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When Different Market Concentration Indices Agree

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Abstract

Market concentration ratios are popular statistics for characterizing the extent of market dominance in an imperfectly competitive market, but these ratios may not agree when comparing two markets. Neither do they necessarily agree with the Herfindahl-Hirschman or entropy indices. This letter compares two Cournot oligopoly markets in which firms have constant unit costs. It is shown that the majorization pre-ordering on normalized marketing margin vectors is both necessary and sufficient for all aforementioned indices to agree on which is the more concentrated market.

Keywords: anti-trust; cost dispersion; majorization.

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1. Introduction

The measurement of market concentration is important for several reasons. In many jurisdictions, market concentration indices are used when determining whether a merger should be allowed and whether an existing firm should be broken up. In addition, some companies use market concentration indices when re-organizing production activities. Furthermore, market structure is believed to affect market efficiency in a variety of ways such as altering incentives to innovate (Aghion *et al.*, 2005).

The most widely considered market concentration indices are the Herfindahl-Hirschman index ($H(S)$), the entropy index ($E(S)$), and the *k*-firm concentration ratio $R(k;S)$, where S is a vector of market shares and we will shortly define each index. If considering *N*-firm industries, we have just enumerated $N+2$ concentration indices. Given a vector of market shares, it is possible to construct a second vector such that $N+1$ of these indices agree in ranking but the remaining index does not. We ask the question: when market structure changes then what conditions must be imposed such that all of these indices agree about the consequences of the change for the extent of market concentration? We do so when working directly with share vectors for any pair of markets. We also do so when working indirectly with normalized marketing margin vectors for a pair of Cournot oligopoly markets possessed of constant unit costs.

2. The set of indices

We adopt the following two conventions. Parentheses () are used in subscripts to identify the lower order statistics for a vector, i.e., for $X = (x_1, x_2, \ldots, x_N)$ then $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(N)}$. Similarly, square brackets [] are used in subscripts to identify the lower order statistics, i.e.,

$x_{[N]} \le x_{[N-1]} \le ... \le x_{[1]}$. Of course, $x_{(n)} \equiv x_{[N-n+1]}$.[1] For market share vector $S = (s_1, s_2, ... , s_N)$ in a N-firm industry, write

$$H(S) = \sum_{n=1}^{N} s_{[n]}^2; \qquad E(S) = -\sum_{n=1}^{N} s_{[n]} \text{Ln}[s_{[n]}]; \qquad R(k;S) = \sum_{n=1}^{k} s_{[n]}. \qquad (1)$$

All of these formulae are symmetric in that one could interchange, say, s_i with s_j in the formula without changing the formula value. Function $H(\cdot)$ is convex, while $-E(\cdot)$ is convex. The concentration ratios are not of uniform curvature. Symmetry is an appealing property because it ensures that the index treats the firms identically. Convexity is appealing because it requires that the marginal contribution to the index increases with firm market share, suggesting that a larger firm is of particular concern when market concentration is an issue.

Pairs of vectors, $S' = (s'_{[1]}, s'_{[2]}, ... , s'_{[N]})$ and $S'' = (s''_{[1]}, s''_{[2]}, ... , s''_{[N]})$, can be readily constructed to show that all but one of the index set $\{H(S'), -E(S'), R(1;S'), ... , R(N-1;S')\}$ is larger than its counterpart in $\{H(S''), -E(S''), R(1;S''), ... , R(N-1;S'')\}$. One question we ask is whether a set of conditions for comparing a pair of market share vectors exists such that all of these indices agree about which represents the more concentrated market structure? Another is whether, given Cournot market structure, a set of conditions on cost primitives exists such that the indices concur.

3. Model

We compare behavior across markets A and B. With unit costs across N_i active firms as

$c_n^i, n \in \{1, 2, ... , N_i\} = \Omega_{N_i}, i \in \{A, B\}$, firm outputs as q_n^i, market outputs as $Q^i = \sum_{n \in \Omega_N} q_n^i$, and

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[1] But the two ways of presenting the same order statistic will prove to be convenient because the order of statistics for a unit cost vector will be the reverse of those for production shares in

inverse demand functions as $P^i(Q^i)$, the standard Cournot oligopoly model asserts that firm output choices satisfy

$$P^i(Q^i) + q_n^i P_Q^i(Q^i) - c_n^i = 0; \tag{2}$$

where $P_Q^i(Q^i) < 0$ is the first derivative.[2] We make the standard assumptions about demand, equilibrium existence and equilibrium uniqueness, and hold that all choices are interior. Optimum choices are characterized as $q_n^{*,i}$.

For $Q^{*,i}$ equal to equilibrium market output, use (2) to write $q_n^{*,i} = [c_n^i - P^i(Q^{*,i})]/P_Q^i(Q^{*,i})$.

Define the unit cost order statistics as $c_{(1)}^i \le c_{(2)}^i \le ... \le c_{(N_i)}^i$. Then, for $q_{[n],c}^{*,i}$ the output for the firm with unit cost $c_{(n)}^i$, it follows that $q_{[N_i],c}^{*,i} \le q_{[N_i-1],c}^{*,i} \le ... \le q_{[1],c}^{*,i}$ and[3]

$$\sum_{n=1}^{k} q_{[n],c}^{*,i} = \frac{\sum_{n=1}^{k}\left[c_{(n)}^i - P^i(Q^{*,i})\right]}{P_Q^i(Q^{*,i})}. \tag{3}$$

Since

$$Q^{*,i} = \sum_{n=1}^{N_i} q_{[n],c}^{*,i} = \frac{\sum_{n=1}^{N_i}\left[c_{(n)}^i - P^i(Q^{*,i})\right]}{P_Q^i(Q^{*,i})}, \tag{4}$$

the value of $Q^{*,i}$ is determined by the values of the average cost over active firms, $\bar{C}^i = N_i^{-1}\sum_{n=1}^{N_i} c_{(n)}^i$, and the number of active firms, N_i. Dividing the partial summations in (3) through by $Q^{*,i}$ in (4) establishes

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Cournot oligopoly.

[2] The analysis to follow can be extended to the context of quadratic costs of form $c_n^i q_n^i + 0.5\gamma(q_n^i)^2, \gamma > P_Q^i(Q^i)$, where the constraining feature is that γ be firm-invariant.

[3] We use square brackets in $q_{[n],c}^{*,i}$ to remind readers that the sequence rank order for firm outputs is the reverse of that for unit costs. So as to avoid reversing the indexation on summations, we sum costs from low to high but sum outputs from high to low.

$$\sum\nolimits_{n=1}^{k} s_{[n],c}^{*,i} = \frac{\sum_{n=1}^{k}\left[P^i(Q^{*,i}) - c_{(n)}^i \right]}{N_i \times \left[P^i(Q^{*,i}) - \overline{C}^i \right]}; \qquad s_{[n],c}^{*,i} = \frac{q_{[n],c}^{*,i}}{Q^{*,i}}. \tag{5}$$

Here, $s_{[N_i],c}^{*,i} \leq s_{[N_i-1],c}^{*,i} \leq ... \leq s_{[1],c}^{*,i}$ so that the order statistics are $s_{(1)}^{*,i} \equiv s_{[N_i],c}^{*,i} \leq s_{(2)}^{*,i} \equiv s_{[N_i-1],c}^{*,i} \leq ... \leq$

$s_{(N)}^{*,i} \equiv s_{[1],c}^{*,i}$. Another way of presenting the above is

$$\sum\nolimits_{n=1}^{k} s_{[n],c}^{*,i} = \frac{k}{N_i} \times \frac{M_k}{M_{N_i}}; \qquad M_k = 1 - \frac{A_k}{P^i(Q^{*,i})}; \qquad A_k = \frac{\sum_{n=1}^{k} c_{(n)}^i}{k}. \tag{6}$$

Now $(N_i/k)\sum\nolimits_{n=1}^{k} s_{[n],c}^{*,i}$ has value 1 when $k = N_i$, a value at most N_i when $k = 1$, and declines in

value as k increases. Expression M_k is the average profit margin on the k lowest-cost firms, so

that ratio M_k/M_{N_i} represents an index of the normalized average profit margin for the k

lowest-cost firms.

Write the vector of equilibrium market shares as $S^{*,i}$. The majorization concept, see

Marshall and Olkin (1979, pp. 10 and 59), is relevant to equations (5)-(6):[4,5]

Definition 1. *Vector $V' \in \mathbb{R}^M$ is <u>majorized</u> by $V'' \in \mathbb{R}^M$ ($V' \prec V''$) if the ordinates satisfy*

$\sum\nolimits_{n=1}^{k} v_{[n]}' \leq \sum\nolimits_{n=1}^{k} v_{[n]}'' \; \forall k \in \Omega_M$ *and* $\sum\nolimits_{n=1}^{M} v_{[n]}' = \sum\nolimits_{n=1}^{M} v_{[n]}''$. *A function $U(Q): \mathbb{R}^M \to \mathbb{R}$ is said to*

be <u>Schur-convex</u> if $U(V') \leq U(V'')$ whenever $V' \prec V''$.

Observe that, since the partial sums are taken from top to bottom, it is possible to compare

vectors of unequal dimensions. Just add a balancing number of zero entries to the lower

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[4] This is a pre-ordering. It is reflexive and transitive but, unlike a partial ordering, the truth of
both $Q' \prec Q''$ and $Q'' \prec Q'$ does not imply $Q' = Q''$ in the sense of equivalence since vector
permutations are distinct. For our purposes, the distinction is of no relevance.
[5] Consider $(1,2,6)$ and $(2,3,4)$ with common sum, 9. But $4 \leq 6$ and $7 \leq 8$ so that $(2,3,4) \prec$
$(1,2,6)$.

dimensional vector. For example, suppose that $V'' = (1,2,6)$ as in footnote 4 while $V' = (1,1,3,4)$. Then extend V'' to $\tilde{V}'' = (0,1,2,6)$. Comparisons $4 \leq 6$, $7 \leq 8$, $8 \leq 9$, and $9 = 9$ then establish $V' \prec \tilde{V}''$. In what is to follow we assume the dimension extension and write, with some informality in notation, that $V' \prec V''$.

The majorization idea has been widely applied when addressing income inequality, see Dasgupta, Sen, and Starrett (1973) or Shorrocks (1983). It has also been used when modeling decision making under uncertainty, as in Chambers and Quiggin (2000) or Lapan and Hennessy (2002). Note that $R(k; S^i) = \sum_{n=1}^{k} s_{[n],c}^{*,i}$, and define

$$\theta_{[n]}^i = \frac{P^i(Q^{*,i}) - c_{(n)}^i}{N_i \times \left[P^i(Q^{*,i}) - \overline{C}^i \right]}. \tag{7}$$

The numerator is the nth largest firm's marketing margin in the ith industry. The denominator is the product of $P^i(Q^{*,i}) - \overline{C}^i$, to scale for the industry's average marketing margin, and N_i where the latter ensures that $\sum_{n=1}^{N_i} \theta_{[n]}^i = 1$. Write $\theta^i = (\theta_1^i, \theta_2^i, \dots, \theta_N^i)$.

When comparing these industries there are $\min[N_A, N_B]$ meaningful concentration ratios, and so (1) provides $\min[N_A, N_B] + 2$ meaningful concentration indices.

Proposition 1. *(a) Industry B is more concentrated than industry A in the sense of all $\min[N_A, N_B]$ concentration ratios in (1) if and only if $S^{*,A} \prec S^{*,B}$. Furthermore, if $S^{*,A} \prec S^{*,B}$ then $H(S)$ and $E(S)$ agree with each of the concentration ratios when ranking market concentrations.*

(b) When we compare Cournot oligopoly markets, agreement across concentration ratios occurs if and only if $\theta^A \prec \theta^B$. Furthermore, if $\theta^A \prec \theta^B$ then $H(S)$ and $E(S)$ agree with each of the concentration ratios.

Proof. The parts assert the same thing, so we will only demonstrate (b). For $H(S)$ and $E(S)$, note that any symmetric, convex function is Schur-convex.[6] For these indices then, $\theta^A \prec \theta^B$ ensures consistency in ranking markets.[7] Now view the $\min[N_A, N_B]$ concentration ratios. By definition 1, (5), (7) and $R(k; S^{*,i}) = \sum_{n=1}^{k} s_{[n],c}^{*,i}$, it follows that $R(k; S^{*,B}) \geq R(k; S^{*,A}) \; \forall k \in \{1, 2, \dots, \min[N_A, N_B]\}$ if and only if $\theta^A \prec \theta^B$. $\qquad\qquad\square$

A special case arises when we consider how a change in the allocation of unit costs across firms in a given industry affects industry concentration, see Salant and Shaffer (1999). Then $N_A = N_B = N$, $P^A(Q) = P^B(Q) = P(Q)$, and $\overline{C}^A = \overline{C}^B = \overline{C}$, so that we can remove the industry identifier in (7) and write

$$\theta_{[n]} = K_1 - K_2 c_{(n)}; \qquad K_1 = \frac{P(Q^*)}{N \times \left[P(Q^*) - \overline{C} \right]}; \qquad K_2 = \frac{1}{N \times \left[P(Q^*) - \overline{C} \right]}. \qquad (8)$$

From definition 1 it is readily seen that $c' \equiv (c_1', c_2', \dots, c_N') \prec c'' \equiv (c_1'', c_2'', \dots, c_N'')$ if and only if $\theta' \equiv (\theta_1', \theta_2', \dots, \theta_N') \prec \theta'' \equiv (\theta_1'', \theta_2'', \dots, \theta_N'')$ because it is readily shown that an affine transformation of the coordinates does not change the majorization relation. We have then[8]

Corollary 1.1. *Let N firms be active in a constant unit cost Cournot oligopoly, where unit costs are represented by vector c'. Consider a rearrangement of unit costs $c' \rightarrow c''$ such that the sum*

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[6] See p. 4 and p. 14 of Marshall and Olkin (1979).

[7] We do not suggest the reverse, that agreement between $H(S)$ and $E(S)$ implies order in the sense of majorization. This reversal is not true in general.

[8] It has been demonstrated in corollary 1 of Salant and Shaffer (1999) that an increase in the variance of unit costs increases all of the Herfindahl-Hirschman index, social welfare, and industry profits. This calls into question the merit of anti-trust enforcement in an industry where homogeneous goods, constant unit costs, Cournot behavior, and restricted entry are believed to adequately approximate the decision environment.

of unit costs does not change and all firms remain active. The industry is more concentrated

under c'' than under c' in the sense of all $N+2$ concentration indices in (1) if and only if

$c' \prec c''$.

As to whether the requirement that all firms remain active can be relaxed, the answer is in the negative. Suppose that $c_{(N)} > P(Q^*)$ ex-post so that the least efficient producer ceases production. Then, by definition 1, the sum of unit costs over remaining firms declines. This reduction in unit cost sums could involve a unit cost reduction in just one small producer. Zhao (2001) has provided an example involving linear demand in which a unit cost reduction for a small producer reduces both social welfare and the Herfindahl-Hirschman index. The reason is that the smallest remaining firm behaves less like a price taker and the resulting decline in competition more than offsets the effects of the cost reduction.

3. Graphical interpretation

Figures 1 and 2 identify points in the unit simplex of market shares such that market share vector α is more concentrated (i.e., majorizes and given in fig. 1) and is less concentrated (i.e., majorized by and given in fig. 2).[9] Fig. 1 was constructed by identifying all six permutations of some share allocation in an industry where three firms are active. If, for example, $\alpha =$ $(0.7, 0.2, 0.1)$ then the other five points would be $(0.7, 0.1, 0.2)$, $(0.1, 0.7, 0.2)$, $(0.1, 0.2, 0.7)$, $(0.2, 0.1, 0.7)$, and $(0.2, 0.7, 0.1)$. These points all provide the same market concentration index levels for all considered indices because the indices are indifferent to which firm is matched with a share. Intuitively, a convex combination of the six points (the blackened area in fig. 1) should

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[9] These convex hull representations of majorization are originally due to Rado (1952). See Ok

be less concentrated because convexification decreases the share of the most dominant firm.

Fig. 2 is constructed from α by asking what market share vectors would be more concentrated than α. These would be outside any convex hull that could be constructed using α. It turns out that only two convex hulls need be considered. One is an equilateral triangle (label it T_1) constructed from adding three smaller equilateral triangles to the smaller sides of the hexagon generated in fig. 1. As it happens, this equilateral triangle is contained inside the unit simplex. The other is also an equilateral triangle (label it T_2), but this time it is obtained from adding three smaller equilateral triangles to the larger sides of the hexagon generated in fig. 1. This time, however, the larger equilateral triangle so constructed is not contained within the unit simplex. Three equilateral triangles can be discarded as irrelevant, see the dotted line extensions. The set complement of $T_1 \bigcup T_2$, but inside the unit simplex are the blackened points in fig. 2. Any point in these blackened areas generates, through symmetrization, five other points the convex hull of which contains α.

Points not blackened in either figures (three trapezoids, each contiguous to the simplex perimeter) are not comparable in the majorization ordering. As such they will lead to inconsistencies among some of the concentration indices. In particular, for the three-firm case the one-firm and two-firm concentration ratios cannot agree on ranking market concentration. For example, with $\alpha = (0.7, 0.2, 0.1)$ and $\beta = (0.75, 0.14, 0.11)$ then the one-firm and two-firm concentration ratios do not agree.

4. Conclusion

The intent of this note has been to identify necessary and sufficient conditions under which

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(1997) for an application in the theory of equitable taxation.

all of the most widely used market concentration indices are in agreement. The condition set, known as the majorization pre-ordering, is restrictive in that it is only partial. Unlike any one of the concentration indices by itself, the relation does not always rank share vectors to be compared. That is the price to be paid for spanning all the functions in the set. For a pair of well-defined markets, if the market share vectors are ordered in the sense of majorization then there can hardly be any technical dispute about which is the more concentrated.

References

Aghion, P., Bloom, N., Blundell, R., Griffith, R., Howitt, P., 2005. Competition and innovation: An inverted-U relationship. Quarterly Journal of Economics 120(2, May), 701–728.

Chambers, R.G., Quiggin, J., 2000. Uncertainty, Production, Choice, and Agency: The State-Contingent Approach. Cambridge University Press, Cambridge, UK.

Dasgupta, P., Sen, A.K., Starrett, D., 1973. Notes on the measurement of inequality. Journal of Economic Theory 6(2), 180–187.

Lapan, H.E., Hennessy, D.A., 2002. Symmetry and order in the portfolio allocation problem. Economic Theory 19(4), 747–772.

Marshall, A.W., Olkin, I., 1979. Inequalities: Theory of Majorization and its Applications. Academic Press, San Diego, CA.

Ok, E.A., 1997. A note on the existence of progressive tax structures. Social Choice and Welfare 14(4, October), 527–543.

Rado, R., 1952. An inequality. Journal of the London Mathematical Society 27(1, January), 1–6.

Salant, S.W., Shaffer, G., 1999. Unequal treatment of identical agents in Cournot equilibrium. American Economic Review 89 (3), 585–604.

Shorrocks, A.F., 1983. Ranking income distributions. Economica 50(197, February), 3–17.

Zhao, J., 2001. A characterization for the negative welfare effects of cost reduction in Cournot oligopoly. International Journal of Industrial Organization 19 (3-4, March), 455–469.
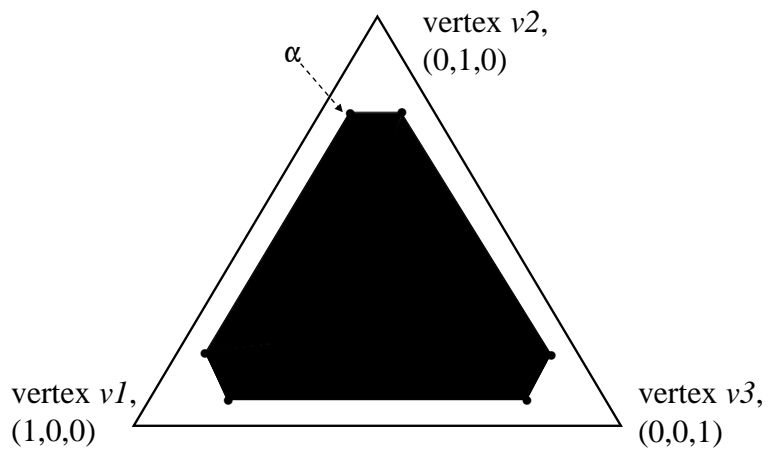
Fig. 1. Convex hull of share vector α, where interior points are less
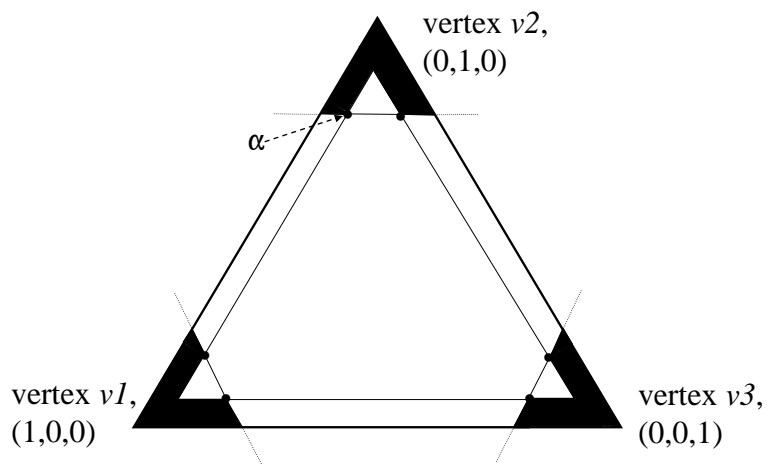concentrated under all considered market concentration indices.

Fig. 2. Blackened area comprise points in simplex that are more
concentrated than α under all considered market concentration indices.