Multiple risky securities valuation II.

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Abstract. In this paper we present valuation of CDO tranches paying primary attention to the equity tranche. Our approach is close to one that was outlined in [1].

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Credit derivative contract can be interpreted as an insurance of a derivative structure of the pooled securities. These securities are usually market-traded instruments such as corporate bonds, loans, swaps or other assets. Marketable securities amidst others hold a credit and market risks. Recall that credit risk is measured by a degree of losses of the possible defaults in a future time. Market risk or systematic risk stipulated by the market effect on instruments price. People sometimes believe that derivative instruments are able to use for perfect hedging different types of risk. There are four primary types of credit derivatives. These are total return swap, credit default swap, credit linked notes, and collateral debt obligations or simply CDOs. In this paper we ray attention to CDO pricing. A CDO is a type of asset-backed securities whose collateral is usually a portfolio bonds or loans.

There are two sides of the CDO deal. One is known as asset side. The low attachment and upper detachment points uniquely define stochastic lifetime of the each tranche. The tranche is activated at the moment when total loss of the underlying portfolio securities rises over the attachment point of the tranche. The tranche is terminated at the moment when losses exceed the detachment point of the tranche.

Simple examples of the risky assets structured by CDO tranches are risky bonds or swaps. There are several main rating agencies that provide investors useful information regarding companies' risk. Rating is an important information related to a chance of default or failing to fulfill the company's obligations with its equity or debt. The main credit risk characteristics of a single entity are probability of default, loss given default, and mitigation risk. There are two significant elements of a portfolio risk. These are the exposure defined as a portion of the portfolio exposed to default.

Our valuation differs from commonly used in two primary aspects. First, in research papers a basket pricing takes place by dealing with expected values of the cash flows generated by assets and coupon payments. Therefore, the solution of the problems represents an estimate of the expected value of the unknown formal solution. We note that in order to get real expected value of the tranche price one needs to present the solution itself and then calculate expectation of the exact solution. The second distinction is a technical one. Dealing with stochastic model we need to remark that a tranche lifetime is a random period.

Assume at first that event of default of the i-th bond with maturity T_i can occurred only at maturity. Denote $R_i(t, T_i)$, $i = 1, 2, \ldots m$, $t \le T_i$ a 0-coupon corporate bond price at date t and $R_i(T_i, T_i) = \$1$ if no default at the bond maturity. Therefore, the value of the i-th bond at maturity is defined as

$$R_i(T_i, T_i) = \begin{cases} 1, & \text{if no default} \\ \Delta_i, & \text{if bond defaults} \end{cases} \qquad (1)$$

where recovery rate $\Delta_i \in [0, 1)$, $i = 1, 2, \ldots, m$. Note that some recovery rates Δ_i can be equal. Let $\{\Omega, F, P\}$ be a probability space. Let D_i denote the default event of the i-th bond at its maturity T_i and $P(D_i)$, $D_i \in F$ be its probability of default. Taking into account (1) the bond price can interpreted as a random process $R_i(t, T_i) =$
$= R_i(t, T_i, \omega)$, $\omega \in \Omega$. Denote $B_0(t, T)$ the value at t of the 0-coupon Treasury bond with no chance of default. Also assume that $B_0(T, T) = \$1$. From (1) it follows that

$$\begin{aligned} R_i(t, T_i, \omega) &= B_0(t, T_i)\,\chi(\omega, \Omega \setminus D_i) + \Delta_i B_0(t, T_i)\,\chi(\omega, D_i) = \\ &= B_0(t, T_i)\,[\,1 - (1 - \Delta_i)\,\chi(\omega, D_i)\,] \end{aligned} \qquad (2)$$

Here $\chi(\omega, D)$ denotes indicator of the event D which is equal to 1 when $\omega \in \Omega$ and is equal to 0 otherwise. Assume for simplicity that risk-free interest rate is deterministic function. Taking expectation in (2) we arrive at equality

$$E\,R_i(t, T_i, \omega) = B_0(t, T_i)\,[\,1 - (1 - \Delta_i)\,P(D_i)\,] \qquad (3)$$

Putting recovery rate $\Delta_i = 0$ from (3) we note that

$$E\,R_i(t, T_i, \omega) = B_0(t, T_i)\,[\,1 - P(D_i)\,]$$

Thus the expected value of the corporate bond is less than Treasury bond with the same maturity on the value of the probability of default.

Let us outline CDO pricing problem. For illustration let us first consider a simple case when underlying risky assets can default at their maturity dates only. Arrange maturity dates of underlying securities $\mathbf{T} = (T_1, T_2, \ldots, T_m)$ in increasing order $T_j \le T_{j+1}$ for all j.. Introduce a portfolio of corporate bonds

$$\Pi(t,T) = \sum_{i=1}^{m} n_i R_i(t, T_i, \omega) = \sum_{i=1}^{m} n_i B_0(t, T_i)[1 - (1 - \Delta_i)\chi(\omega, D_i, T_i)]$$

The portfolio $\Pi(t, T)$ can be rewritten in the following form

$$\Pi(t,T) = n\Pi_n(t,T) = n\sum_{i=1}^{m} \frac{n_i}{n} B_0(t, T_i)[1 - (1 - \Delta_i)\chi(\omega, D_i, T_i)] \tag{4}$$

where $n = \sum_{i=1}^{m} n_i$. Assume that quantity n_i for any i does not change over the lifetime of the bond. Note that q% of the portfolio Π corresponds to q% of the normalized portfolio. The cash flow that corresponds to the normalized portfolio Π_n can be represented by formula

$$cf(t,\omega) = \sum_{i=1}^{m} \frac{n_i}{n} \chi(t = T_i)[1 - (1 - \Delta_i)\chi(\omega, D_i, T_i)] \tag{4′}$$

Note that if ω relates to the 0-default scenario then the portfolio's cash flow coincides with the cash flows generated by the risk-free bonds. The expected present value, EPV of the portfolio (4) is

$$E\Pi_n(t,T) = \sum_{i=1}^{m} \frac{n_i}{n} B_0(t, T_i)[1 - (1 - \Delta_i)P_i(D_i)]$$

Let us briefly outline CDO's tranches counterparties mechanics. A CDO issuer is the SPV and it provides protection on the default risk. The SPV issues tranched securities called CDO notes to investors. Selling tranches the SPV transfers default risk of the protection buyers to investors, the tranche holders. The protection buyers are usually some commercial banks. The SPV pays premium (coupon) to the tranche investors (protection sellers). The CDO buyers (investors) are the tranche holders. They are risk takers also referred to as to the protection sellers.. They do not pay coupons to investors directly. They pay coupons to SPV and it in principle is SPV who pays the tranche premiums to investors.

Consider, for example a synthetic CDO. As long as no defaults take place the SPV pays a regular premium to the tranche holder. In the event of default, the investor has to bear the loss. The next premium would be paid on the remaining notional amount, which is previous notional that will be reduced by the loss amount.

Let $0 < q_1 < q_2 < \ldots < q_p = 1$ be a set of positive numbers. The i-th tranche is by definition is a financial instrument that accumulates total portfolio losses within the interval $[q_{i-1}, q_i]$. The numbers q_{i-1} and q_i are called the attachment and detachment points of the i-th tranche correspondingly. From the right hand side of (4′) one can easy extract the cash flow generated by the portfolio due to defaults. This is the term

$$cf_{loss}(\omega) = \sum_{i=1}^{m} \frac{n_i}{n} (1 - \Delta_i) \chi(\omega, D_i)$$

Let $\tau^{(i)}$ denote a sequence of random stopping times defined by recursively by the inequalities

$$\sum_{i=1}^{\tau^{(1)}-1} (1 - \Delta_i) \chi(\omega, D_i, T_i) \frac{n_i}{n} < q_1$$

$$Q_1(\omega) = \sum_{i=1}^{\tau^{(1)}} (1 - \Delta_i) \chi(\omega, D_i, T_i) \frac{n_i}{n} \geq q_1 \tag{6}$$

Then

$$Q_1(\omega) - q_1 + \sum_{i=\tau^{(1)}}^{\tau^{(2)}-1} (1 - \Delta_i) \chi(\omega, D_i, T_i) \frac{n_i}{n} < q_2$$

$$Q_2 = Q_1(\omega) - q_1 + \sum_{i=\tau^{(1)}}^{\tau^{(2)}} (1 - \Delta_i) \chi(\omega, D_i, T_i) \frac{n_i}{n} \geq q_2$$

Recall that the equity tranche attachment point is 0. Then by induction

$$Q_{j-1}(\omega) - q_{j-1} + \sum_{i=\tau^{(j-1)}}^{\tau^{(j)}-1} (1 - \Delta_i) \chi(\omega, D_i, T_i) \frac{n_i}{n} < q_2$$

$$Q_j = Q_{j-1}(\omega) - q_{j-1} + \sum_{i=\tau^{(j-1)}}^{\tau^{(j)}-1} (1 - \Delta_i) \chi(\omega, D_i, T_i) \frac{n_i}{n} \geq q_2 \tag{7}$$

$j = 1, 2, \ldots p$. On the other hand with the last default of the $(j-1)$-tranche the value of the loss affect two consecutive $(j-1)$ and j tranches. The lifetime of the j-th tranche is the random interval $[\tau^{(j-1)}, \tau^{(j)}]$. During this period the total losses of the portfolio belong to the interval $[q_{j-1}, q_j]$, j = 1, 2, ... p. Recall that 0 tranche is called equity tranche, while the upper tranches are called mezzanine. The highest in our notations is p-tranche is referred to as senior tranche. Note recovery rate of the 0-default bond can be assumed to be equal to 1.

Let us first consider valuation of the equity tranche. Its lifetime is the random interval $[t, \tau^{(1)}(\omega)]$. By construction at the date $\tau^{(1)}(\omega)$ the equity tranche holders would receive from the CDO seller the last payment of

$$q_1 - \sum_{i=1}^{\tau_1 - 1} (1 - \Delta_i)\, \chi(\omega, D_i, T_i) \frac{n_i}{n}$$

If for some scenario ω

$$\sum_{i=1}^{m} (1 - \Delta_i)\, \chi(\omega, D_i, T_i) \frac{n_i}{n} \le q_1$$

then for this scenario the lifetime of the equity tranche coincides with the lifetime of the CDO. The remainder of the equity loss at the date $\tau^{(1)}(\omega)$ is

$$\sum_{i=1}^{\tau^{(1)}} (1 - \Delta_i)\, \chi(\omega, D_i, T_i) \frac{n_i}{n} - q_1$$

would be paid by the holders of the next upper tranche that is the first mezzanine tranche. Let us consider a mezzanine or senior tranche with marked by the points $[q_1, q_2]$. This tranche would be activated at the date $\tau^{(1)}(\omega)$. At this date sellers of the tranche are due to the first payment equal to

$$Q_1(\omega) - q_1 = \sum_{i=1}^{\tau^{(1)}} (1 - \Delta_i) \chi(\omega, D_i, T_i) \frac{n_i}{n} - q_1$$

Let us consider a higher seniority tranche which collects losses $[q_{j-1}, q_j]$, j = 2, ... , p. A mezzanine tranche that is activated at the date $\tau_{j-1}(\omega)$ and the holders of the tranche are due to the first payment of

$$Q_{j-1}(\omega) - q_{j-1} = Q_{j-2}(\omega) - q_{j-2} + \sum_{i=\tau_{j-2}}^{\tau_{j-1}} (1 - \Delta_i) \chi(\omega, D_i, T_i) \frac{n_i}{n} - q_{j-1}$$

The last payment of this tranche will occur at the date $\tau_j(\omega)$, j = 2, 3, … m. The holders of the tranche are responsible for the payment of

$$q_{j-1} - \sum_{i=\tau_{j-1}}^{\tau_j - 1} (1 - \Delta_i)\, \chi(\omega, D_i, T_i) \frac{n_i}{n}$$

Let us return to the equity tranche and consider its default history. Denote $\lambda_j(\omega) = \lambda_{1,j}(\omega) \le \tau_1(\omega)$, $j = 1, 2, \dots, l_1(\omega)$ an increasing sequence of integer-valued random variable such that $T_{\lambda j}$ is the date of j-th successive default of the equity tranche. The value of the loss at this date is

$$(1 - \Delta_i)\frac{n_i}{n}\Big|_{i=\lambda_j(\omega)}$$

From (6) it follows that the present value of the equity tranche is

$$\sum_{i=1}^{l_1}(1 - \Delta_{\lambda_i})\chi(D_{\lambda_i})\frac{n_{\lambda_i}}{n}B(t, T_{\lambda_i}) +$$

$$+ [q_1 - \sum_{i=1}^{l_1}(1 - \Delta_{\lambda_i})\chi(D_{\lambda_i})\frac{n_{\lambda_i}}{n}B(t, T_{\lambda_i})]\, B(t, T_{\lambda_{l_1}}) = \tag{8}$$

$$= (1 - B(t, T_{\lambda_{l_1}}))\sum_{i=1}^{l_1}(1 - \Delta_{\lambda_i})\chi(D_{\lambda_i})\frac{n_{\lambda_i}}{n}B(t, T_{\lambda_i}) + q_1 B(t, T_{\lambda_{l_1}})$$

This is present value of the equity tranche which represents the tranche holders' commitments.

On the other side of the CDO are the periodic payments made by the SPV to the tranche holders during the lifetime of the tranche. Each payment of the tranche holders is equals to a fixed rate of the tranche principal. The tranche principal is a variable decreasing function on time. Immediately after a default the notional principal reduces its value on the value of the defaulted assets. The fixed rate remains unchangeable over lifetime of the equity tranche. Let 'd' denote the period between two consecutive coupon payment dates. Introduce the random variables $\zeta_{1,j}(\omega) = \zeta_j(\omega)$

$$\zeta_j(\omega) = ip\left(\frac{T_{\lambda_j} - T_{\lambda_{j-1}}}{d}\right)$$

$j = 2, 3, \ldots l_1$. Here $ip(x)$ is the largest integer that does not exceed value x. We also omitted index 1 that assign to the equity tranche. The random variables $\zeta_j(\omega)$ represents the number of payments received by the tranche holders between $j - 1$ and j successive defaults of the equity tranche. Let c_1 denote a fixed coupon of the equity tranche. Therefore, there are $\zeta_1(\omega)$ payments of the $c_1 q_1$ dollars during the period $[t, T_{\lambda 1})$. Here q_1 is the original notional principal of the equity tranche. The present value of these series payments is

$$\sum_{i=1}^{\zeta_1} c_1 q_1 B(t, id) \tag{9'}$$

At the time of the first default $T_{\lambda 1}$ occurs the change of the notional principal q_1. The new notional principal is

$$q_{1,2} = q_1 - (1 - \Delta_{\lambda_1}) \frac{n_{\lambda_1}}{n}$$

It would be applied over the next random interval $[T_{\lambda 1}, T_{\lambda 2})$. During this period the number of the fixed rate payments received by the tranche holders is $\zeta_2(\omega)$ and the notional principal is $q_{1,2}$. Thus the present value of the payments received by tranche holders over $[t, T_{\lambda 2})$ is

$$\sum_{i=1}^{\zeta_2} c_1 q_{1,2} B(t, (\zeta_1 + i) d)$$

Over the last period of the equity tranche $[T_{\lambda l-1}, T_{\lambda l})$ the notional principal of the tranche is

$$q_{1,l(\omega)-1} = q_1 - \sum_{i=1}^{l(\omega)-1} (1 - \Delta_{\lambda_i}) \frac{n_{\lambda_i}}{n}$$

and the number of payments is $\zeta_{\lambda l}(\omega)$. Hence, the present value of these payments is

$$c_1 q_{1,l-1}(\omega) \sum_{i=1}^{\zeta_{l-1}} B(t, (\sum_{k=1}^{l-2} \zeta_k + i) d)$$

The last lifetime period is $[T_{\lambda l}, T_{\tau 1})$ and denote the number of the payments to the tranche holders over this period as

$$\zeta_{l+1}(\omega) = \mathrm{ip}\left(\frac{T_{\tau_1} - T_{l_1}}{d}\right)$$

and the notional principal over the last period would be

$$q_{1,l(\omega)} = q_1 - \sum_{i=1}^{l(\omega)} (1 - \Delta_{\lambda_i}) \frac{n_{\lambda_i}}{n}$$

During this period the tranche holders would receive the cash flow which present value is

$$c_1\, q_{1,l}(\omega) \sum_{i=1}^{\zeta_{l+1}} B\left(t, \left(\sum_{k=1}^{l(\omega)} \zeta_k + i\right) d\right)$$

Here, that random index $l(\omega)$ depends on the tranche seniority. Summing up all terms presented fixed rate present value of the equity tranche to its holders we see that

$$c_1 \Big[\sum_{i=1}^{\varsigma_1} q_1 B(t, id) + \sum_{i=1}^{\varsigma_2} q_{1,1} B(t, \zeta_1 + id) + \ldots + \sum_{i=1}^{\varsigma_{l}} q_{1,l-1} B(t, (\sum_{k=1}^{l-1} \zeta_k + i d) +$$

$$+ \sum_{i=1}^{\varsigma_{l+1}} q_{1,l} B(t, (\sum_{k=1}^{l} \zeta_k + id) \Big] = \tag{9}$$

$$= c_1 \Big[\sum_{j=1}^{l} \sum_{i=1}^{\varsigma_j} q_{1,j-1}(\omega) B(t, (\sum_{k=1}^{j-1} \zeta_k + i d) + \sum_{i=1}^{\varsigma_{l+1}} q_{1,l} B(t, (\sum_{k=1}^{l} \zeta_k + id)) \Big]$$

By definition we put here $\zeta_0(\omega) = 0$. The equity spread is the value of c_1 that makes the present value of the cash inflow be equal to the present value of the cash outflow. Hence, the equity spread is equal to

$$c_1 = \frac{\sum_{i=1}^{l} (1-\Delta_i) \frac{n_{\lambda_i}}{n} B(t, T_{\lambda_i}) [1 - B(t, T_{\tau_1})] + q_1 B(t, T_{\tau_1})}{\sum_{j=1}^{l} [\sum_{i=1}^{\zeta_j} q_{1,j-1}(\omega) B(t, (\sum_{k=1}^{j-1} \zeta_k + id) + \sum_{i=1}^{\zeta_{l+1}} q_{1,l}(\omega) B(t, (\sum_{k=1}^{j-1} \zeta_k + id)]} \tag{10}$$

Formula (10) represents the fixed rate of the equity tranche. Note that the value $c_1 = c_1(\omega)$ given by (10) is a random variable. If a nonrandom fixed rate ν is applied as a market spread both counterparties are subjected to risk. The tranche holders risk is that they would receive compensation less than they would pay. This risk is equal to $P\{c_1(\omega) < \nu\}$ and the loss is a random variable. The value of the loss is a random variable

$$c_1(\omega) - \nu = \Big[\sum_{j=1}^{l} \sum_{i=1}^{\varsigma_j} q_{1,j-1}(\omega) B(t, (\sum_{k=1}^{j-1} \zeta_k + i d) + \sum_{i=1}^{\varsigma_{l+1}} q_{1,l} B(t, (\sum_{k=1}^{l} \zeta_k + i d)) \Big]$$

Having analytic representation of the loss one can calculates its statistical characteristics. There is a difference in valuation between equity and upper tranches. The equity tranche contains the “warming up” period during which there is no credit events (defaults) occurred. During this period the tranche holders receive fixed periodic payments on the initial tranche principal. Upper tranche holders actually would pay remainder of the loss of the default from the previous tranche. The holders of the previous tranche pay off the first portion of this default. Along with this payment the lower tranche stops to exist. Other difference between equity and upper tranches is that lifetime of the equity tranche begins at nonrandom moment of time while others tranches would be activated at the random dates. These differences need to be taking into account for upper tranches valuation.

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