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David A. Hennessy, Harvey E. Lapan

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by

David A. Hennessy
Professor
Department of Economics
Iowa State University
Ames, IA 50011-1070

Harvey E. Lapan
University Professor
Department of Economics
Iowa State University
Ames, IA 50011-1070

Mailing address for correspondence

David A. Hennessy
Department of Economics
578 C Heady Hall
Iowa State University
Ames, IA 50011-1070

    Ph: (515) 294-6171
    Fax: (515) 294-6336
    e-mail: hennessy@iastate.edu

An Algebraic Theory of the Multi-Product Firm

Summary: The typical firm produces for sale a plural number of distinct product lines. This paper characterizes the composition of a firm's optimal production vector as a function of cost and revenue function attributes. The approach taken applies mathematical group theory and revealed preference arguments to exploit controlled asymmetries in the production environment. Assuming some symmetry on the cost function, our central result shows that all optimal production vectors must satisfy a dominance relation on permutations of the firm's revenue function. When the revenue function is linear in outputs, then the set of admissible output vectors has linear bounds up to transformations. If these transformations are also linear, then convex analysis can be applied to characterize the set of admissible solutions. When the group of symmetries decomposes into a direct product group with index $\kappa \in \mathbb{N}$, then the characterization problem separates into κ problems of smaller dimension. The central result may be strengthened when the cost function is assumed to be quasiconvex.

1 Introduction

Arguably, the firm that produces multiple products for sale is more representative of observed production activities than the firm that produces a single product for sale. Transactions costs analysis, in focusing attention on firm organization, intra-firm incentives structures and externalities, has explained many salient features of the firm. While the make-or-buy decision has been studied in detail, an item on the Coase (1937) agenda that has received comparatively little attention is the decision to produce more than one product for sale.[1] This is unfortunate in light of the prevalence of the firm attribute.

Firms may produce two or more products for a number of reasons. The transactions costs motives for product diversification emphasize the trade-off between scope economies due to production externalities or technical cost inefficiencies in single-product firms and the organizational costs of internalizing these activities. For developed countries with dense capital and state-contingent markets, the risk management motive for diversification has been widely dismissed. However, academic debates on other plausible merits of a diversified firm have not been resolved. Berger and Ofek (1995) identify a stock market discount on conglomerate firms, in the period 1986 through 1991, relative to an imputed value were the firm's businesses traded as separate stock.[2] They point to agency problems in intra-firm capital allocation as the cause. While not completely discounting this thesis, Klein's (2001) study of conglomerates supports Alchian's (1969) argument that conglomerates can utilize information more efficiently than can external markets when allocating capital to specific uses. In particular, this may have been the case for conglomerates during the 1960s. Maksimovic and Phillips (2002) also find empirical evidence suggesting efficient capital allocation within the conglomerate firm.

Laux (2001) motivates diversification as a mechanism to mitigate an executive

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[1] Coase (p. 402) writes "But it is clearly important to investigate how the number of products produced by a firm is determined, while no theory which assumes that only one product is in fact produced can have very great practical significance."

[2] For concerns about bias in their methodology, see Graham *et al.* (2002).

compensation problem. Limited liability renders it difficult for an owner to encourage performance enhancing decisions by a risk-neutral manager. Rather than deter shirking by increasing the default (efficiency) wage and then penalizing severely for measured output deficiencies, the owner might increase the number of projects under the manager. Rewards on these other projects can be used to bond the manager, and so overcome the limited liability problem while lowering the expected cost of doing so.

Multi-good production might enhance a firm's market power in a number of ways. For instance, Bernheim and Whinston (1990) demonstrated how collusion might be more readily supported when oligopolists compete in two or more markets. Whinston (1990) showed how a monopolist that elects to produce a second good for an imperfectly competitive market may increase profits by a tying strategy that forces competitors to exit. Concerned with the role of incumbency in market dynamics, Carlton and Waldman (2002) proposed some strategic roles for tying. It may be used to preserve a monopoly in the presence of technical change, and it may also be used to dominate a new market.

Exclusively technical motives for multi-product activities are also readily conjectured. In agriculture, crop rotations can act to mitigate pest problems, conserve moisture, promote soil tilth, enhance soil nutrient status, and better sequence time commitments during busy periods. In mining, forestry or fishing, heterogeneity in raw materials imply that production activities may yield a variety of elements, minerals or species. In manufacturing, scope economies in purchasing, advertising or product development can provide motives for manifold outputs. These scope economies can arise in somewhat subtle forms. For example, Mitchell (2000) has pointed to scope economies in learning as a motive for multi-product firms, and he also models the idea that technological dissimilarities limit the firm's incentive to diversify.

The fact that a large fraction of production as final goods originates from non-specialist firms is relevant for a number of government policies. In the case of imperfectly competitive

markets, the effects of power in multiple markets on anti-competitive behavior is a concern. Sometimes too environmental externalities may be at issue. For example, the computation of optimal output and input taxes must acknowledge the effects on all markets. Further, any constraint on a firm's production patterns will have implications for equilibrium output prices.

In all of the above, it is not just the fact that a firm elects to produce two or more goods for sale that is relevant. The comparative intensities of production matter. Yet apart from a long empirical tradition in the duality literature, as in Fuss and McFadden (1978), little is known about the determinants of vector valued outputs. We claim that heterogeneity together with the related phenomenon of symmetry comprise an important determinant of this key facet of firm behavior.

To develop our thesis, this paper provides a framework for analyzing the product mix that a firm elects to produce. We will do so by inferring bounds on the firm's optimal decision vector in output space from the pertaining cost and revenue functions. The key assumption is that some weak symmetry property is possessed by the firm's multi-product technology. This symmetry property can be weak in two senses. First, the arguments of the technology's cost function can undergo symmetry breaking transformations before any concept of symmetry need be invoked on the resulting 'pseudo-cost' functional. Second, the type of symmetry that is required to exist on this pseudo-cost functional may be far weaker than the usual notion of symmetry, permutation symmetry, whereby a function is invariant under any permutation of argument values in the function.

The tools most suitable for our problem originate in group algebra and convex analysis. The latter set of tools are at the foundations of modern production, consumption and equilibrium theory. In general equilibrium theory, the former set of tools have been applied by Balasko (1990) to study sunspot equilibria. Groups have also been used by Koopmans *et al.* (1964), Sato (1976), Vogt and Barta (1997), and others to study time preferences, production, and

aggregation. That line of research, however, is in the tradition of analytic functional equations. It seeks attributes on functions such that certain derivative properties, such as integrability, aggregability, and separability, are satisfied by a function representation of a preference structure or a technology.[3] Their studies relate nothing about level vectors, e.g., when does a firm produce at least twice as many tons of product A than product B?

Our present research seeks to identify structure on the level vectors of production decisions. The approach that we take has much in common with the equivariance theory of statistical analysis.[4] The sorts of questions that are asked in equivariance theory may pertain to the class of objective functions (i.e., expected loss functions) such that a given statistic-conditioned decision rule is invariant to the measurement units. There, the focus is on the performance of a decision rule so as to remove rules that perform poorly. We, however, have our profit maximization decision rule at hand, and are content with it. Our focus is, rather, on the description of that decision rule as it relates to parameters in the objective function. Thus, while the statistics literature restricts the properties that the decision consequences may have in order to refine the set of objective functions that one might wish to consider, we fix on a decision rule and then explore the decision consequences.

The most closely related research is that due to Hennessy and Lapan (2003a). In a general permutation group framework, they identify bounds on the output allocation vector for a multi-plant firm that seeks to minimize the cost of producing a single homogeneous good. Using a similar tool kit, Lapan and Hennessy (2002) and Hennessy and Lapan (2003b) have analyzed the portfolio allocation problem. There, symmetric (2002) and general permutation (2003b) group structures on a vector of random returns were shown to provide bounds on the fund allocation

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[3] We note also that groups on topologies arise in the mathematical study of fixed points (Munkres, 2000). While elements of algebraic topology are widely used in game theory, as in Herings and Peeters (2001), a formal group theory approach does not appear to have been used.

[4] See, e.g., the textbook by Schervish (1995).

vector, as well as inferences on welfare, diversification, and fund separation.

This paper commences with a presentation of the structure that is assumed on the multi-output firm's profit maximization problem. The firm need not be a price-taker, although stronger deductions can be made if price-taking is assumed. Having formalized the problem in Section 2, we overview some of the relevant concepts from finite group theory. Section 4 assumes, apart from some group invariance, only that the pseudo-cost functional is monotone. Our central result is a dominance relation for points on the hull that a particular group generates when it is applied to the optimum decision vector. A particular instance of this result provides conditions under which the ratio of product A to product B that a price-taking firm sells is in excess of some number, say two. Similar results can be generated for a firm that produces in a multi-product monopoly environment, or when product markets have mixed structures.

Section 5 specializes to price-taking and to linear transformations on the pseudo-cost functional's arguments. There, we apply methods from convex analysis to study the structure of admissible regions in output space. These regions are convex, can be decomposed into a particular vector sum of sets, and must contain a specific translated ray. An admissible region need not be a polyhedral cone. The defining conditions for an admissible region are decomposable into a collection of disjoint conditions if the group is decomposable into a direct product of a group family and if the argument sets on which each family member acts are disjoint. The final analysis section makes the additional assumption of cost function quasiconvexity. Then, by use of group majorization relations, the central dominance relation for points on a group-generated hull that includes the optimum can be extended to points in that convex hull. The paper concludes with discussions on extensions and empirical issues.

2 Problem

A profit maximizing multi-product firm receives revenue $R(q_1, q_2, \ldots, q_n)$ where the ith output is

represented by q_i. In order to admit flexibility when modeling cost asymmetries across outputs, we do not specify the cost function directly. Instead we model it as a 'pseudo-cost' functional where outputs have been transformed before entering the functional. This pseudo-cost functional is given by

$$C(\mu_1, \mu_2, \ldots, \mu_n): \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+, \quad \mu_i = H_i(q_i), \quad H_i(q_i): \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+, \quad i \in \Omega_n, \tag{1}$$

where $\Omega_n = \{1, 2, \ldots, n\}$, and $\overline{\mathbb{R}}_+^n$ is the closed n-dimensional positive orthant on real numbers. Input prices are fixed and are suppressed to conserve on notation. The $H_i(\cdot)$ are continuously differentiable on $[0, \infty)$, with $dH_i(q)/dq = h_i(q) \in (0, \infty) \; \forall \, q \in [0, \infty)$. They are also invertible with $q_i = H_i^{-1}(\mu_i) = J_i(\mu_i)$.

By substitution we may define $D(\vec{q}) = C[H_1(q_1), H_2(q_2), \ldots, H_n(q_n)]$, but there is no loss of generality in studying $C(\vec{\mu})$ rather than $D(\vec{q})$ because both have the same solution up to the $H_i(\cdot)$ transformations. The advantage of using the form $C(\vec{\mu})$ is that the transformations $H_i(\cdot)$ allow flexibility in designing $C(\vec{\mu})$ such that it has symmetries of a particular type. These symmetries can be exploited in a manner to be elaborated upon shortly. Any inferences drawn on the optimal choice of $\vec{\mu}$ can then be inverted through the bijections provided by the $H_i(\cdot)$ in order to relate equivalent inferences on the optimal choice of $\vec{q}$.

The transformations capture unidimensional asymmetries in the production technology, but inter-dimensional asymmetries in the technology may also arise. We employ group theory to capture asymmetries across dimensions. To this end, and throughout the paper, we make

Assumption 1. $C(\cdot): \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ is increasing and $\tilde{G}$-symmetric.

In the section to follow, we will define and discuss the sorts of $\tilde{G}$-symmetry properties of

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interest. As an illustration, the most readily described is permutation symmetry which we label as $\tilde{S}_n$-symmetry. When viewed as a group acting on function arguments, as represented by the index set Ω_n, group $\tilde{S}_n$ is called the symmetric group on Ω_n. With this symmetry property, $C(\vec{\mu}) = C(\vec{\mu}_g)$ where $\vec{\mu}_g$ is any one of the $n!$ permutations on a given (transformed) output vector $\vec{\mu} \in \overline{\mathbb{R}}^n_+$. Because the extent of symmetry that $\tilde{S}_n$-symmetry admits is quite large, the set of cost functions that satisfy it is comparatively small. The formal treatment of permutation group theory will impose less extensive symmetry, and so will typically admit larger sets of cost functions that satisfy some $\tilde{G}$-symmetry property.

The agent's revenue function is given by $R[J_1(\mu_1), J_2(\mu_2), \ldots, J_n(\mu_n)]$, and we assert

Assumption 2. $R(q_1, q_2, \ldots, q_n) : \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}_+$ *is increasing.*

The agent's decision problem, which we identify as (P), is then to[5]

$$\max_{\vec{\mu} \in \overline{\mathbb{R}}^n_+} R[J_1(\mu_1), J_2(\mu_2), \ldots, J_n(\mu_n)] - C(\mu_1, \mu_2, \ldots, \mu_n). \tag{2}$$

For a given competitive environment, and assuming an interior optimum, identify the optimum as $\vec{\mu}^*$ or $\vec{q}^*$ where $q_i^* = J_i(\mu_i^*)$. Notice that, while we do not allow strategic interactions among firms, the specification is otherwise quite general. For example, the firm may be a price-taker in all its output markets, a monopolist in all its output markets, or a price-taker in some and a monopolist in the remaining set.

Our interest is in understanding what revealed preferences, together with symmetry property $\tilde{G}$, can relate about the admissible values of $q_i^* = J_i(\mu_i^*)$. A particular case of interest is where

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[5] Note the solution would have been the same, upon transformation, had we instead solved $\max_{\vec{q} \in \overline{\mathbb{R}}^n_+} R(q_1, q_2, \ldots, q_n) - C[H_1(q_1), H_2(q_2), \ldots, H_n(q_n)]$.

the firm is a price-taker in all of its output markets and receives the exogenous unit price $p_i > 0$ for the ith output. Then the problem, which we label as (P'), is

$$\max_{\vec{\mu} \in \overline{\mathbb{R}}^n_+} \sum_{i=1}^{n} p_i J_i(\mu_i) \; - \; C(\mu_1, \mu_2, \ldots, \mu_n). \tag{3}$$

While the optimizations in (P) and (P') rely on tools from analysis, our interest is in the optimized vector. There, we will demonstrate how algebraic properties on the optimized objective function can be used to identify order on the decision vector.

3 Algebraic concepts

Our approach involves exploiting an iso-cost contour in $\vec{q} \in \overline{\mathbb{R}}^n_+$ so that preferences over decision vectors are confined to preferences over revenue function evaluations. When the cost function obliges, then an iso-cost contour can be generated by permutations on the arguments. Cost invariances may be depicted through group operations.

Definition 1. A *group*, $\tilde{G}$, is a set of elements, G, together with a single-valued binary operation, *, such that the structure satisfies all of

I)   closure; G is closed with respect to *,

II)   identity element; $\exists\, e \in G$ such that $g * e = e * g = g \; \forall\, g \in G,$

III) inverse elements; $\forall\, g \in G$ there exists a unique element, labeled $g^{-1} \in G$, such that $g * g^{-1}$
     $= g^{-1} * g = e,$

IV) associativity; $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3 \; \forall\, g_1, g_2, g_3 \in G$, where the operations in
     parentheses occur first.

Thus a group is a set of operations together with a composition operation such that the whole

structure satisfies four structural properties. These properties are ideal for studying invariance, in our case a fixed cost level. To be quite specific about what a group element $g \in G$ does, in our context it acts on a set. This is the set of arguments Ω_n, and the group performs a bijection on Ω_n. We write $g(i) = j$ if group element g replaces the ith argument of a function with the pre-existing j^{th} argument.

Example 1. Pseudo-cost function $C(\vec{\mu})$ is $\tilde{S}_3$-symmetric, i.e., invariant, under permutation group $\tilde{S}_3$ when $C(\mu_1, \mu_2, \mu_3) = C(\mu_2, \mu_1, \mu_3) = C(\mu_3, \mu_2, \mu_1)$. This is true because, by iterating the transposition operations we have $C(\mu_1, \mu_2, \mu_3) = C(\mu_1, \mu_3, \mu_2) = C(\mu_2, \mu_3, \mu_1) = C(\mu_3, \mu_1, \mu_2)$ also. The group has $3! = 6$ elements, one for each permutation under which the pseudo-cost functional is invariant. That is, the cardinality of set S_3, $|S_3|$ and which we refer to as the *order* of the group, is 6. Returning to the invariance $C(\mu_1, \mu_2, \mu_3) = C(\mu_2, \mu_3, \mu_1)$, let the pertinent group element that takes (μ_1, μ_2, μ_3) to (μ_2, μ_3, μ_1) be $\hat{g}$. Then $\hat{g}(1) = 2$, $\hat{g}(2) = 3$, and $\hat{g}(3) = 1$.

Example 2. Cost function $C(q_1, \ldots, q_6) = \hat{C}(q_1 + q_2, q_3 + 2q_4, q_5 + q_6)$ is invariant under $\tilde{G}_a$, the group that allows q_1 to permute with q_2 only and q_5 with q_6 only. The transposition of q_3 with q_4 does not generate an invariance because $q_3 + 2q_4$ is not quite symmetric. The group has four elements. In cycle notation, write e as the identity element, $g_1 = (1,2)$, $g_2 = (5,6)$, and $g_3 = (1,2)(5,6)$ so the set of group operations is $G_a = \{e, g_1, g_2, g_3\}$.[6]

If, however, we re-labeled $\hat{q}_4 = 2q_4$, then the modified cost function is symmetric in transpositions of q_3 with $\hat{q}_4$. In this way, we may extend the group to include also $g_4 = (3,4)$, $g_5 = (1,2)(3,4)$, $g_6 = (3,4)(5,6)$, and $g_7 = (1,2)(3,4)(5,6)$. By checking each of the conditions in Definition 1, it is readily demonstrated that $\tilde{G}_b$ is a group, under the same operation

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[6] Here, the cycle notation (i, j, k) represents the argument bijection $i \to j \to k \to i$. For group element $g_1 = (1,2)$, the other four arguments are fixed and are omitted in the cycle notation. Group element $g_3 = (1,2)(5,6)$ asserts that bijections $1 \leftrightarrow 2$ and $5 \leftrightarrow 6$ occur simultaneously.

as $\tilde{G}_a$, when the element set is $G_b = \{e, g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$.

In Example 2, there is clearly a strong connection between groups $\tilde{G}_a$ and $\tilde{G}_b$.

Definition 2. A *subgroup* $\tilde{H}$ of group $\tilde{G}$ is a group with element set H such that

I) H is a subset of set G

II) G and H are closed under the same operation *.

The subgroup relation is written as $\tilde{H} \sqsubseteq \tilde{G}$.

Thus, $\tilde{G}_a \sqsubseteq \tilde{G}_b$. The large literature on groups has been applied to many object sets. As we have seen in examples 1 and 2, in our case we seek only to permute a set of finite order; the set of arguments in a cost function.[7] The descriptive terminology for this context is given as follows:

Definition 3. Let Ω_n be a finite non-empty set of objects with cardinality n. A bijection of Ω_n onto itself is called a permutation of Ω_n. The set of all such permutations forms a group under the composition of bijections. This is the *symmetric group* of Ω_n, and is denoted by $\tilde{S}_n$. Group $\tilde{S}_n$ is said to *act on* set Ω_n. Any subgroup of $\tilde{S}_n$ is called a *permutation group*.

Example 3. In Example 2, let $G_c = \{e, g_1\}$ so that $\tilde{G}_c \sqsubseteq \tilde{G}_a \sqsubseteq \tilde{G}_b$. All are subgroups of $\tilde{S}_6$.

At this point we are in a position to make assertions about problems (P) and (P').

4 Monotone cost

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[7] Finite permutation groups are easy to work with and illustrate. Our analysis can be extended to model all transformations on an argument vector that preserve a well-defined iso-cost contour. This approach would, perhaps, be best studied using continuous transformation groups.

A simple revealed preference argument readily establishes a system of inequalities that any optimal solution must satisfy.

Theorem 1. For (P) under assumptions 1 and 2, then

$$R[J_1(\mu_1^*), J_2(\mu_2^*), \ldots, J_n(\mu_n^*)] \geq R[J_1(\mu_{g(1)}^*), J_2(\mu_{g(2)}^*), \ldots, J_n(\mu_{g(n)}^*)] \quad \forall\, g \in G. \tag{4}$$

Proof. By revealed preference, we have

$$R[J_1(\mu_1^*), J_2(\mu_2^*), \ldots, J_n(\mu_n^*)] - C(\mu_1^*, \mu_2^*, \ldots, \mu_n^*) \geq$$

$$R[J_1(\mu_{g(1)}^*), J_2(\mu_{g(2)}^*), \ldots, J_n(\mu_{g(n)}^*)] - C(\mu_{g(1)}^*, \mu_{g(2)}^*, \ldots, \mu_{g(n)}^*) \quad \forall\, g \in G. \tag{5}$$

Cancellation, due to cost functional $\tilde{G}$-symmetry, then yields the result for each $g \in G$. □

Notice that the evaluations of the cost functional are on those permutations of $\vec{\mu}^*$ that render the cost functional invariant. It is for this reason that the convexity status of $C(\vec{\mu})$ is of no relevance to the finding. Indeed, even monotonicity on $C(\vec{\mu})$ and on $R(\vec{q})$ are not necessary and are only imposed because they are almost certainly true in the economic context of interest to us.

The inequalities in (4) generate an unconstrained admissible region, $A(\vec{q}^*)$, for solution set $\vec{q}^*$ where 'admissible region' is to be interpreted as the region in which we have found that the solution must live. We do not yet assert that it can live at any point in the admissible region, but we will return to this issue in the section to follow. We can, however, impute implications for the impact of quantity restrictions on firm profits. Let $V(\vec{q})$ be a constraint set on firm production activities such that a chosen output vector must satisfy $\vec{q}^* \in V(\vec{q})$ to comply with laws on firm behavior.

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Corollary 1.1. For (P) under assumptions 1 and 2

I)   if $A(\vec{q}^*) \cap V(\vec{q}) = \varnothing$ then firm profits are policy constrained.

II) if $A(\vec{q}^*) \subseteq V(\vec{q})$, then firm profits are not policy constrained.

This constraint set might involve a prohibition on monoculture wheat production in order to reduce soil erosion externalities. Or it might involve a local government requirement that a home builder construct 20% of houses in a sub-division to be affordable for mid-income families. Upon specializing to price-taking behavior in all markets, we have

Corollary 1.2. For (P') under assumptions 1 and 2, then

$$\sum_{i=1}^{n} p_i J_i(\mu_i^*) \geq \sum_{i=1}^{n} p_i J_i(\mu_{g(i)}^*) \quad \forall g \in G. \tag{6}$$

To obtain a sense of what relation (6) means, suppose that we let $g(i) = j$ so that $J_i(\mu_{g(i)}^*) = H_i^{-1}[H_j(q_j^*)]$. The expression is a quantity in the q_i dimension. It is the quantity of q_i that delivers the same argument value in the ith argument of $C(\vec{\mu})$ as q_j^* does in the jth argument of $C(\vec{\mu})$. We will exploit this observation after presenting an illustration of the Theorem. A further specialization of Corollary 1.2, as given in Example 4 below, is quite instructive and provides a basis for much of the analysis to follow.

Example 4. Suppose that the pseudo-cost functional is $\tilde{S}_n$-invariant, and that $H_i(q_i) = \alpha_i + \beta_i q_i$, $\beta_i > 0 \ \forall i \in \Omega_n$. Permutation group element g_{jk}, which is defined as that under which the pseudo-cost functional's arguments are mapped according to $j \leftrightarrow k$, is an element of set S_n. Therefore, upon applying Corollary 1.2 and cancelling terms, we have

$$\left(\frac{p_j}{\beta_j} - \frac{p_k}{\beta_k}\right)\left(\alpha_j - \alpha_k + \beta_j q_j^* - \beta_k q_k^*\right) \geq 0. \tag{7}$$

This bound might be interpreted as a law of comparative supply; if the 'normalized price' of the jth good exceeds that of the kth good, then the 'normalized level' of the jth good exceeds that of the kth good.

If it is known that $p_j/\beta_j \geq p_k/\beta_k$, then we have the upper bound $q_j^* \geq (\alpha_k + \beta_k q_k^* - \alpha_j)/\beta_j$. Suppose that $\alpha_j > \alpha_k$ while $\beta_k < \beta_j$. Define $b_0 = (\alpha_j - \alpha_k)/\beta_k$ and $b_1 = \beta_j/\beta_k$ so that the upper bound may be written as $q_k^* \leq b_0 + b_1 q_j^*$. Then, together with the two non-negativity requirements $q_j^* \geq 0$ and $q_k^* \geq 0$, the admissible region may be described as the semi-open region A in Figure 1. In this example, the positive parts of all rays through the origin are contained in A whenever the ray's slope is no larger than b_1, while the unbounded convex region has two vertices.

If in addition, it is known that $p_j/\beta_j \geq p_k/\beta_k \ \forall j < k$, $j, k \in \Omega_n$, then $\tilde{S}_n$-invariance allows us to assert $\alpha_j + \beta_j q_j^* \geq \alpha_k + \beta_k q_k^*$ for any pair of ordinates $j < k$. In particular, if $\beta_i = \beta \ \forall i \in \Omega_n$ then $q_j^* \geq q_k^* + (\alpha_k - \alpha_j)/\beta$ where the bounding lines have unit slopes. Instead, if $\alpha_i = \alpha \ \forall i \in \Omega_n$ then $\beta_j q_j^* \geq \beta_k q_k^* \ \forall j < k$, $j, k \in \Omega_n$ whenever $p_j/\beta_j \geq p_k/\beta_k \ \forall j < k$, $j, k \in \Omega_n$. In this case, the β_i, $i \in \Omega_n$, can be thought of as efficiency coefficients and the bounding conditions all pass through the origin. Since $p_j/\beta_j \geq p_k/\beta_k$, we have $p_j q_j^* \geq p_k q_k^* \ \forall j < k$, $j, k \in \Omega_n$ so that the kth product revenue is less than or equal to the jth product revenue. Notice that each output is distinguished by two sources of asymmetries, i.e., supply and demand side. Yet, due to the symmetries of the symmetric group and the fact that the cost function location shifters are common, product revenues may be ranked in ascending order according to a single index, p_i/β_i, $i \in \Omega_n$. Due to the problem's structure this composite index is, in a statistical sense, minimal sufficient because no additional information is obtained about the rank order of revenues by

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considering the demand and supply parameters separately.[8]

The intent of Example 4 was to bring out some of the implications of symmetry that the general transformation functions $H_i(q)$ might obscure. However, stronger assertions may be established without recourse to linear transformations of the arguments. One approach is to view the transformations as positive, finite measures on $\overline{\mathbb{R}}_+$ so that existing work in probability theory may be invoked.

Definition 4. Positive, finite, invertible measure $\mu_i = H_i(q_i)$ is said to be larger in the *dispersive order* than positive, finite, invertible measure $\mu_j = H_j(q_j)$ if

$$H_i^{-1}(\tilde{\mu}) - H_i^{-1}(\hat{\mu}) \geq H_j^{-1}(\tilde{\mu}) - H_j^{-1}(\hat{\mu}) \quad \text{whenever } 0 < \hat{\mu} \leq \tilde{\mu}. \tag{8}$$

The partial order relation is written as $H_i(q) \overset{disp}{\geq} H_j(q)$.

The order, a variant of which is discussed in Shaked and Shanthikumar (1994), is of interest because the expressions $H_i^{-1}(\mu)$ and $H_j^{-1}(\mu)$ enter relation (6) in a linear manner.[9]

Theorem 2. For (P') under assumptions 1 and 2, let $g_{ij} \in G$. Then $\mu_i^* \geq \mu_j^*$ whenever $p_i \geq p_j$ if and only if

I) $H_i(q) \overset{disp}{\geq} H_j(q)$, or equivalently

II) $H_i^{-1}[H_j(q)] - q$ is increasing in q, or equivalently

III) $h_j[H_j^{-1}(\mu)] \geq h_i[H_i^{-1}(\mu)] \ \forall\, \mu \geq 0$.

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[8] On minimal sufficient statistics, see Schervish (1995, p. 92).

[9] This proof is placed in the Appendix, together with other proofs that require some work.

Part III) provides, perhaps, the best intuition for the result. At equivalent levels of pseudo-output, i.e., the levels entering the pseudo-cost functional, the transformations provide a bias toward a lower marginal cost for the ith product than for the jth product.

Example 5. If $H_i(q) = \beta_i q$ and $H_j(q) = \beta_j q$ with $0 < \beta_i < \beta_j$, then $H_i(q) \overset{disp}{\geq} H_j(q)$ because it can be seen from part II) that expression $\beta_j q / \beta_i - q$ must be increasing. Theorem 2 then concludes that $\beta_i q_i^* \geq \beta_j q_j^*$ whenever $p_i \geq p_j$. Notice that this example is consistent with Example 4 when $\alpha_i = \alpha \ \forall \ i \in \Omega_n$. Example 4, in fact, generates somewhat stronger results because it exploits the ray property in $H_i(q) = \beta_i q$. In Example 4, inference $\beta_i q_i^* \geq \beta_j q_j^*$ may be drawn whenever $p_i \geq \beta_i p_j / \beta_j$, so that $p_i < p_j$ is admissible.

That $H_i(q) = \beta_i q$ and $H_j(q) = \beta_j q$ are dispersively ordered is just the assertion of the fact that distinct uniform probability distributions can be dispersively ordered. Other examples of dispersive order are readily confirmed. Two univariate normal distributions with distinct variances are dispersively ordered regardless of their means. Also, for the family of measures $H_i(q) = H(\alpha_i + \beta_i q)$ we have that $H_j(q) \overset{disp}{\geq} H_k(q)$ whenever $\beta_j \leq \beta_k$. This is because we may insert $[H^{-1}(\mu) - \alpha_i] / \beta_i = q = H_i^{-1}(\mu)$ into (8).

5 Linear transformations and monotone cost

In this section we assume that $H_i(q_i) = \alpha_i + \beta_i q_i$ under problem (P'). We do this because findings in the large literature on linear algebra can then be applied. For an arbitrary permutation group, and where the transformations are linear, use of Corollary 1.2 shows that the solution set must satisfy the system of $n + |G|$ inequalities,

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$$\sum_{i=1}^{n} \frac{p_i}{\beta_i} \times \left[\alpha_i - \alpha_{g(i)} + \beta_i q_i^* - \beta_{g(i)} q_{g(i)}^* \right] \geq 0 \quad \forall g \in G, \tag{9}$$

$$q_i^* \geq 0 \quad \forall i \in \Omega_n.$$

For future reference we will write the solution set to the group-generated inequalities,

$$\sum_{i=1}^{n} \frac{p_i}{\beta_i} \times \left[\alpha_i - \alpha_{g(i)} + \beta_i q_i^* - \beta_{g(i)} q_{g(i)}^* \right] \geq 0 \quad \forall g \in G, \tag{10}$$

as $\mathcal{L}(\vec{\alpha}, \vec{\beta}, \tilde{G})$. Notice that for the identity element, e, inequality (10) reduces to the trivial $0 \geq 0$, and may be discarded.

Example 6. To see that the bounds in system (9) may be tight, consider

$$C(\vec{\mu}) = \hat{C}\left(\max[\mu_1, \mu_2, \dots, \mu_n] \right), \tag{11}$$

where $\hat{C}(\cdot): \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is increasing. Then, since output prices are strictly positive, any $\mu_i \neq \mu_j$ would violate profit maximization for interior solutions. Therefore, $\mu_i^* = \mu_j^* \ \forall i, j \in \Omega_n$. In this case, all $|G|$ group $\tilde{G}$ generated bounds in system (9) are satisfied with equality.

Following Solodovnikov (1980), system (9) may be decomposed through successive simplifications. Commence with the inhomogeneous inequalities as given in system (9). Next, develop a set of homogeneous inequalities by removing the constant terms in each equation, i.e., by setting $\alpha_i = \overline{\alpha} \ \forall i \in \Omega_n$. The homogeneous system of inequalities is

$$\sum_{i=1}^{n} \frac{p_i}{\beta_i} \times \left[\beta_i q_i^* - \beta_{g(i)} q_{g(i)}^* \right] \geq 0 \quad \forall g \in G, \tag{12}$$

$$q_i^* \geq 0 \quad \forall i \in \Omega_n.$$

The latter system can be further simplified by imposing equalities for each of the weak inequalities so that

$$\sum_{i=1}^{n} \frac{p_i}{\beta_i} \times \left[\beta_i q_i^* - \beta_{g(i)} q_{g(i)}^* \right] = 0 \quad \forall g \in G,$$

$$q_i^* = 0 \quad \forall i \in \Omega_n.$$

(13)

This homogeneous system of equalities clearly has a unique solution, $q_i^* = 0 \ \forall i \in \Omega_n$. Solodovnikov's analysis of system (9) is of relevance to our study because he makes much use of the concept of a normal system.

Definition 5. [Solodovnikov (1980, p. 30)] A system of linear inequalities is said to be *normal* if the corresponding system of homogeneous equations has <u>only the zero solution</u>.

Thus, system (9) is normal. The solution to the system may be broken into two steps:

STEP 1: Take some n equations from the $n + |G|$ equations in (9), the system of inhomogeneous inequalities. Replace the inequalities with equalities, and solve that sub-system. Do this for all sub-systems comprised of consistent and independent equations, i.e., that give unique solutions as quantity points in $\mathbb{R}^n$. There are at most $(n + |G|)!/(n!\,|G|!)$ such sub-systems. For each solution, return to system (9) and validate that it satisfies the larger system. If it does not, then discard it. The remaining solutions could be called vertices for the system. Let there be m such points, with $m \leq (n + |G|)!/(n!\,|G|!)$, and label them V_i with $i \in \Omega_m$ and $V_i \in \overline{\mathbb{R}}_+^n$. Write the convex hull of these points as $[\![V_1, V_2, \ldots, V_m]\!]$.

STEP 2: Take all sub-systems of $n-1$ equations from the $n + |G|$ equations in (13), the system

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of homogeneous equalities. There are at most $(n+|G|)!/[(n-1)!(|G|+1)!]$ such sub-systems. Solve each sub-system, where each solution set may be written as a line in $\mathbb{R}^n$. If a *non-zero* point on this line satisfies system (12), then place the point in set E. All equations in system (13) pass through the origin. When such a point satisfies system (12), then one half of the line satisfies system (12). Suppose that $r = |E|$ where $(n+|G|)!/[(n-1)!(|G|+1)!] \geq r$. Label the elements of set E as E_i, $i \in \Omega_r$, and construct the convex set of points $[\![E_1, E_2, \ldots, E_r]\!]$ that can be written as $\Sigma_{i=1}^{r} r_i E_i$, $r_i \in \overline{\mathbb{R}}_+$.

Proposition 1. System (9) is normal, and the solution set

I)   is convex,

II)   may be represented as

$$\mathfrak{F} = [\![V_1, V_2, \ldots, V_m]\!] + [\![E_1, E_2, \ldots, E_r]\!]. \tag{14}$$

where the addition operation is the vector addition of sets.

The solution set may or may not be a convex polyhedral cone, a characterization that would be particularly convenient to work with.

Definition 6. [see Solodovnikov (1980, p. 21)] A *convex polyhedral cone* is the intersection of a finite number of half-spaces whose bounding planes all pass through a common point, called the *vertex of the cone*.

Example 7. Returning to Figure 1 and Example 4, where $p_j/\beta_j \geq p_k/\beta_k$, we see that the described region is not a convex polyhedral cone. It has two vertices, $V_1 = (0,0)$ and $V_2 = (0, b_0)$. The homogeneous system solved in step 2 has two edges, E_1 whereby $q_k = 0$ and E_2

whereby $q_k = b_1 q_j$. The equation $q_j = 0$ is not an edge because there exists no non-zero point on $q_j = 0$ that satisfies both $q_k \geq 0$ and $q_k \leq b_1 q_j$. The convex hull of vertices is given by $[\![V_1, V_2]\!] = (0,0)\lambda + (0,b_0)(1-\lambda)$, $\lambda \in [0,1]$. Set $[\![E_1, E_2]\!]$ is the convex polyhedral cone with vertex $(0,0)$ and generated by the pair of halfspaces $q_k \geq 0$, $q_k \leq b_1 q_j$.

Set $[\![V_1, V_2]\!] + [\![E_1, E_2]\!]$ is not a polyhedral cone because the extreme points on segment $(0,0)\lambda + (0,b_0)(1-\lambda)$, $\lambda \in [0,1]$, are both vertices. The vertices only coincide when $\alpha_j = \alpha_k$. If we were to amend Example 4 so that $\alpha_j < \alpha_k$, then $b_0 < 0$ and the non-negativity constraint $q_k \geq 0$ would not bind. In that case the only vertex would be $((\alpha_k - \alpha_j)/\beta_j, 0)$.

The propositions to follow in this section will clarify some of the attributes of solution set (14). Because system (10) is group-generated, the inequalities are not arbitrary. Inspection of group-generated system (10) clarifies that these bounding hyperplanes possess $\vec{q}_0 = (-\alpha_1/\beta_1, -\alpha_2/\beta_2, \ldots, -\alpha_n/\beta_n)$ as a common point. In particular, the point $\vec{q}_0$ is either the unique vertex or a 'shadow' vertex that is inadmissible only due to one or more non-negativity constraint. This endows the solution set with a particular structure. For this reason the solution set could plausibly be a convex polyhedral cone, i.e., all bounding planes among the $n + |G|$ linear constraints pass through a common point. Some development of this observation establishes two results.

Proposition 2. The solution set to system (9) is not bounded. Further, it contains all points on the semi-open line segment

$$\vec{q}^{\,*} = \left(\frac{\theta - \alpha_1}{\beta_1}, \frac{\theta - \alpha_2}{\beta_2}, \ldots, \frac{\theta - \alpha_n}{\beta_n} \right) \quad \forall\, \theta \geq \max[\alpha_1, \alpha_2, \ldots, \alpha_n]. \tag{15}$$

Proposition 3. The admissible region

I)   is given by $\mathcal{L}(\vec{\alpha},\vec{\beta},\tilde{G})\cap\overline{\mathbb{R}}_+^n$ where $\mathcal{L}(\vec{\alpha},\vec{\beta},\tilde{G})$ is a convex polyhedral cone with vertex $\vec{q}_0$.

II) contains the origin if and only if $\sum_{i=1}^{n}(p_i/\beta_i)(\alpha_i - \alpha_{g(i)}) \geq 0 \;\; \forall\, g \in G$. This condition is satisfied if $\alpha_i = \alpha \in \mathbb{R} \;\; \forall\, i \in \Omega_n$, or if $(p_i/\beta_i - p_j/\beta_j)(\alpha_i - \alpha_j) \geq 0 \;\; \forall\, i,j \in \Omega_n$.

III) is the convex polyhedral cone generated by system (10) if $\max[\alpha_1,\alpha_2,\dots,\alpha_n] \leq 0$ and the β_i are all distinct.

That (15) satisfies the system can be verified by insertion to obtain equality for each of the group-generated inequalities. Unboundedness is a consequence of the unboundedness of the semi-open line segment, $\theta \geq \max[\alpha_1,\alpha_2,\dots,\alpha_n]$. Part I) of Proposition 3 re-organizes the set of bounding hyperplanes. Parts II) and III) draw implications when the non-negativity constraints are all binding, as in part II), or are all slack, as in part III).

Example 8. In Figure 2, where the context is the same as in Figure 1 except that now $\alpha_j < \alpha_k$, the admissible region (in dimensions q_j and q_k) is a convex polyhedral cone. This is because the constraint $q_k \geq 0$ does not now bind, as has been explained in Example 7.

Depending on the group order, and also the dimension and values of the parameter vectors $\vec{\alpha}$ and $\vec{\beta}$, it may be difficult to study the solution set for system (9). Therefore, it would be of some interest to ascertain when it is possible to simplify the solution set. It is possible to do so when the group of symmetries is, in some sense, separable. To this end we introduce

Definition 7. [see Hungerford (1974, p. 59)] Let $\tilde{G}^i$, $i \in \Omega_I$, be a family (i.e., a set) of permutation groups with respective orders n_i. Form the direct product $\times_{i=1}^{I}\tilde{G}^i$ with order $\Pi_{i=1}^{I}n_i$. Then $\tilde{G} = \times_{i=1}^{I}\tilde{G}^i$ forms a group under component-wise composition. This is called the *direct*

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product group.

Example 9. For the pseudo-cost functional $C(\vec{\mu}) = C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$, suppose that first two arguments permute, i.e., $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \equiv C(\mu_2, \mu_1, \mu_3, \mu_4, \mu_5)$. This invariance corresponds to a group of order 2, and we call it $\tilde{G}^1$ with element set $G^1 = \{e^1, g_1^1\}$ where the first element is the identity. Suppose also that the invariances related to the other three arguments are given by group $\tilde{G}^2$ with element set $G^2 = \{e^2, g_1^2, g_2^2\}$ where e^2 corresponds to the identity invariance $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \equiv C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$, g_1^2 corresponds to the invariance $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ $\equiv C(\mu_1, \mu_2, \mu_4, \mu_5, \mu_3)$, and g_2^2 corresponds to the invariance $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \equiv$ $C(\mu_1, \mu_2, \mu_5, \mu_3, \mu_4)$. Then the direct product group $\tilde{G} = \times_{i=1}^2 \tilde{G}^i$ is of order 6. It may be represented by the elements (e^1, e^2) [i.e., identity invariance $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \equiv$ $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$], (g_1^1, e^2) [i.e., $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \equiv C(\mu_2, \mu_1, \mu_3, \mu_4, \mu_5)$], (e^1, g_1^2) [i.e., $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \equiv C(\mu_1, \mu_2, \mu_4, \mu_5, \mu_3)$], (g_1^1, g_1^2) [i.e., $C(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \equiv$ $C(\mu_2, \mu_1, \mu_4, \mu_5, \mu_3)$], (e^1, g_2^2), and (g_1^1, g_2^2).

Direct product groups are of interest because they identify fault lines along which the dimensions of problem (9) may be reduced. In particular, in Example 9 note that the path that each argument in a cost functional may take is strictly smaller than argument set Ω_5. To formalize the idea that the argument set may be partitioned according to the paths taken when operated on by group elements, we employ the definitions of the orbit of an argument, a G-space, and a decomposable group.

Definition 8. For group $\tilde{G}$ acting on Ω_n, the orbit of $k \in \Omega_n$, which is labeled O_k, is the subset of Ω_n that $\tilde{G}$ replaces the kth coordinate with. That is, write $i = g(k)$ as the element of Ω_n that group element g replaces k with. Then $O_k = \{i \in \Omega_n : g(k) = i, g \in G\}$.

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Definition 9. [See Cameron (1999, p. 2)] For group $\tilde{G}$ acting on Ω_n, a G-space is a set $\Delta \subseteq \Omega_n$ (called a G-set) together with a function $\gamma : \Delta \times G \to \Delta$ satisfying the conditions

a) $\gamma(\gamma(\alpha, g_i), g_j) = \gamma(\alpha, g_i * g_j) \ \forall \alpha \in \Delta, \ \forall g_i, g_j \in G$

b) $\gamma(\alpha, e) = \alpha \ \forall \alpha \in \Delta$.

In Example 1, the only G-set for $\tilde{S}_3$ acting on Ω_3 is Ω_3 itself when the function is just the group acting on the G-set, $\gamma(\alpha, g_i) = g_i(\alpha)$. No other $\Delta \subseteq \Omega_3$ will do because no other $\Delta \subseteq \Omega_3$ is closed under group $\tilde{S}_3$. In Example 2, $\tilde{G}_a$ has the G-set $\{1,2\}$. It also has the G-sets $\{5,6\}$ and $\{1,2,5,6\}$. Notice that, for permutation groups in general, if an element of Ω_n is in two G-sets for the same group then one of the G-sets must contain the other. This is because a G-set must contain either all the elements of an argument's orbit or none of the elements of the orbit.

Definition 10. [See Hungerford (1974, p. 83)] A permutation group $\tilde{G}$ is said to be *decomposable* if there exists a family of permutation groups, $\tilde{G}^i$, $i \in \Omega_I$, such that $\tilde{G} = \times_{i=1}^I \tilde{G}^i$. This decomposition need not be unique.

Proposition 4. Let $\tilde{G}$ be the group of invariances on the pseudo-cost functional where $\tilde{G}$ is decomposable such that $\tilde{G} = \times_{i=1}^I \tilde{G}^i$, and where each $\tilde{G}^i$ acts only on G-set Δ_i, one for each family member with $\Delta_j \cap \Delta_k = \varnothing \ \forall j, k \in \Omega_I$. Then system (9) decomposes into I sub-systems such that the admissible region for the values of the arguments represented by the G-set Δ_i, $i \in \Omega_I$, is independent of arguments outside that set.

Example 10. Let the direct product group and pseudo-cost functional be as in Example 9. Further, for convenience let the transformations be linear with $\alpha_i = \alpha \ \forall i \in \Omega_5$. Then system (9)

decomposes into the sub-system with non-trivial conditions

$$p_1 q_1^* + p_2 q_2^* \geq \frac{p_1}{\beta_1} \beta_2 q_2^* + \frac{p_2}{\beta_2} \beta_1 q_1^*,$$

$$q_1^* \geq 0, \quad q_2^* \geq 0,$$

(16')

and the sub-system with non-trivial requirements

$$p_3 q_3^* + p_4 q_4^* + p_5 q_5^* \geq \frac{p_3}{\beta_3} \beta_4 q_4^* + \frac{p_4}{\beta_4} \beta_5 q_5^* + \frac{p_5}{\beta_5} \beta_3 q_3^*,$$

$$p_3 q_3^* + p_4 q_4^* + p_5 q_5^* \geq \frac{p_3}{\beta_3} \beta_5 q_5^* + \frac{p_4}{\beta_4} \beta_3 q_3^* + \frac{p_5}{\beta_5} \beta_4 q_4^*,$$

(16")

$$q_3^* \geq 0, \quad q_4^* \geq 0, \quad q_5^* \geq 0.$$

The admissible regions are mutually separated so that the prices and productivities for one set of arguments have no effect on the admissible region for the other set of arguments.

6 Quasiconvex cost

As far as the approach that we have taken is concerned, the quasiconvexity property carries with it particularly useful structure on the level sets. Specifically, quasiconvexity of the cost functional allows a pre-ordering on vectors called *G*-majorization to be employed to advantage.

Assumption 3. $C(\cdot): \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ *is increasing, quasiconvex and $\tilde{G}$-symmetric.*

The group majorization, or *G*-majorization, pre-ordering is the means by which we will

exploit this additional structure.

Definition 11. [From Marshall and Olkin (1979, p. 422)] Let $\tilde{G}$ be a group of linear transformations mapping $\mathbb{R}^n$ to $\mathbb{R}^n$. Then $\vec{z}^{\alpha}$ is *group majorized* by $\vec{z}^{\beta}$ with respect to group $\tilde{G}$, written as $\vec{z}^{\alpha} \preceq_{\tilde{G}} \vec{z}^{\beta}$, if $\vec{z}^{\alpha}$ lies in the convex hull of the orbit of $\vec{z}^{\beta}$ under the group $\tilde{G}$.

For solution vector $\vec{\mu}^{*}$ to problem (P), write the convex hull as $\mathcal{H}(\vec{\mu}^{*})$ where $\mathcal{H}(\vec{\mu}^{*}) = \{\vec{\mu} : \exists \, \vec{\alpha} \in \mathbb{R}^{|G|}, \alpha_i \geq 0 \; \forall \, i \in \Omega_{|G|}, \sum_{i=1}^{|G|} \alpha_i = 1, \vec{\mu} = \sum_{g \in G} \vec{\alpha} \cdot \vec{\mu}_g^{*}\}$. The following generalization of Theorem 1 is readily demonstrated.

Theorem 3. For (P) with assumptions 2 and 3, then

$$R[J_1(\mu_1^{*}), J_2(\mu_2^{*}), \ldots, J_n(\mu_n^{*})] \geq R[J_1(\mu_1), J_2(\mu_2), \ldots, J_n(\mu_n)] \quad \forall \, \vec{\mu} \in \mathcal{H}(\vec{\mu}^{*}). \tag{17}$$

The finding extends Theorem 1 in the following sense. If quasiconvexity on the cost functional is assumed, in addition to $\tilde{G}$-symmetry and monotonicity, then dominance a) relative to all group-generated points that define a convex hull may be strengthened to dominance b) relative to all points on and in the convex hull. That is, any point in the convex hull of $\vec{\mu}^{*}$ garners (weakly) lower revenue.

7 Discussion

The intent of this paper has been to show that a study of a firm's production decision vector which emphasizes the presence and absence of structural symmetries can reveal much about what sorts of decisions are optimal. The appeal of the approach arises from the simple and intuitive consequences that fall out of accepting some form of group symmetry.

Nonetheless, one might wonder how the results might hold up under more general production environments. For example, our treatment of preferences was elementary. Caplin and Nalebuff (1991), who study unimodal measures of product attribute demands, provide a more realistic framework for viewing product preferences. Their analysis, which is founded on what the Brunn-Minkowski theorem has to say about the measure of a convex set in $\mathbb{R}^m$, has a strong geometry and algebra orientation. Much of the recent work on measures of convex sets has employed group theory [Tong (1980); Dharmadhikari and Joag-Dev (1988); Bertin *et al.* (1997)]. It may be feasible to link a variant of the Caplin-Nalebuff demand structure with our production structure to draw inferences about equilibrium pricing decisions for a multi-market monopolist.

A general treatment of the multi-product firm in oligopoly markets will likely be more challenging. Quite apart from the larger set of strategic environments that present themselves, there is the issue of accommodating the strategic interactions themselves. For firms producing a single, homogeneous good, Salant and Shaffer (1999) and others have exploited the symmetric group to develop a theory of equilibrium. Approaches that employ group theory or *G*-majorization theory in a more formal manner may provide the modeler with sufficient lattitude to glean useful additional insights.

Concerning the prospects for empirical inquiry into the forms taken by symmetries and asymmetries in technologies and preferences, we note that the interest among statisticians in symmetry has given rise to a large literature on testing for various symmetries and asymmetries. This literature includes, for example, work by Neuhaus and Zhu (1998) who develop a test for symmetry under reflection groups, and by Koltchinskii and Li (1998) who develop a test for spherical symmetry.

The above line of research seeks to establish a general framework for the study of equilibrium level vectors in economic decision analysis. A central feature of this framework

involves the exploitation of invariance along a level curve. In the theory of comparative statics for economic systems, much use has been made of the related notion of the iso-quant. It would be of interest to establish how a theory of equilibrium level vectors might relate to the theory of monotone comparative statics. Observations on how levels and sensitivities of equilibrium vectors relate will surely assist in establishing a coherent and integrated characterization of equilibrium decisions.

Appendix

Proof of Theorem 2. For the reversed implication in part a), by invoking Corollary 1.2 we apply group element g_{ij} in inequality (6) to obtain

$$p_i \times [H_i^{-1}(\mu_i^*) - H_i^{-1}(\mu_j^*)] \geq p_j \times [H_j^{-1}(\mu_i^*) - H_j^{-1}(\mu_j^*)]. \tag{A.1}$$

This is true for all $p_i \geq p_j$, so that it is true for $p_i = p_j$. Therefore,

$$H_i^{-1}(\mu_i^*) - H_i^{-1}(\mu_j^*) \geq H_j^{-1}(\mu_i^*) - H_j^{-1}(\mu_j^*). \tag{A.2}$$

From Definition 4, it follows that if $H_i(q) \overset{disp}{\geq} H_j(q)$ then $\mu_i^* \geq \mu_j^*$, i.e., $H_i(q_i^*) \geq H_j(q_j^*)$. To see the reversed implication, let $\mu_i^* \geq \mu_j^*$ and $p_i \geq p_j$. Again we can choose over the interval $p_i \geq p_j$ and it is convenient to choose $p_i = p_j$. Again we arrive at inequality (A.2), which must be true for all $\mu_i^* \geq \mu_j^*$. But this is the definition of the dispersive order, so that we have $H_i(q) \overset{disp}{\geq} H_j(q)$.

To verify parts b) and c), we follow the obvious routes as given in Shaked and Shanthikumar (1994). Relation (8) is true if and only if $z(\mu) = H_i^{-1}(\mu) - H_j^{-1}(\mu)$ is increasing in μ. Upon substituting in $\mu = H_j(q)$, the monotonicity of $H_j(q)$ generates the equivalent condition that $H_i^{-1}[H_j(q)] - q$ be increasing in q. Given that we have already assumed that the transformation function derivatives exist, a straightforward differentiation of $z(\mu)$, together with a little algebra, yields the equivalent expression as given in part c). □

Proof of Proposition 1. The property of 'normal' follows from the definition. For convexity, pick points $\vec{q}_a$ and $\vec{q}_b$ that satisfy system (9). Clearly $\vec{q}_c = \lambda \vec{q}_a + (1-\lambda)\vec{q}_b$, $\lambda \in [0,1]$ also satisfies the system. For part b), see page 54 in Solodovnikov (1980). □

Proof of Proposition 3. Part a) is just a re-statement of the conditions that define the admissible region. The *if* implication of the first assertion in part b) follows from the insertion of $\vec{q}^* = \vec{0}$ into system (9). For the *only if* implication, suppose that there exists a $\hat{g} \in G$ such that $\sum_{i=1}^{n}(p_i/\beta_i)(\alpha_i - \alpha_{\hat{g}(i)}) < 0$. Then the origin does not satisfy one inequality in (9), and so is not admissible.

As for the second assertion, if $\alpha_i = \alpha \in \mathbb{R} \ \forall i \in \Omega_n$ then $\sum_{i=1}^{n}(p_i/\beta_i)(\alpha_i - \alpha_{g(i)}) \equiv 0 \ \forall g \in G$. Alternatively, if $(p_i/\beta_i - p_j/\beta_j)(\alpha_i - \alpha_j) \geq 0 \ \forall i,j \in \Omega_n$ then the identity group element maximizes the value of $\sum_{i=1}^{n}(p_i/\beta_i)\alpha_{g(i)}$, $g \in G$. That is, $\sum_{i=1}^{n}(p_i/\beta_i)\alpha_i = \max_{g \in G}\sum_{i=1}^{n}(p_i/\beta_i)\alpha_{g(i)}$ and $\sum_{i=1}^{n}(p_i/\beta_i)(\alpha_i - \alpha_{g(i)}) \geq 0 \ \forall g \in G$.

In part c), if $\max[\alpha_1,\alpha_2,\dots,\alpha_n] \leq 0$ then $\vec{q}_0 \geq \vec{0}$. We will show that the solution to system (9) is in the shifted positive orthant $\vec{q}^* \geq \vec{q}_0$. If we can show this, then the inequalities $\vec{q}^* \geq \vec{0}$ in (9) are slack and may be ignored. The demonstration will be done by perturbations in the locality of the vertex. In particular, consider the point $\vec{q}_1 = \vec{q}_0 + (0,0,\dots,\varepsilon,\dots,0)'$, $\varepsilon > 0$, where the only non-zero entry in vector $(0,0,\dots,\varepsilon,\dots,0)'$ is at the ith ordinate. Suppose too, and without loss of generality, that $\beta_{g(i)} > \beta_i$ for some $g \in G$. For this group element, and for the point $\vec{q}_1$, compute the bound

$$\sum_{i=1}^{n} \frac{p_i}{\beta_i} \times \left[\alpha_i - \alpha_{g(i)} + \beta_i q_i^* - \beta_{g(i)} q_{g(i)}^*\right] \tag{A.3}$$

to obtain

$$\frac{\varepsilon p_i}{\beta_i}\left[\beta_i - \beta_{g(i)}\right] < 0. \tag{A.4}$$

Thus, any hyperplane through vertex $\vec{q}_0$ and parallel to any of the n axes does not intersect the convex polyhedral cone. That is, if the cone's vertex were translated to the origin then the cone

would remain in the positive orthant. The condition $\max[\alpha_1, \alpha_2, \ldots, \alpha_n] \leq 0$ ensures that $\mathcal{L}(\vec{\alpha}, \vec{\beta}, \tilde{G}) \subseteq \overline{\mathbb{R}}^n_+$. $\qquad\square$

Proof of Proposition 4. Choose $g = \{g_{i_1}^1, g_{i_2}^2, \ldots, g_{i_I}^I\}$ as an arbitrary element of $\tilde{G}$. This generates the inequality

$$\sum_{i=1}^{I} \sum_{k \in \Delta_i} \frac{p_k}{\beta_k} \times \left[\alpha_k - \alpha_{g(k)} + \beta_k q_k^* - \beta_{g(k)} q_{g(k)}^* \right] \geq 0 \quad \forall \, g \in G. \tag{A.5}$$

Instead, choose $\hat{g} = \{e^1, e^2, \ldots, g_{i_v}^v, \ldots, e^I\}$ where the elements from all but one of the group family are identity elements. Then (A.5) reduces to

$$\sum_{k \in \Delta_v} \frac{p_k}{\beta_k} \times \left[\alpha_k - \alpha_{\hat{g}(k)} + \beta_k q_k^* - \beta_{\hat{g}(k)} q_{\hat{g}(k)}^* \right] \geq 0 \quad \forall \, \hat{g} \in e^1 \times e^2 \times \ldots \times \tilde{G}^v \times \ldots e^I. \tag{A.6}$$

If (A.6) is true for each $\hat{g} \in G$, then summation demonstrates that (A.5) is true for each $g \in G$. Conversely, if (A.6) is false for any $\hat{g} = \{e^1, e^2, \ldots, g_{i_v}^v, \ldots, e^I\}$ then (A.5) is false for some $g \in G$ because we may choose $g = \hat{g}$. Therefore, each of the I systems represented by (A.6) may be solved separately. $\qquad\square$

Proof of Theorem 3. If $\vec{\mu} \in \mathcal{H}(\vec{\mu}^*)$, then there exists an $\vec{\alpha} = (\alpha_1, \ldots, \alpha_{|G|}) \in [0,1]^{|G|}$ such that $\sum_{i=1}^{|G|} \alpha_i = 1$ and $\vec{\mu} = \sum_{g \in G} \vec{\alpha} \cdot \vec{\mu}_g^*$. By quasiconvexity and invariance on the convex hull, we have that costs are smaller inside convex hull $\mathcal{H}(\vec{\mu}^*)$, i.e., $C(\vec{\mu}^*) = \sum_{g \in G} \vec{\alpha} \cdot C(\vec{\mu}_g^*) \geq$ $C(\sum_{i=1}^{|G|} \alpha_i \mu_{g_i(1)}^*, \sum_{i=1}^{|G|} \alpha_i \mu_{g_i(2)}^*, \ldots, \sum_{i=1}^{|G|} \alpha_i \mu_{g_i(n)}^*)$ where it is understood that $G = \{g_1, g_2, \ldots, g_{|G|-1}, e\}$ with $g_{|G|} = e$ and where it is clear that $(\sum_{i=1}^{|G|} \alpha_i \mu_{g_i(1)}^*, \sum_{i=1}^{|G|} \alpha_i \mu_{g_i(2)}^*, \ldots, \sum_{i=1}^{|G|} \alpha_i \mu_{g_i(n)}^*) = \sum_{g \in G} \vec{\alpha} \cdot \vec{\mu}_g^* \in \mathcal{H}(\vec{\mu}^*)$. Therefore, revenue must be larger in order to justify the weak optimality of $\vec{\mu}^*$ over any $\vec{\mu} \in \mathcal{H}(\vec{\mu}^*)$, i.e.,

$$R[J_1(\mu_1^*), J_2(\mu_2^*), \ldots, J_n(\mu_n^*)] \geq R[J_1(\mu_1), J_2(\mu_2), \ldots, J_n(\mu_n)] \quad \forall \, \vec{\mu} \in \mathcal{H}(\vec{\mu}^*). \tag{A.7}$$

Put another way, $R[J_1(\mu_1^*), \ldots, J_n(\mu_n^*)] = \sup_{\vec{\mu} \in \mathcal{H}(\vec{\mu}^*)} R[J_1(\mu_1), \ldots, J_n(\mu_n)].$ $\qquad\qquad \square$

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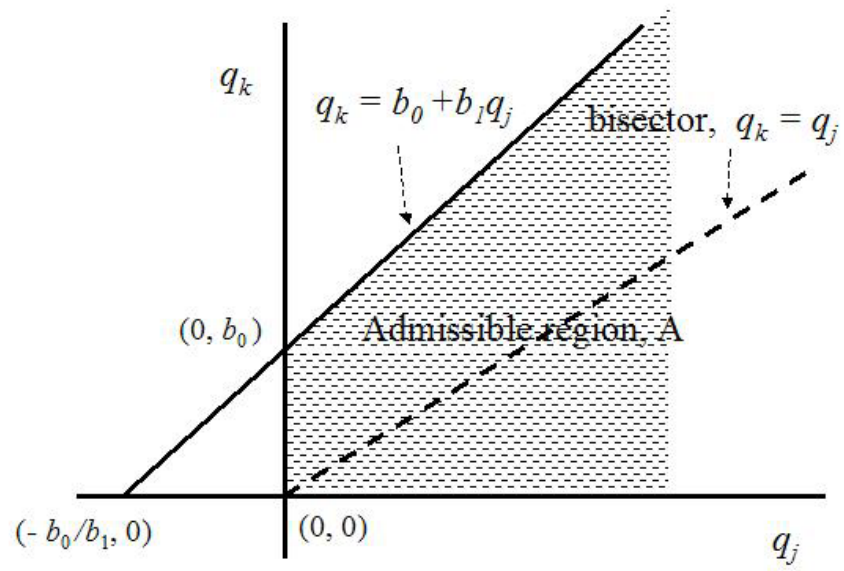Whinston, M. D.: Tying, foreclosure, and exclusion. American Economic Review, September **80**(4), 837–859 (1990)

Figure 1. Multi-product bounds generated by transposition g_{jk}