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Working Paper

Group contest success functionsGroup contest success functions

Discussion papers // WZB, Wissenschaftszentrum Berlin für Sozialforschung, Schwerpunkt II Märkte und Politik, Abteilung Marktprozesse und Steuerung, No. SP II 2008-20

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Suggested citation: Münster, Johannes (2008) : Group contest success functions, Discussion papers // WZB, Wissenschaftszentrum Berlin für Sozialforschung, Schwerpunkt II Märkte und Politik, Abteilung Marktprozesse und Steuerung, No. SP II 2008-20, http://hdl.handle.net/10419/51100

WISSENSCHAFTSZENTRUM BERLIN
FÜR SOZIALFORSCHUNG

SOCIAL SCIENCE RESEARCH
CENTER BERLIN

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Group Contest Success Functions

SP II 2008 – 20

July 2008

ISSN Nr. 0722 – 6748

Research Area
Markets and Politics

Schwerpunkt
Märkte und Politik

Research Unit
Market Processes and Governance

Abteilung
Marktprozesse und Steuerung

Zitierweise/Citation:

Johannes Münster, **Group Contest Success Functions**,
Discussion Paper SP II 2008 – 20, Wissenschaftszentrum
Berlin, 2008.

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Internet: www.wzb.eu

ABSTRACT

Group Contest Success Functions

by Johannnes Münster [*]

This paper extends the axiomatic characterization of contest success functions of Skaperdas (1996) and Clark and Riis (1998) to contests between groups.

Keywords: Contest, conflict, axiom, group

JEL Classification: C70, D72, D74

ZUSAMMENFASSUNG

Group Contest Success Functions

Eine "contest success function" beschreibt, wie in einem Wettkampf die Gewinnwahrscheinlichkeiten von den Einsätzen der Beteiligten abhängen. Dieser Aufsatz verallgemeinert die auf Skaperdas (1996) und Clark und Riis (1998) zurückgehende axiomatische Fundierung von contest success functions, indem er Wettkämpfe zwischen Gruppen untersucht.

* Comments by Dan Kovenock significantly improved the paper and are gratefully acknowledged. I would also like to thank Pavlo Blavatskyy, Aron Kiss, Kai Konrad, Florian Morath, Dana Sisak, participants of the SFB/TR 15 meeting in Gummersbach 2004, the 2004 meeting of the Verein für Socialpolitik in Dresden, and two anonymous referees. Errors are mine. Financial support from the Deutsche Forschungsgemeinschaft through SFB/TR 15 is gratefully acknowledged.

1 Introduction

Contests often take place between groups. In lobbying and rent-seeking contests, many lobbyists work together on the same side. In R&D races, groups of researchers team together in order to develop new technologies earlier than rival teams. Further examples are wars and sport tournaments. By now there is a substantial literature on group contests.[1]

Skaperdas (1996) provides, in an important paper, an axiomatic characterization of contest success functions. He deals with contests between individuals. Clark and Riis (1998) generalize Skaperdas (1996) by dropping the assumption of symmetry. The purpose of the present paper is to extend these axiomatic foundations to contests between groups. In a group contest, each member of a group can invest time, resources, or effort in order to increase the probability of his group winning.[2] I propose a set of axioms for group contests that are close analogues to those studied by Skaperdas (1996) and Clark and Riis (1998). In particular, if each group consists of only one individual, then the axioms are similar to the axioms in Skaperdas (1996). Following Clark and Riis (1998), however, I allow for asymmetries. This is natural for group contests since groups may have different sizes.

Skaperdas (1996, Theorem 1) shows that, under a set of reasonable axioms, the probability of individual i winning a contest is given by

$$\frac{f(x_i)}{\sum_j f(x_j)}$$

whenever the denominator is positive, where x_j is the effort of individual j, and the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is sometimes referred to as the *impact function*. The axioms laid out below generalize this to group contests and

[1] For surveys, see Garfinkel and Skaperdas (2006, Section 7), Corchón (2007, Section 4.2), and Konrad (2007, Sections 6.4 and 7).

[2] A related situation arises in multi-activity contests between individuals as studied in Epstein and Hefeker (2003), Arbatskaya and Mialon (2007), and Caruso (2006). Here, each individual chooses several activities that help in winning the contest.

allow for asymmetries (Theorem 1). Under these axioms, whenever at least one individual chooses a strictly positive effort, the probability of group g winning is given by

$$\frac{f_g(\mathbf{x}_g)}{\sum_k f_k(\mathbf{x}_k)}, \tag{1}$$

where for each group k, $\mathbf{x}_k$ denotes the vector of efforts of the m_k members of group k, and $f_k : \mathbb{R}_+^{m_k} \to \mathbb{R}_+$ is a non-negative and strongly increasing function.

Moreover, Skaperdas (1996, Theorem 2) shows that, if the contest success function if homogenous of degree zero, his axioms imply a Tullock contest success function. The generalization to a group contest given here (Theorem 2) results in a contest success function of the form given in (1), where all the impact functions f_k are homogeneous of the same degree. I also generalize the axiomatic foundation of a logistic contest success function (Theorem 3).

Finally, I consider an axiom stating that the probability of success remains unchanged if the effort of one individual increases by some amount while the effort of another individual belonging to the same group decreases by the same amount. This leads to functional forms for group contest success functions that have frequently been used in the literature.[3] If the contest success function is homogeneous of degree zero, a natural generalization of the Tullock contest success function results: the impact functions in (1) are multiples of a power function of the *sum* of the efforts of the group's members. A similar statement holds for the logistic contest success function.

This paper has grown out of appendix 8.1 of Münster (2004). It is related to other axiomatic work on contest success functions such as Blavatskyy (2004) on contests with ties, Arbatskaya and Mialon (2007) and Rai and Sarin (2007) on multi-activity contests, and to other approaches to contest

[3] A partial list is: Katz, Nitzan and Rosenberg (1990), Nitzan (1991), Katz and Tokatlido (1996), Skaperdas and Syropoulos (1997), Wärneryd (1998), Esteban and Ray (2001), Müller and Wärneryd (2001), Garfinkel (2004a, 2004b), Baik (2007), Inderst, Müller and Wärneryd (2007), Münster (2007), Münster and Staal (2007).

success functions such as Epstein and Nitzan (2006, 2007) and Jia (2008). Rai and Sarin (2007) is perhaps the most closely related; they independently cover similar ground as Theorems 1 and 2 below, but do not discuss the generalization of the logistic contest success function, or the summation case.

2 Main axioms

There are n individuals and G groups. Each individual is a member of exactly one group. Group g has $m_g \geq 1$ members, $\sum_{g=1}^{G} m_g = n$. The set of groups is denoted by $\Gamma = \{1, ..., G\}$. The inter-group contest effort of individual i in group g is $x_{ig} \in \mathbb{R}_+$. For the purpose of the present paper, it is immaterial whether these efforts are chosen by the individuals or enforced by the groups. Let $\mathbf{x}_g = \left(x_{1g}, ..., x_{m_g g}\right)$ be the m_g-vector of efforts of the members of group g, $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_G)$ the $n-$vector that collects all individual efforts, $\mathbf{x}_{-g}$ the $(n - m_g)-$vector of the efforts of all players who do not belong to group g, and $\mathbf{x}_{-ig}$ the $(n-1)-$vector of all efforts except the effort of player i in group g. Sometimes the convenient notation $\left(\mathbf{x}_g, \mathbf{x}_{-g}\right) = (x_{ig}, \mathbf{x}_{-ig}) = \mathbf{x}$ will be used. Moreover, for any $M \subset \Gamma$, let $\mathbf{x}_M$ denote the $\left(\sum_{g \in M} m_g\right)-$vector of the efforts of members of groups g that belong to M.

For any group $g \in \Gamma$, it is assumed that there exists a function $p_g : \mathbb{R}_+^n \to \mathbb{R}_+$, where $p_g(\mathbf{x})$ can be interpreted as the probability that group g wins the contest. Alternatively, $p_g(\mathbf{x})$ can also be interpreted as the share of some rent that group g gets. I will refer to p_g as the contest success function (abbreviated CSF).

Axiom 1 *(Probability)* $\sum_{g=1}^{G} p_g(\mathbf{x}) = 1$ *and* $p_g(\mathbf{x}) \geq 0$ *for all* $g \in \Gamma$.

Axiom 2 *(Monotonicity) For all* $g \in \Gamma$ *and all* $i \in \{1, .., m_g\}$: *if* $\hat{x}_{ig} > x_{ig}$, *then*

i) $p_g(\hat{x}_{ig}, \mathbf{x}_{-ig}) \geq p_g(x_{ig}, \mathbf{x}_{-ig})$, *with strict inequality unless* $p_g(x_{ig}, \mathbf{x}_{-ig}) = 1$, *and*

ii) for all $k \neq g$, $k \in \Gamma$: $p_k(\hat{x}_{ig}, \mathbf{x}_{-ig}) \leq p_k(x_{ig}, \mathbf{x}_{-ig})$.

A1 says that $p_g(\mathbf{x})$ is a probability. A2 says that it is strictly increasing in the effort of any member of the group; the only exception being that the group already wins with probability one, in which case the probability of winning stays constant when the effort of a member increases. Moreover, a group's probability of winning is weakly decreasing in the efforts of the individuals who belong to the other groups. A2 implies that, if $x_{ig} > 0$ for some i in group g, then $p_g(\mathbf{x}) > 0$. This rules out the perfectly discriminating CSF (or all-pay auction) considered in Baik, Kim and Na (2001) and Konrad (2004), where a group wins with probability one if the sum of the efforts of individuals in this group is higher than the sum of the efforts of the members of any competing group. This CSF, however, can be viewed as the limit of the CSF axiomatized in Proposition 2 below.[4]

Skaperdas' (1996) third axiom is a symmetry assumption. I do not impose any symmetry for the main results. For completeness, however, I discuss the impact of assuming symmetry.

Axiom 3 *(Between-group-anonymity) A contest success function satisfies between-group-anonymity if, whenever $m_g = m_k$,*

$$p_g\left(\mathbf{x}_g, \mathbf{x}_k, \mathbf{x}_{\Gamma\setminus\{g,k\}}\right) = p_k\left(\mathbf{x}_k, \mathbf{x}_g, \mathbf{x}_{\Gamma\setminus\{g,k\}}\right).$$

A3 says that the contest is fair between groups of equal size and the identities of the groups do not matter per se. It should be contrasted with anonymity within groups:

Axiom 3' *(Within-group-anonymity) A contest success function satisfies within-group-anonymity if, for any group $g \in \Gamma$ and for any bijection ψ : $\{1, ..., m_g\} \rightarrow \{1, ..., m_g\}$,*

$$p_g\left(\mathbf{x}_g, \mathbf{x}_{-g}\right) = p_g\left(\hat{\mathbf{x}}_g, \mathbf{x}_{-g}\right),$$

[4]Similarly, A2 implies that the impact function cannot be Cobb Douglas, weakest link, or best shot. These impact functions, however, are limit cases of a family of CES functions consistent with Theorem 2 below.

where $\hat{\mathbf{x}}_g = \left(x_{\psi(1)g}, ..., x_{\psi(m_g)g}\right)$ *is the vector of efforts of the members of group* g *after a permutation according to* ψ.

Between-group-anonymity (A3) and within-group-anonymity (A3') are different, even if all groups are of equal size. For example, suppose that $G = m_1 = m_2 = 2$, and consider the CSF

$$p_1(\mathbf{x}) = \begin{cases} \frac{a(bx_{11}+x_{21})}{a(bx_{11}+x_{21})+(bx_{12}+x_{22})}, & \mathbf{x} \neq \mathbf{0}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Here, and in the following, $\mathbf{0} = (0, ..., 0)$ denotes the vector of appropriate length where every component is equal to zero. If $a = b = 1$, both A3 and A3' hold; if $a = 1 \neq b$, only A3 holds; if $a \neq 1 = b$, only A3' holds; finally, if $a \neq 1$ and $b \neq 1$, neither A3 nor A3' holds.[5]

The next two axioms concern the CSF for a contest among fewer groups.

Axiom 4 *(Subcontest consistency) Let* $p_g^M(\mathbf{x})$ *be group* g*'s probability of winning a subcontest played by a subset* $M \subset \Gamma$ *consisting of at least two groups. Then for all* $g \in M$,

$$p_g^M(\mathbf{x}) = \frac{p_g(\mathbf{x})}{\sum_{k \in M} p_k(\mathbf{x})} \quad \forall \mathbf{x} \text{ } s.t. \text{ } \mathbf{x}_M \neq \mathbf{0}.$$

Axiom 5 *(Subcontest independence)* $p_g^M(\mathbf{x})$ *is independent of the efforts of individuals belonging to groups not in* M.

A4 implies that contests among fewer groups or more groups are qualitatively similar. Note that the equation is well defined since $\mathbf{x}_M \neq \mathbf{0}$, i.e. there is some $k \in M$ and $i \in \{1, ..., m_k\}$ s.t. $x_{ik} > 0$, and thus, by A2, $p_k(\mathbf{x}) > 0$. A5 is related to the independence of irrelevant alternatives in the context of individual probabilistic choice.

[5] In the context of a multi-activity contest, within-group-anonymity seems a strong assumption, since the activities may have a different impact on the winning probabilities. One may want to model this in the CSF, as in Epstein and Hefeker (2003).

For the main results, A1, A2, A4 and A5 are assumed to hold. The approach is to derive the CSF for any subcontest of some bigger contest. In this way, A4 and A5 can also be used to derive the CSF for a contest between only two groups.

Axioms 1-5 reformulate the assumptions in Skaperdas (1996) for an inter-group contest. In particular, if there is only one individual in each group, A1-A5 are similar to the corresponding axioms in Skaperdas (1996). For further motivations and discussions of the axioms, see also Clark and Riis (1998) and Corchón (2007).

3 Results

Following Jehle and Reny (2001, p. 437), a function $f : \mathbb{R}_+^k \to \mathbb{R}_+$ is said to be *strongly increasing* whenever $\hat{z}_i \geq z_i$ for all $i \in \{1, ..., k\}$ and $\hat{z}_j > z_j$ for at least one $j \in \{1, ..., k\}$ implies $f(\hat{z}_1, ..., \hat{z}_k) > f(z_1, ..., z_k)$.

Theorem 1 *Suppose the contest success function satisfies A1, A2, A4, and A5. Let M be any proper subset of Γ consisting of at least two groups. Then, for each $g \in M$, there exists a non-negative and strongly increasing function $f_g : \mathbb{R}_+^{m_g} \to \mathbb{R}_+$ such that*

$$p_g^M(\mathbf{x}) = \frac{f_g(\mathbf{x}_g)}{\sum_{k\in M} f_k(\mathbf{x}_k)} \quad \forall \mathbf{x} \text{ s.t. } \mathbf{x}_M \neq \mathbf{0}. \tag{2}$$

Proof. Since M is a proper subset of Γ, there exist a group $a \in \Gamma \backslash M$. Fix any $\mathbf{x}_a = \mathbf{a} \neq \mathbf{0}$. Then by A2, $p_a(\mathbf{x}_a, \mathbf{x}_{-a}) > 0$. Thus by A4,

$$p_g^M(\mathbf{x}) = \frac{\frac{p_g(\mathbf{x}_a,\mathbf{x}_{-a})}{p_a(\mathbf{x}_a,\mathbf{x}_{-a})}}{\sum_{k\in M} \frac{p_k(\mathbf{x}_a,\mathbf{x}_{-a})}{p_a(\mathbf{x}_a,\mathbf{x}_{-a})}} \quad \forall \mathbf{x} \text{ s.t. } \mathbf{x}_M \neq \mathbf{0},\ \mathbf{x}_a = \mathbf{a} \neq \mathbf{0}. \tag{3}$$

By A4, for any group $k \in M$,

$$\frac{p_k^{\{a,k\}}(\mathbf{x}_a, \mathbf{x}_{-a})}{p_a^{\{a,k\}}(\mathbf{x}_a, \mathbf{x}_{-a})} = \frac{\frac{p_k(\mathbf{x}_a,\mathbf{x}_{-a})}{p_k(\mathbf{x}_a,\mathbf{x}_{-a})+p_a(\mathbf{x}_a,\mathbf{x}_{-a})}}{\frac{p_a(\mathbf{x}_a,\mathbf{x}_{-a})}{p_k(\mathbf{x}_a,\mathbf{x}_{-a})+p_a(\mathbf{x}_a,\mathbf{x}_{-a})}} = \frac{p_k(\mathbf{x}_a, \mathbf{x}_{-a})}{p_a(\mathbf{x}_a, \mathbf{x}_{-a})} \quad \forall \mathbf{x} \text{ s.t. } \mathbf{x}_a = \mathbf{a} \neq \mathbf{0}.$$

All the expressions are well defined since $\mathbf{x}_a = \mathbf{a} \neq \mathbf{0}$. By A5, $p_k^{\{a,k\}}(\mathbf{x}_a, \mathbf{x}_{-a})$ and $p_a^{\{a,k\}}(\mathbf{x}_a, \mathbf{x}_{-a})$ depend only on $\mathbf{x}_a$ and $\mathbf{x}_k$. By fixing $\mathbf{x}_a = \mathbf{a} \neq \mathbf{0}$, one can thus define, for each $k \in M$, a non-negative function $f_k : \mathbb{R}_+^{m_k} \to \mathbb{R}_+$ by

$$f_k(\mathbf{x}_k) = \frac{p_k(\mathbf{a}, \mathbf{x}_{-a})}{p_a(\mathbf{a}, \mathbf{x}_{-a})}.$$

Next I show that f_k is strongly increasing. Suppose $\hat{x}_{ik} > x_{ik}$. From A2, $p_a(\hat{x}_{ik}, \mathbf{x}_{-ik}) \leq p_a(x_{ik}, \mathbf{x}_{-ik})$, moreover

$$p_k(\hat{x}_{ik}, \mathbf{x}_{-ik}) > p_k(x_{ik}, \mathbf{x}_{-ik}) \quad \forall \mathbf{x}_{-ik} \text{ s.t. } \mathbf{x}_a = \mathbf{a} \neq \mathbf{0},$$

since $p_k(x_{ik}, \mathbf{x}_{-ik}) < 1$ follows from $\mathbf{x}_a = \mathbf{a} \neq \mathbf{0}$ by A2 and A1. Therefore,

$$\frac{p_k(\hat{x}_{ik}, \mathbf{x}_{-ik})}{p_a(\hat{x}_{ik}, \mathbf{x}_{-ik})} > \frac{p_k(x_{ik}, \mathbf{x}_{-ik})}{p_a(x_{ik}, \mathbf{x}_{-ik})} \quad \forall \mathbf{x}_{-ik} \text{ s.t. } \mathbf{x}_a = \mathbf{a} \neq \mathbf{0}.$$

Thus f_k is strongly increasing.

By (3),

$$p_g^M(\mathbf{x}) = \frac{f_g(\mathbf{x}_g)}{\sum_{k\in M} f_k(\mathbf{x}_k)} \quad \forall \mathbf{x} \text{ s.t. } \mathbf{x}_M \neq \mathbf{0}, \mathbf{x}_a = \mathbf{a} \neq \mathbf{0}.$$

By A5, $p_g^M(\mathbf{x})$ does not depend on $\mathbf{x}_a$, therefore

$$p_g^M(\mathbf{x}) = \frac{f_g(\mathbf{x}_g)}{\sum_{k\in M} f_k(\mathbf{x}_k)} \quad \forall \mathbf{x} \text{ s.t. } \mathbf{x}_M \neq \mathbf{0}.$$

■

In the literature on contests between individuals, the function f_g is sometimes called the impact function. I follow this terminology. In the present setting, the impact function aggregates the individual efforts chosen by members of a group to a single number.

Note that, if a CSF is of form (2), and for the case $\mathbf{x}_M = \mathbf{0}$ some tie-breaking rule that is consistent with the axioms is assumed, then the CSF fulfills A1, A2, A4, and A5 everywhere on its domain. Similar converse statements hold for all the results derived below.

The implications of adding anonymity to A1, A2, A4 and A5 are straightforward. Between-group-anonymity (A3) requires that, if $m_g = m_k$, then $f_g(\mathbf{z}) = f_k(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}_+^{m_g}$. In contrast, within-group-anonymity (A3') requires that, for each group, the impact function f_g is symmetric.

3.1 Homogeneity

To derive a Tullock CSF, Skaperdas (1996) assumes that the CSF is homogeneous of degree zero. A6 generalizes this for group contests.

Axiom 6 *For all $\lambda > 0$ and all $g \in \Gamma$, $p_g(\lambda \mathbf{x}) = p_g(\mathbf{x})$.*

A6 implies that, if all individuals double their efforts, the probabilities of success remain unchanged. Moreover, the CSF is independent of the unit of measurement, which seems an attractive property.

Theorem 2 *If the contest success function satisfies A1, A2, A4, A5, and A6, then it satisfies (2) and the impact functions f_k are homogeneous of the same degree $r > 0$.*

Proof. From A4 and A6, if $\mathbf{x}_M \neq \mathbf{0}$, then $p_k^M(\lambda \mathbf{x}) = p_k^M(\mathbf{x})$ for all $\lambda > 0$ and all $k \in M$. Thus by Theorem 1, for all $\mathbf{x}_k \in \mathbb{R}_+^{m_k} \setminus \{\mathbf{0}\}$ and all $\lambda > 0$, the impact functions satisfy

$$\frac{f_k(\lambda \mathbf{x}_k)}{f_k(\mathbf{x}_k)} = \frac{f_k(\lambda \mathbf{1})}{f_k(\mathbf{1})},$$

where $\mathbf{1} = (1, ..., 1)$ is the m_k-vector where every component is equal to one. Define $F(\mathbf{x}_k) = f_k(\mathbf{x}_k) / f_k(\mathbf{1})$. Then

$$F(\lambda \mathbf{x}_k) = F(\lambda \mathbf{1}) F(\mathbf{x}_k). \tag{4}$$

In particular, if $\mathbf{x}_k = t\mathbf{1}$ where $t > 0$,

$$F(\lambda t\mathbf{1}) = F(\lambda \mathbf{1}) F(t\mathbf{1}).$$

Define $G(\lambda) = F(\lambda \mathbf{1})$. Note that G is a strictly increasing function of a single variable. Moreover, $G(\lambda t) = G(\lambda) G(t)$.

In order to transform this equation into a Cauchy equation (cf. Aczél 1969), substitute $\lambda = \exp(\lambda')$ and $t = \exp(t')$ to get

$$G(\exp(\lambda' + t')) = G(\exp(\lambda')) G(\exp(t')).$$

Let $H(s) = G(\exp(s))$. Then $H(\lambda' + t') = H(\lambda') H(t')$. Finally, let $h(s) = \ln(H(s))$ to get

$$h(\lambda' + t') = h(\lambda') + h(t'). \tag{5}$$

Since G is strictly increasing, h is strictly increasing and thus continuous almost everywhere. Under this condition, the only solution to (5) is given by $h(s) = rs$ where $r > 0$ (Aczél 1966, p. 34). Thus $H(s) = \exp(rs)$,

$$G(s) = H(\ln s) = \exp(r \ln(s)) = s^r,$$

and $F(\lambda \mathbf{1}) = G(\lambda) = \lambda^r$. Inserting this in (4) gives $F(\lambda \mathbf{x}_k) = \lambda^r F(\mathbf{x}_k)$. By definition of $F(\mathbf{x}_k)$,

$$f_k(\lambda \mathbf{x}_k) = F(\lambda \mathbf{x}_k) f_k(\mathbf{1}) = \lambda^r F(\mathbf{x}_k) f_k(\mathbf{1}) = \lambda^r f_k(\mathbf{x}_k).$$

The above argument shows that, for any group $k \in M$, whenever $\mathbf{x}_k \neq \mathbf{0}$, $f_k(\lambda \mathbf{x}_k) = \lambda^r f(\mathbf{x}_k)$, where $r > 0$. From A6, r is the same for all groups.

Now suppose that $\mathbf{x}_k = \mathbf{0}$ for some $k \in M$. Fix some $\mathbf{x}_g \neq \mathbf{0}$ for each $g \neq k$, and some $\lambda \neq 1$. Then $p_k^M(\lambda \mathbf{x}) = p_k^M(\mathbf{x})$ and

$$\frac{f_k(\mathbf{0})}{f_k(\mathbf{0}) + \lambda^r \sum_{g \in M, g \neq k} f_g(\mathbf{x}_g)} = \frac{f_k(\mathbf{0})}{f_k(\mathbf{0}) + \sum_{g \in M, g \neq k} f_g(\mathbf{x}_g)}.$$

Since $\lambda^r \neq 1$ and $\sum_{g \in M, g \neq k} f_g(\mathbf{x}_g) > 0$, it follows that $f_k(\mathbf{0}) = 0$. Therefore, f_k is homogeneous of degree r on $\mathbb{R}_+^{m_k}$. ■

To see the relation to the axiomatic foundation of a Tullock CSF for contests between individuals (Skaperdas 1996, Theorem 2), note that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ of a single variable is homogeneous if and only if it can be written as $f(x) = ax^r$, where $a = f(1)$ and r is the degree of homogeneity (Carter 2001, p. 351). Thus, if every group consists of only one individual, Theorem 2 immediately gives a Tullock CSF.

A1-A6 are compatible with several functional forms of the impact functions. For example, the impact functions can be CES functions

$$f_g(\mathbf{x}_g) = \left(\sum_{i=1}^{m_g} x_{ig}^{\alpha} \right)^{\frac{r}{\alpha}}, \ r > 0, \ \alpha \neq 0. \tag{6}$$

Other potentially interesting impact functions can be seen as limit cases, since they are limits of (6) but lead to a violation of A2. A case in point is the Cobb-Douglas function, which violates the strict inequality in A2 (i), but only when $p_g(\mathbf{x}) = 0$. Other cases are $f_g(\mathbf{x}_g) = \min\{x_{1g}, ..., x_{m_g g}\}$ and $f_g(\mathbf{x}_g) = \max\{x_{1g}, ..., x_{m_g g}\}$, which are related to Hirshleifer's (1983) weakest-link and best-shot technologies for the production of public goods.[6]

[6] Some of the examples of private supply of public goods given by Hirshleifer (1983) to motivate these technologies are actually about contests between groups. Consider missile defence: only one rocket needs to hit an incoming ICBM in order to destroy it. See also Clark and Konrad (2007).

3.2 An alternative to homogeneity

Relying on data from Dupuy (1987), Hirshleifer (1989, 1991) argues for a logistic CSF. He points out that the logistic CSF fits some 'stylized facts' from military warfare, in particular, that being just a little stronger than one's rival provides a big advantage. For contests between individuals, the logistic CSF can be derived by assuming that the probabilities of success do not change if a constant is added to the effort of each individual (Skaperdas 1996, Theorem 3). To generalize this to group contests, consider the following alternative to A6. Let $\mathbf{1} = (1, ..., 1)$ denote the vector of appropriate length where every component is equal to one.

Axiom 7 *For all $\lambda > 0$, and all $g \in \Gamma$,*

$$p_g (\mathbf{x} + \lambda \mathbf{1}) = p_g (\mathbf{x}) .$$

Theorem 3 *If the contest success function satisfies A1, A2, A4, A5, and A7, then it satisfies (2) and the impact functions f_k satisfy, for all $\lambda > 0$,*

$$f_k (\lambda \mathbf{1} + \mathbf{x}_k) = \exp (r\lambda) f_k (\mathbf{x}_k) \tag{7}$$

where $r > 0$ is a parameter.

Proof. From A4 and A7, for all $\lambda > 0$ and all $k \in M$,

$$p_k^M (\mathbf{x} + \lambda \mathbf{1}) = p_k^M (\mathbf{x}) \quad \forall \mathbf{x} \text{ s.t. } \mathbf{x}_M \neq \mathbf{0}. \tag{8}$$

Suppose that $f_k(\mathbf{0})=0$. Then one can derive a contradiction as follows: whenever $\mathbf{x}_h \neq \mathbf{0}$ for some group $h \in M\backslash\{k\}$,

$$\begin{aligned} 0 &= \frac{f_k(\mathbf{0})}{f_k(\mathbf{0})+\sum_{g\neq k, g\in M} f_g(\mathbf{x}_g)} \\ &= p_k^M(\mathbf{0}, \mathbf{x}_{-k}) \\ &= p_k^M(\mathbf{1}, \mathbf{x}_{-k}+\mathbf{1}) \\ &= \frac{f_k(\mathbf{1})}{f_k(\mathbf{1})+\sum_{g\neq k, g\in M} f_g(\mathbf{x}_g+\mathbf{1})} \\ &> 0. \end{aligned}$$

The right hand side of the first line is well defined since $\mathbf{x}_h \neq \mathbf{0}$. The second line is from Theorem 1, the third from (8), the fourth from Theorem 1, and the inequality from the strong monotonicity of f_k. It follows that $f_k(\mathbf{0}) > 0$.

By Theorem 1 and (8), for all $\mathbf{x}_k \in \mathbb{R}_+^{m_k}$,

$$\frac{f_k(\lambda\mathbf{1}+\mathbf{x}_k)}{f_k(\mathbf{x}_k)} = \frac{f_k(\lambda\mathbf{1})}{f_k(\mathbf{0})}.$$

Define $F(\mathbf{s}) = f_k(\mathbf{s})/f_k(\mathbf{0})$. Thus

$$F(\lambda\mathbf{1}+\mathbf{x}_k) = F(\lambda\mathbf{1})\,F(\mathbf{x}_k). \tag{9}$$

For $\mathbf{x}_k = k\mathbf{1}$ where $k > 0$, $F((\lambda+k)\mathbf{1}) = F(\lambda\mathbf{1})\,F(k\mathbf{1})$. Let $G(s) = F(s\mathbf{1})$. Then $G(\lambda+k) = G(\lambda)\,G(k)$. Finally, let $H(s) = \ln(G(s))$ to get $H(\lambda+k) = H(\lambda)+H(k)$. Since H is strictly monotone, it is continuous almost everywhere, and the only solution is $H(s) = rs$. Thus

$$F(s\mathbf{1}) = G(s) = \exp(H(s)) = \exp(rs).$$

Inserting this in (9), $F(\lambda\mathbf{1}+\mathbf{x}_k)=\exp(r\lambda)F(\mathbf{x}_k)$. Thus by definition of F,

$$\begin{aligned} f_k(\lambda\mathbf{1}+\mathbf{x}_k) &= F(\lambda\mathbf{1}+\mathbf{x}_k) f_k(\mathbf{0}) = \exp(r\lambda) F(\mathbf{x}_k) f_k(\mathbf{0}) \\ &= \exp(r\lambda) f_k(\mathbf{x}_k). \end{aligned}$$

■

The relation between Theorem 3 and the corresponding axiomatization of a logistic CSF for contests between individuals is pointed out in the following lemma.

Lemma 1 *Suppose that f is a function of a single variable, $f(0)>0$, and f satisfies (7). Then, for all $t\geq 0$, $f(t)=a\exp(rt)$, where $a=f(0)$ is a positive constant.*

Proof. In (7), let $x_k=0$ to get $f(\lambda)=\exp(r\lambda)f(0)$. The lemma follows by substituting t for λ. ■

By Lemma 1 and Theorem 3, in the case $m_g=1$ for all g, the only CSF satisfying A1, A2, A4, A5, and A7 is the logistic CSF proposed by Hirshleifer (1989, 1991). With groups consisting of several players, (7) is satisfied, for example, by $f_k(\mathbf{x}_k)=\exp\left(\sum_{i=1}^{m_k}x_{ik}\right)$. This functional form will be studied in more detail in the next section. It is, however, not the only functional form satisfying (7). Another example is as follows. Let $m_g=2$ and $f_g(x_{1g},x_{2g})=\exp\left(\sin(x_{1g}-x_{2g})+(x_{1g}+x_{2g})\frac{r}{2}\right)$ where $r>2$.

3.3 Summation

Many papers have assumed that only the sum of the efforts of the individuals in the same group matters. To give an axiomatic foundation for this, consider the following axiom.

Axiom 8 *Fix any $\Delta>0$ such that $\Delta\leq x_{ig}$ for all $i\in\{1,...,m_g\}$. Define*

$$\hat{\mathbf{x}}_g^{ij}:=\left(x_{1g},...,x_{(i-1)g},x_{ig}-\Delta,x_{(i+1)g},...,x_{(j-1)g},x_{jg}+\Delta,x_{(j+1)g},...,x_{m_g g}\right).$$

Then

$$p_g\left(\mathbf{x}_g,\mathbf{x}_{-g}\right)=p_g\left(\hat{\mathbf{x}}_g^{ij},\mathbf{x}_{-g}\right)$$

for all $i,j\in\{1,...,m_g\}$ *and all* $g\in\Gamma$.

In words, A8 says that if one member of group g puts in more effort while another member of the same group reduces his efforts by the same amount, the probability of group g winning is unaffected. A8 seems reasonable, for example, when efforts are amounts of money that are pooled within a given group. It is less reasonable in other applications, such as team sports. A8 implies within-group-anonymity (A3'), but not vice versa: for example, (6) with $\alpha\neq 1$ satisfies A3', but not A8.

Proposition 1 *If the contest success function satisfies A1, A2, A4, A5, and A8, then it satisfies (2) and*

$$f_g\left(\mathbf{x}_g\right)=\phi_g\left(\sum_{i=1}^{m_g}x_{ig}\right)\tag{10}$$

where $\phi_g:\mathbb{R}_+\rightarrow\mathbb{R}_+$ *is non-negative and strictly increasing.*

Proof. By Theorem 1 and A8,

$$f_g\left(\mathbf{x}_g\right)=f_g\left(\frac{1}{m_g}\sum_{i=1}^{m_g}x_{ig},...,\frac{1}{m_g}\sum_{i=1}^{m_g}x_{ig}\right).$$

Defining

$$\phi_g\left(\sum_{i=1}^{m_g}x_{ig}\right)=f_g\left(\frac{1}{m_g}\sum_{i=1}^{m_g}x_{ig},...,\frac{1}{m_g}\sum_{i=1}^{m_g}x_{ig}\right)$$

completes the proof. ■

A CSF as characterized in Proposition 1 has been used in Skaperdas (1998) and in Inderst, Müller, and Wärneryd (2007). Adding Homogeneity (A6) results in the following generalization of the Tullock CSF:

Proposition 2 *If the contest success function satisfies A1, A2, A4, A5, A6, and A8, then it satisfies (2) and*

$$f_g(\mathbf{x}_g) = a_g \left(\sum_{i=1}^{m_g} x_{ig} \right)^r \tag{11}$$

where $a_g, r > 0$ are parameters.

Proof. From Proposition 1 and Theorem 2, we have (10) and that ϕ_g is homogeneous. Proposition 2 follows from the fact that a homogeneous function of one variable is a multiple of a power function (Carter 2001, p. 351). ■

CSFs as characterized in Proposition 2 have been used, for example, in Skaperdas (1998), Garfinkel (2004a, 2004b), Inderst, Müller, and Wärneryd (2005), and Münster (2007). The limiting case where $r \to \infty$ is the all-pay auction considered in Baik, Kim and Na (2001) and in Konrad (2004). Münster and Staal (2007) use a logistic CSF as characterized in the following Proposition 3.

Proposition 3 *If the contest success function satisfies A1, A2, A4, A5, A7, and A8, then it satisfies (2) and*

$$f_g(\mathbf{x}_g) = a_g \exp\left(r \sum_{i=1}^{m_g} x_{ig} \right) \tag{12}$$

where $a_g, r > 0$ are parameters.

Proof. From Theorem 3, Proposition 1, and Lemma 1. ■

Assuming A8 can make a difference for equilibrium characterizations and comparative static results in models of group contests. For each group, the impact $f_g(\mathbf{x}_g)$ can be thought of as a public good for group g. If it is assumed that groups cannot enforce individual efforts, given $\mathbf{x}_{-g}$, members of group g play a game of private provision of a public good. The impact function f_g describes the production technology of the public good; it is similar to what is called the social composition function in the literature on private provision of public goods. Properties of the social composition function are important for results concerning free riding and comparative statics, in particular concerning inequality (see Hirshleifer 1983, Cornes 1993, Ray, Baland, and Dagnelie 2007). Clearly, this is relevant for models of group contests.

4 Conclusion

This paper extends Skaperdas' (1996) and Clark and Riis' (1998) axiomatic foundation of contest success functions to contests between groups. It thereby gives a foundation to many contest success functions that have frequently been used in the literature on group contests.

References

[1] Aczél, J., 1969, On applications and theory of functional equations, Birkhäuser Verlag, Basel.

[2] Aczél, J., 1966, Lectures on functional equations and their applications, Academic Press, New York.

[3] Arbatskaya, M. and H. M. Mialon 2007, Multi-Activity Contests, working paper.

[4] Baik, K.H., 2008, Contests with group-specific public-good prizes, Social Choice and Welfare, 30, 103–117.

[5] Baik, K.H., I. -G. Kim, S. Na 2001, Bidding for a group-specific public-good prize, Journal of Public Economics, 82(3), 415-429.

[6] Blavatskyy, P. 2004, Contest success function with the possibility of a draw: axiomatization, working paper.

[7] Carter, M. 2001, Foundations of mathematical economics, MIT Press, Cambridge, Massachusetts.

[8] Caruso, R. 2006, Conflict and Conflict Management with Interdependent Instruments and Asymmetric Stakes (The Good-Cop and the Bad-Cop Game), Peace Economics, Peace Science, and Public Policy, Volume 12, Issue 1 Article 1.

[9] Clark, D. J., and K. A. Konrad, 2007, Asymmetric conflict: Weakest link against best shot, Journal of Conflict Resolution, 51(3), 457-469.

[10] Clark, D. J., and C. Riis, 1998, Contest success functions: an extension, Economic Theory, vol. 11(1), 201-204.

[11] Corchón, L. C., 2007, The theory of contests: a survey, Review of Economic Design, 11(2), 69-100.

[12] Cornes, R. 1993, Dyke Maintenance and Other Stories: Some Neglected Types of Public Goods, The Quarterly Journal of Economics, Vol. 108, No. 1 (Feb., 1993), pp. 259-271.

[13] Dupuy, T.N. 1987, Understanding war: history and theory of combat, New York: Paragon House Publishers.

[14] Epstein, G.S., and C. Hefeker, 2003, Lobbying contests with alternative instruments, Economics of Governance 4(1), 81-89.

[15] Epstein, G.S., and S. Nitzan, 2007, The politics of randomness, Social Choice and Welfare 27, 423-433.

[16] Epstein, G.S., and S. Nitzan, 2008, Endogenous Public Policy and Contests, Springer, Berlin.

[17] Esteban, J. and D. Ray, 2001, Collective Action and the Group Size Paradox, American Political Science Review Vol. 95(3), 663-672.

[18] Garfinkel, M. 2004a, On the stability of group formation: managing the conflict within. Conflict Management and Peace Science 21, 1-26.

[19] Garfinkel, M. 2004b, Stable alliance formation in distributional conflict, European Journal of Political Economy.

[20] Garfinkel, M. and S. Skaperdas, 2006, Economics of Conflict: An Overview, forthcoming in T. Sandler and K. Hartley (eds.), Handbook of Defense Economics, Vol. 2, Chapter 22.

[21] Hirshleifer, J. 1983, From weakest-link to best-shot: The voluntary provision of public goods, Public Choice 41, 371-386.

[22] Hirshleifer, J. 1989, Conflict and rent-seeking success functions: Ratio vs. difference models of relative success. Public Choice 63, 101-112.

[23] Hirshleifer, J. 1991, The technology of conflict as an economic activity. American Economic Review Papers and Proceedings 81, 130-134.

[24] Inderst, R., H. Müller, and K. Wärneryd, 2007, Distributional conflict in organizations, European Economic Review, vol. 51(2), 385-402.

[25] Inderst, R., H. Müller, and K. Wärneryd, 2005, Influence costs and hierarchy, Economics of Governance, Springer, vol. 6(2), pages 177-197.

[26] Jehle, G. und P. Reny, 2001, Advanced Microeconomic Theory, 2nd ed., Addison Wesley, Boston.

[27] Jia, H., 2008, A stochastic derivation of the ratio form of contest success functions, Public Choice 135, 125–130.

[28] Katz, E., S. Nitzan, and J. Rosenberg, J. 1990, Rent-seeking for pure public goods, Public Choice 65: 49–60.

[29] Katz, E. and J. Tokatlidu 1996, Group competition for rents. European Journal of Political Economy 12, 599-607.

[30] Konrad, K. A. 2007, Strategy in Contests – an Introduction, WZB Discussion Paper SP II 2007-01.

[31] Konrad, K. A. 2004, Bidding in hierarchies, European Economic Review 48, 1301-1308.

[32] Müller, H. M. and K. Wärneryd 2001, Inside versus outside ownership: a political theory of the firm, RAND Journal of Economics 32(3), 527-541.

[33] Münster, J. 2004, Simultaneous inter- and intragroup conflicts, mimeo. Available at:
http://www.socialpolitik.org/tagungshps/2004/Papers/Muenster.pdf

[34] Münster, J. 2007, Simultaneous inter- and intragroup conflicts, Economic Theory, 32(2), 333-352.

[35] Münster, J., and K. Staal, 2007, War with outsiders makes peace inside, mimeo.

[36] Nitzan S. 1991, Collective rent dissipation, Economic Journal, 101, 1522-1534.

[37] Nitzan, S. 1994, Modelling rent-seeking contests, European Journal of Political Economy 10(1), 41-60.

[38] Rai, B. K., and R. Sarin 2007, Generalized contest success functions, forthcoming in Economic Theory.

[39] Ray, D., Baland, J.-M., and Dagnelie, O. 2007, Inequality and Inefficiency in Joint Projects, Economic Journal, Vol. 117, No. 522, pp. 922-935.

[40] Skaperdas, S. 1996, Contest success functions, Economic Theory 7, 283-290.

[41] Skaperdas, S. 1998, On the formation of alliances in conflict and contests, Public Choice 96: 25–42.

[42] Skaperdas, S. and Syropoulos, C., 1997, The Distribution of Income in the Presence of Appropriative Activities, Economica, February 1997, 64, 101-117.

[43] Wärneryd, K. 1998: Distributional conflict and jurisdictional organization, Journal of Public Economics 69, 435-450.