Determining The Value-at-risk In The Shadow Of The Power Law: The Case Of The SP-500 Index

Dominique, C-Rene; Rivera-Solis, Luis Eduardo and Des Rosiers, Francois

Independent Researcher, Dowling College, Laval University

09. May 2010

DETERMINING THE VALUE AT RISK IN THE SHADOW OF THE POWER LAW:

THE CASE OF THE SP-500 INDEX

C-René Dominique, Luis Eduardo Rivera-Soli, and François Des Rosiers*

ABSTRACT: In extant financial market models, including the Black-Scholes' contruct, the dramatic events of October 1987 and August 2007 are totally unexpected, because these models are based on the assumptions of 'independent price fluctuations' and the existence of some 'fixed-point equilibrium'. This paper argues that the convolution of a generalized fractional Brownian motion (into an array in frequency or time domain) and their corresponding amplitude spectra describes the surface of the attractor driving the evolution of prices. This more realistic approach shows that the SP-500 Index is characterized by a high long term Hurst exponent and hence by a 'black noise' with a power spectrum proportional to $f^{-\beta}$ ($\beta > 2$). In that set up, the above dramatic events are expected and their frequencies are determined. The paper also constructs an exhaustive frequency-variation relationship which can be used as practical guide to assess the 'value at risk'.

Keywords: Market Collapse, Fractional Brownian Motion, Fractal Attractors, Maximum Hausdorff Dimension of Markets and Affine Profiles, Hurst Exponent, Power Spectrum Exponent, Value at Risk. JEL Classification System: C90, G10.

I - INTRODUCTION

The process known as Brownian motion (B(t)), first observed by Robert Brown (1773-1858), the botanist, has found applications in many other areas, mainly due to its scale and time invariance properties. For example, it has inspired Karl Weierstrass (1815-1897) to develop a whole new class of functions that bear his name today. B (t) was next introduced to mathematical physics by Albert Einstein (1879-1953) and Marian Smoluchowski (1872-1917), and refined by another physicist Norbert Wiener (1894-1964), giving what is known today as a 'Wiener Process'. B (t) is also applied in medicine, chemistry, telecommunication, thermodynamics, etc. Its generalization is the foundation of the field of tomography where it is successfully used to model a variety of natural phenomena such as attractors' surfaces, terrains, coastlines, clouds, etc.

Bachelier (1900) was perhaps the first to import the regular B (t) into mathematical finance. That idea resurfaced in Chicago in the early 1950s where financial theorists began to use it to model a variety of stochastic processes. However, its importation *in extenso* into mathematical finance has not always been an enlightening experience.

To begin with, Bachelier was so convinced that commodity prices evolve according to a B (t) that he concluded, rather boldly, that "the mathematical expectation of a speculator is zero." That belief was resuscitated in the 1950s to support the construction of stochastic models based on continuous and discrete-time random walk (Kendall, 1953; Osborne, 1959; and more recently, Malkiel, 1973, among others). That line of research subsequently gave rise to the notion of Efficient Market Hypothesis (EMH) (Fama and Blume, 1966) and to more sophisticated stochastic partial

----

*Formerly Titular Professor of Applied Economics, Laval University, rdom1@Netzero.net

Professor of Finance, Dowling College, RiveraL@dowling.edu

Professor of Finance, Insurance and Real Estate, Laval University, francois.desrosiers@fsa.ulaval.ca.

differential equations describing optimal investment or stock and derivative pricing strategies (Merton, 1969, 1973; Black and Scholes, 1973; Cox et al., 1979).

Despite numerous criticisms (Basu, 1977; Shiller, 1981; Lo, 1999), however, these constructs (the later vintage in particular) remain the foundation of financial analyses until the cumulated effects of 'black Tuesday' in October 1987 and the collapse of the credit market in August 2007 sounded their death knell.

Many factors underline this dramatic failure. First, the naïve but convenient assumption of 'independent increments' is perhaps the most damaging, because it led theorists straight to the normal distribution to assess risk. A close second is the failure to focus on the properties of the often unknown attractor driving the evolution of any market, including the capital market. Last but not least, there is the ubiquitous data problem that plagues socioeconomic times series.

Admittedly, the later problem is less severe in financial analyses where the signal-to-noise ratio is higher than in macroeconomic time series. However, financial data may well escape the pitfalls of aggregation and arbitrary sampling intervals, but they appear to be sensitive to 'inherent' noise and to policy and technology shocks; some of these may permanently alter the statistical properties of these series. It goes without saying that financial time series, as any socio-economic ones, should be well prepared before they are analyzed. However, in so doing, one stumbles on the necessity to question the supposition that real world data must satisfy the mathematical requirements of stochastic calculus.

The weakness of models based on independent increments are well documented in Mandelbrot (1971, 1977), in Greene and Fielitz, 1977), in Cutland *et al*., 1995, and in Peters, 1994, among others. Mandelbrot and van Ness (1968), for example, have long ago and convincingly argued for the use of 'fractional Brownian motion', (B^H(t)), in financial analyses. A few have paid heed to that recommendation and have developed new constructs that fall into two categories. Namely, 'fractional stochastic volatility' models along the line of Heston (1993) and Comte and Renault (1998), and 'extension of the Black-Scholes'' model to B^H (t) (Sottinen, 2001; Cheredito, 2003, among others). There are problems with both categories. In the first category, the main difficulty stems from the fact that B^H (t) and volatility are not Markovian, while the second category allows for arbitrage opportunities. There is a third category that tries to exclude arbitrage opportunities (e. g. Guasoni, 2006), but models falling into that category do not seem to have any clear economic interpretation. Therefore, beyond the mere data problem, it is quite proper to ask also whether or not these drawbacks are mainly due to the fact that B^H (t) is neither a semi martingale nor a Markov process?

The extant tools of stochastic calculus were developed for time series meeting three strong conditions to which we will return. It suffices now to say that this study will show that socio-economic time series cannot faithfully satisfy these conditions. Yet, theorists in both economics and finance rely on these very socio-economic time series to gauge the properties of the capital market, although they approach the analysis in a different manner, which we will now try to reconcile into a simpler format.

Economists are primarily interested in distinguishing deterministic chaos from pure randomness, employing three procedures. That is, the determination of the correlation dimension (d^c) (see Grassberger and Procaccia, 1983), the computation of Lyapunov exponents (L_c) (Lyapunov, 1907; Osedelec, 1968), and the Singular Spectrum Analysis (SSA) (Medio, 1992). All three procedures happen to be sensitive to the noise inherent in macro-economic time series, and they all fail to distinguish unambiguously between randomness and the fractal nature of socio-economic processes.

The presence of inflation and growth in socio-economic time series requires filtering which, in turn, requires efficient algorithms and analytical skill. When the law of motion of a dynamical system is unknown, the SSA method appears more reliable, but it is still not known how the filtering of SSA affects the statistics sought after. According to Medio (1992), there are two remaining pitfalls. The averaging implicit in the projection of a series onto a new basis could eliminate certain deterministic components of the motion, in particular the high frequency components. The second potential drawback relates to the effect of noise on the estimation of the pertinent statistics. The reason is that the embedding space of the attractor is divided into a 'deterministic' subspace (where orbits would stay in the absence of noise) and a 'stochastic' subspace (in which motion is due to noise). If all the noise has not been eliminated by filtering, all the positive eigenvalues will be raised by an amount corresponding to the 'noise floor'; meaning that the computed properties of the attractor will be affected. We should add a third minor drawback relative to the notions of accuracy and precision to which we will return.

In financial analysis, on the other hand, i. e., in models with which we are familiar, no explicit reference is made to the attractor driving the process. Yet, the reconstruction of the unknown attractor should be the ultimate goal of the analysis of time series. This paper will attempt to address this omission. In fact, attempting to identify the properties of the attractor may solve an extant puzzle. For, it is observed that regardless of the size of the embedding dimension, the attractor driving the dynamic process is always confined within dimensions two and three. Why? We believe that focusing on the attractor may provide an answer to this heretofore unanswered question.

The main purpose of this paper is twofold. To show first that in man-made processes such as the financial market, the three strong conditions allowing processes to be described by a regular Brownian motion are violated. Price variations evolve according to an imperfect B^H (t)) instead; hence, it is improper to assess risk on the normal distribution. The second objective is to then construct a relationship between frequencies and the intensities of variations within a self-affine distribution that can serve as a practical guide to assess the 'value- at- risk'.

The paper is organized as follows. Part I states the problem. Part II examines the reasons why the above three conditions, namely, scale invariance, time invariance, and independence of increments are violated, followed by the derivation of the three fundamental 'descriptive statistics' necessary to uncover the fractal nature of the capital (or any) market. At the same time, we will attempt to assess the impacts of both 'weak stationarity' and 'noise' on the estimates. Part III discusses the data. Part IV brings to the fore the potential negative impact of the noise floor present in the SP-500 Index, shows that that process is governed by a power law whose spectrum exponent $\beta > 2$, and constructs the relationship frequency-intensity. Part V discusses our overall results, and summarizes our conclusions.

II- THEORETICAL CONSIDERATIONS

Let $\xi(.), p))\ \varepsilon\ \Re^n$ be an unknown discrete-time dynamical system, represented by a system of difference equations, embedded in a space of dimension n. Taken's (1981) Theorem asserts that it is possible to extract all the pertinent information needed for decision-making from a univariate time series {p (t)} of dimension $m < n$, as long as {p (t)} is an output of the system ξ (.).

Consider then the sequence:

$$p\ (t) = \{P_\tau (t)\}^{m-1}{}_{k=1}, \quad (1)$$

where P (t), P (t + τ_k), P (t + $2\tau_k$), . . ., P (t + m-$1\tau_k$) are elements of a vector defining a point in $\Re^m$. Following the evolution of that vector, one obtains an orbit in a subspace $F \subset \Re^m$. If m is large enough relative to n and if τ is sufficiently small, then the m-dimensional image obtained from (1) will be a true image of the unknown attractor of the system. This is the first step that legitimizes the use of time series in both economics and finance.

Assuming that p (t) evolves as a fractional Brownian motion, B^H (t), it can be shown that three descriptive statistics suffice to describe both the appropriateness of the assumption and the properties of the unknown attractor. They are:

The Hurst's (1951) Exponent H, defined as:

$$H = \log (R/S) / \log (\Delta t), \quad (2)$$

where R is the range, S is the standard deviation, Δt is the interval, and $0 < H < 1$. H is a global characteristic (or a long memory process) that indicates in this case the ruggedness of the attractor's surface. H is positively related to the spectral exponent, β.

The Power Spectrum. Statistical time series are governed by the homogeneous power law in the form of $f^{-\beta}$, where β is the squared magnitude of the Fourier Transform. We will have more to say about β in the next section, but for now it suffices to note that β is negatively related to the Hausdorff dimension, d^H, over the range of H.

The Hausdorff (or fractal) dimension is defined as:

$$d^H = \lim_{r\to 0} [\log (B(r, p_i) / \log (1/r)], \quad (3)$$

where B (.) stands for the number of balls of radius r necessary to cover (1). d^H is often a non-integer dimension-like quantity that measures the way orbits fill up the phase space under the action of the flow of the attractor. Thus, a non-integer dimension indicates that orbits do not fill up all the available space. It is also the upper limit of the correlation dimension, d^c. Often d^H and d^c are used interchangeably in informal discussion, but it should be borne in mind that d^c ($\leq d^H$) is a probability measure indicating the frequency with which orbits visit different parts of the attractor. d^H is negatively related to both β and H, and it is the upper limit of d^c.

These descriptive statistics, which are easily computable, suffice to describe the attractor. But before computing them, it would be instructive to show why such a computation is a preferred alternative due to the limitations of socioeconomic time series in meeting the requirements of stochastic calculus.

Fractional Brownian Motion $B^H(t)$ is a generalized Gaussian process with mean functional zero and covariance functional as a fractional integral or a fractional integral derivative of the Delta- function. Using the integral form introduced by Mandelbrot and van Ness (1968) for H ε [0, 1], (s, t) ε $\Re$, $t \geq 0$, It can easily be shown that B^H (t) must satisfy three basic conditions. That is, scale invariance, time invariance, and independence of increments. However, dealing with real world data presents a number of difficulties. Consider the following:

Contrarily to the ease of mathematics, the self-similarity property or the scale invariance property of natural processes such markets is limited to finite ranges. Real world self-similar process has a largest and a smallest scale. It is now known that a regular B (t) scales over a range of 10 to one. But what is it for a B^H (t)? No one seems to know the answer.

If the statistical properties of (1) do not vary over time, then it is stationary and the spectral density function has a physical interpretation of a power frequency distribution. But, socio-economic time series are subject to noise, to policy and technology shocks, which may have long lasting effects, and which would make (1) non-stationary. Then, as such, (1) would not admit a power spectrum in the conventional sense. Some sort of average spectrum equal to $\sigma^2/ |\omega|$ must be assigned to it. Non-stationarity also affects all three descriptive statistics. For example, Casti (1995, 256) reports on a value of H = 0.78 for the SP-500 Index, based on monthly returns from January 1950 to July 1988. Bayraktur *et al*. (2003) use Wavelet analysis to estimate H from price data, over different segments of stationarity, from January 1989 to May 2000, sampled at one-minute intervals. For segments of length 2^{12}, H = 0.6156 with a standard deviation of σ = 0.0531. For segments of length 2^{14}, the value of H falls to 0.6011; for segments of length 2^{15}, H = 0.60008, etc.; however, the level of the noise floor is not mentioned in these two studies. James Preciado, *et al*. (2008), estimate H of the SP-500, from August 2007 to April 2008, at 0.7936, but form January 2003 to July 2007, the value falls to 0.6946. These differences may be due to noise, non-stationarity, or due to the fact that H is not constant in the short run. The H exponent seems to vary with the sampling interval, with the range, and with series' length. In fact, we will assume that series' length is crucial and that H is long term characteristic. We will then test for the stationarity of the SP-500 by analyzing it in three segments. That is, from (1928-1945), from (1946-2009) and from (1928-2009), via two different transforms (Fourier and Wavelet). Hopefully, this will shed light on both the constancy of H and the extent of non-stationarity if any. Anyway, for our estimates, we will rely on the longer segment as series' length is an important factor for the stability of the estimates. In addition, we believe that the attractor is a multi-fractal and that the spectra of its frequency intervals follow the power law with different exponents; hence, with different H values.

According to the form introduced by Mandelbrot and van Ness (1968), denoting E (.) the mathematical expectation, then the mean functional is $E(B^H(t)) = 0, \forall t \varepsilon \Re$. The covariance kernel is therefore:

$$\text{(4)} \qquad \mathrm{Cov}(t_1, t_2) = E[B^H(t_1) B^H(t_2)] = 1/2 [|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}].$$

This property refers to disjoint increments that are either positively or negatively correlated. This usually referred to as 'long range dependence' or persistence if $H > 1/2$ or anti-persistence if $H < 1/2$. In other words, if $H \neq 1/2$, because economic agents remember the past, $B^H(t)$ is neither a Markovian process nor a semi-martingale. A semi-martingale is a natural process for which stochastic calculus can be developed, and can be expressed as a local martingale which has a finite quadratic variation and the sum of a bounded variation process. $B^H(t)$ cannot be one, because for $H < ½$, the quadratic variation $\rightarrow \infty$, and for $H > ½$, it is zero. Hence, the conventional tools of stochastic calculus are not directly applicable.

These non-conformities lead to a more fundamental question. That is, what do accuracy and precision mean in the social sciences? In a domain such as physics, the 'accuracy' of measurements and 'closeness' to actual values of entities are defined in terms of 'fixed references' such as the wavelength or the speed of light, atomic vibrations, the Plank constant, etc. In the social sciences, there are no fixed references to go by. These domains, including the brain of agents (see Falconer, 1990), belong to "fractal-land" where forms are irregular and measurements are often fuzzy. What comes close to a constant is the Hausdorff dimension of a fractal entity. Alternatively, it is the ratio of the logarithm of the increase in the linear dimension to the logarithm of the decrease in the unit of measurement. Hence, the dimension of a fractal entity depends on the size of the unit used to measure it.

For all these reasons, the $B^H(t)$ referred to in natural processes are pseudo-fractional Brownian motions. We could test for these properties and even force them to comply, but we risk destroying their subtle characteristics. This also explains why we can only deal with approximations 'reasonable' enough to guide decision-making. With this qualification duly noted, we will now turn to the derivation of the descriptive statistics.

The Generalization of $B^H(t)$. If (1) evolves as a pseudo $B^H(t)$, it can then be generalized to a $B^H(\mathbf{t})$, where **t** is now a vector in $\Re^2$. If next we consider positive increments Δt_1 and Δt_2, $B^H(\mathbf{t})$ becomes a function $g: \Re^2 \rightarrow \Re$, giving the elevation of a point $(\Delta t_1, \Delta t_2)$ on the surface of $B^H(\mathbf{t})$. Then,

$$g\{[(t_1 + \Delta t_1), (t_2 + \Delta t_2)] - [g(t_1, t_2)]\}$$

has a centered Gaussian distribution with variance given by:

$$\text{(5)} \qquad \sigma_B^2 = [(\Delta t_1)^2 + (\Delta t_2)^2]^H.$$

The exponent is H for $B^H(.)$ instead of 1/2 for a regular B (.). It follows that $B^H(\mathbf{t})$ has a surface index H which accounts for the difference between d^H_s of the surface and the embedding dimension, m_s. Positing $\beta = 2H + 1$, the Hausdorff dimension, $d^H_{s,}$, of the *surface,* which is identical to that of the attractor, is:

(6) $$d^H_s = m_s - (\beta - 1) / 2 = (2\, m_s + 1 - \beta) / 2.$$

If $m_s = 3$, $H = ½$, and β (B (.)) = 2 for an ordinary B (**t**), then d^H_s cannot exceed 2.5. Indeed, we expect d^H_s to be lower for H > 1/2 in B^H (**t**).

The *affine trace* is obviously in $\Re^2$, then, its fractal dimension is:

(7) $$d^H_a = (2m_a + 1 - \beta) / 2.$$

This last equation follows from (6), which incidentally is identical to Schroeder's (1992, 135-137), except that here the derivation is more straightforward.

The Equivalent Operation of Convolution. In practice, i. e., in the field of tomography, for example, (6) is obtained by convolving arrays in frequency domain (f) or in time domain (t) and the amplitude spectra of B^H (.) (see, Moreira,1994). Let $\mathbf{f} \in \Re^2$ and $\mathbf{t} \in \Re^2$ stand for the frequency and time vectors, respectively, with lengths $(f_1^2 + f_2^2)^{1/2}$ and $(t_1^2 + t_2^2)^{1/2}$. Define now two arrays **M** (f_1, f_2) and **M** (t_1, t_2) and let their corresponding amplitude spectra be as given below. Then:

(8)
$$\begin{array}{ccc}
\mathbf{M}(f_1, f_2) * \left| \mathbf{f}^{-(H+1/2)} \right| & \rightarrow \Gamma_k = \Sigma_{n=0}^{N-1} \Gamma_n \exp(-2\pi kn/N) & \rightarrow \text{time domain} \\
\Gamma \downarrow \quad \Gamma^{-1} \uparrow & & \Gamma \downarrow \quad \Gamma^{-1} \uparrow \\
\mathbf{M}(t_1, t_2) * \left| \mathbf{t}^{-(H+1/2)} \right| & \rightarrow \Gamma_n = (1/N)\, \Sigma_{k=0}^{N-1} \Gamma_k \exp(2\pi kn/N), & \rightarrow \text{frequency domain}
\end{array}$$

where k = 0, 1, . . . , N-1; n = 1, 2, . . . , N-1; Γ_k is the forward Discrete Fourier transform, Γ^{-1} is the inverse discrete Fourier transform, and $*$ stands for the convolution operation.

Either operation can be selected, but one must bear in mind that the transform from spatial domain to frequency domain quantifies the higher frequencies more coarsely. The Discrete Fast forward Fourier transform should be used to transform from frequency to time domain if the series is stationary. On the other hand, as Wavelet Transform with multi-resolution is a time-scale method and is more precise and convenient, in particular when high frequency components are present; it should be used to compute the Hurst exponent. We will use both transforms to check their performance first and, second, because the product of the convolution or the Fourier transform describes the surface of the attractor. The validity of this last assertion then explains why the fractal dimension (d^H_s) is confined between dimensions 2 and 3. We dare say in passing that it will always be so regardless of the dimension in which the attractor itself is embedded. We should note, however, that (6) applies in the general case; e. g., the Hausdorff dimension of a B (t) in an embedding space m is: (m – ½).

Let us now turn to Eqs. (6) and (7), as $H \in [0, 1]$, the connection between the three descriptive statistics and the color scheme used in physics is given in Table 1 below. The waveform in the color metaphor is another indication of the difference between a B (t) and a B^H (t). According to Schroeder (1992), a $\beta > 2$ is a black noise accounting for violent swings in market processes and other natural disasters. Anyhow, if (1) evolves as a B^H (t), the data will show that $H \in (1/2, 1]$.

Table 1: Descriptive Statistics and the Color Scheme

H	β	d^H_s	d^H_a	Noise Color
-	0	-	-	white
0	1	3	2	pink
1/2	2	2.5	1.5	brown
1	3	2	1	black

With the theoretical values of Table 1 in mind, we will now proceed to determine empirically the β of the SP-500 Index. If it found to be higher than 2, that will explain the violent and unexpected swings in prices and returns in October 1987 and August 2007.

III THE DATA

The SP-500 Index is a market value weighted index of stock prices times the number of shares outstanding in which each stock's weight is proportional to its market value. We examine its monthly closing prices, sampled at regular intervals from December 1928 to December 2009, from 1928-1945, and from 1946-2009 in two software packages. That is, the $\text{Benoit}_{\text{TM}}$ which is particularly well suited for Rescaled/range and power spectrum analyses, using both Fourier and Wavelet transforms. Benoit also makes use of various methods of filtration for white noise. One of them works by conducting a transform by modification of the transform coefficients and then making the reverse transform, thereby removing data points that contribute uncorrelated noise to the data set. The coefficients are modified by 3 parameters. It should be noted, however, that the combination horizontal/hard, the most common in scientific applications, removes high frequencies from the data. While horizontal/soft does the opposite. Since we would like to examine both high and low frequencies, we will use both Fourier and Wavelet transforms to size up differences in the estimates. Thus, with the Benoit software, we will first compute the power spectrum and the Hurst exponent. We will do likewise with the two transforms and relate their values to the Hausdorff dimensions of both the surface of the attractor and the self-affine profile so as to determine the concordance of the estimates relative to the constancy of H and as an indication of the impact of non-stationarity.

The other software we use is the SPSS package which is more suitable to the computation of the frequency distribution. Financial data is reputed to have high signal-to-noise ratio. To verify that assertion, we de-trend the SP-500 series using logarithmic difference. This series is identified as $\textit{Log-SP500}_{d2}$. Next we filtered $\textit{Log-SP500}_{d2}$ for white noise; that new series is identified as $\textit{Log-SP500}_{d2\ ftrd}$. If the series is found to be at least trend-stationary, we will use the *weighted means* to center both series. Fluctuations about the means are recorded and tallied in three fashions, i. e., absolute, positive and negative. This will allow us to size up the impact of the noise floor, if any, on the estimates. However, in any event, for the computation of the Hurst exponent, power spectrum exponent, Hausdorff dimension, and the frequency-.deviation relationship, the stationary, de-trended and filtered series, i. e., $Log\text{-}SP500_{d2ftrd}$ will be used.

The distributions of deviations in both series will be compared to the normal distribution. The purpose here is to show that the close resemblance of the distributions may be misleading as it may obscure some subtle differences. But, more importantly, we wish to stress the difference between evaluating risk on the normal curve and risk assessment based on fractal distributions.

IV- THE RESULTS

As regards stationarity, the procedure recommended by Yaglom (1962) is not convenient in this case. Instead, we employ the SDP method on various segments at the beginning and end of the series, although Figure 5 shows that $Log\text{-}SP500_{d2flrd}$ is practically trend-free. No fluctuations are detected in the mean-variance relationship in the segments 1928-2009 and 1946-2009, but there were some fluctuations in the segment 1928-1945. We assign that difference to the insufficiency of the segment length for monthly sampling interval. The segment 1928-2009 shows a slight drift. We attempt to correct sporadic entries using the formula:

$$e_t = \log p(t) + [\ \log (p_t) - \mu] / \mu, \text{ where } \mu \text{ is the weighted mean.}$$

The difference in entry values was about $1/100^{th}$. Since we restrict ourselves to just one decimal place for the descriptive statistics, the additional correction changes nothing. On the basis of these findings, we concluded that the series was clean and stationary enough for the present purpose. The overall results are given in the various tables and figures listed below.

More specifically, Table 2 present the results of the estimation of the power spectrum of the SP-500 series, sampled at one month intervals, in the three segments. Figure 1(a and b) presents the result of the Fourier transform in which the power spectrum β lies between 2.63 and 2.79 and $0.82 \le H \le 0.89$. In Figure 2 which uses Wavelet analysis, $2.72 \le \beta \le 2.76$ and $0.86 \le H \le 0.88$. As it can be seen, the difference is rather insignificant beside the fact that the Fourier Transform uses 984 data points, while the Wavelet Transform uses fewer data points as power of 2. We believe that the differences are small enough to uphold the criteria of quasi-self-similarity and even the property of weak stationarity either from 1928-2009 and 1945-2009.

According to Table 2, the SP-500 evolves as a fractional Brownian motion, governed by a homogeneous power law in the form of $f^{-\beta}$ as a function of frequency f. Its power spectrum is proportional to $f^{-2.6}$ or $f^{-2.7}$. For a β within that range, the system falls into the category of a *black noise* according to Schroeder (1992). This means

Table 2: Descriptive Statistics of the SP-500, 1928-2009, 1928-1945, 1946-2009

	Fourier Transform			Wavelet Transform		
	1928-2009	**1928-1945**	**1946-2009**	**1928-2009 (1)**	**1928-1945 (2)**	**1946-2009 (1)**
H	0.82	0.62	0.89	0.88	0.78	0.86
β	2.63	2.23	2.79	2.76	2.56	2.72
d^H_s	2.18	2.38	2.11	2.12	2.22	2.14
d^H_a	1.18	1.39	1.11	1.12	1.22	1.14

***H** = Hurst Exponent; β = Power Spectrum Exponent; d^H_s = Hausdorff Dimension of the Surface of the Attractor; d^H_a = Hausdorff Dimension of the Affine Profile; (1) 2^9 data points; (2) 2^7data points.

that violent price swings in both positive and negative directions are to be expected. For additional insurance, we present the regression of absolute deviations and frequencies in Table 8, showing a reasonable fit (β = 2.296) despite the division into broad classes of frequencies. And in conformity with experimental findings, the surface of the fractal attractor of the SP-500 Index falls between dimensions 2 and 3. Up to now, there was no explanation for that phenomenon.

From the descriptive statistics, it can be deduced that the segment 1928-2009 is not representative of (1); the segment length appears too short for the sampling interval. Normally, in such cases, the covariance kernel would have to undergo some modification. However, as the spectral properties of segments 1928-2009 and 1946-2009 are compatible, the series is considered an oscillatory discrete parameter process with a kernel representation of the form: $K (s, t) = \int^{\infty}_{-\infty} e^{i\omega(t-s)} dV (\omega)$, where $i = (-1)^{1/2}$. Also, in view of what was said before regarding the notions of accuracy and precision, we consider the statistics for the whole series are sufficient for decision-making.

The simplicity of this approach is now obvious. For, knowing any one of these descriptive statistics allows one to compute the other two. For example, Falconer (2003) reports that for parameters r = 28, b = 4 and Prandlt number σ = 16, the Hausdorff dimension d^H_s of the Lorenz deterministic fractal attractor is 2.06. Then, using Equation (7) or (8), its H = 0.94, and its β = 2.88. We can then conclude that its surface is quite smooth and highly persistent on any one of its three manifolds.

Table 3 and Figure 3 present the statistics and the histogram of $Log\text{-}SP500_{d2}$, while Table 4 and Figure 4 give the same type of information but for $Log\text{-}SP500_{d2ftrd}$. These values can now be compared to determine the impact of the noise floor. Both distributions can also be compared to the normal distribution superimposed. The fractal distributions are clearly leptokurtic. Figure 5, showing the time profile of $Log\text{-}SP500_{d2ftrd}$, is added for completeness.

Figure 1: Power Spectrum of *Log-SP500*$_{d2ftrd}$, 1928-2009, 1946-2009

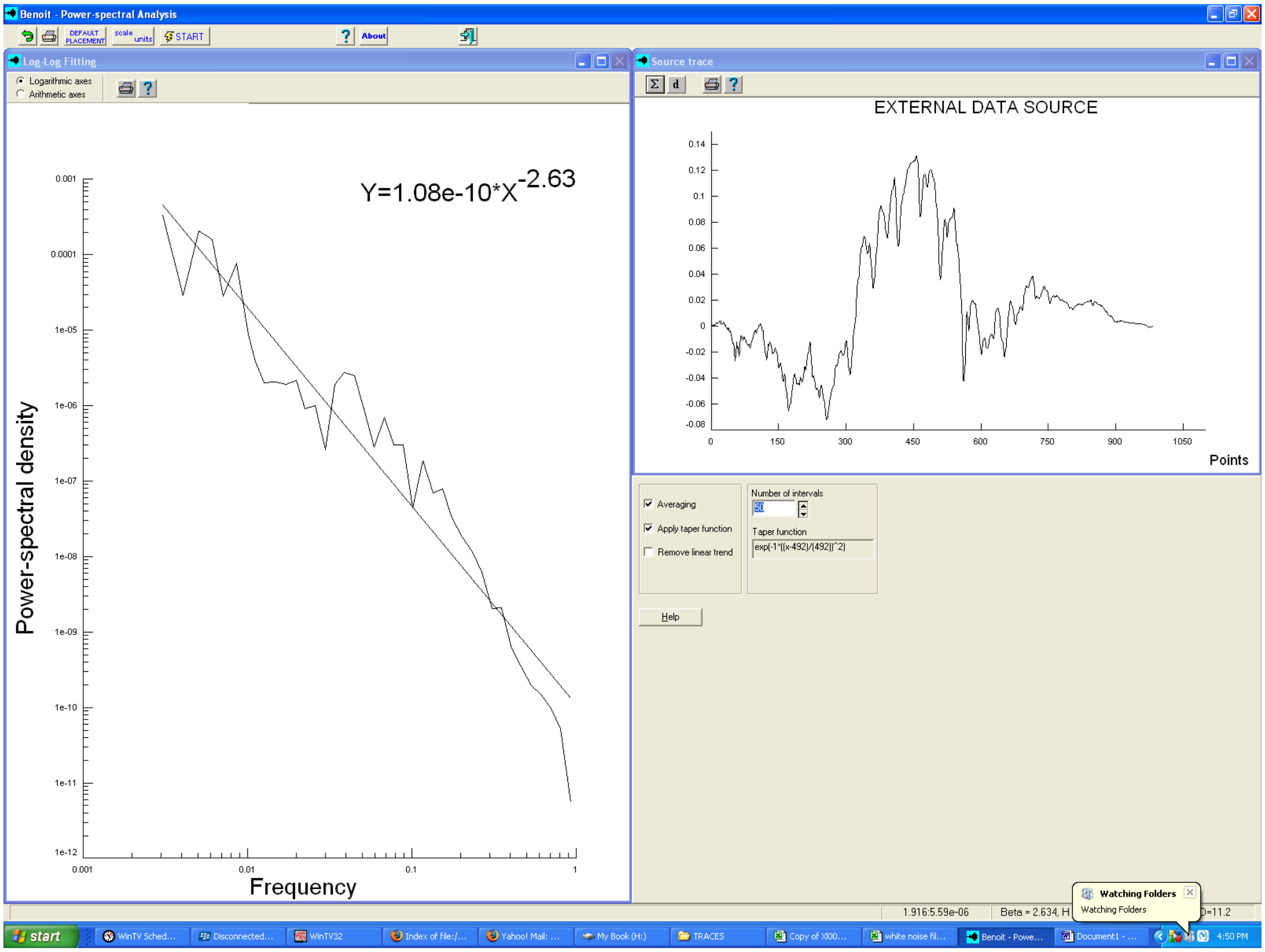


Plate a: 1928-2009, H= 0.817, Beta = 2.634

Plate b: 1946-3009, H = 0.889, Beta = 2.779

Figure 2: Wavelet Analysis of *Log-SP500*$_{d2ftrd}$, 1928 -2009, 1946-2009

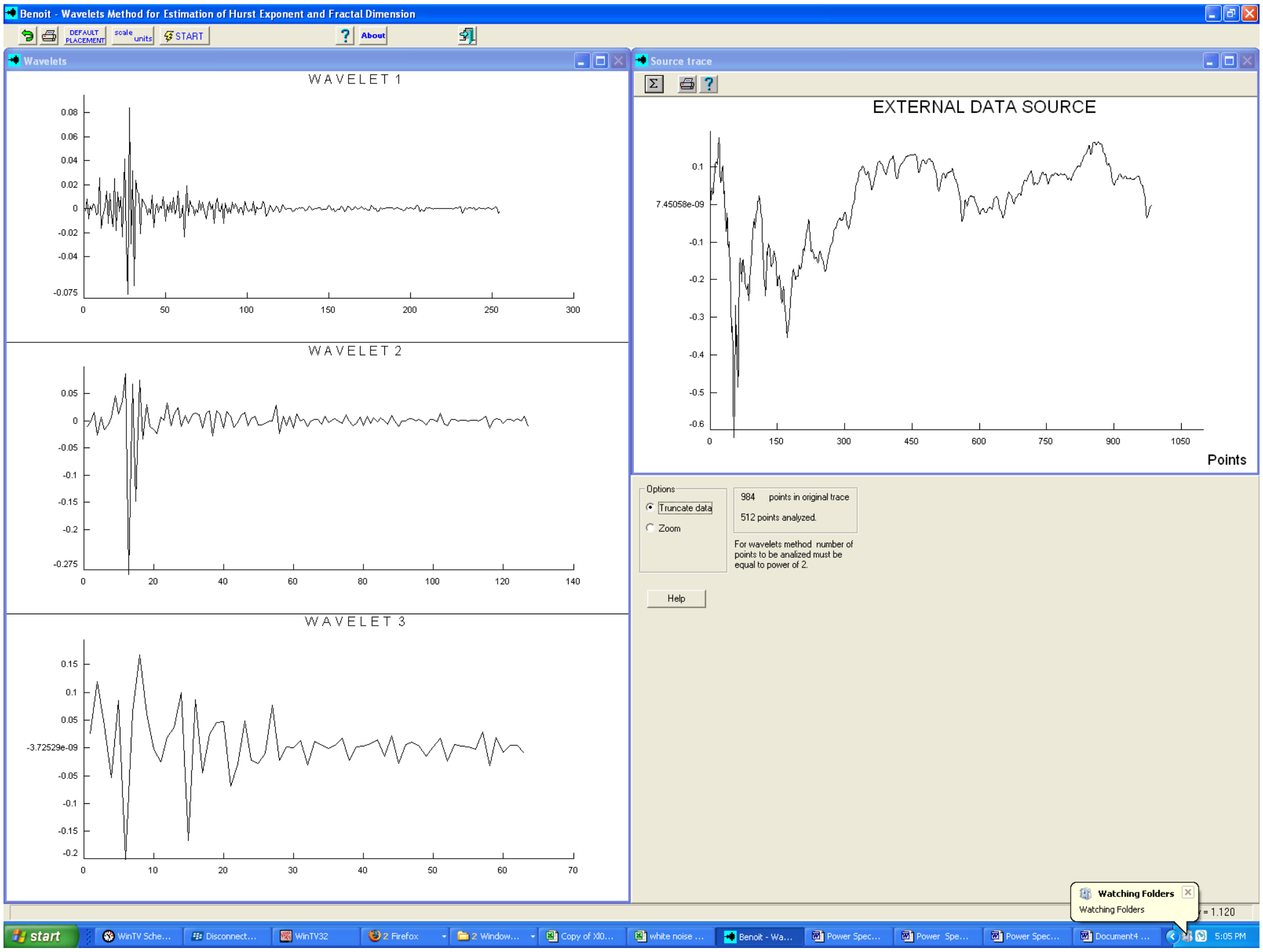


Plate a: 1928-2009, H= 0.880, Beta = 2.76

Plate b: 1946-2009, H = 0.854, Beta = 2.71

The impact of the noise floor can be seen much more clearly at the level of frequency distributions. Tables 6 and 7 show how frequencies are distributed. We compute and tally frequencies grouped in intervals of 9/1000, meaning: 0.000 to 0.009, 0.01 to 0.019, etc. We next compute positive, negative and absolute deviations from the weighted mean on which the series are centered. Table 5 is only for a comparison with Table 6 and 7, which show the more reliable distributions coming from $\textit{Log-SP500}_{d2fted}$. For example, Table 5 shows that there are 521 positive deviations against 463 negative ones. There are also positive deviations at 20, 18, 16, 9 percent, etc; the first negative one appears at 15 percent. It must be borne in mind, however, that in Table 5 the stochastic subspace is not completely empty. When these values are compared with those of Tables 6 and 7, which show the more reliable estimates, the impact of the noise floor can easily be assed. We should also remark that in Table 6 there are 540 positive deviations compared with 444 negative ones. There exists then an upward drift, which we could seek to eliminate, but it is not worth the trouble; after all the waveform of a black noise does not have to be symmetrical. We consider the small drift as a long run average rate of endogenous growth. The difference between the two tables, however, quantifies the impact of the noise floor.

An important point that should be stressed is that the information depicted in Table 6 may be interpreted within the context of the attractor. The weighted mean may be seen as the inner boundary set between a 'gain-manifold' and a 'loss-manifold'. It can be inferred that in continuous time, orbits would spend 54.8 percent of the time on the gain-manifold and 45.2 percent on the negative one, crossing the boundary in both directions 830 times during the sampling interval, or about every 1.2 months; while remembering however that deviations are grouped. We, therefore, conclude that an investor who was patient enough to stay in the market during the sampling period would have come out ahead.

Another reason why we let the drift stand is that we do not know how an additional correction, i. e., approaching the boundary of the two manifolds, might affect the spectral exponent of the process as a whole. We are aware of at least one study carried out by Yanhui Liu *et al*. (1999) of the Center of Polymer Studies and Department of Physics of Boston University that shows a significant fall in that statistic as one approaches the center. Also, we are interested in preserving the outliers or what Nicholas Thales, the author of the *Black Swan*, has shown to be important.

What we did instead, was to subject the attractor to the multi-fractal test. A multi-fractal is one more generalization of the fractal concept whose purpose is to include phenomena and structures with more than one scaling exponents. The way positive and negative frequencies are distributed is one such applications, but according to Schroeder (1992, 187-190), multi-fractal distributions apply to the distribution of people and minerals on the earth surface, to energy dissipation in turbulence, to fractal resistor and computer networks, to games of chance, to strange attractors of non-linear dynamic systems, etc. From the distribution of positive and negative frequencies of $\textit{log-SP500}_{d2ftrd}$, the probability of gains and losses are p = 0.548 and 0.452, respectively. We then apply the multi-fractal test to *classes* of variations over the unit interval. However, given the amount of work involved, we show the results

Table 3: De-trended & De-trended and Filtered SP-500, 1928-2009

	LOG_SP500_d2	LOG_SP500_d2_Fltrd
N	984	984
Mean	1.00112	1.00112
Median	1.00170	1.00162
Mode	1.00200	.99408
Std. Deviation	.02044	.01536
Skewness	.906	2.895
Std. Error of Skewness	.078	.078
Kurtosis	30.250	67.157
Std. Error of Kurtosis	.156	.156
Range	.35790	.35684
Minimum	.84930	.86703
Maximum	1.20720	1.22387

Figure 3: Histogram of the De-trended SP500, 1928-2009

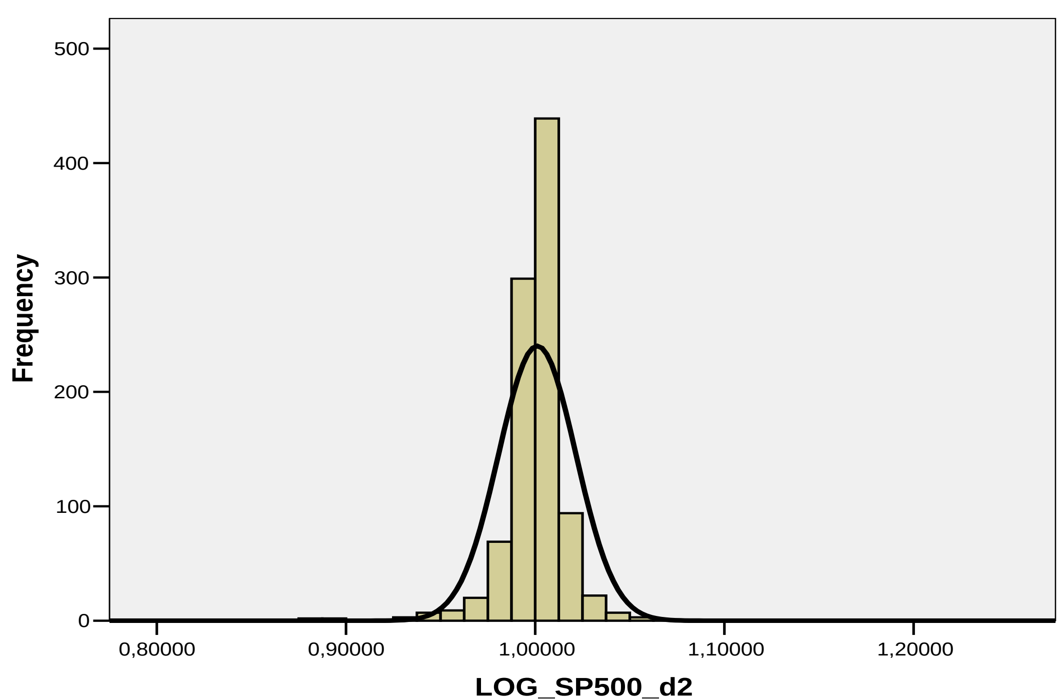
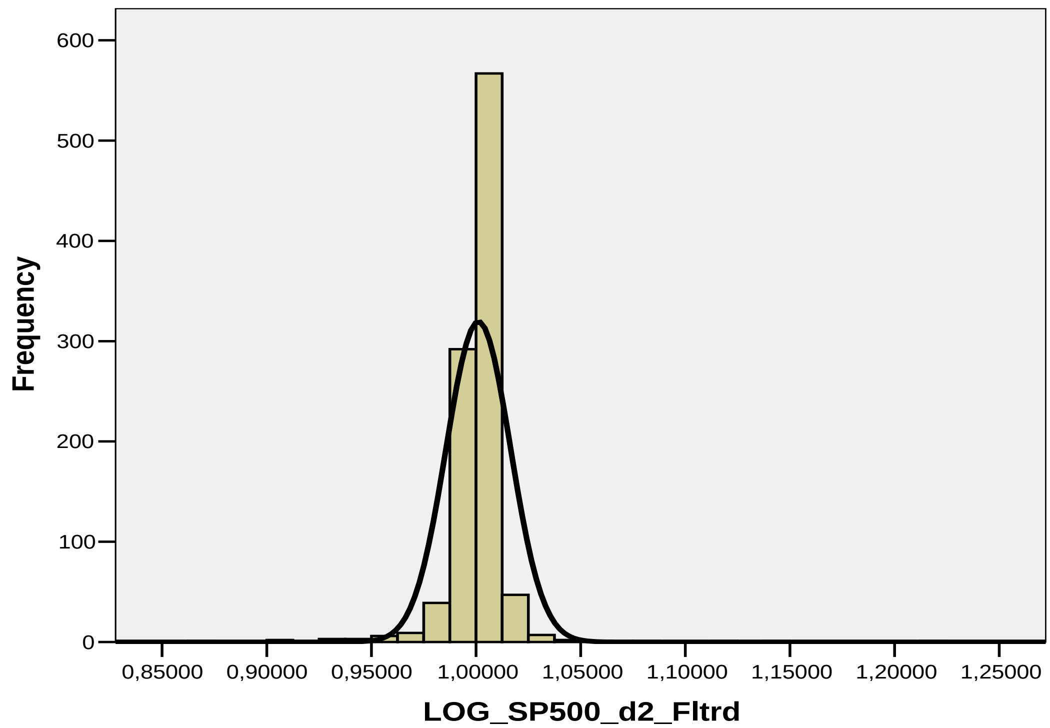


Figure 4: The Histogram of De-trended and Filtered SP500, 1928-2009



of the first two iterations. Iterating this bisecting process should approach an asymptotically self-affine distribution, but at the same time, we should remember being in fractal-land and we have broad classes of frequencies. Nonetheless, this partial result shown below is sufficient to show that the SP-500 is almost surely a multi-fractal.

Table 4: Multi-Fractal Test for the Distribution of Positive and Negative Frequencies: Two Iterations

Division of Unit Interval	**$(1-p)^2$**	**(1-p) p**	**p (1-p)**	**p^2**
Theoretical Values	0.204	0.247	0.247	0.300
1st Iteration (492 points)	0.180	0.243	0.243	0.337
2nd Iteration (246 points)	0.239	0.252	0.252	0.262

A final factor to consider is impact of inflation on the index. From December 1989 to December 2009, prices grew at an average rate of 10.7 percent in nominal terms. That is, during Mr. Greenspan's era, banks were receiving almost free money. They pull additional resources from the rest of the economy to play with and they headed to the casino, creating bubbles after bubbles; the last one busted in August 2007. Figure 5, on the other hand, shows only regular movements, and Figure 1and 2 seem to reflect the busting of the bubble. In Figure 1, for example, one can see sharp drops in power whenever bubbles busted; drops that could not be ascribed to spectral leakages.

Figure 5: The Time Profile of $Log\text{-}SP500_{d2ftrd}$, 1928-2009.

Figure 6: Plot of Frequency-Absolute Deviations of $Log\text{-}SP500_{d2ftrd}$, 1928-2009.

Figure 7: Plot of Frequency-Positive Deviations of $Log\text{-}SP500_{d2ftrd}$, 1928-2009.

Figure 8: Plot of Frequency-Negative Deviations of $Log\text{-}SP500_{d2ftrd}$, 1928-2009.

Table 5: Absolute, Positive and Negative Deviations in $Log\text{-}Sp500_{d2}$, 1928-2009

Adj. Dev. Cat_Abs (Absolute Dev.)	Pos._Frequencies	Neg._Frequencies
0.20	1	0
0.19	0	0
0.18	1	0
0.17	0	0
0.16	1	0
0.15	0	1
0.14	0	0
0.13	0	1
0.12	0	0
0.11	0	3
0.10	0	1
0.09	2	0
0.08	0	0
0.07	0	2
0.06	1	2
0.05	2	6
0.04	6	7
0.03	10	15
0.02	38	28
0.01	97	92
0.00	362	305
TOTAL	521	463

Table 6: The Frequency-Deviation Intensity Relationship in *Log-SP500*$_{d2ftrd}$, 1928-2009

Frequency	Positive Moves	Occurrence in (0, T) (# of months)	Negative Moves	Occurrence in (0, T) (# of months)
0.22	1	984	-	-
0.21	-	-	-	-
0.20	-	-	-	-
0.19	-	-	-	-
0.18	-	-	-	-
0.17	-	-	-	-
0.16	-	-	-	-
0.15	-	-	-	-
0.14	1	984	-	-
0.13	-	-	1	984
0.12	-	-	-	-
0.11	1	984	-	-
0.10	-	-	1	984
0.09	-	-	1	984
0.08	-	-	-	-
0.07	1	984	1	984
0.06	1	984	3	328
0.05	1	984	2	492
0.04	1	984	6	164
0.03	3	328	4	246
0.02	14	70.2	19	51.7
0.01	47	20.9	45	21.8
0.00	469	2.0	361	2.7
Total:	**540**		**444**	

Table 7: The Frequency-Absolute-Deviation-Intensity Relationship in *Log-SP500*$_{d2ftrd}$, 1927-2009

Frequency	**Absl. Deviation**	**Percentage**	**Occurrence in (0, T)**
0.22	1	0.10	984
0.21	-	-	-
0.20	-	-	-
0.19	-	-	-
0.18	-	-	-
0.17	-	-	-
0.16	-	-	-
0.15	-	-	-
0.14	1	0.10	984
0.13	1	0.10	984
0.12	-	-	-
0.11	1	0.10	984
0.10	1	0.10	984
0.09	1	0.10	984
0.08	-	-	-
0.07	2	0.20	492
0.06	4	0.41	246
0.05	3	0.30	328
0.04	7	0.71	140
0.03	7	0.71	140
0.02	33	3.3	39.8
0.01	92	9.3	10.7
0.00	830	84.3	1.2
Total	**984**		

Table8: Regression Statistics of Frequency vs. Abs. Dev. In *Log-SP500*$_{d2ftrd}$: Panels a, b, c and d.

V- CONCLUDING COMMENTS

This study focuses on the SP-500 Index from 1928 to 2009. With the help of the Benoit and SPSS softwares, we demonstrate that the index is powered by a fractal attractor or a power law whose power spectrum is proportional to $f^{-2.6}$ or $f^{-2.7}$, depending on the transform used. For these values of β, the color metaphor of the index is 'black', meaning that the unexpected events during the sampling period were expected. In an attempt to preserve both high and low frequencies, we used the Fast Discrete Fourier Transform and Wavelet analysis in a reasonably long series. The two analyses are in good agreement for the two segments: 1928-2009 and 1946-2009.

The approach adopted for the analysis allows the identification of the properties of the fractal attractor driving the index quite easily. The Hurst exponent reveals an almost smooth surface with an H exponent lying between 0.82and 0.89 and corresponding Hausdorff dimension between 2.18 and 2.11 and affine trace between 1.18 and 1.11. The surface of the attractor is between dimensions 2 and 3 in agreement with empirical tests. The high value found for H indicates that economic agents remember the past, contrarily to the prevailing assumptions underlying analyses in the financial sector. To double check the result of the power spectrum analysis, we run the regression supposing that variation intensity are proportional to frequency raised to some exponent β^{g} for grouped intervals. The result of the regressions indicate a reasonable fit, and a $\beta^{g} = 2.296$, a value close to 2.6.

Our study also demonstrates that for a fractional Brownian motion, there does not exist a fixed-point equilibrium. Orbits only cross the mean but do not stay there. This calls to question the practice in both economics and finance to suppose that orbits will always return to the mean to stay. And it goes against the notion that information arrives in the market at random. Information is only an intermediate input into the demand-supply interplay. It appears random because the dynamic process whose attractor we have tried to describe is hidden from view. It becomes also obvious, at the same time, that risk should not be assessed on a normal distribution; the second part of our results shows why.

The other purpose of the study was to make an exhaustive analysis of the distribution of frequencies. We centered the series on their weighted mean and tallied frequencies by classes of $9/1000^{th}$ of the interval. Variations, absolute, positive, and negative were tabulated for both the *de-trended* and *de-trended and filtered series*; the idea here was to assess the impact of the noise floor in the de-trended-only series. From Table 6 and 7, the present distribution can more easily be compared to the normal distribution. Peter Bernstein (1996) reports on a similar analysis of the SP-500, from January 1926 to December 1995; that is, 840 monthly price-change observations; Bernstein does not indicate whether or not his series was filtered. The mean (arithmetic?) and standard deviation were 0.6 percent and 5.8 percent, respectively. Bernstein then uses the normal distribution to conclude that a price decline of 10 percent can be expected every 15 months. He then multiplies that figure by 2 to conclude that an absolute deviation of 10 percent is likely every 30 months. Although our measures are different, the same principle holds. Table 6 shows that a 10 percent decline occurred only once in 984 months. Since there was no 10 percent increase in this case, an absolute deviation of 10 percent happened also only once in the 984-month period (Table 7) even though our series is longer by 184 months. This shows why we should be mindful of the difference between the normal and fractal distributions. More specifically, according to the distribution of frequencies over the unit interval, ours is a self-affine distribution in which deviations are not equally likely.

The displays of Tables 6 and 7 reveal that orbits spend more time on the gain-manifold. We deem this drift natural. This then shows that: *The mathematical expectation of an investor is not zero*, in perfect agreement with empirical observations.

To summarize, our main findings are that the capital market is governed by a power law with a spectrum exponent that locates it in the category of *black noise*. As such, violent and unexpected fluctuations in prices happen at computable frequencies. In other words, price changes between 20 and 15 percent were extremely rare; it did not happen once during in the 82-year period that we have examined. But, there was a 22 percent positive change. In negative territory, there were price declines at 14 percent, at 10, at 9 percent, and so on. The most frequent deviations, however, were confined to a reduced place in the invariant equilibrium set, that is, between 1 and 4 percent. But, once in a while, orbits venture far away from the center of the attractor. Obviously, the future is not going to duplicate the past, not completely, as the data is subject to permanent shocks. However, the data say something about the short and intermediate runs. It goes without saying that it is better to be prepared.

REFERENCES

Bachelier, L. (1900). Théorie *de la speculation: Annales de l'École Normale Supérieure*, 3^{e} série, 2186.

Basu, S. (1977). "Investment Performance of Common Stocks in Relation to Their Price Earnings Ratios: A Test of the Efficiency Market Hypothesis." *Journal of Finance*, 33, June, 663-682.

Bayraktur, E, *et al*. (2003). "Estimating the Fractal Dimension of the SP-500 Index Using Wavelet Transform Analysis." *www.princeton.edu/-Sircar/ Public/ Article/bps.pdf.*

Bernstein, P. (1996). *Against the Gods*. New York: John Wiley & Sons.

Black, F. and Scholes, M. (1973). " The Pricing of Options and Corporate Liabilities." *Jour. Of Pol. Econ*.,81, 637-659.

Casti, J. (1995). *Complexification*. New York: Harper Perennial.

Cheredito, P. (2003). "Arbitrage in Fractional Brownian Motion Models." *Finance and Statistics*, 7, 533-553.

Comte, F. and Renault, E. (1998). Long-memory in Continuous-Time Stochastic Volatility Models." *Mathematical Finance*, 8, 291-323.

Cox, J. *et al*. (1979). "Option Pricing: A Simplified Approach. *Jour. Of Financial Econ*., 7, 229-263.

Cutland, N. J., *et al*. (1995)."Stock Price Retrurns and the Joseph Effect: A Fractal Version of the Black-Scholes Model." *Pro-Gress in Probability*, 36, 327-351.

Fama, E. and Blume, M. (1966). "Filter Rule and Stock Market Trading.", A *Supplement of the Jour. Of Business*, 39, Jan., 226-241.

Falconer, K. (2003). *Fractal Geometry*. John Wiley & Sons: New York.

Grassberger, P. and Procaccia, I. (1983). “Measuring The Strangeness of Strange Attractors.” *Physica* ,9D, 189-208.

Greene, M. T. and Fielitz, B. D. (1977). “ Long-Term Dependence in Common Stock Returns.” *Jour. of Financial Econ.*, 4, 339-349.

Guasoni, P. (2006). “ No Arbitrage Under Transaction Costs With Fractional Brownian Motion and Beyond.” *Mathematical Finance*, 16, 569-582.

Heston, S. L. (1993). “A Close-form Solution for Options With Stochastic Volatility With Applications to Bonds and Currency Options.” *Review of Financial Studies*, 6, 327-343.

Hurst, H. E. *et al*. (1951). *Long-term Storage: An Experimental Study*. London: Constable.

Kendall, M. (1954). “ The Analysis of Economic Time Series, Part I: Prices.” *Jour. Of the Royal Stat. Soc.*, 96, 11-25.

Yanhui, L. *et al*. (1999). “Statistical Properties of the Volatility of Price Flustuations.”*Physical Review E*, 60, 1890-1400.

Lo, A. W. (1999). *A Non-Random Walk Down Wall Street*. New Jersey: Princeton University Press.

Lyapunov, A. M. (1907). “ Problème général de stabilité de mouvement.” Annuaire de la Faculté des sciences, Université de Toulouse, 9, 203-475.

Malkiel, B. G. (1973). *A Random Walk Down Wall Street*. New York: W. W. Norton.

Mandelbrot, B. (1971). “ When Can Price Be Arbitraged Efficiently? A Limit to the Volatility of the Random Walk and Martingale Models.” Rev. *of Econ. and Stat.*, 53, 225-236.

Mandelbrot, B. (1977). *Fractal and Scaling in Finance*. New York: Springer Verlag.

…………… and van Ness, J. W. (1968). “Fractional Brownian Motion, Fractional Noises and Applications.” *SIAM Rev.*, 10, 422-437.

Merton, R. C. (1969). “Life Time Portfolio Selection Under Uncertainty.” *Rev. of Econ. and Stat*., 51, 247-257.

…………….. (1973). “Theory of Rational Option Pricing.” *Bell Jour.of Econ. and Management* Sc., 4, 141-183.

Medio, A. (1992). *Chaotic Dynamics: Theory and Applications to Economics*. New York: Cambridge Univ. Press.

Moreira, J. G. (1994). “On the Fractal Dimension of Self-Affine Profiles.” *Jour. Of Physics A*, 27, 8079-8089.

Osborne, M. (1954). “Brownian Motion in the Stock Market.” *Operation Res*., 7, March-April, 145-173.

Osedelec, V. I. (1968). “A Multiplicative Egodic Theorem: Lyapunov Characteristic Numbers for Dynamic Systems.” *Trans. of the Moscow Math. Soc.*, 19, 197-231.

Peters, E. (1991). “A Chaotic Attractor for the SP-500.” *Financial Analyst Jour*., March-April, 55-81.

……………….. (1994). *Fractal Analysis: Applying Chaos to Investment and Economics*. New York: John Wiley & Sons.

Preciado, James *et al*. (2008). “Proceedings of the Congress on Engineering and Computer Science 2008.” *WCECS 2008*, October 22-24, San Francisco.

Shiller, R. I. (1981). “Do Stock Prices Move too Much to Be Justified By Subsequent Changes in Dividends.” *Amer. Econ. Rev.*, 71, 421-436.