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ON IDENTIFIABILITY OF *MAP* PROCESSES

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Abstract

Two types of transitions can be found in the Markovian Arrival process or *MAP*: with and without arrivals. In *transient* transitions the chain jumps from one state to another with no arrival; in *effective* transitions, a single arrival occurs. We assume that in practice, only arrival times are observed in a *MAP*. This leads us to define and study the *Effective Markovian Arrival process* or *E-MAP*. In this work we define identifiability of *MAP*s in terms of equivalence between the corresponding *E-MAP*s and study conditions under which two sets of parameters induce identical laws for the observable process, in the case of 2 and 3-states *MAP*. We illustrate and discuss our results with examples.

Keywords: Batch Markovian Arrival process, Hidden Markov models, Identifiability problems.

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1 Introduction

In Neuts (1979), the versatile Markovian point process (*VMPP*) is introduced. This is a rich class of point processes which contains many familiar arrival processes as very special cases. In each case, arrivals are allowed to occur in batches where different types of arrivals can have different batch size distribution. In Lucantoni et al (1990) the terms *MAP* (Markovian Arrival Process) and *BMAP* (Batch Markovian Arrival process) are introduced to describe the *VMPP* by simpler notation. Later, it was realized that the *VMPP* and *BMAP* are equivalent process (Lucantoni 1990, and Lucantoni 1991). The *BMAP* includes as special cases both phase type renewal processes (which include the Erlang and hyperexponential renewal process) and non-renewal processes such as the Markov modulated Poisson process (*MMPP*) and many other processes in the applied probability literature. In particular, the *MAP* is a *BMAP* with all batch sizes equal to one. The different processes obtained from the *BMAP* are reviewed in Neuts (1979).

The *BMAP* may be defined as a two-dimensional Markov process $\{J(t), N(t)\}$, where $J(t)$ is an underlying continuous time Markov process and $N(t)$ is a count process. When $J(t)$ jumps from one state to another, $N(t)$ may vary or not: at some transitions that we will call *effective transitions*, an arrival of batch size k is produced and $N(t)$ increases by k, for some value of k larger than 1. There exist another transitions, called *transient transitions* in which although the chain varies, the count process remains the same. The holding times at every state are exponentially distributed but the time between arrivals are not; they are sum of r exponentials, where r is the number of transient transitions until the arrival occurs.

The idea of a *BMAP* is to keep the tractability of the Poisson arrival process but significantly generalizes it in ways to allow the inclusion of dependent interarrival times, non-exponential interarrival-time distributions, and correlated batch sizes. This makes the *BMAP* a more effective and powerful traffic model than the simple or batch Poisson process: it is able to capture dependence and correlation, one of the main features in Internet-related data. Several works where the *BMAP* is used in the modeling of teletraffic data can be found: Heffes (1980), Heffes and Lucantoni (1986), Nielsen (2000), Heyman and Lucantoni (2003), Klemm et al (2003), and Scott and Smyth (2003). When fitting a *BMAP* model to real data, only arrival times and batch sizes of arrivals (arrival times of *packets* and their lengths, in the teletraffic context), are observable. All state changes in the underlying Markov chain are not observable and thus, cannot be derived from measured trace data (Klemm et al, 2003).

The *BMAP* can be understood as a generalization of *Hidden Markov process* (*HMP*). The Hidden

Markov processes are discrete-time bivariate parametric processes, characterized by an underlying finite-state homogeneous Markov chain (not observable) which determines the values of a second process (observable). This second process is formed by a sequence of conditionally independent random variables, given the values of the Markov chain. At any given time, the distribution of each random variable depends on the Markov chain only through its value at that time. For a detail study of Hidden Markov models we refer the reader to Ephraim and Merhav (2002). In *BMAP*s there exists an underlying Markov process controlling both the time between transitions and the number of arrivals (or batch sizes) at each transition. The observable sequence would be in this case, a bivariate process: time between transitions and batch sizes.

When dealing with inference for *HMP*s it is very common to encounter *identifiability* problems. These occur when different set of parameters give rise to the same probability distribution. Identifiability conditions for *HMP*s are studied in Leroux (1992) and Rydén (1994). The particular case of *MMPP* was undertaken by Rydén (1996). The result was that a *MMPP* is identifiable, up to state permutations, if and only if all Poisson rates are distinct.

In this work, we study identifiability conditions for the *MAP* with two and three states, denoted by MAP_2 and MAP_3, and delineate conditions for the general MAP_m. We define identifiability in terms of the observable process, which will be called *Effective Markovian Arrival process* and *E-MAP*.

The paper is organized as follows. In Section 2 a theoretical background of the Markovian Arrival process, definitions and key properties are briefly reviewed. In Section 3, the Effective Markovian Arrival process or *E-MAP* is defined and some of its properties are illustrated. Section 4 deals with the concept of identifiability for *MAP*s, in terms of equivalence between E-*MAP*s. We define when a *MAP* is identifiable and derive some results from the definition. In Section 5 study in depth the case for the MAP_2 and give conditions under which the process is identifiable. We also extend the results to the MAP_3. In Section 6 we provide conclusions and delineate some possible directions for future research.

2 The Markovian Arrival process

Let us consider an irreducible continuous Markov chain $J(t)$ with state space $\mathcal{S} = \{1, \ldots, m\}$ and generator matrix D. The process $N(t)$ represents the cumulate number of arrivals in $(0, t]$. A *MAP* process behaves as follows: the initial state $i_0 \in \mathcal{S}$ is given by an initial probability vector $\boldsymbol{\alpha}$ and at the end of a sojourn time in a transient state i, exponentially distributed with parameter $\lambda_i > 0$, there are two possible cases for state transitions. With probability p_{ij1} the *MAP* enters state $j \in \mathcal{S}$ and a single arrival occurs. The selection of state j, which may be the same $j = i$, is uniquely determined by p_{ij1}. On the other hand, with probability p_{ij0} the *MAP* enters state j without arrivals, this time $j \neq i$. If we define the matrices P_0, and P_1 as those containing elements p_{ij0} ($p_{ii0} = 0$) and p_{ij1}, respectively, then the *MAP* process is characterized by the set $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ and $P = P_0 + P_1$ is the stochastic matrix that determines the state changes either an arrival has been occurred or has not. We therefore have, for $1 \leq i \leq m$,

$$\sum_{j=1, j\neq i}^{m} p_{ij0} + \sum_{j=1}^{m} p_{ij1} = 1. \tag{2.1}$$

It may be convenient to represent the evolution of the system in terms of a sequence of matrices $\{D_0, D_1\}$, by letting $(D_0)_{ii} = -\lambda_i$, $(D_0)_{ij} = \lambda_i p_{ij0}$, $1 \leq i,\ j \leq m,\ j \neq i$ and $(D_1)_{i,j} = \lambda_i p_{ij1}$, $1 \leq i, j \leq m$. This definition implies that $D_0 + D_1 = D$ is the infinitesimal generator of the underlying Markov chain. Intuitively, we think of D_0 as governing transitions that do not generate arrivals, and D_1 as the rate of single arrivals.

The matrix D_0 has strictly negative diagonal elements, nonnegative offdiagonal elements, row sums less that or equal to zero and it is assumed to be nonsingular. In other words, D_0 is a stable matrix (all of its eigenvalues have negative real parts). This implies that the interarrival times are finite with probability one (see Lemma 2.2.1 of Neuts, 1981) and that the arrival process does not terminate.

Let $\boldsymbol{\pi}$ be the stationary probability vector of the Markov process with generator D, i.e, $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi} D = \mathbf{0}, \quad |\boldsymbol{\pi}| = 1, \tag{2.2}$$

where $|\mathbf{x}|$ denotes the sum of values of vector $\mathbf{x}$. Thus, the component π_j for $j = 1, \ldots, m$, represents the stationary probability that the arrival process is in state j. The stationary arrival rate of the process is defined as

$$\lambda^{\star} = \boldsymbol{\pi} D_1 \mathbf{e},$$

where $\mathbf{e}$ is a column vector of 1's. The reciprocal $1/\lambda^{\star}$ is the mean interarrival time in the stationary *MAP*.

Let T_1 denote the time to the first arrival in a *MAP* with parameter matrices $\{\boldsymbol{\alpha}, D_0, D_1\}$. Then the probability distribution of T_1 is given by

$$F_{T_1}(t) = \boldsymbol{\alpha}(I - e^{D_0 t})(-D_0)^{-1} D_1 \mathbf{e}, \quad \text{for } t \geq 0, \tag{2.3}$$

or

$$F_{T_1}(t) = \boldsymbol{\alpha}(I - e^{D_0 t})(-D_0)^{-1} L, \quad \text{for } t \geq 0, \tag{2.4}$$

where

$$L = \begin{pmatrix} \lambda_1 \left(1 - \sum_{j \neq 1} p_{1j0}\right) \\ \lambda_2 \left(1 - \sum_{j \neq 2} p_{2j0}\right) \\ \vdots \\ \lambda_m \left(1 - \sum_{j \neq m} p_{mj0}\right) \end{pmatrix}.$$

By taking $\boldsymbol{\alpha} = \boldsymbol{\pi}$, $F(\cdot)$ becomes the distribution function for the stationary version. If instead, $\boldsymbol{\alpha} = (\boldsymbol{\pi} D_1 \mathbf{e})^{-1} \boldsymbol{\pi} D_1$, $F(\cdot)$ becomes the distribution function of the time between two successive arrivals in the stationary version, denoted by T.

3 The *Effective MAP* process or *E-MAP*

In practice, the *MAP* can be used to fit data where only arrivals times are observed. Thus, the state changes and holding exponentially times are not observable. This fact leads us to define a new process, the observable process in a *MAP*, which we will call *Effective* Markovian Arrival Process or *E-MAP*.

The *E-MAP* process behaves at the following way: at the end of a sojourn time in a transient state i, (which is distributed as a sum of exponential distributions, the first one with rate λ_i) there are m possible cases for state transitions: with probability $p^\star_{ij}$, for $j = 1, \ldots, m$, an arrival occurs and the process is instantaneously restarted in state j. The selection of state $j \in \{1, \ldots, m\}$ is uniquely determined by $p^\star_{ij}$. The probabilities $p^\star_{ij}$ depend on the *MAP*s probabilities p_{ij0}, p_{ij1} in the sense

$$p^\star_{ij} = \overbrace{p_{ij1}} + \overbrace{\sum_{j_1 \neq i}^{m} p_{ij_1 0} p_{j_1 j 1}} + \overbrace{\sum_{j_1 \neq i}^{m} \sum_{j_2 \neq j_1}^{m} p_{ij_1 0} p_{j_1 j_2 0} p_{j_2 j 1}} + \ldots = \sum_{r=0}^{\infty} \sum_{j_1 \neq i}^{m} \sum_{j_2 \neq j_1}^{m} \cdots \sum_{j_r \neq j_{r-1}}^{m} p_{ij_1 0} p_{j_1 j_2 0} \cdots p_{j_{r-1} j_r 0} p_{j_r j 1} \tag{3.1}$$

And

$$\sum_{j=1}^{m} p^\star_{ij1} = 1, \quad \text{for all} \quad 1 \leq i \leq m.$$

The existing relationship (3.1) between transition probabilities of *MAP*s and *E-MAP*s can be also stated in a matricial way,

$$P^\star = \sum_{r=0}^{\infty} P_0^r P_1, \tag{3.2}$$

where r represents the possible number of transient transitions in the *MAP*. Thus, $P^\star$ represents the transition probabilities in a *E-MAP*, where at each transition, an arrival occurs. The stationary distribution associated to matrix $P^\star$ will be denote by $\boldsymbol{\phi}$,

$$\boldsymbol{\phi} P^\star = \boldsymbol{\phi}, \quad |\boldsymbol{\phi}| = 1. \tag{3.3}$$

In a *MAP* we will assume that an effective arrival is always produced with probability 1. So, given an initial state i,

$$\lim_{r\to\infty} p_{ij1}^{(r)} = 1, \quad \text{for some } j \in \{1,\ldots,m\},$$

where $p_{ij1}^{(r)}$ is the probability of an effective transition (with a single arrival) from state i to state j, after r non-effective or transient transitions.

Because of that,

$$\lim_{n\to\infty} p_{ij0}^{(n)} = 0, \quad \text{for all } j \in \{1,\ldots,m\}$$

or

$$\lim_{n\to\infty} P_0^n = 0, \quad \text{for all } j \in \{1,\ldots,m\}\ ,$$

which is a necessary and sufficient condition to

$$\sum_{r=0}^{\infty} P_0^r = (I - P_0)^{-1} \tag{3.4}$$

Because of (3.4), (3.2) can be stated as,

$$P^\star = (I - P_0)^{-1} P_1. \tag{3.5}$$

Let us consider a transition in the *E-MAP*, starting in state i. Let us define, for every $i = 1,\ldots,m$, the random variable H_i = *holding time in state i*. This can be written as a sum of exponential variables,

$$H_i = \mathcal{E}_1 + \ldots + \mathcal{E}_{N_i}, \quad \text{where } \mathcal{E}_1 \sim Exp(\lambda_i), \tag{3.6}$$

and N_i is the random variable expressing *number of non-effective transitions from i to the next arrival.* Moreover,

$$\mathcal{E}_n | J_{n-1} = j \sim Exp(\lambda_j), \quad \text{for } n > 1.$$

The probability function for N_i is

$$\mathbb{P}(N_i = r) = \xi_i P_0^r P_1 \mathbf{e} \quad \text{for } n > 0$$

and

$$\mathbb{P}(N_i = 0) = \xi_i P_1 \mathbf{e},$$

where ξ_i is a zeros row vector with a single 1 in the ith position and $\mathbf{e}$ is a column vector of 1's. Then, the expected value for N_i is given by

$$\mathbb{E}(N_i) = \sum_{n=1}^{\infty} n \xi_i P_0^n P_1 \mathbf{e} = \xi_i P_0 \left(\sum_{n=1}^{\infty} n P_0^{n-1} \right) P_1 \mathbf{e} = \xi_i P_0 \left[(I - P_0)^2 \right]^{-1} P_1 \mathbf{e}.$$

We found that variable H_i can be expressed in terms of the rest of $(H_j)_{j \neq i}$. This yields to the definition of H_i as a mixture:

$$H_i = \begin{cases} \mathcal{E}_i & \sum_{j=1}^{m} p_{ij1} \\ \mathcal{E}_i + H_1 & p_{i10} \\ \vdots & \\ \mathcal{E}_i + H_{i-1} & p_{i,i-1,0} \\ \mathcal{E}_i + H_{i+1} & p_{i,i+1,0} \\ \vdots & \\ \mathcal{E}_i + H_m & p_{im0} \end{cases} \tag{3.7}$$

Let us denote by $\varphi_{H_i}(s)$ the moment generating function of H_i, for $i = 1, \ldots, m$. Then from (3.7),

$$\varphi_{H_i}(s) = \sum_{j=1}^{m} p_{ij1} \varphi_{\mathcal{E}_i}(s) + \sum_{j=1, j \neq i}^{m} p_{ij0} \varphi_{\mathcal{E}_i}(s) \varphi_{H_j}(s) \tag{3.8}$$

In order to express (3.8) in a matricial way, let the column vector $\boldsymbol{\mathcal{E}}$ be defined as

$$\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \ldots, \mathcal{E}_m)',$$

such that every $\mathcal{E}_j \sim Exp(\lambda_j)$.

Let $\mathbf{H}$ be defined as

$$\mathbf{H} = (H_1, \ldots, H_m)',$$

thus the matricial expression of (3.8) is

$$\varphi_{\mathbf{H}}(s) = \left[I - \Delta_\varepsilon(s)P_0\right]^{-1} \Delta_\varepsilon(s) S_{P_1}$$

where

$$\varphi_{\mathbf{H}}(s) = \begin{pmatrix} \varphi_{H_1}(s) \\ \vdots \\ \varphi_{H_m}(s) \end{pmatrix}, \qquad \Delta_\varepsilon(s) = \begin{pmatrix} \varphi_{\varepsilon_1}(s) & 0 & 0 & \cdots \\ 0 & \varphi_{\varepsilon_2}(s) & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \varphi_{\varepsilon_m}(s) \end{pmatrix},$$

and

$$S_{P_1} = \begin{pmatrix} \sum_{j=1}^m p_{1j1} \\ \vdots \\ \sum_{j=1}^m p_{mj1} \end{pmatrix} = \begin{pmatrix} 1 - \sum_{j\neq i} p_{1j0} \\ \vdots \\ 1 - \sum_{j\neq i} p_{mj0} \end{pmatrix}.$$

We would like to point out that given a *MAP* defined by $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$, then there exist a single E-*MAP* associated to it, say $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$, where $P^\star = (I - P_0)^{-1}P_1$. However, given a *E-MAP* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$, there are multiple *MAP*s which have this *E-MAP* associated to them: it is possible to find many combinations of matrices $\{P_0, P_1\}$ such that $P^\star = (I - P_0)^{-1}P_1$.

4 Identifiability of *MAP*

Given a set of parameters $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$ characterizing a *MAP* process, we assume that only observe an effective *MAP* process, that is, times at which effective arrivals occur. If inference is done given the real data, one would try to recover the original *MAP* parameters. But, sometimes different estimates from the original can be obtained, say $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$. Our purpose is to know if both give rise to the same probability distribution in the *MAP*. Let us define the random variable

$$T_n\text{= holding time in the } (n-1)th \text{ transition in a E-MAP.}$$

Equivalently, the variable T_n also represents the time between the $(n-1)$th and nth arrival in a *MAP*.

Definition 4.1. Two *E-MAP*s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ and $\{\boldsymbol{\alpha}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$ are equivalent if

$$T_n \stackrel{d}{=} \widetilde{T}_n, \quad \forall n \geq 1, \tag{4.1}$$

where $\stackrel{d}{=}$ mean equality in distribution.

Definition 4.2. Two *MAP*s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$ and $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ are equivalent if and only if the corresponding *E-MAP* processes $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ and $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$ are equivalent.

Definition 4.3. We will say that a *MAP* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$ with corresponding *E-MAP* process $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ is identifiable if there does not exist a different *E-MAP* $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$ equivalent to $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$.

Next we study conditions under which (4.1) holds. First, let us compute the moment generating function of T_1,

$$\begin{aligned}
\varphi_{T_1}(s) &= \sum_{i=1}^{m} \alpha_i \varphi_{H_i}(s) \sum_{j=1}^{m} p_{ij}^\star \\
&= \sum_{i=1}^{m} \alpha_i \varphi_{H_i}(s) \\
&= \boldsymbol{\alpha} \varphi_{\mathbf{H}}(s)
\end{aligned}$$

As a result, condition (4.1) with $n = 1$ is, *MAP* processes to be equivalent is:

$$\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s.$$

In general, for any n, the moment generating function of T_{n+1} is

$$\begin{aligned}
\varphi_{T_{n+1}}(s) &= \sum_{i=1}^{m} \alpha_i^{(n)} \varphi_{H_i}(s) \sum_{j=1}^{m} p_{ij}^\star \\
&= \boldsymbol{\alpha}^{(n)} \varphi_{\mathbf{H}}(s),
\end{aligned}$$

where

$$\boldsymbol{\alpha}^{(n)} = (\alpha_1^{(n)}, \ldots, \alpha_m^{(m)}) = \boldsymbol{\alpha}(P^\star)^{(n)}, \tag{4.2}$$

represents the initial probability vector after n transitions.

In order to prove that two *MAP*s are equivalent it is necessary to verify 'infinite' equalities:

$$\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \tag{4.3}$$

$$\boldsymbol{\alpha}P^\star\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\alpha}}\widetilde{P^\star}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \tag{4.4}$$

$$\begin{aligned}
\boldsymbol{\alpha}(P^\star)^2\varphi_{\mathbf{H}}(s) &= \widetilde{\boldsymbol{\alpha}}(\widetilde{P^\star})^2\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \\
&\vdots \\
\boldsymbol{\alpha}(P^\star)^n\varphi_{\mathbf{H}}(s) &= \widetilde{\boldsymbol{\alpha}}(\widetilde{P^\star})^n\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \\
&\vdots
\end{aligned} \tag{4.5}$$

In next section we will derive necessary and sufficient conditions under which (4.3), (4.4), etc, hold.

4.1 Necessary and Sufficient conditions for identifiability

In this section we give equivalent conditions to

$$\boldsymbol{\alpha}(P^{\star})^{n}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}(\widetilde{P}^{\star})^{n}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s, \quad \forall n \geq 0, \tag{4.6}$$

Let us define the matrix $\mathbf{\Phi} = (\boldsymbol{\phi}; \ldots; \boldsymbol{\phi})$ that is, where each row is the stationary distribution $\boldsymbol{\phi}$ defined in (3.3).

Result 1: If $\boldsymbol{\alpha} = \boldsymbol{\phi}$, $\widetilde{\boldsymbol{\alpha}} = \widetilde{\boldsymbol{\phi}}$ and $\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}, \forall s$, then the identifiability condition (4.6) holds.

Proof:

Condition (4.6) when $n = 0$ holds by assumption. When, $n \geq 1$, (4.6) also is verified because $\boldsymbol{\phi} = \boldsymbol{\phi}(P^{\star})^{n}$, and $\widetilde{\boldsymbol{\phi}} = \widetilde{\boldsymbol{\phi}}\widetilde{P}^{\star}$, for all n □.

Result 2: If two *MAP*s are equivalent, then

$$\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s.$$

Proof:

We assume $\boldsymbol{\alpha}(P^{\star})^{n}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}(\widetilde{P^{\star}})^{n}\varphi_{\widetilde{\mathbf{H}}}$, $\forall s$,and $\forall n \geq 0$. Then, $\lim_{n\to\infty}(P^{\star})^{n} = (P^{\star})^{\infty} = \mathbf{\Phi}$. In the limit,

$$\boldsymbol{\alpha}(P^{\star})^{\infty}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}(\widetilde{P}^{\star})^{\infty}\varphi_{\widetilde{\mathbf{H}}}(s)$$

or

$$\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s). \quad \square$$

Result 3: If $\widetilde{\boldsymbol{\alpha}} = \widetilde{\boldsymbol{\phi}}$ and condition (4.6) holds, then $\boldsymbol{\alpha} = \boldsymbol{\phi}$.

Proof:

From (4.6) when $n = 0$, $\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\mathbf{H}}(s)$, $\forall s$. From (4.6) when $n \to \infty$, $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\mathbf{H}}(s)$, $\forall s$. Thus,

$$\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s)=\boldsymbol{\phi}\varphi_{\mathbf{H}}(s).$$

Equivalently,

$$\alpha_1\varphi_{H_1}(s) + \ldots \alpha_m\varphi_{H_m}(s) = \phi_1\varphi_{H_1}(s) + \ldots + \phi_m\varphi_{H_m}(s),$$

equivalent to

$$(\alpha_1 - \phi_1)\varphi_{H_1}(s) + \ldots + (\alpha_m - \phi_m)\varphi_{H_m}(s) = 0,$$

for all s. This implies that $\boldsymbol{\alpha} = \boldsymbol{\phi}$. $\square$

Result 4: If two MAPs are equivalent, then

$$\boldsymbol{\alpha}\left(I - \boldsymbol{\Phi}\right)\left(I - P^\star\right)^{-1}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\alpha}}\left(I - \widetilde{\boldsymbol{\Phi}}\right)\left(I - \widetilde{P}^\star\right)^{-1}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s.$$

Proof:

Summing up all equivalence conditions (4.6) for all $n \geq 0$,

$$\boldsymbol{\alpha}[I + P^\star + \ldots + (P^\star)^n + \ldots + \boldsymbol{\Phi}]\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\alpha}}[I + \widetilde{P}^\star + \ldots + (\widetilde{P}^\star)^n + \ldots + \widetilde{\boldsymbol{\Phi}}]\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s.$$

But

$$I + P^\star + \ldots + (P^\star)^n + \ldots = \left(I - \boldsymbol{\Phi}\right)\left(I - P^\star\right)^{-1},$$

which proves the result. $\square$

Result 5: If conditions (4.3) and (4.4) hold, and $P^\star = (P^\star)^2$ and $\widetilde{P}^\star = (\widetilde{P}^\star)^2$, then condition (4.6) holds.

Proof:

Condition (4.6) will also hold for $n \geq 2$ because if $P^\star = (P^\star)^2$, then $P^\star = (P^\star)^n$, $\forall n \geq 2$. $\square$

Result 6: If conditions (4.3) and (4.4) hold, and $\boldsymbol{\alpha}P^\star(P^\star - I) = \widetilde{\boldsymbol{\alpha}}\widetilde{P}^\star(\widetilde{P}^\star - I) = 0$, then the two MAPs are equivalent.

Proof: The proof is straightforward using that if $\boldsymbol{\alpha}P^\star = \boldsymbol{\alpha}(P^\star)^2$, then $\boldsymbol{\alpha}P^\star = \boldsymbol{\alpha}(P^\star)^n$, $\forall n \geq 2$. $\square$

5 Identifiability for the two-states MAP or MAP_2

In this section we consider the MAP with two states, which will be denoted MAP_2 and study identifiability conditions for equivalence between two MAP_2s. The process will be determined by the initial probability vector, $\boldsymbol{\alpha} = (\alpha, 1-\alpha)$, the exponential holding times rates $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, and matrices P_0 and P_1 given by

$$P_0 = \begin{pmatrix} 0 & p_{120} \\ p_{210} & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} p_{111} & p_{121} \\ p_{211} & p_{221} \end{pmatrix},$$

where p_{ij0} and p_{ij1} denote the probability of visiting j from i without arrivals and with a single arrival, respectively.

The corresponding *E-MAP*$_2$ process will be determined by $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$, where in this case,

$$P^\star = \begin{pmatrix} p^\star_{11} & p^\star_{12} \\ p^\star_{21} & p^\star_{22} \end{pmatrix}$$

and thus, the stationary probability vector is given by

$$\phi = \frac{p^\star_{21}}{1 - p^\star_{11} + p^\star_{21}}. \tag{5.1}$$

The existing relationship between the stationary probability $\boldsymbol{\phi}$, and $\boldsymbol{\pi}$, the stationary probability in a *MAP* defined in (2.2) is shown in next result. This implies Corolary (5.2), which will be used to illustrate graphical examples in the following sections.

Proposition 5.1. *Let $\boldsymbol{\phi}$ be the stationary probability associated to the transition matrix $P^\star$ in a E-MAP$_2$ and let $\boldsymbol{\pi}$ be the stationary probability defined in (2.2) in a MAP$_2$. Then,*

$$\boldsymbol{\phi} = (\boldsymbol{\pi} D_1 \mathbf{e})^{-1} \boldsymbol{\pi} D_1.$$

Proof: Because $P^\star = (I - P_0)^{-1} P_1$, the stationary probability $\boldsymbol{\phi} = (\phi, 1-\phi)$, can be also expressed as

$$\phi = \frac{p_{211} + p_{111}p_{210}}{(1 - p_{120}p_{210}) - p_{111} - p_{211}p_{120} + p_{211} + p_{111}p_{210}}.$$

In addition, $\boldsymbol{\pi} = (\pi, 1-\pi)$ verifies,

$$\pi = \frac{\lambda_2(p_{211} + p_{210})}{\lambda_2(p_{211} + p_{210}) + \lambda_1(1 - p_{111})}.$$

From (2.1), and expressing D_1 in terms of P_1, it is easy to calculate $(\boldsymbol{\pi} D_1 \mathbf{e})^{-1} \boldsymbol{\pi} D_1$ and see that this is effectively, equal to $\boldsymbol{\phi}$. □

Corollary 5.2. *Given a MAP$_2$ characterized by $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$ where $\boldsymbol{\alpha} = \boldsymbol{\phi}$, then the distribution function of T_1, time to the first arrival, coincides with the distribution function of T, time between two arrivals in the stationary version.*

Proof: Because of the definition (2.3) and if $\boldsymbol{\alpha} = (\boldsymbol{\pi} D_1 \mathbf{e})^{-1} \boldsymbol{\pi} D_1$, $F(\cdot)$ becomes the distribution function of the time between two successive arrivals in the stationary version, denoted by T. □

Next results, concerning the stationary probability $\boldsymbol{\phi}$, will also be used later.

Lemma 5.3. *Let $\mathbf{x} = (x_1, x_2)$ denote a probability vector, $\mathbf{x} \neq \boldsymbol{\phi}$. Then, $\mathbf{x}P^\star = \boldsymbol{\phi}$ if and only if $P^\star = \boldsymbol{\Phi}$, the matrix with every row is equal to $\boldsymbol{\phi}$.*

Proof:

Let us first assume that $P^\star = \boldsymbol{\Phi}$,

$$P^\star = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1 & \phi_2 \end{pmatrix}.$$

Then, since $|\mathbf{x}| = 1$, $\mathbf{x}P^\star = \boldsymbol{\phi}$, $\forall \mathbf{x}$.

Let us suppose that $\mathbf{x} \neq \boldsymbol{\phi}$, verifies $\mathbf{x}P^\star = \boldsymbol{\phi}$. Then,

$$\begin{aligned} x_1 p_{11}^\star + (1 - x_1) p_{21}^\star &= \phi, \\ \phi p_{11}^\star + (1 - \phi) p_{21}^\star &= \phi \end{aligned}$$

where the second equation comes from the fact that $\boldsymbol{\phi}$ is the stationary probability associated to $P^\star$. The system can be solved to find

$$p_{11}^\star = p_{21}^\star = \phi. \qquad \square$$

Lemma 5.4. *Let $P^\star$ be the transition probability matrix with vector of stationary probabilities $\boldsymbol{\phi}$. If all the rows of $P^\star$ are equal, then $P^\star = \Phi$.*

Proof:

The proof is straightforward once the equation $\boldsymbol{\phi} P^\star = \boldsymbol{\phi}$ is solved, where

$$P^\star = \begin{pmatrix} p_{11}^\star & 1 - p_{11}^\star \\ p_{11}^\star & 1 - p_{11}^\star \end{pmatrix}. \qquad \square$$

The elements of $(P^\star)^n$, which appear in (4.6), will be noted by $p_{ij}^{\star(n)}$, and thus

$$\alpha^{(n)} = \alpha(p_{11}^{\star(n)} - p_{21}^{\star(n)}) + p_{21}^{\star(n)},$$

where

$$\boldsymbol{\alpha}(P^{\star})^n \equiv (\alpha^{(n)}, 1-\alpha^{(n)}).$$

5.1 First condition for equivalence

In Section 4 we defined when two MAP_2 are equivalent, in terms of the distribution of T_n and $\widetilde{T}_n$, the interarrival times of the associated E-MAP_2s processes. In this section we study conditions under which $T_1 \stackrel{d}{=} \widetilde{T}_1$, for a given set of parameters $\{\mathbf{x}, \boldsymbol{\lambda}, P_0, P_1\}$ determining the distribution of T_1. In other words, we find values $\{\widetilde{\mathbf{x}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ such that

$$\mathbf{x}\varphi_{\mathbf{H}}(s) = \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \tag{5.2}$$

for fixed values $\{\mathbf{x}, \boldsymbol{\lambda}, P_0, P_1\}$. In the case of two-states, condition (5.2) can be alternatively expressed as

$$x_1\varphi_{H_1}(s) + x_2\varphi_{H_2}(s) = \widetilde{x}_1\varphi_{\widetilde{H}_1}(s) + \widetilde{x}_2\varphi_{\widetilde{H}_2}(s), \quad \forall s \tag{5.3}$$

where $\mathbf{x} = (x_1, 1-x_1)$ and from (3.8),

$$\begin{aligned} \varphi_{H_1}(s) &= (1-p_{120})\varphi_{\mathcal{E}_1}(s) + p_{120}\varphi_{\mathcal{E}_1}(s)\varphi_{H_2}(s), \\ \varphi_{H_2}(s) &= (1-p_{210})\varphi_{\mathcal{E}_2}(s) + p_{210}\varphi_{\mathcal{E}_2}(s)\varphi_{H_1}(s), \end{aligned} \tag{5.4}$$

where it is known that for the exponential distribution $\varphi_{\mathcal{E}_i}(s) = \lambda_i/(\lambda_i - s)$. It can be easily seen that the solution to (5.4) is

$$\varphi_{H_1}(s) = (1-p_{120})\varphi_{\mathcal{E}_1}(s) + p_{120}\varphi_{\mathcal{E}_1}(s)\varphi_{H_2}(s),$$

where

$$\varphi_{H_2}(s) = \frac{(1-p_{210})\varphi_{\mathcal{E}_2}(s) + (1-p_{120})p_{210}\varphi_{\mathcal{E}_1}(s)\varphi_{\mathcal{E}_2}(s)}{1-p_{120}p_{210}\varphi_{\mathcal{E}_1}(s)\varphi_{\mathcal{E}_2}(s)}.$$

An easy computation shows

$$\begin{aligned} \mathbf{x}\varphi_{\mathbf{H}}(s) &= x_1(1-p_{120})\varphi_{\mathcal{E}_1}(s) + \frac{(1-p_{210})\varphi_{\mathcal{E}_2}(s) + (1-p_{120})p_{210}\varphi_{\mathcal{E}_1}(s)\varphi_{\mathcal{E}_2}(s)}{1-p_{120}p_{210}\varphi_{\mathcal{E}_1}(s)\varphi_{\mathcal{E}_2}(s)}\left[x_1p_{120}\varphi_{\mathcal{E}_1}(s) + (1-x_1)\right] \\ &= \frac{x_1(1-p_{120})\varphi_{\mathcal{E}_1}(s) + (1-x_1)(1-p_{210})\varphi_{\mathcal{E}_2}(s) + \left[x_1(p_{120}-p_{210}) + p_{210}(1-p_{120})\right]\varphi_{\mathcal{E}_1}(s)\varphi_{\mathcal{E}_2}(s)}{1-p_{120}p_{210}\varphi_{\mathcal{E}_1}(s)\varphi_{\mathcal{E}_2}(s)} \\ &= \frac{[x_1\lambda_1(p_{120}-1) + \lambda_2(x_1 + p_{210} - 1 - x_1p_{210})]s + \lambda_1\lambda_2(1-p_{120}p_{210})}{(\lambda_1 - s)(\lambda_2 - s) - p_{120}p_{210}\lambda_1\lambda_2}. \end{aligned}$$

Thus, $\mathbf{x}\varphi_{\mathbf{H}}(s)$ can be expressed as

$$\mathbf{x}\varphi_{\mathbf{H}}(s) = \frac{a_1 s + d_0}{s^2 + d_1 s + d_0},$$

where

$$\begin{aligned} a_1 &= x_1\lambda_1(p_{120}-1)+\lambda_2(x_1+p_{210}-1-x_1p_{210}), \\ d_1 &= -(\lambda_1+\lambda_2), \\ d_0 &= \lambda_1\lambda_2(1-p_{120}p_{210}). \end{aligned}$$

It can be seen that d_1 is a function of $\boldsymbol{\lambda}$, d_0 depends on $\boldsymbol{\lambda}$ and P_0, and finally a_1 depends on all parameters $\mathbf{x}$, $\boldsymbol{\lambda}$ and P_0. For clarity, we will sometimes express a_1 as $(a_1)_x$, to indicate which the initial probability is.

Since the elements of P_1 are not involved in the parameters a_1, d_1 and d_0, then solving (5.2) will imply just finding the values of $\{\widetilde{\mathbf{x}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$, assuming $\{\mathbf{x}, \boldsymbol{\lambda}, P_0\}$ are fixed.

Next, we show some results concerning the form of function $\mathbf{x}\varphi_H(s)$.

Lemma 5.5. *The values a_1, d_1 and d_0 verify the constraints,*

$$d_1 < 0, \quad 0 \le d_0 \le \lambda_1\lambda_2, \quad a_1 \le 0.$$

Proof:

The first and second constraints are straightforward given the definitions of d_1 and d_0. To prove the last one, $a_1 \le 0$, we express a_1 as

$$a_1 = x_1[\lambda_1(p_{120}-1)] + (1-x_1)[\lambda_2(p_{210}-1)]. \tag{5.5}$$

Since x_1, p_{120}, and p_{210} are probabilities and $\lambda_1, \lambda_2 > 0$, both terms are negative, so the sum is. □

Proposition 5.6. *An arrival in a MAP$_2$ is produced with probability 1 if and only if $a_1 < 0$.*

Proof:

By Lemma (5.5) $a_1 \le 0$. Let us study the case $a_1 = 0$, or from (5.5),

$$x_1\lambda_1(p_{120}-1) = -(1-x_1)\lambda_2(p_{210}-1).$$

As the first expression is negative, and the second is positive, this is equivalent to

$$x_1\lambda_1(p_{120}-1) = (1-x_1)\lambda_2(p_{210}-1) = 0,$$

if and only if

$$\begin{aligned} x_1 = 0 \quad &\text{and} \quad p_{210} = 1, \quad \text{or} \\ p_{120} = 1 \quad &\text{and} \quad p_{120} = 1, \quad \text{or} \\ p_{120} = 1 \quad &\text{and} \quad x_1 = 1. \end{aligned}$$

In all cases, either $p_{120} = 1$ or $p_{210} = 1$. If $p_{120} = 1$, then $p_{111} = p_{121} = 0$ (similarly, if $p_{210} = 1$) and thus if the initial state is $i = 1$, $\lim_{r\to\infty} p_{ij}^{(r)} \neq 1$ which is in contradiction with the fact that an effective arrival is always produced with probability 1. □

From now on, we will assume that a_1 is a strictly negative value.

Proposition 5.7. *The variable T_1, time to the first arrival in a MAP$_2$, or time until the first transition in a E-MAP$_2$, can be expressed as a mixture of two exponential variables with rates given by the roots of $s^2 + d_1 s + d_0$.*

Proof:

The moment generating function of T_1, *time to the first arrival* can be expressed as,

$$\begin{aligned} \varphi_{T_1}(s) \equiv \mathbf{x}\varphi_{\mathbf{H}}(s) &= \frac{a_1 s + d_0}{s^2 + d_1 s + d_0} \\ &= \frac{d_0 + a_1 r_1}{r_1(r_2 - r_1)} \frac{r_1}{r_1 - s} + \frac{d_0 + a_1 r_2}{r_2(r_1 - r_2)} \frac{r_2}{r_2 - s} \\ &= p_1 \frac{r_1}{r_1 - s} + p_2 \frac{r_2}{r_2 - s}, \end{aligned}$$

where r_1, r_2 are the roots of $s^2 + d_1 s + d_0$ and it can be easily shown that $p_1 + p_2 = 1$. It follows that, T_1 is a mixture of exponentials with rates r_1, r_2,

$$T_1 = \begin{cases} \mathcal{E}(r_1), & p_1 = \frac{d_0 + a_1 r_1}{r_1(r_2 - r_1)} \\ \mathcal{E}(r_2), & p_2 = \frac{d_0 + a_1 r_2}{r_2(r_1 - r_2)} \end{cases}$$

□

We would like to point out that there exist cases, where $p_1 < 0$ or $p_2 < 0$, for instance if $\boldsymbol{\lambda} = [20, 100]$, $p_{120} = 0.9$, $p_{210} = 0.1$, and $x_1 = 0.99$, then $p_{120} = -0.177$.

Let us continue deriving conditions under which (5.2) holds,

$$\begin{aligned}
\mathbf{x}\varphi_{\mathbf{H}}(s) &= \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \quad \text{if and only if,}\\
\frac{a_1 s + d_0}{s^2 + d_1 s + d_0} &= \frac{\widetilde{a}_1 s + \widetilde{d}_0}{s^2 + \widetilde{d}_1 s + \widetilde{d}_0}, \quad \forall s \quad \text{if and only if,}\\
a_1 s^3 + (d_0 + a_1\widetilde{d}_1)s^2 + (a_1\widetilde{d}_0 + d_0\widetilde{d}_1)s + d_0\widetilde{d}_0 &= \widetilde{a}_1 s^3 + (\widetilde{d}_0 + \widetilde{a}_1 d_1)s^2 + (\widetilde{a}_1 d_0 + \widetilde{d}_0 d_1)s + \widetilde{d}_0 d_0, \quad \forall s.
\end{aligned}$$

Therefore, condition (5.2) holds if and only if

$$a_1 = \widetilde{a}_1 \tag{5.6}$$

$$d_0 + a_1\widetilde{d}_1 = \widetilde{d}_0 + \widetilde{a}_1 d_1 \tag{5.7}$$

$$a_1\widetilde{d}_0 + d_0\widetilde{d}_1 = \widetilde{a}_1 d_0 + \widetilde{d}_0 d_1. \tag{5.8}$$

Equation (5.6) can be expressed as the equation of a line, in terms of the initial probability vectors, $\mathbf{x}$ and $\widetilde{\mathbf{x}}$:

$$\beta_1 x_1 + \widetilde{\beta}_1\widetilde{x}_1 + \beta = 0, \tag{5.9}$$

where,

$$\begin{aligned}
\beta_1 &= \lambda_1(p_{120} - 1) + \lambda_2(1 - p_{210}),\\
\widetilde{\beta}_1 &= \widetilde{\lambda}_1(1 - \widetilde{p}_{120}) + \widetilde{\lambda}_2(\widetilde{p}_{210} - 1),\\
\beta &= \lambda_2(p_{210} - 1) - \widetilde{\lambda}_2(\widetilde{p}_{210} - 1).
\end{aligned} \tag{5.10}$$

The system (5.6)-(5.8) is a three-equations linear system with unknowns, $\widetilde{a}_1, \widetilde{d}_0, \widetilde{d}_1$, where the augmented matrix is

$$C_2 = \left(\begin{array}{ccc|c} 1 & 0 & 0 & a_1\\ d_1 & 1 & -a_1 & d_0\\ d_0 & d_1 - a_1 & -d_0 & 0 \end{array}\right).$$

The first equation is isolated, and thus $\widetilde{a}_1 = a_1$. For the last two rows, we use the Gaussian elimination method:

$$\left(\begin{array}{cc|c} 1 & -a_1 & d_0 - d_1 a_1\\ d_1 - a_1 & -d_0 & -d_0 a_1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -a_1 & d_0 - d_1 a_1\\ 0 & -d_0 + a_1(d_1 - a_1) & d_1[-d_0 + a_1(d_1 - a_1)] \end{array}\right).$$

If

$$-d_0 + a_1(d_1 - a_1) \neq 0 \tag{5.11}$$

the solutions is unique:

$$\widetilde{a}_1 = a_1 \tag{5.12}$$

$$\widetilde{d}_0 = d_0 \tag{5.13}$$

$$\widetilde{d}_1 = d_1. \tag{5.14}$$

However, if $-d_0 + a_1(d_1 - a_1) = 0$, there are infinite solutions:

$$\widetilde{a}_1 = a_1 \tag{5.15}$$

$$\widetilde{d}_0 = a_1\widetilde{d}_1 + d_0 - \widetilde{a}_1 d_1, \quad \widetilde{d}_1 \in \mathbb{R}^- \tag{5.16}$$

Next results establishes the relationship between the uniqueness of solution in the system (5.6)-(5.8) and the roots of polynomials $P(s)$ and $Q(s)$, where $\mathbf{x}\varphi_{\mathbf{H}}(s) = P(s)/Q(s)$.

Proposition 5.8. *The system of equations (5.6)-(5.8) has got infinite solutions if and only if the polynomials* $P(s) = a_1 s + d_0$ *and* $Q(s) = s^2 + d_1 s + d_0$ *share the root* $r = -d_0/a_1$.

Proof:

Let us first prove the right to left implication. Let us suppose that r is a root of $Q(s)$. Then $Q(s) = (s - r)(s - r_2)$, where (because of the relation between the roots of a second-order polynomial and its coefficients)

$$d_1 = -(r + r_2),$$

$$d_0 = r r_2$$

This implies, given the known value of r that $r_2 = -a_1$ which implies $d_1 = -(r + r_2) = a_1 + d_0/a_1$ and thus,

$$d_1 = \frac{a_1^2 + d_0}{a_1} \quad \text{if and only if}$$

$$-d_0 + a_1(d_1 - a_1) = 0,$$

the condition under which the system (5.6)-(5.8) has got infinite solutions.

Let us now prove the left to right implication. Assume both $P(s)$ and $Q(s)$ do not share any root. Then, $Q(s) = (s - r_1)(s - r_2)$, where $r_1 \neq r$ and $r_2 \neq r$. This implies that $a_1 \neq -r_1$ and $a_1 \neq -r_2$. To see this, let us consider the situation where $a_1 = -r_1$. Then, $r = d_0/r_1$, or equivalently, $d_0 = rr_1$. But since, $d_0 = r_1 r_2$, we would obtain $r = r_2$ which yields a contradiction with the initial assumption $r \neq r_2$. To see $a_1 \neq -r_2$ we would proceed similarly. As $a_1 \neq -r_1$ and $a_1 \neq -r_2$, then, $(a_1+r_1)(a_1+r_2) \neq 0$ or $a_1^2+(r_1+r_2)a_1+r_1r_2 \neq 0$, which is equivalent to expression (5.11), which implies a unique solution for (5.6)-(5.8). □

Corollary 5.9. *If $P(s)$ and $Q(s)$ share the root $r = -d_0/a_1$, then the other root of $Q(s)$ is $r_2 = -a_1$, and thus,*

$$Q(s) = (s + d_0/a_1)(s + a_1).$$

Proof:

The condition so that the rank is 2 is

$$a_1^2 - d_1 a_1 + d_0 = 0,$$

equivalent to

$$-Q(-a_1) = 0. \qquad \square$$

5.1.1 Examples

Next, we present numerical examples of MAP_2s verifying (5.2).

1. Consider a MAP_2 with parameters $\{x_1, \lambda_1, \lambda_2, p_{120}, p_{210}\} = \{0.504, 0.5, 20, 0.3, 0.3\}$. In this case, $a_1 = -7.1204$, $d_1 = -20.5$ and $d_0 = 9.1$ Polynomials $P(s)$ and $Q(s)$ do not share any root, and thus, system (5.6)-(5.8) has got a single solution $\widetilde{a}_1 = a_1, \widetilde{d}_1 = d_1, \widetilde{d}_0 = d_0$. Next MAP_2s are examples such that (5.2) holds:

    (a) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.366, 0.5, 20, 0.2, 0.45\}$.

    (b) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.0884, 1, 19.5, 0.888, 0.6\}$.

    (c) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.0029, 1.2, 19.3, 0.9636, 0.63\}$.

    (d) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.201, .8, 19.7, 0.7683, 0.55\}$.

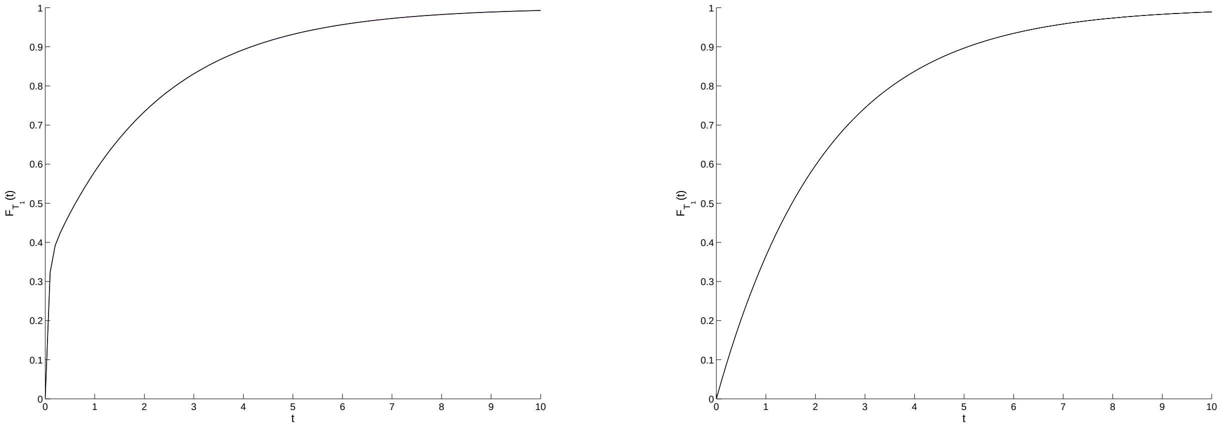

Figure 1: CDF of T_1 in the examples 1 and 2.

Notice that although the solution is unique in terms of $\widetilde{a}_1, \widetilde{d}_1, \widetilde{d}_0$, there exist more than one MAP (only three equations for five unknowns forming a non-linear system equation) verifying the first condition of equivalence.

2. Consider a MAP_2 with parameters $\{x_1, \lambda_1, \lambda_2, p_{120}, p_{210}\} = \{0.9924, 0.5, 20, 0.3, 0.3\}$. In this case, $a_1 = -0.454$, $d_1 = -20.5$ and $d_0 = 9.1$. Polynomials $P(s)$ and $Q(s)$ share the root $r = 20.04$, and thus, system (5.6)-(5.8) has got infinite solutions $\widetilde{a}_1, \widetilde{d}_1, \widetilde{d}_0$. Next MAP_2s are examples such that (5.2) holds:

   (a) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.9945, 0.5, 12, 0.1266, 0.7\}$.

   (b) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.963, 0.5, 12, 0.8861, 0.1\}$.

   (c) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.6644, 5, 10, 0.9644, 0.9\}$.

   (d) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.905, 1, 8, 0.7924, 0.65\}$.

   (e) $\{\widetilde{x}_1, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{0.5702, 30, 40, 0.9936, 0.98\}$.

It can be easily seen that each pair of values, characterizing the random variable, T_1, *time to the first arrival*, give rise to the same distribution function, $F_{T_1}(t)$ defined in (2.3). Figure 1 depicts the distribution function of T_1 for the previous examples. $\Diamond$

5.2 Stationary case

We have shown in Result 2 that

$$\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \tag{5.17}$$

is a necessary condition for equivalence of two given MAPs, where $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$ are the stationary distributions associated to matrices $P^\star$ and $\widetilde{P}^\star$, respectively. According to the definition of $P^\star$, $P^\star = (I - P_0)^{-1}P_1$, the value of $\boldsymbol{\phi}$ will be determined by P_0 and P_1. On the other hand, we have seen that $\varphi_{\mathbf{H}}(s)$ just depends on $\boldsymbol{\lambda}$ and P_0 (respectively, $\widetilde{\boldsymbol{\phi}}$ and $\varphi_{\widetilde{\mathbf{H}}}(s)$).

In the future sections we will consider next problem:

> *Given two MAP_2s characterized by values $\{\boldsymbol{\lambda}, P_0, P_1\}$ and $\{\widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$, such that the stationary equivalence condition (5.17) holds, find initial probabilities $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$, so that the MAP_2s are equivalent, according to (4.6):*
>
> $$\boldsymbol{\alpha}(P^\star)^n\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}(\widetilde{P}^\star)^n\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s, \quad \forall n \geq 0$$

Since we assume (5.17) holds, then $\{\widetilde{\boldsymbol{\phi}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ is a solution of the equations system (5.6)-(5.8), given the values $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$. Thus, one of next two events happens:

(a) The values $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$ make the system (5.6)-(5.8) to have only one solution (if and only if the numerator and denominator of $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)$ do not share any root).

(b) The values $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$ make the system (5.6)-(5.8) to have infinite solutions (or the numerator and denominator of $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)$ share the root).

However, in both cases, the equation $(a_1)_\phi = (\widetilde{a}_1)_{\widetilde{\phi}}$ must hold. From (5.9), this is equivalent to

$$\beta_1\phi + \widetilde{\beta}_1\widetilde{\phi} + \beta = 0,$$

where β_1, $\widetilde{\beta}_1$, β were defined in (5.10).

In next section we will solve the previously stated problem, in terms of the division (a) and (b). Our results will also depend on the facts that $P^\star = \boldsymbol{\Phi}$ and $\widetilde{P}^\star = \widetilde{\boldsymbol{\Phi}}$ or not, and if $\beta_1 = 0$ or $\widetilde{\beta}_1 = 0$ or not.

5.3 Main results

In this section we show identifiability results for the MAP_2. Given two MAP_2s characterized by different sets of parameters, we find initial probabilities, such that if the stationary equivalence condition (5.17) holds, then they are equivalent according to our equivalence definition, from Section 3.

Our proofs will be based mainly on the fact that $\mathbf{x}\varphi_{\mathbf{H}}(s) = \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s)$, for all s, is equivalent to solve the system of linear equations (5.6)-(5.8), where $(a_1)_x = (\widetilde{a}_1)_{\widetilde{x}}$ (a line equation) needs to be solved whatever the system has a single or infinite equations.

According to our empirical experience, in most MAP_2s it is found that $\beta_1 \neq 0$, the system (5.6)-(5.8) has got a unique result (because usually $-d_0 + a_1(d_1 - a_1) \neq 0$) and $P^\star$ is different from $\mathbf{\Phi}$. This leads us to state the first result, a Theorem that provides a general criterion, in terms of initial probabilities, under which two MAP_2s are equivalent. The following Lemmas and Propositions are valid only for special forms of β_1 and $P^\star$.

For every obtained result, we present an illustrative example.

Theorem 5.10 (General Result for $m = 2$.). *Let $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ define two MAP_2s, with corresponding E-MAP_2s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ and $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$, where $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$ are the stationary probabilities associated to $P^\star$ and $\widetilde{P}^\star$. Assume,*

(i) $\beta_1 \neq 0$, *and* $\widetilde{\beta}_1 \neq 0$,

(ii) $P^\star \neq \mathbf{\Phi}$ *or* $\widetilde{P}^\star \neq \widetilde{\mathbf{\Phi}}$.

Then, the MAP_2s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ are equivalent if and only if

1. $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$ *and*

2. $(\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}}) = (\boldsymbol{\phi}, \widetilde{\boldsymbol{\phi}})$.

Proof:

1. *1* and *2* $\rightarrow$ Equivalence.

   Let us first assume that both *1* and *2* hold. We want to prove equivalence given by (4.6). As

$(\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}}) = (\boldsymbol{\phi}, \widetilde{\boldsymbol{\phi}})$ and given that $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$, then all equivalence conditions (4.6) hold because $\boldsymbol{\alpha}P^\star = \boldsymbol{\phi}P^\star = \boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\alpha}}\widetilde{P}^\star = \widetilde{\boldsymbol{\phi}}\widetilde{P}^\star = \widetilde{\boldsymbol{\phi}}$.

2. Equivalence $\rightarrow$ *1* and *2*.

   Given equivalence of two given MAP_2s, Result 2 implies *1*. Let us deduce *2* from equivalence; as $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$, then the pair $(\phi, \widetilde{\phi})$ verifies equation (5.9) (or $(a_1)_\phi = (\widetilde{a}_1)_{\widetilde{\phi}}$),

$$\beta_1\phi + \widetilde{\beta}_1\widetilde{\phi} + \beta = 0.$$

   Because of equivalence, $\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$, and thus the pair $(\alpha, \widetilde{\alpha})$ also satisfies (5.9),

$$\beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha} + \beta = 0.$$

   Both equations imply,

$$\beta_1\phi + \widetilde{\beta}_1\widetilde{\phi} = \beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha},$$

   or equivalently, using (5.1),

$$\beta_1\frac{p^\star_{21}}{1 - p^\star_{11} + p^\star_{21}} + \widetilde{\beta}_1\frac{\widetilde{p}^\star_{21}}{1 - \widetilde{p}^\star_{11} + \widetilde{p}^\star_{21}} = \beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha}. \tag{5.18}$$

   Because of equivalence, condition (4.6) holds for all $n \geq 0$. Taking $n = 1$ yields to

$$\beta_1\frac{p^\star_{21}}{1 - p^\star_{11} + p^\star_{21}} + \widetilde{\beta}_1\frac{\widetilde{p}^\star_{21}}{1 - \widetilde{p}^\star_{11} + \widetilde{p}^\star_{21}} = \beta_1\alpha^{(1)} + \widetilde{\beta}_1\widetilde{\alpha}^{(1)}, \tag{5.19}$$

   where we know that

$$\alpha^{(1)} = \alpha(p^\star_{11} - p^\star_{21}) + p^\star_{21}, \qquad \widetilde{\alpha}^{(1)} = \widetilde{\alpha}(\widetilde{p}^\star_{11} - \widetilde{p}^\star_{21}) + \widetilde{p}^\star_{21},$$

   and thus we need to solve for $(\alpha, \widetilde{\alpha})$ in the next systems of linear equations

$$\begin{aligned} \beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha} &= \beta_1\frac{p^\star_{21}}{1 - p^\star_{11} + p^\star_{21}} + \widetilde{\beta}_1\frac{\widetilde{p}^\star_{21}}{1 - \widetilde{p}^\star_{11} + \widetilde{p}^\star_{21}} \\ \beta_1(p^\star_{11} - p^\star_{21})\alpha + \widetilde{\beta}_1(\widetilde{p}^\star_{11} - \widetilde{p}^\star_{21})\widetilde{\alpha} &= \beta_1\left(\frac{p^\star_{21}}{1 - p^\star_{11} + p^\star_{21}} - p^\star_{21}\right) + \widetilde{\beta}_1\left(\frac{\widetilde{p}^\star_{21}}{1 - \widetilde{p}^\star_{11} + \widetilde{p}^\star_{21}} - \widetilde{p}^\star_{21}\right), \end{aligned} \tag{5.20}$$

   whose coefficient matrix is

$$D_2 = \begin{pmatrix} \beta_1 & \widetilde{\beta}_1 \\ \beta_1(p^\star_{11} - p^\star_{21}) & \widetilde{\beta}_1(\widetilde{p}^\star_{11} - \widetilde{p}^\star_{21}) \end{pmatrix}.$$

   It can be easily seen that $\boldsymbol{\alpha} = \boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\alpha}} = \widetilde{\boldsymbol{\phi}}$ solves the system. We need to determine the uniqueness of this solution. But this comes from the fact that $\beta_1, \widetilde{\beta}_1 \neq 0$ and Lemma (5.4): as $P^\star \neq \mathbf{\Phi}$ or $\widetilde{P}^\star \neq \widetilde{\mathbf{\Phi}}$, then $p^\star_{11} - p^\star_{21} \neq 0$ or $\widetilde{p}^\star_{11} - \widetilde{p}^\star_{21} \neq 0$. In consequence the rank of C_2 is 2.□

The theorem shows that under general conditions, two MAP_2s equivalent in the limit, will be equivalent, if and only if the initial probabilities are, in fact, the stationary ones.

Example 5.1.

Consider the MAP_2 defined by

$$\{\lambda_1, \lambda_2, p_{120}, p_{210}, p_{111}, p_{121}, p_{211}, p_{221}\} = \{0.5, 20, 0.3, 0.3, 0.6148, 0.0852, 0.0886, 0.6114\}$$

and initial probability $\alpha = \phi = 0.504$. In this case,

$$P^{\star} = \begin{pmatrix} 0.7048 & 0.2952 \\ 0.3 & 0.7 \end{pmatrix} \neq \mathbf{\Phi}.$$

Consider another MAP_2 with parameters

$$\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}, \widetilde{p}_{111}, \widetilde{p}_{121}, \widetilde{p}_{211}, \widetilde{p}_{221}\} = \{0.8, 19.7, 0.7683, 0.55, 0.0513, 0.1804, 0.0873, 0.3627\}$$

and initial probability $\alpha = \phi = 0.201$. In this case,

$$\widetilde{P}^{\star} = \begin{pmatrix} 0.21 & 0.79 \\ 0.2 & 0.8 \end{pmatrix} \neq \widetilde{\mathbf{\Phi}}.$$

It can be seen that $\beta_1 = 13.65 \neq 0$ and $\widetilde{\beta}_1 = -8.6796 \neq 0$ and $\boldsymbol{\phi}\varphi_H(s) = \widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{H}}(s)$, for all s, that is $\widetilde{\phi}$, $\widetilde{\boldsymbol{\lambda}}$ and $\widetilde{P}_0$ solves the system (5.6)-(5.8) given the values of ϕ, $\boldsymbol{\lambda}$ and P_0. We are thus in the assumptions of Theorem (5.10). This assures us the processes are equivalent, as Figure 2, which depicts the CDF of the time between two arrivals in the stationary version for both MAP_2s, shows. $\Diamond$

Next proposition is similar to Theorem (5.10), but for the case where $P^{\star} = \mathbf{\Phi}$ and $\widetilde{P}^{\star} = \widetilde{\mathbf{\Phi}}$. In addition, it assumes that $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$ determines a single-solution-system (5.6)-(5.8).

Proposition 5.11. *Let* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$, $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *define two MAP_2s, with corresponding E-MAP_2s* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^{\star}\}$ *and* $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^{\star}\}$, *where* $\boldsymbol{\phi}$ *and* $\widetilde{\boldsymbol{\phi}}$ *are the stationary probabilities associated to* $P^{\star}$ *and* $\widetilde{P}^{\star}$. *Assume,*

(i) the set $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$ *makes the system (5.6)-(5.8) to have only one solution,*

(ii) $\beta_1 \neq 0$, *and* $\widetilde{\beta}_1 \neq 0$,

(iii) $P^{\star} = \mathbf{\Phi}$ *and* $\widetilde{P}^{\star} = \widetilde{\mathbf{\Phi}}$.

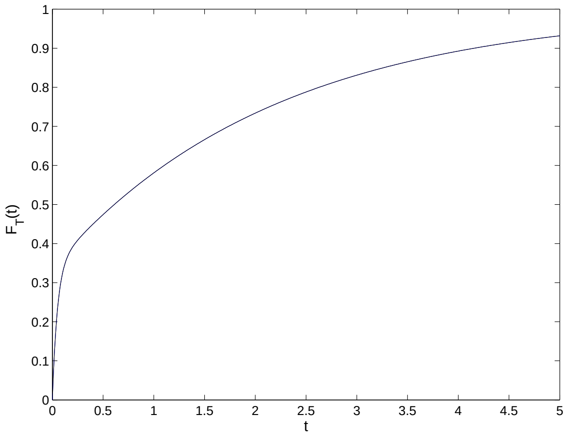

Figure 2: CDF of T, time until next arrival in the stationary version, in the example 5.1. As $\boldsymbol{\alpha} = \boldsymbol{\phi}$, then $T \stackrel{d}{=} T_1$ (similarly, $\widetilde{T} \stackrel{d}{=} \widetilde{T}_1$), and thus $T \stackrel{d}{=} T_1 \stackrel{d}{=} \widetilde{T} \stackrel{d}{=} \widetilde{T}_1$.

Then, the MAP$_2$s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ are equivalent if and only if

1. $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s){=}\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, *$\forall s$ and*

2. $(\alpha, \widetilde{\alpha})$ *verifies equation (5.2):*

$$\beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha} + \beta = 0.$$

Proof: We assume the system (5.6)-(5.8) has only one solution when the initial probability is $\boldsymbol{\phi}$. This implies that the values $\{\widetilde{\boldsymbol{\phi}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ solve $(a_1)_\phi = (\widetilde{a}_1)_{\widetilde{\phi}}$, and the values $\{\widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ solve $d_1 = \widetilde{d}_1$ and $d_0 = \widetilde{d}_0$.

1. <u>*1* and *2*$\rightarrow$ Equivalence</u>.

   Let us first assume that both *1* and *2* hold. We want to prove equivalence given by (4.6).

   We use the fact that (5.6)-(5.8) has only one solution when the initial probability is $\boldsymbol{\phi}$. By *1*, the values $\{\widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ make $d_1 = \widetilde{d}_1$ and $d_0 = \widetilde{d}_0$, which do not depend on initial probabilities. So, if $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$ are such that they satisfy $\beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha} + \beta = 0$ or (equivalently, $(a_1)_\alpha = (\widetilde{a}_1)_{\widetilde{\alpha}}$), then $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ will solve the (new) system (5.6)-(5.8) (now depending on α's and not on ϕ's), and thus,

$$\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s){=}\widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s), \qquad \forall s.$$

   From Lemma (5.3), since $P^\star = \boldsymbol{\Phi}$ and $\widetilde{P}^\star = \widetilde{\boldsymbol{\Phi}}$ then, $\boldsymbol{\alpha}P^\star = \boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\alpha}}\widetilde{P}^\star = \widetilde{\boldsymbol{\phi}}$, and then all equivalence conditions (4.6) hold from *1*.

2. Equivalence $\rightarrow$ *1* and *2*.

   Given equivalence of two given MAP_2s, Result 2 implies *1*. Because of equivalence, $(a_1)_\alpha = (\widetilde{a}_1)_{\widetilde{\alpha}}$, and thus,

$$\beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha} + \beta = 0. \quad \square$$

Example 5.2.

Consider the MAP_2 be defined by

$$\{\lambda_1, \lambda_2, p_{120}, p_{210}, p_{111}, p_{121}, p_{211}, p_{221}\} = \{0.5, 20, 0.3, 0.3, 0.3528, 0.3472, 0.3528, 0.3472\}$$

and stationarity probability $\phi = 0.504$. In this case,

$$P^\star = \begin{pmatrix} 0.504 & 0.496 \\ 0.504 & 0.496 \end{pmatrix} = \mathbf{\Phi}.$$

Consider another MAP_2 with parameters

$$\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}, \widetilde{p}_{111}, \widetilde{p}_{121}, \widetilde{p}_{211}, \widetilde{p}_{221}\} = \{0.8, 19.7, 0.7683, 0.55, 0.0466, 0.1851, 0.0905, 0.3595\}$$

and stationarity probability $\phi = 0.201$. Then,

$$\widetilde{P}^\star = \begin{pmatrix} 0.201 & 0.799 \\ 0.201 & 0.799 \end{pmatrix} = \widetilde{\mathbf{\Phi}}.$$

In this case, $\beta_1 = 13.65 \neq 0$ and $\widetilde{\beta}_1 = -8.6796 \neq 0$, $\beta = -5.135$ and $\boldsymbol{\phi}\varphi_H(s) = \widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{H}}(s)$, for all s, that is $\widetilde{\phi}$, $\widetilde{\boldsymbol{\lambda}}$ and $\widetilde{P}_0$ solves the system (5.6)-(5.8) given the values of ϕ, $\boldsymbol{\lambda}$ and P_0. We are thus, in the assumptions of Proposition (5.11) which assures that given α and $\widetilde{\alpha}$ such that (5.2):

$$\beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha} + \beta = 0,$$

then, the MAP_2s with these initial probabilities are equivalent. Several pairs of values $(\alpha, \widetilde{\alpha})$ were found to verify the former equation, shown in Table 1.

Figure 3 depicts the distribution function of T_1, time to the first arrival, when $(\alpha, \widetilde{\alpha}) = (0.4398, 0.1)$ (solid line) and $(\alpha, \widetilde{\alpha}) = (0.8849, 0.8)$ (dotted line). The distribution of T_1 is not the same that of T, time between two consecutive arrivals in the stationary version. That is because, the initial probabilities are

α	0.4398	0.5034	0.5670	0.6305	0.6941	0.7577	0.8213	0.8849	0.9485
$\widetilde{\alpha}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

Table 1: Different initial MAP_2 probabilities verifying (5.2)

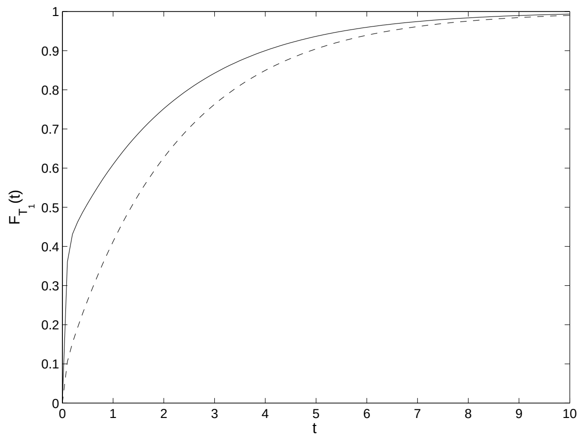

Figure 3: CDF of T_1 in the example 5.2. The solid line is for $(\alpha, \widetilde{\alpha}) = (0.4398, 0.1)$ and the dotted line is for $(\alpha, \widetilde{\alpha}) = (0.8849, 0.8)$. Here, as $\boldsymbol{\alpha} \neq \boldsymbol{\phi}$, then $F_{T_1}(t) \neq F_T(t)$ (similarly, $F_{\widetilde{T}_1}(t) \neq F_{\widetilde{T}}(t)$) .

different from the stationary ones. However, all the MAP_2s with the initial probabilities given by Table 1, possess the same distribution for T, which was actually the same that was depicted in Figure 2.$\Diamond$

Next result applies in the case where the system (5.6)-(5.8) has got infinite solutions when the initial probability is $\boldsymbol{\phi}$. In addition, we also need to impose that the system (5.6)-(5.8) has got infinite solutions when the initial probability is $\boldsymbol{\alpha}$. However, it is valid whatever the form of $P^\star$ and $\widetilde{P}^\star$.

We would like to remark that the fact that initial probabilities are involved in equation (5.16) will prevent $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$ from taking values different from the stationary probabilities, as will be seen in the proof.

Proposition 5.12. *Let* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *define two* MAP_2*s, with corresponding E-*MAP_2*s* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ *and* $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$, *where* $\boldsymbol{\phi}$ *and* $\widetilde{\boldsymbol{\phi}}$ *are the stationary probabilities associated to* $P^\star$ *and* $\widetilde{P}^\star$. *Assume,*

(i) the set $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$ *makes the system (5.6)-(5.8) to have infinite solutions,*

(ii) the set $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0\}$ *makes the system (5.6)-(5.8) to have infinite solutions,*

(iii) $\beta_1 \neq 0$, *and* $\widetilde{\beta}_1 \neq 0$.

Then, the MAP_2s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ are equivalent if and only if

1. $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$ *and*
2. $(\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}}) = (\boldsymbol{\phi}, \widetilde{\boldsymbol{\phi}})$.

Proof:

We assume the system (5.6)-(5.8) has infinite solutions when the initial probability is $\boldsymbol{\phi}$. This implies $\{\widetilde{\boldsymbol{\phi}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ solves $(a_1)_\phi = (\widetilde{a}_1)_{\widetilde{\phi}}$ and $\widetilde{d}_0 = a_1\widetilde{d}_1 + d_0 - \widetilde{a}_1 d_1$, for $\widetilde{d}_1 \in \mathbb{R}^-$.

1. *1* and *2* $\rightarrow$ Equivalence.

   The proof is the same that it was for Theorem (5.10).

2. Equivalence $\rightarrow$ *1* and *2*.

   Given equivalence of two given MAP_2s, Result 2 implies *1*.

   The values $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$ and $\{\widetilde{\boldsymbol{\phi}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ verify equations (5.15) and (5.16),

$$(a_1)_\phi = (\widetilde{a}_1)_{\widetilde{\phi}} \quad \Leftrightarrow \quad \beta_1\phi + \widetilde{\beta}_1\widetilde{\phi} + \beta = 0, \tag{5.21}$$

$$\widetilde{d}_0 = (a_1)_\phi(\widetilde{d}_1 - d_1) + d_0. \tag{5.22}$$

   By assumption, the values $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0\}$ and $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ also verify equations (5.15) and (5.16), that is,

$$(a_1)_\alpha = (\widetilde{a}_1)_{\widetilde{\alpha}}, \quad \Leftrightarrow \quad \beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha} + \beta = 0, \tag{5.23}$$

$$\widetilde{d}_0 = (a_1)_\alpha(\widetilde{d}_1 - d_1) + d_0, \tag{5.24}$$

   where, $d_0, d_1, \widetilde{d}_0$ and $\widetilde{d}_0$ are the same in both equations (5.22) and (5.24), because they just depend on $\boldsymbol{\lambda}, \widetilde{\boldsymbol{\lambda}}, P_0$ and $\widetilde{P}_0$. Since (5.22) and (5.24) are linear in ϕ (and α), we solve for α in (5.24) and get $\alpha = \phi$. Then, in (5.23) we solve for $\widetilde{\alpha}$ and because of (5.21) we obtain $\widetilde{\alpha} = \widetilde{\phi}$.

□

Example 5.3.

Consider all the MAP_2s where

$$\{\lambda_1, \lambda_2, p_{120}, p_{210}\} = \{0.5, 20, 0.3, 0.3\}$$

and initial probability equal to the stationary one α=ϕ = 0.9924. It can be seen that these values makes the system (5.6)-(5.8) to have infinite solutions.

Consider now all the MAP_2s with parameters

$$\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\} = \{5, 10, 0.9644, 0.9\}$$

and initial probability $\widetilde{\alpha}$=$\widetilde{\phi} = 0.201$. It can be also seen that these values solve the equations (5.6)-(5.8) given the previous $\{\lambda_1, \lambda_2, p_{120}, p_{210}\}$. Thus, we can apply Proposition (5.12) to deduce that they are equivalent. Figure 4 shows the distribution function of T, which is the same for both sets of MAP_2s. $\diamondsuit$

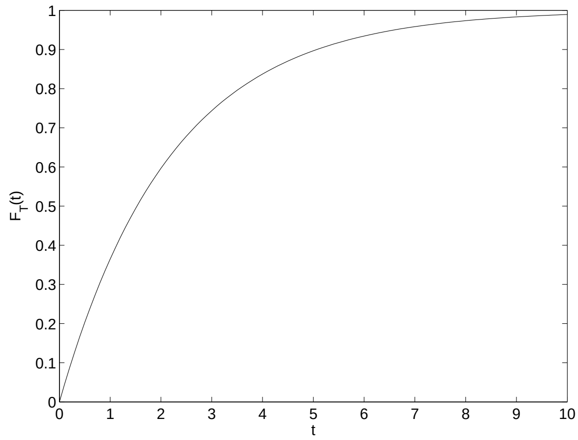

Figure 4: CDF of T, time until next arrival in the stationary version, in the example 5.3. As $\boldsymbol{\alpha} = \boldsymbol{\phi}$, then $T \stackrel{d}{=} T_1$ (similarly, $\widetilde{T} \stackrel{d}{=} \widetilde{T}_1$), and thus $T \stackrel{d}{=} T_1 \stackrel{d}{=} \widetilde{T} \stackrel{d}{=} \widetilde{T}_1$.

Next results apply in the specific cases where any of or both parameters β_1 and $\widetilde{\beta}_1$ defined in (5.10) are zero.

Lemma 5.13. *Let us assume that a MAP_2, defined by $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$ verifies $\beta_1 = 0$ or equivalently,*

$$\lambda_1(p_{120} - 1) + \lambda_2(1 - p_{210}) = 0.$$

Then,

1. *The system (5.6)-(5.8) defined by $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$ has got infinite solutions, and*

2. $\mathbf{x}\varphi_H(s) = \widetilde{\mathbf{x}}\varphi_H(s)$, *for all s, and for all probability vectors $\mathbf{x}$ and $\widetilde{\mathbf{x}}$.*

Proof:

Let us first prove implication *1.* Under the assumption $\beta_1 = 0$ it can be seen that

$$a_1 = \lambda_2(p_{210} - 1).$$

We need to prove $d_0 - a_1(d_1 - a_1) = 0$, or substituting d_0, a_1 and d_1 by their values,

$$\lambda_1\lambda_2(1 - p_{120}p_{210}) + \lambda_2(p_{210} - 1)[\lambda_2(p_{210} - 1) + \lambda_1 + \lambda_2] = 0, \quad \Longleftrightarrow$$

$$\lambda_1\lambda_2 p_{210}(1 - p_{120}) + \lambda_2^2 p_{210}(p_{210} - 1) = 0.$$

As $\lambda_1(p_{120} - 1) = \lambda_2(p_{210} - 1)$, we get

$$\lambda_1\lambda_2 p_{210}(1 - p_{120}) + \lambda_2 p_{210}(p_{120} - 1) = 0,$$

which proves part 1.

Part 2 is deduced from the fact that if $\beta_1 = 0$, then a_1 does not depend on the initial probability, $\mathbf{x}$. $\square$

Proposition 5.14. *Let* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$ *and* $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *determine two* MAP_2s*, where* $\beta_1 = \widetilde{\beta}_1 = 0$*, that is*

$$\lambda_1(1 - p_{120}) = \lambda_2(1 - p_{210}), \quad \textit{and} \quad \widetilde{\lambda}_1(1 - \widetilde{p}_{120}) = \widetilde{\lambda}_2(1 - \widetilde{p}_{210}).$$

Then, they are equivalent if and only if

$$\mathbf{x}\varphi_{\mathbf{H}}(s) = \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s,$$

for any initial probability vectors $\mathbf{x}$, $\widetilde{\mathbf{x}}$.

Proof:

The basic argument in this proof will be that if $\beta_1 = 0$, then $a_1 = \lambda_2(p_{210} - 1)$ will not depend on initial probabilities (similarly for $\widetilde{a}_1$).

If the given *MAP*s are equivalent, then $\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$. As $\beta_1 = \widetilde{\beta}_1 = 0$, by part 2 in Lemma (5.13), $\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s) = \mathbf{x}\varphi_{\mathbf{H}}(s)$, and $\widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s) = \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s)$, whatever the initial probabilities $\mathbf{x}$, and $\widetilde{\mathbf{x}}$ are. Thus,

$$\mathbf{x}\varphi_{\mathbf{H}}(s) = \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s),$$

for any initial probability vectors $\mathbf{x}$, $\widetilde{\mathbf{x}}$.

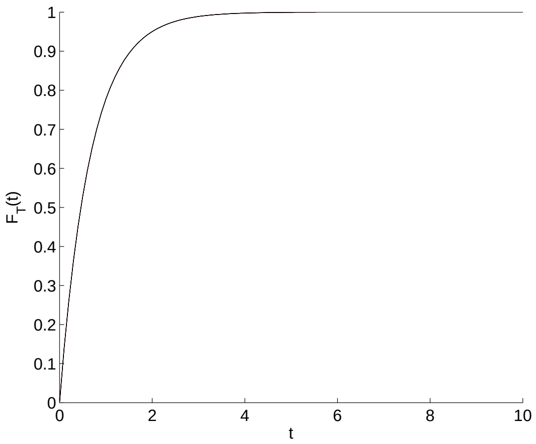

Figure 5: CDF of T in the example 5.4. By Lemma (5.13), $T \stackrel{d}{=} T_1$ (similarly, $\widetilde{T} \stackrel{d}{=} \widetilde{T}_1$), and thus $T \stackrel{d}{=} T_1 \stackrel{d}{=} \widetilde{T} \stackrel{d}{=} \widetilde{T}_1$.

On the other hand, if $\mathbf{x}\varphi_{\mathbf{H}}(s)$=$\widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s), \forall s$, then the equation system (5.6)-(5.8) is satisfied. Since in this case, it does not depend on initial probabilities, the system will also be solved if any other initial probability vectors $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$ (and $\boldsymbol{\alpha}^{(n)}$, $\widetilde{\boldsymbol{\alpha}}^{(n)}$, $\forall n$) are used. Thus, equivalence conditions (4.6) hold $\Box$.

Example 5.4.
Let two MAP_2s have parameters $\{\lambda_1, \lambda_2, p_{120}, p_{210}\} = \{2, 10, 0.25, 0.85\}$, and $\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\}$=$\{2.5, 6, 0.4, 0.75\}$. These values imply $\beta_1 = \widetilde{\beta}_1 = 0$ and solve the equations (5.6)-(5.8), for all initial probabilities. Then, the two MAP_2s are equivalents. Figure 5 depicts the distribution function of the time between two arrivals in the former MAP_2s. Because of Lemma (5.13), it also represents the distribution function of the time until the first arrival, whatever the initial probability is. $\Diamond$

Proposition 5.15. *Let* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *define two* MAP_2*s, with corresponding E-*MAP_2*s* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ *and* $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$*, where* $\boldsymbol{\phi}$ *and* $\widetilde{\boldsymbol{\phi}}$ *are the stationary probabilities associated to* $P^\star$ *and* $\widetilde{P}^\star$*. Assume that* $\widetilde{\beta}_1 = 0$*. Then, the* MAP_2*s* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *are equivalent if and only if*

1. $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)$=$\widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\quad \forall s$*, whatever the initial probability vector* $\widetilde{\mathbf{x}}$ *is, and*
2. $\boldsymbol{\alpha} = \boldsymbol{\phi}$.

Proof:

Firstly, given equivalence of two given MAP_2s, Result 2 implies $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$. Since $\widetilde{\beta}_1 = 0$, we know from Lemma (5.13) that the value $\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$ does not depend on $\widetilde{\boldsymbol{\phi}}$, and thus $\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s) = \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s)$, for any initial probability vector $\widetilde{\mathbf{x}}$. This proves part *1.*

In addition, because $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, and $\widetilde{\beta}_1 = 0$, equation (5.2) implies

$$\beta_1\phi = -\beta.$$

We will prove now that $\boldsymbol{\alpha} = \boldsymbol{\phi}$. The processes are assumed to be equivalent and thus $\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$, or equivalently, $(\alpha, \widetilde{\alpha})$ also satisfy (5.2),

$$\beta_1\alpha = -\beta.$$

Last two equations imply $\boldsymbol{\alpha} = \boldsymbol{\phi}$ and part *2* is proven.

On the other hand, let us prove that $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s)=\widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$, $\forall \widetilde{\mathbf{x}}$, and $\boldsymbol{\alpha} = \boldsymbol{\phi}$ implies equivalence. Condition (4.6) holds for $n = 0$ because $\boldsymbol{\alpha} = \boldsymbol{\phi}$ and by Lemma (5.13), $\widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s) = \widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s)$, for all $\widetilde{\mathbf{x}}$ and $\widetilde{\boldsymbol{\alpha}}$. The following conditions for $n > 0$ also hold applying the same reasoning and $\boldsymbol{\alpha} = \boldsymbol{\phi} = \boldsymbol{\alpha}^{(n)}$, $\forall n$. Thus, the processes are equivalent.

□

Equivalent to Proposition (5.15) we find,

Proposition 5.16. *Let* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *define two* MAP_2*s, with corresponding E-*MAP_2*s* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ *and* $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$*, where* $\boldsymbol{\phi}$ *and* $\widetilde{\boldsymbol{\phi}}$ *are the stationary probabilities associated to* $P^\star$ *and* $\widetilde{P}^\star$*. Assume that* $\beta_1 = 0$*. Then, the* MAP_2*s* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *are equivalent if and only if*

1. $\mathbf{x}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\quad\forall s$*, whatever the initial probability vector* $\mathbf{x}$ *is, and*

2. $\widetilde{\boldsymbol{\alpha}} = \widetilde{\boldsymbol{\phi}}$*.*

Example 5.5.

Let a set of MAP_2s have parameters $\{\lambda_1, \lambda_2, p_{120}, p_{210}\} = \{2, 4, 0.3125, 0.5\}$ and initial probability equal to the stationary one, $\alpha = \phi = 0.8$.

On the other hand, let another set of MAP_2s has $\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\}=\{2.5, 6, 0.4, 0.75\}$, which implies $\widetilde{\beta}_1 = 0$. Then, all MAP_2s with parameters $\{\lambda_1, \lambda_2, p_{120}, p_{210}\} = \{2, 4, 0.3125, 0.5\}$ and $\alpha = \phi = 0.8$ will be equivalent to all MAP_2s with parameters $\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{p}_{120}, \widetilde{p}_{210}\}=\{2.5, 6, 0.4, 0.75\}$.

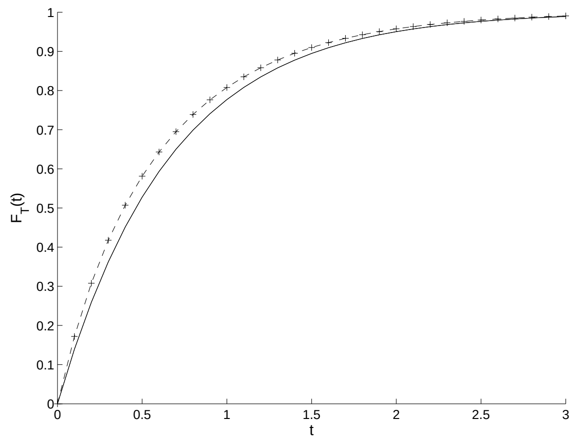

Figure 6: In solid line, CDF of T in the example 5.5; as $\boldsymbol{\alpha} = \boldsymbol{\phi}$, then $T \stackrel{d}{=} T_1$. In dotted line, the CDF of T_1 when $\alpha = 0.1$. Since $\alpha \neq \phi$, $T_1 \stackrel{d}{\neq} T$.

Figure 6 depicts in solid line the distribution function of the time between two arrivals (or again, the time until the first arrival) in the previous MAP_2s. The dotted line represents the distribution function of the time until the first arrival in the first MAP_2 when $\alpha \neq 0.8$, in this case $\alpha = 0.1$. $\Diamond$

As a summary, we illustrate in Table 2, the possible situations that can be found when comparing two MAP_2s, and the value of the initial probabilities $(\alpha, \widetilde{\alpha})$ such that, if $\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$ then, both MAP_2s are equivalent.

5.4 Extension to $\boldsymbol{MAP_3}$

In this section, following the undertaken methodology for the MAP_2 we will derive results equivalent to Theorem (5.10) and Proposition (5.11) for the MAP_3 case. The parameters that define a MAP_3 are the vector of initial probabilities, given by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 1 - \alpha_1 - \alpha_2)$, the exponential rates, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and matrices P_0 and P_1,

$$P_0 = \begin{pmatrix} 0 & p_{12} & p_{13} \\ p_{21} & 0 & p_{23} \\ p_{31} & p_{32} & 0 \end{pmatrix}, \quad P_1 \begin{pmatrix} p_{111} & p_{121} & p_{131} \\ p_{211} & p_{221} & p_{231} \\ p_{311} & p_{321} & p_{331} \end{pmatrix}.$$

$P^\star$	$\widetilde{P}^\star$	β_1	$\widetilde{\beta}_1$	Solution to $(5.6)_\phi$,(5.7),(5.8)	Result	$(\alpha, \widetilde{\alpha})$
$\neq \mathbf{\Phi}$	-	$\neq 0$	$\neq 0$	-	**Theorem (5.10)**	$(\phi, \widetilde{\phi})$
-	$\neq \widetilde{\mathbf{\Phi}}$	$\neq 0$	$\neq 0$	-	**Theorem (5.10)**	$(\phi, \widetilde{\phi})$
$\mathbf{\Phi}$	$\widetilde{\mathbf{\Phi}}$	$\neq 0$	$\neq 0$	unique	**Prop. (5.11)**	$\beta_1\alpha + \widetilde{\beta}_1\widetilde{\alpha} + \beta = 0$ (*)
-	-	$\neq 0$	$\neq 0$	infinite	**Prop. (5.12)**	$(\phi, \widetilde{\phi})$
-	-	0	0	-	**Prop. (5.14)**	$(\cdot, \cdot)$
-	-	$\neq 0$	0	-	**Prop. (5.15)**	$(\phi, \cdot)$
-	-	0	$\neq 0$	-	**Prop. (5.16)**	$(\cdot, \widetilde{\phi})$

Table 2: Possible situations when comparing two MAP_2s in terms of the probability matrices $P^\star$, $\widetilde{P}^\star$, parameters β_1, $\widetilde{\beta}_1$ and type of system (5.6)-(5.8) characterized by $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$. The corresponding Result and the values of initial probabilities, such that if $\boldsymbol{\alpha}\varphi_{\mathbf{H}}(s)=\widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$, then equivalence is obtained, are also given. In the case (*), the condition (5.17) is also necessary for equivalence.

The corresponding *E-MAP*$_3$ process will be characterized by $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ where,

$$P^\star = \begin{pmatrix} p^\star_{11} & p^\star_{12} & p^\star_{13} \\ p^\star_{21} & p^\star_{22} & p^\star_{23} \\ p^\star_{31} & p^\star_{32} & p^\star_{33} \end{pmatrix},$$

with associated stationary probability vector $\boldsymbol{\phi}$.

5.4.1 First condition for equivalence

In this section we will find values for $\{\widetilde{\mathbf{x}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0\}$ such that

$$\mathbf{x}\varphi_{\mathbf{H}}(s)=\widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \tag{5.25}$$

for fixed values $\{\mathbf{x}, \boldsymbol{\lambda}, P_0\}$.

In the case of 3-states, condition (5.25) can be alternatively expressed as

$$x_1\varphi_{H_1}(s) + x_2\varphi_{H_2}(s) + (1 - x_1 - x_2)\varphi_{H_3}(s)=\widetilde{x}_1\varphi_{\widetilde{H}_1}(s) + \widetilde{x}_2\varphi_{\widetilde{H}_2}(s) + (1 - \widetilde{x}_1 - \widetilde{x}_2)\varphi_{\widetilde{H}_3}(s), \quad \forall s$$

where

$$\begin{aligned}
\varphi_{H_1}(s) &= (1-p_{12}-p_{13})\varphi_{\mathcal{E}_1}(s) + p_{12}\varphi_{\mathcal{E}_1}(s)\varphi_{H_2}(s) + p_{13}\varphi_{\mathcal{E}_1}(s)\varphi_{H_3}(s), \\
\varphi_{H_2}(s) &= (1-p_{21}-p_{23})\varphi_{\mathcal{E}_2}(s) + p_{21}\varphi_{\mathcal{E}_2}(s)\varphi_{H_1}(s) + p_{23}\varphi_{\mathcal{E}_2}(s)\varphi_{H_3}(s), \\
\varphi_{H_3}(s) &= (1-p_{31}-p_{32})\varphi_{\mathcal{E}_3}(s) + p_{31}\varphi_{\mathcal{E}_3}(s)\varphi_{H_1}(s) + p_{32}\varphi_{\mathcal{E}_3}(s)\varphi_{H_2}(s)
\end{aligned} \tag{5.26}$$

Solving the equations system (5.26) it can be seen that

$$\mathbf{x}\varphi_{\mathbf{H}}(s) = \frac{a_2 s^2 + a_1 s + d_0}{d(s)}, \tag{5.27}$$

where

$$d(s) = -s^3 + d_2 s^2 + d_1 s + d_0,$$

and

$$\begin{aligned}
d_2 &= \lambda_1 + \lambda_2 + \lambda_3, \\
d_1 &= p_{21}\lambda_2 p_{12}\lambda_1 - \lambda_3\lambda_1 - \lambda_2\lambda_3 + p_{31}\lambda_3 p_{13}\lambda_1 + p_{23}p_{32}\lambda_2\lambda_3 - \lambda_1\lambda_2, \\
d_0 &= \lambda_1\lambda_2\lambda_3(1 - p_{31}p_{12}p_{23} - p_{31}p_{13} - p_{21}p_{12} - p_{21}p_{13}p_{32} - p_{23}p_{32}).
\end{aligned}$$

In addition,

$$\begin{aligned}
a_2 &= \lambda_1 x_1(1-p_{12}-p_{13}) + \lambda_2 x_2(1-p_{21}-p_{23}) + \lambda_3(1 + x_1p_{31} + x_1p_{32} + x_2p_{31} + x_2p_{32} - x_1 + x_2 - p_{31} - p_{32}), \\
a_1 &= \lambda_3\lambda_1 - \lambda_3x_2\lambda_1 - \lambda_3x_1\lambda_2 - \lambda_3\lambda_1p_{32} + \\
&+ \lambda_3x_1\lambda_2p_{31} + \lambda_3x_1\lambda_1p_{32} + \lambda_3x_1\lambda_1p_{31}p_{12} - \lambda_3\lambda_2p_{32}p_{21} - \lambda_3\lambda_1p_{31}p_{12} + \\
&+ \lambda_3x_2\lambda_1p_{31}p_{13} + x_1\lambda_1\lambda_2 + \lambda_3x\lambda_1p_{31}p_{12} + \\
&+ \lambda_3x_2\lambda_2p_{32}p_{21} + x_2\lambda_2\lambda_1 + \lambda_3x_1\lambda_2p_{32}p_{23} - p_{31}\lambda_3p_{13}\lambda_1 + \\
&+ \lambda_3x_1\lambda_2p_{32}p_{21} - \lambda_3\lambda_2p_{31} + \lambda_3x_2\lambda_2p_{31} + \\
&+ \lambda_3x_2\lambda_1p_{32} - x_2\lambda_2\lambda_3p_{23}p_{31} - x_1\lambda_1\lambda_2p_{12}p_{21} - \\
&- x_2\lambda_2\lambda_1p_{21}p_{13} - x_2\lambda_2\lambda_1p_{21}p_{12} - x_2\lambda_2\lambda_1p_{23} - \\
&- x_2\lambda_2\lambda_3p_{21} - x_1\lambda_1\lambda_2p_{13} - x_1\lambda_1\lambda_3p_{12} - x_1\lambda_1\lambda_3p_{13}p_{32} - \\
&- x_1\lambda_1\lambda_2p_{12}p_{23} - p_{23}p_{32}\lambda_2\lambda_3 + \lambda_2\lambda_3.
\end{aligned}$$

According to (5.27),

$$\begin{aligned} \mathbf{x}\varphi_{\mathbf{H}}(s) &= \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s \quad \text{if and only if,} \\ \frac{a_2 s^2 + a_1 s + d_0}{-s^3 + d_2 s^2 + d_1 s + d_0} &= \frac{\widetilde{a}_2 s^2 + \widetilde{a}_1 s + \widetilde{d}_0}{-s^3 + \widetilde{d}_2 s^2 + \widetilde{d}_1 s + \widetilde{d}_0}, \quad \forall s \quad \text{if and only if,} \\ c_4 s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0 &= 0, \quad \forall s \end{aligned} \tag{5.28}$$

where

$$\begin{aligned} c_4 &= a_2 - \widetilde{a}_2 \\ c_3 &= a_1 - a_2\widetilde{d}_2 - \widetilde{a}_1 + \widetilde{a}_2 d_2, \\ c_2 &= d_0 - a_1\widetilde{d}_2 - a_2\widetilde{d}_1 - \widetilde{d}_0 + \widetilde{a}_1 d_2 + \widetilde{a}_2 d_1 \\ c_1 &= \widetilde{d}_0 d_2 + \widetilde{a}_1 d_1 + \widetilde{a}_2 d_0 - d_0\widetilde{d}_2 - a_1\widetilde{d}_1 - a_2\widetilde{d}_0 \\ c_0 &= \widetilde{d}_0 d_1 + \widetilde{a}_1 d_0 - d_0\widetilde{d}_1 - a_1\widetilde{d}_0 \end{aligned}$$

Equation (5.28) holds for all s if and only if $c_4 = c_3 = c_2 = c_1 = c_0 \equiv 0$, or,

$$\begin{aligned} a_2 &= \widetilde{a}_2 \\ a_1 - a_2\widetilde{d}_2 &= \widetilde{a}_1 - \widetilde{a}_2 d_2 \\ d_0 - a_1\widetilde{d}_2 - a_2\widetilde{d}_1 &= \widetilde{d}_0 - \widetilde{a}_1 d_2 - \widetilde{a}_2 d_1 \\ \widetilde{d}_0 d_2 + \widetilde{a}_1 d_1 + \widetilde{a}_2 d_0 &= d_0\widetilde{d}_2 + a_1\widetilde{d}_1 + a_2\widetilde{d}_0 \\ \widetilde{d}_0 d_1 + \widetilde{a}_1 d_0 &= d_0\widetilde{d}_1 + a_1\widetilde{d}_0 \end{aligned}$$

As it happened in the case where $m = 2$, we obtain a linear equations system with the same number of equations and unknowns $(\widetilde{a}_1, \widetilde{a}_2, \widetilde{d}_0, \widetilde{d}_1, \widetilde{d}_2)$, where the coefficient matrix is

$$C_3 = \begin{pmatrix} a_1 - d_1 & d_0 & 0 & -d_0 & 0 \\ a_2 - d_2 & a_1 & d_0 & -d_1 & -d_0 \\ 1 & a_2 & a_1 & -d_2 & -d_1 \\ 0 & 0 & a_2 & 1 & -d_2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.29}$$

If the rank of C_3 is 5, then the solution is unique:

$$\widetilde{a}_2 = a_2 \tag{5.30}$$

$$\widetilde{a}_1 = a_1 \tag{5.31}$$

$$\widetilde{d}_0 = d_0 \tag{5.32}$$

$$\widetilde{d}_1 = d_1 \tag{5.33}$$

$$\widetilde{d}_2 = d_2 \tag{5.34}$$

Notice that only equations (5.30) and (5.31) involve the initial probabilities, $\mathbf{x}, \widetilde{\mathbf{x}}$. They can be equivalently expressed as,

$$\beta + \beta_1 x_1 + \beta_2 x_2 - \widetilde{\beta} - \widetilde{\beta}_1 \widetilde{x}_1 - \widetilde{\beta}_2 \widetilde{x}_2 = 0, \tag{5.35}$$

$$\gamma + \gamma_1 x_1 + \gamma_2 x_2 - \gamma - \widetilde{\gamma}_1 \widetilde{x}_1 - \widetilde{\gamma}_1 \widetilde{x}_1 = 0, \tag{5.36}$$

where

$$\begin{aligned} a_2 &= \beta + \beta_1 \phi_1 + \beta_2 \phi_2, \\ a_1 &= \gamma + \gamma_1 \phi_1 + \gamma_2 \phi_2 \end{aligned}$$

(respectively, $\widetilde{a}_2, \widetilde{a}_1$) and

$$\begin{aligned} \beta &= \lambda_3(1 - p_{31} - p_{32}), \\ \beta_1 &= \lambda_1(1 - p_{13} - p_{12}) + \lambda_3(p_{31} + p_{32} - 1), \\ \beta_2 &= \lambda_2(1 - p_{21} - p_{23}) + \lambda_3(p_{31} + p_{32} - 1), \\ \gamma &= \lambda_1\lambda_3(p_{32} + p_{12}p_{31} + p_{31}p_{13} - 1) + \lambda_2\lambda_3(p_{31} + p_{21}p_{32} + p_{23}p_{32} - 1), \\ \gamma_1 &= \lambda_1\lambda_2(p_{12}p_{23} + p_{13} + p_{12}p_{21} - 1) + \\ &\quad \lambda_2\lambda_3(1 - p_{23}p_{32} - p_{21}p_{32} - p_{31}) + \\ &\quad \lambda_1\lambda_3(p_{12} + p_{13}p_{32} - p_{31}p_{12} - p_{32}), \\ \gamma_2 &= \lambda_1\lambda_2(p_{12}p_{21} + p_{23} + p_{13}p_{21} - 1) + \\ &\quad \lambda_2\lambda_3(p_{21} - p_{32}p_{21} + p_{23}p_{31} - p_{31}) + \\ &\quad \lambda_1\lambda_3(1 - p_{32} - p_{31}p_{12} - p_{31}p_{13}), \end{aligned}$$

(respectively $\widetilde{\beta}$, $\widetilde{\beta}_1$,$\widetilde{\beta}_2$, $\widetilde{\gamma}$, $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$).

5.4.2 Main results

In order to introduce the general results for the three states case, $m = 3$, we need to define matrix D_3, which plays the role of matrix D_2, in the proof of Theorem (5.10),

$$D_3 = \begin{pmatrix} \beta_1 & \beta_2 & \widetilde{\beta}_1 & \widetilde{\beta}_2 \\ \beta_1' & \beta_2' & \widetilde{\beta}_1' & \widetilde{\beta}_2' \\ \gamma_1 & \gamma_2 & \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \\ \gamma_1' & \gamma_2' & \widetilde{\gamma}_1' & \widetilde{\gamma}_2' \end{pmatrix} \tag{5.37}$$

where

$$\begin{aligned} \beta_1' &= \beta_1(p^\star_{11} - p^\star_{31}) + \beta_2(p^\star_{12} - p^\star_{32}), \quad (5.38) \\ \beta_2' &= \beta_1(p^\star_{21} - p^\star_{31}) + \beta_2(p^\star_{22} - p^\star_{32}), \\ \gamma_1' &= \gamma_1(p^\star_{11} - p^\star_{31}) + \gamma_2(p^\star_{12} - p^\star_{32}), \\ \gamma_2' &= \gamma_1(p^\star_{21} - p^\star_{31}) + \gamma_2(p^\star_{22} - p^\star_{32}), \end{aligned}$$

respectively, $\widetilde{\beta}_1'$, $\widetilde{\beta}_2'$, $\widetilde{\gamma}_1'$ and $\widetilde{\gamma}_2'$.

Theorem 5.17 (General Result for $m = 3$). *Let $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ define two MAP$_3$s, with corresponding E-MAP$_3$s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ and $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$, where $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$ are the stationary probabilities associated to $P^\star$ and $\widetilde{P}^\star$. Assume,*

(i) $\beta_1 \neq 0$, $\beta_2 \neq 0$, $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, $\widetilde{\beta}_1 \neq 0$, $\widetilde{\beta}_2 \neq 0$, $\widetilde{\gamma}_1 \neq 0$, $\widetilde{\gamma}_1 \neq 0$,

(ii) the rank of D_3 is 4,

Then, the MAP$_2$s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ are equivalent if and only if

1. $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$ *and*

2. $(\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}}) = (\boldsymbol{\phi}, \widetilde{\boldsymbol{\phi}})$.

Condition *(i)* implies that the initial probabilities appear in equations (5.30) and (5.31)) and is equivalent to $\beta_1 \neq 0$ and $\widetilde{\beta}_1 \neq 0$ for the 2-states case. We have to point out that conditions *(i)* and *(ii)* in Theorem

(5.10) implied $rank(D_2) = 2$. However, it not so straightforward to prove the same for the 3-states case, and thus we have substituted condition $P^\star \neq \mathbf{\Phi}$ or $\widetilde{P}^\star \neq \widetilde{\mathbf{\Phi}}$ by $rank(D_3) = 4$.

Proof:

1. <u>*1* and *2*$\rightarrow$ Equivalence.</u>

   The proof is the same that it was for Theorem (5.10).

2. <u>Equivalence $\rightarrow$ *1* and *2*.</u>

   Because of equivalence, $\boldsymbol{\alpha}\varphi_H(s) = \widetilde{\boldsymbol{\alpha}}\varphi_{\widetilde{H}}(s)$, for all s, needs to hold, which implies, that both $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$ satisfy (5.35) and (5.36)

$$\begin{aligned} \beta + \beta_1\alpha_1 + \beta_2\alpha_2 - \widetilde{\beta} - \widetilde{\beta}_1\widetilde{\alpha}_1 - \widetilde{\beta}_2\widetilde{\alpha}_2 &= 0, \\ \gamma + \gamma_1\alpha_1 + \gamma_2\alpha_2 - \gamma - \widetilde{\gamma}_1\widetilde{\alpha}_1 - \widetilde{\gamma}_1\widetilde{\alpha}_1 &= 0. \end{aligned} \tag{5.39}$$

   Moreover because of *1*, $\boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\phi}}$ also satisfy (5.35) and (5.36), and thus,

$$\begin{aligned} \beta_1\phi_1 + \beta_2\phi_2 + \widetilde{\beta}_1\widetilde{\phi}_1 + \widetilde{\beta}_2\widetilde{\phi}_2 &= \beta_1\alpha_1 + \beta_2\alpha_2 + \widetilde{\beta}_1\widetilde{\alpha}_1 + \widetilde{\beta}_2\widetilde{\alpha}_2 \\ \gamma_1\phi_1 + \gamma_2\phi_2 + \widetilde{\gamma}_1\widetilde{\phi}_1 + \widetilde{\gamma}_2\widetilde{\phi}_2 &= \gamma_1\alpha_1 + \gamma_2\alpha_2 + \widetilde{\gamma}_1\widetilde{\alpha}_1 + \widetilde{\gamma}_2\widetilde{\alpha}_2 \end{aligned}$$

   Since $\boldsymbol{\alpha}^{(1)}$, $\widetilde{\boldsymbol{\alpha}}^{(1)}$ given by

$$\boldsymbol{\alpha}^{(1)} = \boldsymbol{\alpha}P^\star, \qquad \widetilde{\boldsymbol{\alpha}}^{(1)} = \widetilde{\boldsymbol{\alpha}}\widetilde{P}^\star,$$

   must also verify $\boldsymbol{\alpha}^{(1)}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\alpha}}^{(1)}\varphi_{\widetilde{\mathbf{H}}}(s)$, for all s, then we get a 4 linear equations system where the unknowns are α_1, α_2, $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$,

$$\begin{aligned} \beta_1\alpha_1 + \beta_2\alpha_2 + \widetilde{\beta}_1\widetilde{\alpha}_1 + \widetilde{\beta}_2\widetilde{\alpha}_2 &= \beta_1\phi_1 + \beta_2\phi_2 + \widetilde{\beta}_1\widetilde{\phi}_1 + \widetilde{\beta}_2\widetilde{\phi}_2 \\ \beta_1\alpha_1^{(1)} + \beta_2\alpha_2^{(1)} + \widetilde{\beta}_1\widetilde{\alpha}_1^{(1)} + \widetilde{\beta}_2\widetilde{\alpha}_2^{(1)} &= \beta_1\phi_1 + \beta_2\phi_2 + \widetilde{\beta}_1\widetilde{\phi}_1 + \widetilde{\beta}_2\widetilde{\phi}_2 \\ \gamma_1\alpha_1 + \gamma_2\alpha_2 + \widetilde{\gamma}_1\widetilde{\alpha}_1 + \widetilde{\gamma}_2\widetilde{\alpha}_2 &= \gamma_1\phi_1 + \gamma_2\phi_2 + \widetilde{\gamma}_1\widetilde{\phi}_1 + \widetilde{\gamma}_2\widetilde{\phi}_2 \\ \gamma_1\alpha_1^{(1)} + \gamma_2\alpha_2^{(1)} + \widetilde{\gamma}_1\widetilde{\alpha}_1^{(1)} + \widetilde{\gamma}_2\widetilde{\alpha}_2^{(1)} &= \gamma_1\phi_1 + \gamma_2\phi_2 + \widetilde{\gamma}_1\widetilde{\phi}_1 + \widetilde{\gamma}_2\widetilde{\phi}_2 \end{aligned} \tag{5.40}$$

   The unknowns are actually α_1, α_2, $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ because, $\alpha_1^{(1)}$, and $\alpha_2^{(1)}$ can be expressed in terms of α_1 and α_2, (respectively $\widetilde{\alpha}_1^{(1)}$, and $\widetilde{\alpha}_2^{(1)}$),

$$\begin{aligned} \alpha_1^{(1)} &= \alpha_1(p^\star_{11} - p^\star_{31}) + \alpha_2(p^\star_{21} - p^\star_{31}) + p^\star_{31}, \\ \alpha_2^{(1)} &= \alpha_1(p^\star_{12} - p^\star_{32}) + \alpha_2(p^\star_{22} - p^\star_{32}) + p^\star_{32} \end{aligned}$$

It is clear that $\boldsymbol{\alpha} = \boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\alpha}} = \widetilde{\boldsymbol{\phi}}$ solves system (5.40), which can be alternatively written as,

$$\begin{aligned}
\beta_1\alpha_1 + \beta_2\alpha_2 + \widetilde{\beta}_1\widetilde{\alpha}_1 + \widetilde{\beta}_2\widetilde{\alpha}_2 &= \beta_1\phi_1 + \beta_2\phi_2 + \widetilde{\beta}_1\widetilde{\phi}_1 + \widetilde{\beta}_2\widetilde{\phi}_2 \qquad (5.41)\\
\beta_1'\alpha_{11} + \beta_2'\alpha_2 + \widetilde{\beta}_1'\widetilde{\alpha}_1 + \widetilde{\beta}_2'\widetilde{\alpha}_2 &= \beta_1(\phi_1 - p^\star_{31}) + \beta_2(\phi_2 - p^\star_{32}) + \widetilde{\beta}_1(\widetilde{\phi}_1 - \widetilde{p}^\star_{31}) + \widetilde{\beta}_2(\widetilde{\phi}_2 - \widetilde{p}^\star_{32})\\
\gamma_1\alpha_1 + \gamma_2\alpha_2 + \widetilde{\gamma}_1\widetilde{\alpha}_1 + \widetilde{\gamma}_2\widetilde{\alpha}_2 &= \gamma_1\phi_1 + \gamma_2\phi_2 + \widetilde{\gamma}_1\widetilde{\phi}_1 + \widetilde{\gamma}_2\widetilde{\phi}_2\\
\gamma_1'\alpha_1 + \gamma_2'\alpha_2 + \widetilde{\gamma}_1'\widetilde{\alpha}_1 + \widetilde{\gamma}_2'\widetilde{\alpha}_2 &= \gamma_1(\phi_1 - p^\star_{31}) + \gamma_2(\phi_2 - p^\star_{32}) + \widetilde{\gamma}_1(\widetilde{\phi}_1 - \widetilde{p}^\star_{31}) + \widetilde{\gamma}_2(\widetilde{\phi}_2 - \widetilde{p}^\star_{32}),
\end{aligned}$$

where β_1', β_2', γ_1' and γ_2' were defined in (5.38). The coefficient matrix associated to system (5.41) is D_3. By assumption, the rank is complete, thus the solution to (5.40) is unique, and $\boldsymbol{\alpha} = \boldsymbol{\phi}$ and $\widetilde{\boldsymbol{\alpha}} = \widetilde{\boldsymbol{\phi}}$.
□

Finally, we present the result equivalent to Proposition (5.11), whose proof follows the same lines that that of Proposition (5.11).

Proposition 5.18. *Let* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *define two MAP*$_3$*s, with corresponding E-MAP*$_3$*s* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ *and* $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$*, where* $\boldsymbol{\phi}$ *and* $\widetilde{\boldsymbol{\phi}}$ *are the stationary probabilities associated to* $P^\star$ *and* $\widetilde{P}^\star$*. Assume,*

(i) the set $\{\boldsymbol{\phi}, \boldsymbol{\lambda}, P_0\}$ *makes that* $rank(C_3) = 5$*,*

(ii) $\beta_1 \neq 0$, $\beta_2 \neq 0$, $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, $\widetilde{\beta}_1 \neq 0$, $\widetilde{\beta}_2 \neq 0$, $\widetilde{\gamma}_1 \neq 0$, $\widetilde{\gamma}_1 \neq 0$,

(iii) $P^\star = \boldsymbol{\Phi}$ *and* $\widetilde{P}^\star = \widetilde{\boldsymbol{\Phi}}$.

Then, the MAP$_3$*s* $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ *are equivalent if and only if*

1. $\boldsymbol{\phi}\varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\phi}}\varphi_{\widetilde{\mathbf{H}}}(s)$, $\forall s$ *and*

2. $(\alpha, \widetilde{\alpha})$ *verifies equation (5.39):*

$$\begin{aligned}
\beta + \beta_1\alpha_1 + \beta_2\alpha_2 - \widetilde{\beta} - \widetilde{\beta}_1\widetilde{\alpha}_1 - \widetilde{\beta}_2\widetilde{\alpha}_2 &= 0,\\
\gamma + \gamma_1\alpha_1 + \gamma_2\alpha_2 - \gamma - \widetilde{\gamma}_1\widetilde{\alpha}_1 - \widetilde{\gamma}_1\widetilde{\alpha}_1 &= 0.
\end{aligned}$$

5.4.3 General MAP_m

Here, we delineate some remarks concerning the general case the MAP with m states or MAP_m. For the general case with m states, the system to be solved for the first condition of equivalence, $\mathbf{x}\varphi_{\mathbf{H}}(s) = \widetilde{\mathbf{x}}\varphi_{\widetilde{\mathbf{H}}}(s)$, is

$$\frac{a_{m-1}s^{m-1} + \ldots + a_1 s + d_0}{(-1)^m s^m + d_{m-1}s^{m-1} + \ldots + d_1 s + d_0} = \frac{\widetilde{a}_{m-1}s^{m-1} + \ldots + \widetilde{a}_1 s + \widetilde{d}_0}{(-1)^m s^m + \widetilde{d}_{m-1}s^{m-1} + \ldots + \widetilde{d}_1 s + \widetilde{d}_0}$$

which becomes a linear system of $2m-1$ (because of the product of $s^{m-1} \times s^m$) equations and $2m-1$ unknowns $\widetilde{a}_{m-1}, \ldots, \widetilde{a}_1, \widetilde{d}_{m-1}, \ldots, \widetilde{d}_1, \widetilde{d}_0$. This will define a coefficient matrix C_m similar to (5.29) that, if has maximum rank, then the solution to the system is unique:

$$a_i = \widetilde{a}_i, \quad \text{for } i = 1, \ldots, m-1, \text{ and } d_j = \widetilde{d}_j, \quad \text{for } j = 0, \ldots, m-1,$$

where only the values of a_i and $\widetilde{a}_i$ will depend on initial probabilities. The $(m-1)$ equations $a_i = \widetilde{a}_i$ wi ll be expressed as $(m-1)$ hyperplanes equations, like (5.9) and (5.39), whose coefficients $(\beta_1, \ldots, \beta_{m-1})$, $(\gamma_1, \ldots, \gamma_{m-1})$, $(\delta_1, \ldots, \delta_{m-1})$, etc... will be likely different from 0. Actually, Theorem (5.17) can be generalized assuming as hypothesis that the set of these coefficients are different from zero, and the rank of D_m is maximum, where D_m is the matrix equivalent to (5.37) of order $(2m-2) \times (2m-2)$.

If $P^\star = \mathbf{\Phi}$ and $\widetilde{P}^\star = \widetilde{\mathbf{\Phi}}$, then in order to generalize Proposition (5.18), it will be necessary to assume that C_m has maximum rank and the set of coefficients $(\beta_1, \ldots, \beta_{m-1})$, $(\gamma_1, \ldots, \gamma_{m-1})$, $(\delta_1, \ldots, \delta_{m-1})$, etc... are different from zero.

6 Conclusions

In this work we have discussed when two Markovian Arrival processes or *MAP*s are equivalent, in the sense that they share the same probability function, for the variable *time between one arrival and the next one.* The *MAP* can be understood as a Hidden Markov process which very commonly present identifiability problems.

Since in most of applications, the *MAP* is not entirely observed, a new process, the *observable MAP* can be defined. It is named *Effective Arrival Process* and noted *E-MAP*. We have derived some probabilistic results concerning the *E-MAP*, such us, the transition probability matrix $P^\star$ and the moment generating function of the *holding times*, which is not exponentially distributed but distributed like a sum of exponential variables.

We have defined when a *MAP* process is identifiable in terms of the associated *E-MAP*, and present some results derived from this definition.

The main contribution of this paper is the undertaken depth study for the case of MAP_2s, the two-states *MAP*. We have derived expressions for the moment generating function of T_1, MAP_2s must solve in order to be equivalent. The MAP_2 is not identifiable, and depending on its parameters we show how to find other MAP_2s equivalent. We have shown the important role that the matrix P_0, concerning transitions with no arrivals, and the stationary probability $\boldsymbol{\phi}$ associated to the $E\text{-}MAP_2$, play in the identifiability problem, whereas matrix P_1 turns out to be less important. We have illustrated our results with numerical examples.

The general result for MAP_2s presented in the work, stating which conditions must be hold by two MAP_2 to be equivalent, has been extended for the 3-states *MAP* or MAP_3 and briefly commented for the general case MAP_3.

In our future work we plan to develop in detail the results for the general case MAP_m We also hope to address the identifiability problem for the general case of *MAP*, the Batch Markovian Arrival process or *BMAP*, where arrivals are allow to occur in batches.

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