{\{|x|>h\}}\upsilon_j^{n,C}(du,dx)\\
&\quad+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}2a\int_{\mathbb{R}^d}|\delta|\,1_{\{|x|>h\}}\upsilon_j^{n,C}(\{u\},dx)\\
&\quad+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}\left(\int_{\mathbb{R}^d}|\delta|\,1_{\{|x|>h\}}\upsilon_j^{n,C}(\{u\},dx)\right)^2\\
&\leq \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta^2 1_{\{|x|>h\}}\upsilon_j^{n,C}(du,dx)\\
&\quad+2a\sum_{j\leq\eta(t)}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^d}|\delta|\,1_{\{|x|>h\}}d\upsilon_j^{n,C}\\
&\quad+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}\int_{\mathbb{R}^d}\delta^2 1_{\{|x|>h\}}\upsilon_j^{n,C}(\{u\},dx)
\end{aligned}$$

and finally, we obtain the inequality

$$\sup_{0\leq t\leq T^\star}\left|\left\langle M^{n,C,h}\right\rangle_t-\left\langle M^{n,C}\right\rangle_t\right|\leq 4\sum_{j\leq\eta(t)}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^d}\delta^2 1_{\{|x|>h\}}d\upsilon_j^{n,C}$$

This means that conditions 1.4 and $4'$ are equivalent.

Therefore, all the conditions of Theorem 8.2.1 in **Liptser and Shiryaev** (1989) pp.608 − 609 are satisfied and we have

$$G^{C,n}\overset{\mathbb{P}^n}{\to}G^C$$

for every $C \in \mathcal{K}^{(m)}$, given $m \geq \mathbb{N}_+$, in the Skorohod space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$.

*Second*, we have to prove the **convergence of the finite-dimensional distributions**

$$\left(G^{n,C_1}, G^{n,C_2}, ..., G^{n,C_m}\right) \underset{n\to\infty}{\overset{\mathbb{P}^n}{\to}} \left(G^{C_1}, G^{C_2}, ..., G^{C_m}\right)$$

in the Skorohod space $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$ for every $m \geq 2$.

Let us define $Y^n = \left(G^{n,C_1}, G^{n,C_2}, ..., G^{n,C_m}\right)$ and $Y = \left(G^{C_1}, G^{C_2}, ..., G^{C_m}\right)$. Then, it is enough to prove the weak convergence of $Y^n$ to $Y$ in $D_{[0,T^\star]}\left(\mathbb{R}^d, \mathcal{K}\right)$. Additionally, let us define

$$\bar{B}_{\eta(t)}\left(Y\right) = \left(\alpha_{\eta(t)}(t, C_1, G^{C_1}), ..., \alpha_{\eta(t)}(t, C_m, G^{C_m})\right)$$

and

$$\bar{C}_{\eta(t)}\left(Y\right) = \begin{pmatrix} \sigma_{\eta(t)}(t, C_1, G^{C_1}) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{\eta(t)}(t, C_m, G^{C_m}) \end{pmatrix}$$

such that

$$Y_t = Y_0 + \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \bar{B}_{\eta(s)}\left(Y\right) ds + \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \bar{C}_{\eta(t)}\left(Y\right) d\left(W^{C_1}, ..., W^{C_m}\right)$$

According to **Liptser and Shiryaev** (1989) section 8.3.5 there exists a Brownian motion in $\mathbb{R}^m$ $\left(\tilde{W}^1, ..., \tilde{W}^m\right)$ such that the process $Y$ satisfies the following stochastic differential equation

$$Y_t = Y_0 + \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \bar{B}_{\eta(s)}\left(Y\right) ds + \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \bar{D}_{\eta(t)}\left(Y\right)^{1/2} d\left(\tilde{W}^1, ..., \tilde{W}^m\right)$$

where $\bar{D}_{\eta(t)}\left(Y\right)^{1/2}$ is the square root of the semidefinite-positive matrix $\bar{D}_{\eta(t)}\left(Y\right) = \left(\bar{D}_{\eta(t)}\left(Y\right)(i,j)\right)_{1\leq i,j\leq m}$ given by

$$\bar{D}_{\eta(t)}\left(Y\right)(i,j) = \sigma_{\eta(t)}\left(t, C_i, G^{C_i}\right)\sigma_{\eta(t)}\left(t, C_j, G^{C_j}\right) cov\left(G_1^{C_i}, G_1^{C_j}\right)$$

On the other hand, let us define $\mu^n$ as the jump-measure of $Y^n$ and let $\upsilon^n$ be the compensator. Then $\mu^n$ and $\upsilon^n$ are random measures on $\mathbb{R}^+ \times (\mathbb{R}^m \backslash \{0\})$ and we can choose a version of $\upsilon^n$ such that for every $t \in \mathbb{R}^+$

$$\upsilon^n\left(\{t\} \times (\mathbb{R}^m \backslash \{0\})\right) \leq 1$$

Additionally let us denote as $\mathcal{B}_t^n = \left(B_t^{n,C_1}, ..., B_t^{n,C_m}\right)$, and $\mathcal{C}_t^n$ as the $m \times m$ matrix of predictable quadratic variation at $t \in \mathbb{R}^+$. This basically represents the continuous martingale part of $Y^n$ such that for any $h > 0$

$$\begin{aligned} \mathcal{M}_t^{n,h} &= \mathcal{C}_t^n + \sum_{j\leq\eta(t)} \int_{T_j}^{t\wedge T_{j+1}} \int_{\mathbb{R}^m\backslash\{0\}} \delta\delta^\top 1_{\{\|x\|\leq h\}} \upsilon_j^{n,C}\left(du, dx\right) \\ &\quad - \sum_{j\leq\eta(t)} \sum_{T_j\leq u\leq t\wedge T_{j+1}} \int_{\mathbb{R}^m\backslash\{0\}} \delta 1_{\{\|x\|\leq h\}} \upsilon_j^{n,C}\left(\{u\}, dx\right) \times \int_{\mathbb{R}^m\backslash\{0\}} \delta 1_{\{\|x\|\leq h\}} \upsilon_j^{n,C}\left(\{u\}, dx\right) \end{aligned} \tag{3.11}$$

where $\| \; \|$ is the Euclidean norm in $\mathbb{R}^m$. Therefore, for any $i = 1, ..., m$ and any $j = 1, ..., m$ we denote $\mathcal{M}_t^{n,h}(i,j)$ as the $(i,j)$-th value in the matrix $\mathcal{M}_t^{n,h}$.

Let us recall the conditions established in Theorem 8.3.3 in **Liptser and Shiryaev** (1989) in order to prove the weak convergence of $\left(Y^n\right)_{n\geq1}$ to $Y$, for any $h \in (0,1]$, any $t \leq T \in \mathbb{R}_+$ and any $i, j = 1, ..., m$ :

1′′) $|Y_0^n - Y_0| \xrightarrow{\mathbb{P}^n} 0$

2′′) $\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}} 1_{(\|x\|>h)} d\upsilon_j^n \xrightarrow{\mathbb{P}^n} 0$

3′′) $\sup_{0\leq t\leq T^\star}\left\|\mathcal{B}_t^n - \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\bar{B}_{\eta(s)}(Y)\,ds\right\| \xrightarrow{\mathbb{P}^n} 0$

4′′) $\sup_{0\leq t\leq T^\star}\left|\mathcal{M}_t^{n,h}(i,j) - \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\bar{D}_{\eta(s)}(Y)(i,j)\,ds\right| \xrightarrow{\mathbb{P}^n} 0$

In turn, according to the **Cramer-Wold** theorem (in **Billingsley** (1986) p.397), 1′′) it is equivalent to prove for any $(\lambda_1,...,\lambda_m)\in\mathbb{R}^m$ that

$$\sum_{i\leq m}\lambda_i G_0^{n,C_i} \xrightarrow{\mathbb{P}^n} \sum_{i\leq m}\lambda_i G_0^{C_i}$$

which is equivalent to the condition 1 of group 1 (condition 1.1).

As to 2′′), noting that for any $h>0$

$$\{x=(x_1,x_2,...,x_m):\|x\|>h\}\subset\bigcup_{q\leq m}\left\{x=(x_1,x_2,...,x_m):x_q^2>\frac{h^2}{m}\right\}$$

we have

$$\begin{aligned}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}} 1_{(\|x\|>h)} d\upsilon_j^n &\leq \sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}} 1_{\left(\left\{x_1^2>\frac{h^2}{m}\right\}\cup...\cup\left\{x_m^2>\frac{h^2}{m}\right\}\right)} d\upsilon_j^n\\ &= \sum_{q\leq m}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}} 1_{\left(\left\{x_q^2>\frac{h^2}{m}\right\}\right)} d\upsilon_j^n\end{aligned}$$

Therefore, according to Proposition 7

$$\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}} 1_{(\|x\|>h)} d\upsilon_j^n \leq \sum_{q\leq m}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}} 1_{\left(\left\{|x_q|>\frac{h^2}{m}\right\}\right)} d\upsilon_j^{n,C_q}$$

This proves that 2′′) is given by the hypotheses 2 in the group of conditions 1 (condition 1.2).

To check the condition 3′′) let us recall the inequality $(a+b)^{1/2}\leq a^{1/2}+b^{1/2}$ for every $a,b>0$ and taking into account this result, it is easy to see

$$\sup_{0\leq t\leq T^\star}\left\|\mathcal{B}_t^n - \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\bar{B}_{\eta(s)}(Y)\,ds\right\| \leq \sum_{q\leq m}\sup_{0\leq t\leq T^\star}\left|B_t^{n,C_q} - \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\alpha_{\eta(s)}(t,C_q,G^{n,C_q})ds\right|$$

Finally notice that the condition 3 in group 1 (condition 1.3) implies the equivalent condition 3′) and consequently 3′′).

Before proving condition 4′′) we have to check the following result: for any $h>0$

$$\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\setminus\{0\}}\|x\|^2\,1_{(\|x\|>h)} d\upsilon_j^n \xrightarrow{\mathbb{P}^n} 0 \qquad (3.12)$$

Notice that we have

$$\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}\|x\|^2\,1_{(\|x\|>h)}dv_j^n=\sum_{q\leq m}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}x_q^2 1_{(\{|x_q|>h\})}dv_j^n$$

$$=\sum_{q\leq m}\left(\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}x_q^2 1_{(\{\|x\|>h\}\cap\{|x_q|>h\})}dv_j^n+\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}x_q^2 1_{(\{\|x\|>h\}\cap\{|x_q|\leq h\})}dv_j^n\right)$$

$$\leq\sum_{q\leq m}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}x_q^2 1_{(\{|x_q|>h\})}dv_j^n+a^2\sum_{q\leq m}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}1_{(\{\|x\|>h\})}dv_j^n$$

$$\leq\sum_{q\leq m}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}x_q^2 1_{(\{|x_q|>h\})}dv_j^{n,C_q}+a^2m\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}1_{(\{\|x\|>h\})}dv_j^n$$

We have proved that condition 1.2 implies condition $2'')$ and therefore

$$a^2m\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}1_{(\{\|x\|>h\})}dv_j^n\overset{\mathbb{P}^n}{\to}0$$

On the other hand, using condition also 1.2 we have that for every $q=1,...,m$

$$\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}\backslash\{0\}}x_q^2 1_{(\{|x_q|>h\})}dv_j^{n,C_q}\overset{\mathbb{P}^n}{\to}0$$

This shows that (3.12) is true.

The next step is to prove condition $4'')$ using (3.12). According to expression (3.11), for any $p,q\in\{1,...,m\}$ we have

$$\sup_{0\leq t\leq T^\star}\Bigg|\left\langle\tilde{G}^{n,C_p},\tilde{G}^{n,C_q}\right\rangle_t+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^m\backslash\{0\}}x_px_q1_{\{\|x\|\leq h\}}v_j^n(du,dx)$$
$$-\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}\int_{\mathbb{R}^m\backslash\{0\}}x_p1_{\{\|x\|\leq h\}}v_j^{n,C}(\{u\},dx)\times\int_{\mathbb{R}^m\backslash\{0\}}x_q1_{\{\|x\|\leq h\}}v_j^{n,C}(\{u\},dx)$$
$$-\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\sigma_{\eta(s)}(t,C_p,G^{C_p})\sigma_{\eta(s)}(t,C_q,G^{C_q})cov\left(W_1^{C_p},W_1^{C_q}\right)ds\Bigg|\overset{\mathbb{P}^n}{\to}0$$

Notice that

$$cov\left(W_1^{C_p},W_1^{C_q}\right)s=\left\langle W^{C_p},W^{C_q}\right\rangle_s$$

Taking into account the condition group (2), it is enough to prove

$$\sup_{0\leq t\leq T^\star}\Bigg|\left\langle M^{n,C_p},M^{n,C_q}\right\rangle_t-\left\langle\tilde{G}^{n,C_p},\tilde{G}^{n,C_q}\right\rangle_t+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^m\backslash\{0\}}x_px_q1_{\{\|x\|\leq h\}}v_j^n(du,dx)$$
$$+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}\int_{\mathbb{R}^m\backslash\{0\}}x_p1_{\{\|x\|\leq h\}}v_j^{n,C}(\{u\},dx)\times\int_{\mathbb{R}^m\backslash\{0\}}x_q1_{\{\|x\|\leq h\}}v_j^{n,C}(\{u\},dx)\Bigg|\overset{\mathbb{P}^n}{\to}0$$

For this note that

$$\sup_{0\le t\le T^\star}\left|\left\langle M^{n,C_p},M^{n,C_q}\right\rangle_t-\left\langle \tilde{G}^{n,C_p},\tilde{G}^{n,C_q}\right\rangle_t+\sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}x_px_q1_{\{\|x\|\le h\}}\upsilon_j^n(du,dx)\right.$$
$$\left.+\sum_{j\le\eta(t)}\sum_{T_j\le u\le t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}x_p1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)\times\int_{\mathbb{R}^m\setminus\{0\}}x_q1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)\right|$$

$$\le \sup_{0\le t\le T^\star}\left|\sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}x_px_q1_{\{\|x\|\le h\}}\upsilon_j^n(du,dx)-\sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}x_px_q\upsilon_j^n(du,dx)\right.$$
$$+\sum_{j\le\eta(t)}\sum_{T_j\le u\le t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}x_p1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)\times\int_{\mathbb{R}^m\setminus\{0\}}x_q1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)$$
$$\left.-\sum_{j\le\eta(t)}\sum_{T_j\le u\le t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}x_p\upsilon_j^n(\{u\},dx)\times\int_{\mathbb{R}^m\setminus\{0\}}x_q\upsilon_j^n(\{u\},dx)\right|$$

$$\le \sum_{j\le n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x_px_q|\,1_{\{\|x\|>h\}}\upsilon_j^n(du,dx)+\sum_{j\le n}\sum_{T_j\le u\le T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x_p|\,1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)$$
$$\times\left[\left|\int_{\mathbb{R}^m\setminus\{0\}}x_q1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)-\int_{\mathbb{R}^m\setminus\{0\}}x_q\upsilon_j^n(\{u\},dx)\right|\right]$$
$$+\sum_{j\le n}\sum_{T_j\le u\le T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x_q|\,1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)$$
$$\times\left[\left|\int_{\mathbb{R}^m\setminus\{0\}}x_p1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)-\int_{\mathbb{R}^m\setminus\{0\}}x_p\upsilon_j^n(\{u\},dx)\right|\right]$$
$$+\sum_{j\le n}\sum_{T_j\le u\le T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x_q|\,1_{\{\|x\|>h\}}\upsilon_j^n(\{u\},dx)$$
$$\times\left[\left|\int_{\mathbb{R}^m\setminus\{0\}}x_p1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)-\int_{\mathbb{R}^m\setminus\{0\}}x_i\upsilon_j^n(\{u\},dx)\right|\right]$$

Notice that we have assumed that $\upsilon^n(\{t\}\times(\mathbb{R}^m\setminus\{0\}))\le 1$. Therefore

$$\sup_{0\le t\le T^\star}\left|\left\langle M^{n,C_i},M^{n,C_j}\right\rangle_t-\left\langle \tilde{G}^{n,C_i},\tilde{G}^{n,C_j}\right\rangle_t+\sum_{j\le\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}x_px_j1_{\{\|x\|\le h\}}\upsilon_j^n(du,dx)\right.$$
$$\left.+\sum_{j\le\eta(t)}\sum_{T_j\le u\le t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}x_p1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)\times\int_{\mathbb{R}^m\setminus\{0\}}x_j1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)\right|$$
$$\le \sum_{j\le n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x_px_j|\,1_{\{\|x\|>h\}}\upsilon_j^n(du,dx)+a\sum_{j\le n}\sum_{T_j\le u\le T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x_j|\,1_{\{\|x\|\le h\}}\upsilon_j^n(\{u\},dx)$$
$$+a\sum_{j\le n}\sum_{T_j\le u\le T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x_p|\,1_{\{\|x\|>h\}}\upsilon_j^n(\{u\},dx)$$
$$+\sum_{j\le n}\sum_{T_j\le u\le T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x_p|\,1_{\{\|x\|>h\}}\upsilon_j^n(\{u\},dx)\int_{\mathbb{R}^m\setminus\{0\}}|x_q|\,1_{\{\|x\|>h\}}\upsilon_j^n(\{u\},dx)$$

Using the inequality $ab \leq \frac{1}{2}\left(a^2+b^2\right)$ valid for every $a, b > 0$, we have

$$\begin{aligned}
&\sup_{0\leq t\leq T^\star}\Bigg|\left\langle M^{n,C_p}, M^{n,C_q}\right\rangle_t - \left\langle \tilde{G}^{n,C_p}, \tilde{G}^{n,C_q}\right\rangle_t + \sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} x_p x_q 1_{\{\|x\|\leq h\}}\upsilon_j^n(du,dx)\\
&+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} x_p 1_{\{\|x\|\leq h\}}\upsilon_j^n(\{u\},dx)\times\int_{\mathbb{R}^m\setminus\{0\}} x_q 1_{\{\|x\|\leq h\}}\upsilon_j^n(\{u\},dx)\Bigg|
\end{aligned}$$

$$\begin{aligned}
\leq\;& \frac{1}{2}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} x_p^2 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx) + \frac{1}{2}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} x_q^2 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx)\\
&+a\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} |x_p|\, 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx) + a\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} |x_q|\, 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx)\\
&+\frac{1}{2}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} x_p^2 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx) + \frac{1}{2}\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} x_q^2 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx)
\end{aligned}$$

$$\begin{aligned}
\leq\;& 2\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} \|x\|^2 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx) + 2\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} a\,\|x\|\, 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx)\\
\leq\;& 4\sum_{j\leq n}\int_{T_j}^{T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}} \|x\|^2 1_{\{\|x\|>h\}}\upsilon_j^n(du,dx)
\end{aligned}$$

and using the expression (3.12) we achieve the condition $4'')$.

Finally, in order to complete the proof of Theorem 11 we just need to show that the conditions of group 3) imply then condition 2) in Theorem 10. Notice that $G^{n,C'}$ and $G^{n,C}$ are special semimartingales. Therefore

$$G_\tau^{n,C} = G_0^{n,C} + \tilde{B}_\tau^{n,C} + M_\tau^{n,C}$$

and

$$G_\tau^{n,C'} = G_0^{n,C'} + \tilde{B}_\tau^{n,C'} + M_\tau^{n,C'}$$

whence

$$\left|G_\tau^{n,C} - G_\tau^{n,C'}\right| \leq \left|G_0^{n,C} - G_0^{n,C'}\right| + \left|\tilde{B}_\tau^{n,C} - \tilde{B}_\tau^{n,C'}\right| + \left|M_\tau^{n,C} - M_\tau^{n,C'}\right|$$

Let us recall the following inequality, for every $a > 0$, $b > 0$ and $p \geq 1$ we have

$$(a+b)^p \leq (p+1)\left(a^p + b^p\right)$$

Therefore,

$$\begin{aligned}
\mathbb{E}^n\left(\left|G_\tau^{n,C} - G_\tau^{n,C'}\right|^p\right) \;\leq\;& (p+1)^2\left[\mathbb{E}^n\left(\left|G_0^{n,C} - G_0^{n,C'}\right|^p\right) + \mathbb{E}^n\left(\left|\tilde{B}_\tau^{n,C} - \tilde{B}_\tau^{n,C'}\right|^p\right)\right.\\
&\left.+\mathbb{E}^n\left(\left|M_\tau^{n,C} - M_\tau^{n,C'}\right|^p\right)\right] \qquad (3.13)
\end{aligned}$$

Notice that if $G^{n,C'}$ and $G^{n,C}$ are special semimartingales (locally square integrable), then the local martingales $M^{n,C'}$ and $M^{n,C}$ are square integrable. Additionally, let us define $\tilde{\mu}^{n,C,C'}$ as the jump measure of $M^{n,C}-M^{n,C'}$ and $\tilde{\upsilon}^{n,C,C'}$ its compensator. Therefore we can use directly Lemma 4 to prove the following inequality for any $p\geq 2$

$$\mathbb{E}^n\left(\left|M_\tau^{n,C}-M_\tau^{n,C'}\right|^p\right)\leq c_p\mathbb{E}^n\left(\left\langle M^{n,C}-M^{n,C'}\right\rangle_\tau^{p/2}+\sum_{j\leq\eta(t)}\int_{T_j}^{t\wedge T_{j+1}}\int_{\mathbb{R}^m\setminus\{0\}}|x|^p\,d\tilde{\upsilon}^{n,C,C'}\right)\quad(3.14)$$

and using Proposition 7 we have

$$\begin{aligned}M^{n,C}-M^{n,C'} &= G^{n,C}-G^{n,C'}+\sum_{j\leq\eta(t)}\int_{T_j}^{\cdot\wedge T_{j+1}}\int_{\mathbb{R}\setminus\{0\}}xd\left(\mu_j^{n,C}-\upsilon_j^{n,C}\right)\\ &\quad-\sum_{j\leq\eta(t)}\int_{T_j}^{\cdot\wedge T_{j+1}}\int_{\mathbb{R}\setminus\{0\}}yd\left(\mu_j^{n,C'}-\upsilon_j^{n,C'}\right)\\ &= G^{n,C}-G^{n,C'}+\sum_{j\leq\eta(t)}\int_{T_j}^{\cdot\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)\,d\left(\mu_j^{n,C,C'}-\upsilon_j^{n,C,C'}\right)\end{aligned}$$

According to **Liptser and Shiryaev** (1989) Theorem 3.5.1 we get

$$\begin{aligned}\left\langle M^{n,C}-M^{n,C'}\right\rangle_\tau &= \left\langle G^{n,C}-G^{n,C'}\right\rangle+\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)^2\,d\left(\upsilon_j^{n,C,C'}\right)\\ &\quad-\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)\,\upsilon_j^{n,C,C'}\left(\{u\},dx,dy\right)^2\end{aligned}$$

and again, using the fact that $\upsilon^n\left(\{t\}\times\left(\mathbb{R}^m\setminus\{0\}\right)\right)\leq 1$ we have

$$\begin{aligned}\left\langle M^{n,C}-M^{n,C'}\right\rangle_\tau &= \left\langle G^{n,C}-G^{n,C'}\right\rangle_\tau+\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)^2\,d\left(\upsilon_j^{n,C,C'}\right)\\ &\quad+\sum_{j\leq\eta(t)}\sum_{T_j\leq u\leq\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)^2\,\upsilon_j^{n,C,C'}\left(\{u\},dx,dy\right)^2\\ &\leq \left\langle G^{n,C}-G^{n,C'}\right\rangle_\tau+\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)^2\,d\left(\upsilon_j^{n,C,C'}\right)\end{aligned}$$

Recall the inequality $(a+b)^p\leq(p+1)\left(a^p+b^p\right)$, for every $a>0$, $b>0$ and $p\geq 1$, and similarly as before

$$\begin{aligned}\mathbb{E}^n\left(\left\langle M^{n,C}-M^{n,C'}\right\rangle_\tau^{p/2}\right) &\leq \left(\frac{p}{2}+1\right)\left[\mathbb{E}^n\left(\left\langle G^{n,C}-G^{n,C'}\right\rangle_\tau^{p/2}\right)\right.\\ &\quad\left.+2^{p/2}\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)^2\,d\left(\upsilon_j^{n,C,C'}\right)\right)\right]\end{aligned}\quad(3.15)$$

On the other hand, we have

$$\begin{aligned}\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}|x|^p\,d\left(\tilde{\upsilon}_j^{n,C,C'}\right)\right) &= \mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}|x|^p\,d\left(\tilde{\mu}_j^{n,C,C'}\right)\right)\\ &= \mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\left|\Delta M_s^{n,C}-\Delta M_s^{n,C'}\right|^p\right)\end{aligned}$$

so that

$$\begin{aligned}
&\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\left|\Delta M_s^{n,C}-\Delta M_s^{n,C'}\right|^p\right)\\
=\;&\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\sum_{T_j\leq u\leq\tau\wedge T_{j+1}}\left|\int_{\mathbb{R}\setminus\{0\}}x\mu_j^{n,C}\left(\{u\},dx\right)-\int_{\mathbb{R}\setminus\{0\}}x\upsilon_j^{n,C}\left(\{u\},dx\right)\right.\right.\\
&\left.\left.-\int_{\mathbb{R}\setminus\{0\}}y\mu_j^{n,C'}\left(\{u\},dx\right)+\int_{\mathbb{R}\setminus\{0\}}y\upsilon_j^{n,C'}\left(\{u\},dx\right)\right|^p\right)\\
=\;&\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\sum_{T_j\leq u\leq\tau\wedge T_{j+1}}\left|\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)\,\mu_j^{n,C,C'}\left(\{u\},dx,dy\right)\right.\right.\\
&\left.\left.-\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)\,\upsilon_j^{n,C,C'}\left(\{u\},dx,dy\right)\right|^p\right)\\
\leq\;&(p+1)\left[\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\sum_{T_j\leq u\leq\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}|x-y|_j^p\,\mu^{n,C,C'}\left(\{u\},dx,dy\right)\right)\right.\\
&\left.-\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\sum_{T_j\leq u\leq\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}|x-y|^p\,\upsilon_j^{n,C,C'}\left(\{u\},dx,dy\right)\right)\right]\\
\leq\;&2\,(p+1)\,\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}|x-y|^p\,d\upsilon_j^{n,C,C'}\right) \qquad (3.16)
\end{aligned}$$

Finally, insert the results (3.16), (3.15) and (3.14) in (3.13). We obtain for every $p\geq 2$ the existence of a constant $c_p$, such that

$$\begin{aligned}
\mathbb{E}^n\left(\left|G_\tau^{n,C}-G_\tau^{n,C'}\right|^p\right)\;\leq\;&c_p\left[\mathbb{E}^n\left(\left|G_0^{n,C}-G_0^{n,C'}\right|^p\right)+\mathbb{E}^n\left(\left|\tilde{B}_\tau^{n,C}-\tilde{B}_\tau^{n,C'}\right|^p\right)\right.\\
&+\mathbb{E}^n\left(\left\langle G^{n,C}-G^{n,C'}\right\rangle_\tau^{p/2}\right)\\
&+\mathbb{E}^n\left(\left(\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}(x-y)^2\,d\upsilon_j^{n,C,C'}\right)^{p/2}\right)\\
&\left.+\mathbb{E}^n\left(\sum_{j\leq\eta(\tau)}\int_{T_j}^{\tau\wedge T_{j+1}}\int_{\mathbb{R}^2\setminus\{0,0\}}|x-y|^p\,d\upsilon_j^{n,C,C'}\right)\right]
\end{aligned}$$

Now, the proof is complete, because it is easy to see that condition 2) of Theorem 10 follows immediately from the group of conditions 3 in Theorem 11 and Lemma 4. □

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