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Abstract

The welfare dominance of ad valorem taxes over unit taxes in a single-market Cournot oligopoly is well-known. This article extends the analysis to multi-market oligopoly. Provided all ad valorem taxes are positive, unit costs are constant, firms are active in all considered markets, and a representative consumer has convex preferences, it is shown that ad valorem taxes dominate in multi-product equilibrium. We discuss the role of unit cost covariances across multi-product firms in determining the extent of cost efficiencies arising under ad valorem taxation. The issue of merger under oligopoly is also considered. Conditions are identified under which a merger increases the sum of consumer and producer surpluses while also increasing the revenue yield from a set of unit taxes. If not all firms are active in all considered markets, then it is also shown that additional conditions are required to ensure the dominance of ad valorem taxes. In multi-input Cournot oligopsony, however, unit taxation welfare dominates. This is because ad valorem taxes on inputs reduce demand elasticities, amplifying market power distortions.

Keywords: ad valorem tax; imperfect competition; oligopoly merger; quantity-setting game; specific tax; tax efficiency; tax revenue.

JEL Classification Numbers: H21; D43

1. Introduction

The choice between per unit taxation and ad valorem taxation in imperfectly competitive markets has received continuing attention since Suits and Musgrave (1953). It is an important issue because imperfect competition characterizes many markets and sales taxes are an important source of government revenue at state and local levels throughout the world. As Skeath and Trandel (1994) point out, both tax structures have commonly been used. Delipalla and Keen (1992), for same-cost firms, and Denicolò and Matteuzzi (2000), for heterogeneous-cost firms, confirm the welfare dominance of ad valorem taxes under Cournot behavior in single product oligopoly. Thus an ad valorem tax that holds aggregate output the same as a specific tax both reduces firm costs and increases tax receipts. Anderson, de Palma, and Kreider (2001) have extended consideration to product differentiation and price-setting behavior to find that ad valorem taxes no longer necessarily dominate. Kitahara and Matsumura (2006) show that the choice of tax system can affect firm location decisions, and so consumer surplus, for price-setting firms when products are location differentiated. Blackorby and Murty (2007) show equivalence between the taxes under monopoly in general equilibrium when profits are taxed and revenues are returned to consumers. They show that Pareto optima are the same under either tax.

Missing from the literature are models acknowledging that most firms in imperfectly competitive markets are multi-product. We develop such a model in the Cournot setting. Our model assumes constant unit costs, as with most of the literature, but marginal costs can differ across firms and products. The demand side is characterized by a representative consumer with convex preferences. Making the assumption that all firms are active in all considered markets, we show that dominance of the ad valorem tax system extends to multi-markets. We also show that the extent of dominance depends on cost correlations across firms and on whether goods substitute or complement in demand.

Our multi-product model also allows for consideration of conglomerate mergers. For specific tax levels that hold market outputs fixed, we find conditions under which conglomerate mergers increase the tax yield. We also identify conditions under which mergers both decrease industry costs and increase tax revenues, so that conglomerate mergers can then be viewed as welfare dominating.

Our assumption that all firms are active in all considered markets is scrutinized. Relaxing it, we show that it is probably best for conglomerate firms (rather than single-product firms) to have most of the variability in unit costs. This is because they are best positioned to adjust outputs while market level outputs are not affected by the shift in cost variability. For ad valorem taxes to dominate specific taxes when firms are inactive in some markets, additional conditions are needed. This is because the effects of taxes on outputs depend on what other products a firm markets through demand-side interactions. Since multi-product Cournot markets in which some firms are inactive in some considered markets is a plausible economic environment, we conclude that welfare dominance of ad valorem taxes is questionable.

The final main section develops on Hamilton's (1999) observation that dominance is reversed when market power is on the part of a single market monopsonist. We find dominance reversal to be true for multi-input Cournot oligopsonists. The reason is that ad valorem taxes then dampen, rather than magnify, the effective elasticity the firms with market power face. This modifies distortions such that aggregate revenues contract. We conclude with a brief summary.

2. Framework

The model of multi-product Cournot behavior for a representative consumer is based on that in Lapan and Hennessy (2006). Each of N firms is active in all of M markets while there are H price-taking consumers. Firms are denoted as $n \in \{1, 2, \ldots, N\} = \Omega_N$ while markets and consumers are identified similarly. The aggregate level of the *m*th good available for consumption is X_m and the vector of goods available for consumption is $X = (X_1, \ldots, X_M)$. Direct utility from consuming goods in the M markets is $U(X)$, a concave function. Prices are $P_m(X), m \in \Omega_M$. Aggregate income available across all consumers is I and z is the numeraire good so that $z = I - \sum_{m\in\Omega_M} P_m(X)X_m$ and the price-taking representative consumer's problem is $\max_X I + U(X) - \sum_{m\in\Omega_M} P_m X_m$. This consumer-level optimization gives inverse demand functions as $P_m(X) = \partial U(X)/\partial X_m \equiv U_m(X)$.

On the production side, c_m^n is the constant unit cost for the *n*th firm when producing the *m*th good. The *n*th firm produces vector $(x_1^n, x_2^n, \ldots, x_M^n)$ where $x_m^n > 0\ \forall m \in \Omega_M$. The firm solves $\max_{(x_1^n, x_2^n, \ldots, x_M^n)} \sum_{m\in\Omega_M} \left[U_m(X) - c_m^n\right] x_m^n$ under Nash-Cournot conjectures, i.e., assuming the outputs of other firms are fixed. Firm reaction functions are described by

$$U_m(X) - c_m^n + U_{mm}(X)x_m^n + \sum_{\substack{s\in\Omega_M, \\ s\neq m}} U_{ms}(X)x_s^n = 0 \quad \forall m \in \Omega_M, \forall n \in \Omega_N. \tag{1}$$

The summation term in (1) captures demand-side interaction effects that the producer seeks to exploit. It is assumed that (1) has a unique pure-strategy interior solution. Choice variables in the solution are described by $\hat{x}_m^n$.[1] In summary, our assumptions are

Assumption set <u>AS</u>. *Consumer preferences are quasi-linear so that a representative consumer exists. Preferences are also convex and consumers are price-takers. Unit costs are constant, but firm-specific, in each considered market and firms are active in all considered markets. Firms are quantity-setting Nash-Cournot decision-makers, and there exists a unique equilibrium.*

3. Comparing tax systems

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[1] As with single-product oligopoly models, but perhaps more so, uniqueness and stability are concerns in multi-product oligopoly models. These issues are developed in Okuguchi and

Suppose a government raises taxes by imposing a uniform tax, be it unit or ad valorem, on each unit of a good produced. This is unit tax t_m in the *m*th market or ad valorem tax τ_m in that same market. For the sake of concision in algebra the ad valorem tax is written in mark-down form, i.e., *n*th firm net firm revenue is $\sum_{m\in\Omega_M}(1-\tau_m)U_m(X)x_m^n$. The question we ask is whether one can identify one of the tax systems as being more efficient. Anderson, de Palma, and Kreider (2001) looked at this question for a single output market where firms compete in Cournot manner and have heterogeneous unit production costs. They demonstrated that of the two systems, the ad valorem tax can (*a*) raise more revenue while at the same time (*b*) reduce industry costs and (*c*) deliver the same output level. We extend their approach to a multi-product oligopoly.

Under alternative tax systems, the private Nash optimality conditions (1) become

$$\text{Unit}: \quad U_m(X)-c_m^n-t_m+\sum_{s\in\Omega_M}U_{ms}(X)x_s^n=0 \quad \forall m\in\Omega_M, \forall n\in\Omega_N. \qquad (2)$$

$$\text{Ad val}: \quad U_m(X)(1-\tau_m)-c_m^n+\sum_{s\in\Omega_M}U_{ms}(X)(1-\tau_s)x_s^n=0 \quad \forall m\in\Omega_M, \forall n\in\Omega_N. \qquad (3)$$

Upon evaluating at equilibrium, averaging, and differencing, we find that market-by-market total output is the same under either tax system whenever taxes are chosen to satisfy

$$t_m=\tau_m U_m(X)+\sum_{s\in\Omega_M}\tau_s U_{ms}(X)x_s^{av} \quad \forall m\in\Omega_M; \qquad x_s^{av}=\frac{1}{N}\sum_{n\in\Omega_N}x_s^n. \qquad (4)$$

Given the τ vector and equilibrium x_m^n choices denoted by $\hat{x}_m^n$, one can solve directly for the specific tax levels. Conversely, given the specific tax vector, one can solve for the ad valorem tax equivalents by inverting system (4). The system immediately conveys the following:

Proposition 1. *Under* <u>*AS*</u>*, suppose that*

(a) all goods are complements in demand in that $U_{ms}\geq 0\ \forall m,s\in\Omega_M, s\neq m$. *If the ad valorem taxes are positive then the specific tax solutions to (4) are also positive.*

Szidarovszky (1990) and Szidarovszky and Li (2000).

(b) some pair of goods are substitutes in demand in that $U_{ms} < 0$ *for some* $m, s \in \Omega_M, s \neq m$.

Then there exists a strictly positive ad valorem tax vector such that, for the specific tax vector that yields the same output, at least one specific tax is strictly negative.

We have not asserted that positive specific taxes satisfying (4) ensure positive ad valorem taxes under complementary goods, nor do we believe this to be the case for all such specific tax vectors and demand structures. The proof of part (*a*) is immediate from (4), while that of part (*b*) follows from setting τ_s to be orders of magnitudes larger than the other ad valorem taxes. We have next:

Proposition 2. *Under AS, suppose a government compares two tax policies, unit taxes and ad valorem taxes, that lead to the same aggregate output levels. Then, provided that all ad valorem tax rates are non-negative, a vector of commodity-specific ad valorem taxes,* τ_m*: (a) raises more revenue; and (b) results in lower costs, and hence higher producer surplus and economic efficiency than the corresponding set of commodity-specific unit taxes,* t_m.

Corollary 2.1. *Under the circumstances of Proposition 2, consider a monopolist in a single output market, i.e.,* $N = M = 1$. *Then inferences (a) and (b) apply.*[2]

The corollary was first shown as Theorem 1 in Skeath and Trandel (1994). The proof of Proposition 2, as with others that are not immediate, is provided in the Appendix. The ad valorem system penalizes high cost firms more than low cost firms, as scaling by $1-\tau_m$ in (3) leads to the map $c_m^n \to c_m^n/(1-\tau_m)$.[3] This shifts production toward lower-cost firms. Since

[2] For part (*b*), costs are the same across taxes in monopoly since firm output is the same.
[3] Another useful way of thinking about how equations (2) and (3) compare is to note that the $1-\tau_s$ effectively increase the demand elasticities that firms face. As a consequence, they behave more like competitive firms when making supply choices. Yet total outputs remain fixed,

aggregate production remains fixed, this shift in production reduces industry costs. It is likely to be particularly effective in doing so when all goods complement in demand. Then $c_m^n \to c_m^n/(1-\tau_m)$ encourages production by low cost firms in the *m*th good and, through (1), also in the *s*th good. Cost covariances across firms also matter in determining cost efficiencies. Example 1 illustrates how this is so.

Example 1. Let $\sigma_{ms}^c = \mathrm{Cov}(c_m, c_s)$, i.e., it is the covariance across unit costs for the *m*th and *s*th outputs. Setting $M = 2$, TC^u as the total cost under unit taxes when (4) applies and TC^v as the total cost under ad valorem taxes, eqn. (A11) in the Appendix shows that the cost difference may be written as[4]

$$TC^u - TC^v = -\frac{N}{\Phi}\left[\frac{\tau_1 U_{22}\sigma_{11}^c}{1-\tau_1} - \frac{\tau_1 U_{12}\sigma_{12}^c}{1-\tau_1} - \frac{\tau_2 U_{12}\sigma_{12}^c}{1-\tau_2} + \frac{\tau_2 U_{11}\sigma_{22}^c}{1-\tau_2}\right]; \qquad \Phi = U_{11}U_{22} - (U_{12})^2. \quad (5)$$

Since concavity ensures that $\Phi > 0$, $U_{11} < 0$, and $U_{22} < 0$, it follows that $d[TC^u - TC^v]/d\sigma_{ii}^c > 0$ when $i \in \{1,2\}$ and $\tau_i > 0$. In words, an increase in the variability of unit costs for some good increases the relative cost efficiency of the ad valorem tax, as compared with the output-equivalent unit tax, provided the ad valorem tax for that good is positive.

But $d[TC^u - TC^v]/d\sigma_{12}^c \overset{sign}{=} (\tau_1 + \tau_2 - 2\tau_1\tau_2)U_{12}/[(1-\tau_1)(1-\tau_2)]$. For complements, or $U_{12} > 0$, when both ad valorem taxes are non-negative and one is strictly positive then the following applies: an increase in the covariance between unit costs increases the cost efficiency of production under ad valorem taxes when compared with unit taxes. This is because both supply and demand side interactions align to promote more production by efficient firms while the sum

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implying that low-cost producers take market share from high-cost producers. Given market outputs, high-cost producers were producing too much, so this shift is socially beneficial.

[4] A general analysis of how unit cost statistical moments affect social welfare in multi-product Cournot oligopoly is provided in Lapan and Hennessy (2007). Determinants of how unit cost variance affects aggregate costs in single market oligopoly have been developed by Bergstrom and Varian (1985) and Salant and Shaffer (1999).

of unit costs do not change so that aggregate outputs are unaffected. By contrast, if $U_{12} < 0$ and ad valorem taxes are positive then a reduction in the covariance between unit costs promotes comparative cost efficiency under the ad valorem tax system.

The cost difference in (5) is always positive, but the size of the gap depends on preferences and the distribution of technologies. Observe in (5) that what really matters when it comes to product interactions is whether $U_{12}\sigma_{12}^{c} > 0$. When this is true and when ad valorem taxes are positive then the demand and supply interactions combine to increase the magnitude of cost efficiency of the ad valorem tax system over the unit tax system. These interactions are also relevant as determinants of cost efficiency gains in the event of conglomerate mergers, and we will encounter them again in the next section.

4. Mergers, taxation, and welfare

In this section, we ask what effect conglomerate mergers will have on tax revenue and on welfare measures. We consider two scenarios. Scenario I is *separation* where there are *NM* separate firms, each producing only one good. Scenario II is *merger* and it is the paper's baseline model where there are *N* firms, each producing *M* goods. Consider the case of a sector-wide specific (unit) tax, namely t_m^{sep} for Scenario I or t_m^{mer} for Scenario II.[5] From (2) under separation, the equilibrium conditions for Scenario I are $U_m - c_m^n - t_m^{sep} + U_{mm}\hat{x}_m^{n,sep} = 0$ where $\hat{x}_m^{sep}$ is the separated firm's action under Cournot equilibrium. Separated firms do not take account of demand-side interactions. With $c_m^{av} = (1/N)\sum_{n\in\Omega_N} c_m^n$ and $\hat{x}_m^{av,sep} = (1/N)\sum_{n\in\Omega_N} \hat{x}_m^{n,sep}$, then the aggregated equilibrium conditions are:

$$\text{Scen I}: \quad U_m - c_m^{av} - t_m^{sep} + U_{mm}\hat{x}_m^{av,sep} = 0 \quad \forall m \in \Omega_M. \tag{6}$$

Note that, even though there is no cross ownership, each equilibrium level of aggregate

[5] It can readily be seen that the analysis for ad valorem taxes would be less tractable, and we do not consider it.

output, label it $\hat{X}_m^{sep}$, depends on the whole vector of cost averages $(c_1^{av}, \ldots, c_M^{av})$, provided $U_{ij} \neq 0$. That is, although the linkages exist they are not exploited by firms. For Scenario II (our main model), the aggregate equilibrium conditions are:

$$\text{Scen II}: \qquad U_m - c_m^{av} - t_m^{mer} + \sum_{s\in\Omega_M} U_{sm}\hat{x}_s^{av,mer} = 0 \quad \forall m \in \Omega_M, \tag{7}$$

where $\hat{x}_m^{n,mer}$ is the merged firm's Cournot equilibrium action and $\hat{x}_m^{av,mer} = (1/N)\sum_{n\in\Omega_N} \hat{x}_m^{n,mer}$.

In general, for arbitrary taxes (including zero), output will differ under the two regimes. However, suppose specific taxes adjust for the (on average) marginal incentive consequences of merger, i.e.,

$$t_m^{sep} = t_m^{mer} - \sum_{s\in\Omega_M, s\neq m} U_{ms}\hat{x}_s^{av,mer} \quad \forall m \in \Omega_M, \tag{8}$$

where the relation emerges from differencing (6) and (7). Then outputs under the two regimes will be equal, $\hat{X}_m^{sep} = \hat{X}_m^{mer} \equiv \sum_{n\in\Omega_N} \hat{x}_m^{n,mer}$, and we can deduce

Proposition 3. *A set of specific taxes that yields the same equilibrium output levels will generate more (less) tax revenue under the merged regime if all goods are complements (substitutes) in the inverse demand sense. That is, if taxes are chosen under the two regimes to yield equal outputs then the difference in tax revenue,* $\sum_{m\in\Omega_M}(t_m^{mer} - t_m^{sep})\hat{X}_m^{mer}$, *is:*

$$N\sum_{m\in\Omega_M}\sum_{s\in\Omega_M, s\neq m} U_{sm}\hat{x}_s^{av,mer}\hat{x}_m^{av,mer} \begin{cases} \geq 0 & \text{if } U_{sm} \geq 0 \; \forall m,s \in \Omega_M, s\neq m; \\ \leq 0 & \text{if } U_{sm} \leq 0 \; \forall m,s \in \Omega_M, s\neq m. \end{cases} \tag{9}$$

Note that while (9) looks like a quadratic form, it is not because the diagonal elements are omitted. When all goods complement in demand, then merger strengthens the incentive for efficient firms to expand. This allows the tax authority to levy higher unit taxes and still achieve the same output levels. Tax receipts increase for each good. If some goods are substitutes, and some complements, then the result is ambiguous. Notice too that the conditions in (9) are not necessary. The possibility exists for the government to raise more tax revenue, at the same output levels, under firm mergers even when there are some substitution interactions in demand. Indeed, if we are willing to assume the sub-utility function $U(X)$ is a homogeneous function and also assume a bound on own-price responsiveness then this is the case.

Corollary 3.1. *Suppose* $U(X)$ *is homogenous of degree* $h<1$. *Then, comparing taxes that yield the same outputs under the two regimes, the specific tax for good m under Scenario I (i.e., separation) will be higher (lower) than that under Scenario II (i.e., merger) if* $-U_m/[\hat{X}_m^{mer}U_{mm}]$ *is greater (less) than* $1/(1-h)$.

Expression $-U_m/[\hat{X}_m^{mer}U_{mm}]$ might, loosely, be viewed as the (absolute) own-price elasticity of demand.[6] In the case of homothetic preferences, the cross-good effects under merger resolve into own-good effects. So the revenue authority is really considering two constrained maximization problems where the only difference is the own-good market elasticity. If the effective elasticity is lower under merger, then the unit tax will be higher. Consequently if $\max_{m\in\Omega_M} -U_m/[\hat{X}_m^{mer}U_{mm}]<1/(1-h)$ under homogeneous $U(X)$ then $\sum_{m\in\Omega_M}(t_m^{mer}-t_m^{sep})\hat{X}_m^{mer}$ ≥ 0, and this is true regardless of the signs of the cross elasticities. We also have

Corollary 3.2. *Consider a unit tax vector under*

(a) Scenario I (i.e., separation) such that $t_m^{sep}\geq 0\ \forall m\in\Omega_M$ *. Then the corresponding (i.e., that preserves aggregate output) output tax vector under the merged regime may entail a negative tax (i.e., a subsidy) to some sector.*

(b) Scenario II (i.e., merger) such that $t_m^{mer}\geq 0\ \forall m\in\Omega_M$ *. Then the corresponding tax vector under the separation regime may entail a negative tax to some sector.*

From (8) we see that tax rates which leave consumers indifferent are higher (lower) under

[6] Bear in mind, though, that $B_{mm}\not\approx 1/U_{mm}$.

merger than under separation when all goods complement (substitute) in the inverse demand sense. To confirm part (*a*), suppose that $t_m^{sep} = 0$. Then $t_m^{mer} = \sum_{s \in \Omega_M, s \neq m} U_{ms} \hat{x}_s^{av,mer}$, from (8). If all goods are substitutes, then this expression must be strictly negative. Then, by continuity, there exists a scalar $\varepsilon > 0$ such that if $0 < t_m^{sep} \leq \varepsilon$ the resulting tax will still be negative.

We turn now to the effects of merger on industry costs. We will focus on a two-product oligopoly setting, $\Omega_M = \{1,2\}$.

Proposition 4. *Under assumption set* <u>*AS*</u> *with* $M = 2$, *let unit taxes be levied in the post-merger situation such that aggregate outputs under merger are the same as under separation. Then industry costs are smaller under merger than under separation if either of the following pair of condition sets applies;*

$$\begin{aligned} &\text{sign}(U_{12}) = + \ \text{ and } \ \frac{\sigma_{12}^c}{\sqrt{\sigma_{11}^c \sigma_{22}^c}} \geq -\left| \sqrt{\frac{U_{12}^2}{U_{11}U_{22}}} \right|; \\ &\text{sign}(U_{12}) = - \ \text{ and } \ \frac{\sigma_{12}^c}{\sqrt{\sigma_{11}^c \sigma_{22}^c}} \leq \left| \sqrt{\frac{U_{12}^2}{U_{11}U_{22}}} \right|. \end{aligned} \tag{10}$$

Thus, if *N* good *A* producers match off with *N* good *B* producers, then knowing the sign of U_{12} is not enough to ascertain the effect on industry costs. If goods complement in demand, then a non-negatively correlated matching of pre-merger firms suffices to ensure a lower industry cost post-merger. If goods substitute, then a non-positively correlated matching does so. As a corollary, $\sigma_{12}^c = 0$ suffices independent of $\text{sign}(U_{12})$ so long as $U_{12} \neq 0$. In the case of complementary demand, market power considerations encourage a firm to produce more of one good when it is already producing more of the other good. When a firm with a low unit cost for producing one good also tends to have a low unit cost for producing the other good, then demand complementarity guides firm production patterns toward lower total costs.

Indeed, (10) allows the cost correlation to be negative when $\text{sign}(U_{12}) = +$. If the cost correlation is negative, then the demand-side complementarity must be sufficiently strong to over-ride the contrary supply-side tendency. By concavity, we know that $|U_{12}/(U_{11}U_{22})^{0.5}|<1$ so that if the cost correlation is strongly negative then the first pair of conditions in (10) will not apply under demand complementarity. Concavity condition $|U_{12}/(U_{11}U_{22})^{0.5}|<1$ ensures downward curvature along a line in $\mathbb{R}^2$ and, as such, $|U_{12}/(U_{11}U_{22})^{0.5}|\text{sign}(U_{12})$ can be viewed as an index of substitution in demand between the two goods. In the case of substitution in demand the interpretation of (10) is similar except that the disposition is toward firms emphasizing one or other of the two goods.

Upon reviewing Proposition 3, an implication of Proposition 4 is:

Corollary 4.1. *Under assumption set AS with $M=2$, let unit taxes be levied in the post-merger situation such that aggregate outputs under merger are the same as under separation. Under merger, if*

(a) $U_{12}\geq 0$ and $\sigma_{12}^{c}\geq 0$, then both i) the sum of consumer surplus and industry profits and ii) tax revenue are larger;

(b) $U_{12}\leq 0$ and $\sigma_{12}^{c}\leq 0$, then i) the sum of consumer surplus and industry profits is larger while ii) tax revenue is smaller.

All other assumptions recognized, when $U_{12}\geq 0$ and $\sigma_{12}^{c}\geq 0$ then the corollary would favor a lax anti-trust approach toward cross-sector mergers.

5. Some firms inactive in some markets

In this section we relax the requirement that all firms be active in all markets. We do so in order to develop inferences on how our findings on taxation need to be modified. Just two products are considered so as to facilitate tractability and clarity, i.e., $M=2$. Suppose there are

N^J firms, labeled as $n \in \Psi_J$, that produce both goods. They have unit cost vector (c_1^n, c_2^n), $n \in \Psi_J$. There are also N_m^S single-market firms in the *m*th market. These firms are labeled as $n \in \Psi_{m,S}$, $m \in \{1,2\}$, and each has unit cost c_m^n.

Including both specific and ad valorem taxes, first-order conditions for the three considered firm types are

$$\begin{aligned} &n \in \Psi_J: \quad (1-\tau_m)U_m - c_m^n - t_m + (1-\tau_1)\hat{x}_1^n U_{1m} + (1-\tau_2)\hat{x}_2^n U_{2m} = 0 \quad \forall m \in \{1,2\}; \\ &n \in \Psi_{m,S}: \quad (1-\tau_m)U_m - c_m^n - t_m + (1-\tau_m)\hat{x}_m^n U_{mm} = 0 \quad \forall m \in \{1,2\}. \end{aligned} \tag{11}$$

With

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{12} & U_{22} \end{pmatrix}^{-1}, \tag{12}$$

work in the appendix shows that for joint-product firms, i.e., $\forall n \in \Psi_J$:

$$\begin{aligned} \hat{x}_1^n - \hat{x}_1^{av,J} &= \frac{B_{11}\left(c_1^n - c_1^{av,J}\right)}{1-\tau_1} + \frac{B_{12}\left(c_2^n - c_2^{av,J}\right)}{1-\tau_1}; \\ \hat{x}_2^n - \hat{x}_2^{av,J} &= \frac{B_{12}\left(c_1^n - c_1^{av,J}\right)}{1-\tau_2} + \frac{B_{22}\left(c_2^n - c_2^{av,J}\right)}{1-\tau_2}; \end{aligned} \tag{13}$$

where $c_m^{av,J} = (1/N^J)\sum_{n\in\Psi_J} c_m^n$ and $\hat{x}_m^{av,J} = (1/N)\sum_{n\in\Psi_J} \hat{x}_m^n$. Using (13), total costs of production for jointly producing firms amount to

$$\begin{aligned} &\sum_{n\in\Psi_J}\sum_{m\in\{1,2\}} c_m^n \hat{x}_m^n = N^J \sum_{m\in\{1,2\}} c_m^{av,J}\hat{x}_m^{av,J} + \sum_{n\in\Psi_J}\sum_{m\in\{1,2\}} (c_m^n - c_m^{av,J})(\hat{x}_m^n - \hat{x}_m^{av,J}) \\ &= c_1^{av,J}\hat{X}_1^J + c_2^{av,J}\hat{X}_2^J + \frac{N^J}{1-\tau_1}\left(B_{11}\sigma_{11}^{c,J} + B_{12}\sigma_{12}^{c,J}\right) + \frac{N^J}{1-\tau_2}\left(B_{12}\sigma_{12}^{c,J} + B_{22}\sigma_{22}^{c,J}\right); \\ &\sigma_{ms}^{c,J} = \frac{1}{N^J}\sum_{n\in\Psi_J} (c_m^n - c_m^{av,J})(c_s^n - c_s^{av,J}). \end{aligned} \tag{14}$$

Turning to single-product firms, some work with (11) leads to

$$\hat{x}_1^n - \hat{x}_1^{av,S} = \frac{c_1^n - c_1^{av,S}}{(1-\tau_1)U_{11}} \quad \forall n \in \Psi_{1,S}; \qquad \hat{x}_2^n - \hat{x}_2^{av,S} = \frac{c_2^n - c_2^{av,S}}{(1-\tau_2)U_{22}} \quad \forall n \in \Psi_{2,S}; \tag{15}$$

where $c_m^{av,S} = (1/N_m^S)\sum_{n\in\Psi_{m,S}} c_m^n$ and $\hat{X}_m^S = \sum_{n\in\Psi_{m,S}} \hat{x}_m^n$. Insert into the expression for the sum of single-product firm costs to find that costs for these firms aggregate to

$$\sum_{m\in\{1,2\}}\sum_{n\in\Psi_{m,S}} c_m^n \hat{x}_m^n = c_1^{av,S}\hat{X}_1^S + c_2^{av,S}\hat{X}_2^S + N_1^S \frac{\sigma_{11}^{c,S}}{(1-\tau_1)U_{11}} + N_2^S \frac{\sigma_{22}^{c,S}}{(1-\tau_2)U_{22}}. \qquad (16)$$

Finally, equilibrium outputs are $\hat{X}_1^{total} = \hat{X}_1^J + \hat{X}_1^S$ and $\hat{X}_2^{total} = \hat{X}_2^J + \hat{X}_2^S$ while total cost is the sum of (14) and (16):

$$\begin{aligned} C^{total} &= c_1^{av,S}\hat{X}_1^S + c_1^{av,J}\hat{X}_1^J + c_2^{av,S}\hat{X}_2^S + c_2^{av,J}\hat{X}_2^J + N_1^S \frac{\sigma_{11}^{c,S}}{(1-\tau_1)U_{11}} \\ &+ \frac{N^J}{1-\tau_1}\left(B_{11}\sigma_{11}^{c,J} + B_{12}\sigma_{12}^{c,J}\right) + N_2^S \frac{\sigma_{22}^{c,S}}{(1-\tau_2)U_{22}} + \frac{N^J}{1-\tau_2}\left(B_{12}\sigma_{12}^{c,J} + B_{22}\sigma_{22}^{c,J}\right). \end{aligned} \qquad (17)$$

In contrast with the case where all firms were active in all markets, now any change in tax structure or the allocation of unit costs across firms that preserves overall unit cost variance at given levels will, in general, alter costs. Total production within each group (the group of single-product firms and the group of joint-product firms) and average cost within each group will, in general, have to be preserved in order to preserve the value of $\sum_{m\in\{1,2\}}(c_m^{av,J}\hat{X}_m^J + c_m^{av,S}\hat{X}_m^S)$. In addition, the extent of reduction in total cost due to cost heterogeneity depends in general on where the cost heterogeneity occurs. Only when $U_{12} = 0$ will the allocation of cost variability between the group of single-product firms and the group of multi-product firms be irrelevant.

For $c_m^{av} = (N_m^S c_m^{av,S} + N_m^J c_m^{av,J})/(N_m^S + N_m^J)$, we have now

Proposition 5. *With* $\zeta > 0$, *suppose that* $\sum_{n\in\Psi_J}(c_m^{n,J} - c_m^{av})^2 \to \sum_{n\in\Psi_J}(c_m^{n,J} - c_m^{av})^2 + \zeta$ *and* $\sum_{n\in\Psi_{m,S}}(c_m^{n,S} - c_m^{av})^2 \to \sum_{n\in\Psi_{m,S}}(c_m^{n,S} - c_m^{av})^2 - \zeta$. *Thus mean unit cost within each firm groups remain fixed, total variance remains fixed, variance for joint-product firms increases, variance for single-product firms decreases, while all other variances and covariances remain the same.*

Then industry profit increases whenever $U_{12} \neq 0$ *(because* $B_{mm} < 1/U_{mm}$*).*

The general intuition for why cost variability among multi-product firms is preferred over variability among specialist firms is as follows. Multi-product firms are less constrained in adapting to the variability so as to reduce firm costs. While these firms do seek to exploit market power, the overall market power effects cancel because unit cost sums within each firm category are held fixed. So the net effect is only a reduction in industry costs.

For more specific intuition we note that there is a strong connection between the cost variance transfers discussed above and the LeChatelier principle (Milgrom and Roberts, 1996). The 'principle' identifies conditions under which the unconstrained response of an input to its own price exceeds the response when some means of response have been curtailed. In Samuelson's (1947, pp. 36-39) early proof, the relation $B_{mm} \leq 1/U_{mm}$ for inverted matrices was invoked. Here, B_{mm} captures the total own-price response of a factor when all other factors can adapt while $1/U_{mm}$ captures the partial response when only the factor itself can adapt.

In our case, higher variance of unit costs for some product means that (in the aggregate) there are more opportunities to adapt so that aggregate costs should fall. But multi-product firms are better positioned to adapt. Single-product firms may be considered to be multi-product firms acting under the constraint set that all outputs except one be set at zero. The context is different in that we are considering responses in the aggregate by firms engaged in strategic interactions in the face of downward sloping demands, and not responses by an individual price-taking firm. But the mathematical formalization is the same. Theorem 2 in Milgrom and Roberts (1996) corresponds to our Proposition 5.[7]

We turn now to comparing ad valorem and specific taxes when some firms are inactive in some markets. Write aggregate output under specific taxes only as $X^J(t)+X^S(t)$ and that under ad valorem taxes only as $X^J(\tau)+X^S(\tau)$. From (11), together with some work, equality of total output of the *m*th product under specific taxes with total output of the *m*th product under ad valorem taxes occurs if and only if $\forall m \in \{1,2\}$:

$$\frac{N_m^S\left[(1-\tau_m)t_m-\tau_m c_m^{av,S}\right]}{U_{mm}}+N^J\sum\nolimits_{i\in\{1,2\}}B_{im}\times\left[(1-\tau_m)t_i-\tau_m c_i^{av,J}+(\tau_m-\tau_i)U_i\right]=0. \tag{18}$$

The first left-hand term represents the effect of the change in tax regime, moving from ad valorem to specific regime, on *m*th good production by the single output group. The second left-hand term represents the effect of the tax regime change on production of that good by the multi-product group of firms. In general the values of B_{im} are relevant when determining how taxes should relate to average costs such that (18) is true. There are circumstances, though, under which demand interactions are not relevant.

Proposition 6. *Suppose that* $c_1^{av,S}=c_1^{av,J}$ *and* $c_2^{av,S}=c_2^{av,J}$. *Suppose further that, in the situation where ad valorem taxes are applied then they are common across goods, i.e.,* $\tau_1=\tau_2=\tau$. *Then there is a unique vector of specific taxes that holds total output of each good constant regardless of the values of the* B_{ij}. *It is* $(t_1,t_2)=(\lambda c_1^{av,J},\lambda c_2^{av,J}),\lambda=\tau/(1-\tau)$. *This tax vector also holds constant the total output of the group of multi-product firms and that of the group of single-product firms.*

[7] A result for more than two output markets, analogous to Theorem 1 of Milgrom and Roberts, is available from the authors. But the assumption that all utility function cross-products must be non-negative is required, i.e., the representative consumer's utility function must be

Corollary 6.1. *Suppose that* $c_1^{av,S} = c_1^{av,J}$ *and* $c_2^{av,S} = c_2^{av,J}$ *while* $\tau_1 = \tau_2 = \tau$ *in the situation where ad valorem taxes are applied, i.e., they are common across goods. Then, relative to this ad valorem tax, the vector of specific taxes given by* $(t_1, t_2) = (\lambda c_1^{av,J}, \lambda c_2^{av,J}), \lambda = \tau/(1-\tau)$, *decreases tax revenue, increases industry costs, and maintains consumer surplus.*

The proof of this corollary follows that in Proposition 2 above. In what follows we continue to assume that $c_1^{av,S} = c_1^{av,J}$ and $c_2^{av,S} = c_2^{av,J}$, i.e., for each product the average cost over the group of single-product firms equals the average cost over the group of multi-product firms. But ad valorem taxes are allowed to differ across the two goods. We will show that production within groups changes under the different tax regimes even if total production is fixed, i.e., the conclusion in Proposition 6 fails to hold when ad valorem tax rates vary across products. For the sake of notational convenience and at no real loss of generality, we assume that $N_1^S = N_2^S$. And we write $\vartheta = N_1^S / N^J$. Equation (18) then becomes $\forall m \in \{1,2\}$

$$\frac{\vartheta\left[(1-\tau_m)t_m - \tau_m c_m^{av,J}\right]}{U_{mm}} + \sum\nolimits_{i\in\{1,2\}} B_{im} \times \left[(1-\tau_m)t_i - \tau_m c_i^{av,J} + (\tau_m - \tau_i)U_i\right] = 0. \qquad (19)$$

Define

$$t_m = t_m^* + \xi_m; \quad t_m^* \equiv \frac{\tau_m c_m^{av,J}}{1-\tau_m}; \qquad (20)$$

where we may view t_m^* as the specific tax that secures the same aggregate output from within the group of single-product firms as does the ad valorem tax. Value $\xi_m = t_m - t_m^*$ is just the difference between the specific tax that fixes aggregate output from single-product firms and the specific

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supermodular.

tax that fixes total output, i.e., satisfies (19). After some work, the following can be established:

Proposition 7. *In two-product oligopoly, suppose* $c_1^{av,S} = c_1^{av,J}$, $c_2^{av,S} = c_2^{av,J}$, $N_1^S = N_2^S$, $1 > \tau_1 > \tau_2 = 0$ *, and solutions are interior. If*

(a) goods complement in demand (i.e., $U_{12} > 0$ *and so* $B_{12} < 0$ *) then* $\xi_1 < 0 < \xi_2$ *. That is, for the aggregate outputs to be held fixed at the same level as under the ad valorem tax system then a positive specific tax is required on good 2 and a specific tax less than tax equivalent* t_1^* *under non-joint production is required on good 1.*

(b) goods are independent in demand (i.e., $U_{12} = 0$ *) then* $\xi_1 = 0 = \xi_2$.

(c) goods substitute in demand then either $\xi_1 > 0$ *or* $\xi_2 < 0$ *or both.*

Corollary 7.1. *Under the conditions of Proposition 7, when the two goods*

(a) complement in demand then the aggregate output of the single-product producers of good 1 (good 2) is higher (lower) under the ad valorem tax than under the specific tax.

(b) substitute in demand then either the aggregate output of the single-product producers of good 1 is lower under the ad valorem tax than under the specific tax or the aggregate output of good 2 is higher or both.

In part (*a*) of the corollary, since fixed total output ensures fixed prices, single product producers of good 1 are likely to gain on average upon moving to the ad valorem system. Single product producers of good 2 are likely to lose on average. We cannot be sure, however, as the covariance effects that arose in Proposition 2 (see Example 1) also need to be accounted for. Proposition 7 and Corollary 7.1 indicate that it is, in general, not possible to construct a specific tax system that has the same impact on each group as does an ad valorem tax system. Thus, an analogy of Proposition 2 to establish the superiority of the ad valorem system over the unit tax system will in general be a challenge to obtain in a multi-product oligopoly when not all firms are active in all markets.

6. Oligopsony

Hamilton (1999) has noted that when the market power is in favor of the buyer, in his case monopsony for a single input, then specific taxes may be the preferred instrument. This is because ad valorem taxes now increase marginal cost rather than dampen marginal revenue. In so doing, they make the power-endowed firm face a less elastic supply curve rather than a more elastic demand curve. In what is to follow we adapt our earlier model to show that Hamilton's insight extends to oligopsony and to a multi-input environment.

Each of N firms uses M inputs, and the firms have market power in these local input markets. The marginal social benefit for use of the *m*th input is common across all firms at constant value p_m while the *n*th firm's use level for that input is z_m^n. With $Z_m = \sum_{n\in\Omega_N} z_m^n$, cost over inputs is given by the convex function $H(Z)$ where $Z = (Z_1, \dots, Z_M)$. The supply of inputs is competitive in each case so that the *m*th input price is $H_m = \partial H(Z)/\partial Z_m$. Under specific input taxes, a firm's objective is to $\max_{(z_1^n, \dots, z_M^n)} \sum_{m\in\Omega_M} \left[p_m - t_m - H_m(Z) \right] z_m^n$. Under ad valorem taxes, the objective is to $\max_{(z_1^n, \dots, z_M^n)} \sum_{m\in\Omega_M} \left[p_m - (1+\tau_m) H_m(Z) \right] z_m^n$.

For the two tax systems, and using previously defined notation, the private Nash optimality conditions are

$$\text{Unit}: \quad p_m - t_m - H_m(\hat{Z}) - \sum_{s\in\Omega_M} H_{ms}(\hat{Z}) \hat{z}_m^n = 0 \quad \forall m \in \Omega_M, \forall n \in \Omega_N; \tag{21}$$

$$\text{Ad val}: \quad p_m - (1+\tau_m) H_m(\hat{Z}) - \sum_{s\in\Omega_M} H_{ms}(\hat{Z})(1+\tau_s) \hat{z}_s^n = 0 \quad \forall m \in \Omega_M, \forall n \in \Omega_N. \tag{22}$$

The analog of Proposition 2, part (*a*), is:

Proposition 8. *Suppose the input cost function is of aggregated form* $H(Z)$*, a convex function.*

Suppose too that input suppliers are price takers. Let unit benefits to all input users be constant at p_m *in the mth market, and assume input using firms are active in all considered markets. Firms choose inputs according to Nash behavior and there exists a unique equilibrium. Then, provided that all ad valorem tax rates are non-negative, a vector of commodity-specific ad valorem taxes,* τ_m*, that holds market outputs fixed raises less revenue than the corresponding vector of commodity-specific unit taxes,* t_m*.*

Since there are no revenue or cost heterogeneities, consumer surplus and producer profits do not differ under the alternative taxes. So the specific tax raises more revenue at no social welfare loss. The key distinction between the oligopoly and oligopsony contexts is that the proportional tax is now of form $1+\tau_m$ rather than $1-\tau_m$, as in oligopoly. It acts to exaggerate, rather than attenuate, market power on the part of the firms.

7. Conclusion

In the literature, ad valorem taxes have been shown to dominate specific taxes when only a single market is considered under Cournot behavior. But most firms produce for plural markets and often compete against the same firms in these different markets. For this reason and in light of the popularity of these tax instruments, whether ad valorem taxes dominate specific taxes and the extent of such dominance are matters of considerable policy relevance. Our multi-product model assumes Cournot behavior, firm participation in all markets, a representative consumer demand structure, and convex preferences. We have shown that the ad valorem tax dominance result does extend, and we have explored determinants of the extent of cost efficiencies that arise. We also describe how relaxing the active-in-all-markets assumption affects the results. Finally, we have shown that specific tax dominance for monopsony extends to multi-input oligopsony.

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Appendix

Proof of Proposition 2. By (4), outputs in each market do not depend on the tax system so that consumer surplus is fixed as well.

Part (*a*): With $\hat{X}_m = \sum_{n\in\Omega_N} \hat{x}_m^n$ and $\hat{X} = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_M)$, total tax revenue under the ad valorem tax system ($T^{adv} = \sum_{m\in\Omega_M} \tau_m U_m(\hat{X})\hat{X}_m$) is (weakly) larger than under the unit tax system ($T^u = \sum_{m\in\Omega_M} t_m \hat{X}_m$) if

$$\sum_{m\in\Omega_M} \left[\tau_m U_m(\hat{X}) - t_m\right]\hat{X}_m \geq 0. \tag{A1}$$

Using (4), this may be written as the requirement

$$-\sum_{m\in\Omega_M} \left[\sum_{s\in\Omega_M} \tau_s U_{ms}(\hat{X})\hat{X}_s\right]\hat{X}_m \geq 0. \tag{A2}$$

The left-hand side of (A2) can, in turn, be written as:

$$-\frac{1}{N}\begin{pmatrix}\hat{X}_1 & \hat{X}_2 & \cdots & \hat{X}_M\end{pmatrix}\begin{pmatrix}\tau_1 U_{11} & \tau_2 U_{12} & \cdots & \tau_M U_{1M} \\ \tau_1 U_{12} & \tau_2 U_{22} & \cdots & \tau_M U_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1 U_{1M} & \tau_2 U_{2M} & \cdots & \tau_M U_{MM}\end{pmatrix}\begin{pmatrix}\hat{X}_1 \\ \hat{X}_2 \\ \vdots \\ \hat{X}_M\end{pmatrix}, \tag{A3}$$

where concave $U(\cdot)$ ensures its $M \times M$ matrix Hessian is negative definite. The determinants of the principal minors for the square matrix in (A3) are:

$$\begin{gathered}|\tau_1 U_{11}| = \tau_1 |U_{11}| < 0; \qquad \begin{vmatrix}\tau_1 U_{11} & \tau_2 U_{12} \\ \tau_1 U_{12} & \tau_2 U_{22}\end{vmatrix} = \tau_1\tau_2\begin{vmatrix}U_{11} & U_{12} \\ U_{12} & U_{22}\end{vmatrix} > 0; \\ \begin{vmatrix}\tau_1 U_{11} & \tau_2 U_{12} & \tau_3 U_{13} \\ \tau_1 U_{12} & \tau_2 U_{22} & \tau_3 U_{23} \\ \tau_1 U_{13} & \tau_2 U_{23} & \tau_3 U_{33}\end{vmatrix} = \tau_1\tau_2\tau_3\begin{vmatrix}U_{11} & U_{12} & U_{13} \\ U_{12} & U_{22} & U_{23} \\ U_{13} & U_{23} & U_{33}\end{vmatrix} < 0; \qquad \cdots \quad \cdots \quad ;\end{gathered} \tag{A4}$$

if the ad valorem tax rates are positive, since matrix A is negative definite. Thus, provided that the ad valorem equivalent tax rates are non-negative then the ad valorem tax system generates more revenue.

Part (*b*): We must establish that the ad valorem tax also reduces true costs as compared with costs under the unit tax. Total cost, under either regime, can be written as

$$TC = \sum_{m\in\Omega_M} c_m^{av}\hat{X}_m + \sum_{n\in\Omega_M}\sum_{m\in\Omega_M}(c_m^n - c_m^{av})(\hat{x}_m^n - \hat{x}_m^{av}); \qquad c_m^{av} = \frac{1}{N}\sum_{m\in\Omega_M} c_m^n; \tag{A5}$$

where the $\hat{x}_m^n$ pertain to the relevant tax regime. Remember that aggregate outputs are the same under the two regimes, see condition (4). So the difference in costs between the two regimes is given by comparing the double summation under each tax system. Let $\hat{x}_m^{n,u}$ be equilibrium firm output under the unit tax system, and $\hat{x}_m^{n,v}$ under the ad valorem tax system. Using (2)-(3) we have:

$$\sum_{m\in\Omega_M} U_{ms}(\hat{x}_m^{n,u} - \hat{x}_m^{av}) = c_s^n - c_s^{av}; \qquad \sum_{m\in\Omega_M} U_{ms}(1-\tau_m)(\hat{x}_m^{n,v} - \hat{x}_m^{av}) = c_s^n - c_s^{av}. \tag{A6}$$

Define G as the $M\times M$ matrix with element $G_{ms} = U_{ms}(1-\tau_s)$. With $U_{ij}(X) \equiv \partial^2 U(X)/\partial X_i \partial X_j$ as elements in the $M\times M$ matrix of second derivatives for the utility function, write

$$B \equiv U^{-1} \equiv \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1M} \\ U_{12} & U_{22} & \cdots & U_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ U_{1M} & U_{2M} & \cdots & U_{MM} \end{pmatrix}^{-1} \equiv \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{12} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ B_{1M} & B_{2M} & \cdots & B_{MM} \end{pmatrix}. \tag{A7}$$

Also, write

$$\hat{x}^{n,i} = \begin{pmatrix} \hat{x}_1^{n,i} \\ \hat{x}_2^{n,i} \\ \vdots \\ \hat{x}_M^{n,i} \end{pmatrix}, \quad i\in\{u,v\}; \qquad \hat{x}^{av} = \frac{1}{N}\begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \vdots \\ \hat{X}_M \end{pmatrix}; \qquad c^n = \begin{pmatrix} c_1^n \\ c_2^n \\ \vdots \\ c_M^n \end{pmatrix}; \qquad c^{av} = \begin{pmatrix} c_1^{av} \\ c_2^{av} \\ \vdots \\ c_M^{av} \end{pmatrix}. \tag{A8}$$

Then, from (A6):

$$\hat{x}^{n,u} - \hat{x}^{av} = B\Delta_n; \qquad \hat{x}^{n,v} - \hat{x}^{av} = G^{-1}\Delta_n \equiv R\Delta_n; \qquad \Delta_n \equiv c^n - c^{av}; \tag{A9}$$

where $R = G^{-1}$. Define D as a diagonal matrix with elements $D_{ii} = 1-\tau_i$ (and $D_{ij} = 0$ for $i \neq j$). Then, by construction:

$$G = UD \to R = D^{-1}U^{-1} = EB; \qquad E \equiv D^{-1}; \qquad E_{ii} = 1/(1-\tau_i); \qquad E_{ij} = 0, i \neq j. \tag{A10}$$

Thus, with superscript *tr* as the transpose operation and I_M as the $M \times M$ identity matrix, use (A9) and (A10) to establish the difference in (A5) costs across the two tax regimes as:

$$TC^u - TC^v = \sum_{n\in\Omega_N} \Delta_n^{tr}(B-R)\Delta_n = \sum_{n\in\Omega_N} \Delta_n^{tr}(I_M - E)B\Delta_n. \tag{A11}$$

Now, $K \equiv I_M - E$ is a diagonal matrix with elements $K_{ii} = -\tau_i/(1-\tau_i)$ and $K_{ij} = 0, i \neq j$. Since *B* is negative definite, and K is negative semi-definite provided that $\tau_i \geq 0\ \forall i \in \Omega_M$, it follows that cost difference (A11) is a positive semi-definite quadratic form, and hence the evaluation of (A11) must be a non-negative scalar. To confirm this, the reader may want to multiply out matrix product KB and then evaluate principal minors in the manner of (A4). The non-negativity of (A11) means costs are smaller under the ad valorem system. Q

Proof of Corollary 3.1. From (9):

$$t_m^{sep} = t_m^{mer} + U_{mm}\hat{x}_m^{av,mer} - \sum_{s\in\Omega_M} U_{ms}\hat{x}_s^{av,mer} = t_m^{mer} + U_{mm}\hat{x}_m^{av,mer} - \frac{(h-1)}{N}U_m, \tag{A12}$$

where the last equality follows from applying Euler's theorem to function U_m. The theorem states that U_m is homogenous of degree $h-1$ whenever $U(X)$ is homogeneous of degree h. Thus:

$$t_m^{sep} - t_m^{mer} = \overbrace{(1-h)}^{+}\overbrace{U_{mm}\hat{x}_m^{av,mer}}^{-}\left(\frac{1}{1-h} + \frac{U_m}{U_{mm}\hat{X}_m^{mer}}\right), \tag{A13}$$

so that the term in parentheses determines the sign of $t_m^{mer} - t_m^{sep}$. Q

Proof of Proposition 4. We use (6) and its firm-level analog to calculate aggregate costs under separation. Under the separation scenario, industry cost deviation from that under firm-invariant production at output means is:

$$\sum_{n\in\Omega_N} c_1^n(\hat{x}_1^{n,sep} - \hat{x}_1^{av,sep}) + \sum_{n\in\Omega_N} c_2^n(\hat{x}_2^{n,sep} - \hat{x}_2^{av,sep}) = N\frac{\sigma_{11}^c}{U_{11}} + N\frac{\sigma_{22}^c}{U_{22}}, \tag{A14}$$

where the σ_{ii}^c are as defined in Example 1. Now use (7) and its firm-level analog to calculate aggregate costs under merger. Under the merger scenario and $\Omega_M = \{1,2\}$, then industry cost deviation from average production is:

$$\begin{aligned}&\sum\nolimits_{n\in\Omega_N} c_1^n(\hat{x}_1^{n,mer} - \hat{x}_1^{av,mer}) + \sum\nolimits_{n\in\Omega_N} c_2^n(\hat{x}_2^{n,mer} - \hat{x}_2^{av,mer}) \\ &= N\left(B_{11}\sigma_{11}^c + 2B_{12}\sigma_{12}^c + B_{22}\sigma_{22}^c\right) = N\left(\frac{U_{22}}{\Phi}\sigma_{11}^c - 2\frac{U_{12}}{\Phi}\sigma_{12}^c + \frac{U_{11}}{\Phi}\sigma_{22}^c\right);\end{aligned} \tag{A15}$$

where $\Phi = U_{11}U_{22} - (U_{12})^2$ and the B_{ij} are as given in (A7). In (A5), total costs are decomposed into two parts, namely (*a*) $\sum\nolimits_{m\in\Omega_M} c_m^{av}\hat{X}_m$, and (*b*) $\sum\nolimits_{n\in\Omega_M}\sum\nolimits_{m\in\Omega_M}(c_m^n - c_m^{av})(\hat{x}_m^n - \hat{x}_m^{av})$. The former, i.e., part (*a*), are common under merger and separation since taxes are such that the $\hat{X}_m$ are common under merger and separation. Thus, the cost difference is given by subtracting the right-most expression set in (A14) from that in (A15) to obtain

$$\begin{aligned}&N\left(\frac{U_{22}}{\Phi}\sigma_{11}^c - 2\frac{U_{12}}{\Phi}\sigma_{12}^c + \frac{U_{11}}{\Phi}\sigma_{22}^c - \frac{\sigma_{11}^c}{U_{11}} - \frac{\sigma_{22}^c}{U_{22}}\right) \\ &= N\frac{(U_{12})^2}{\Phi}\left(\frac{\sigma_{11}^c}{U_{11}} + \frac{\sigma_{22}^c}{U_{22}} - 2\frac{\sigma_{12}^c}{U_{12}}\right) \overset{sign}{=} \frac{\sigma_{11}^c}{U_{11}} + \frac{\sigma_{22}^c}{U_{22}} - 2\frac{\sigma_{12}^c}{U_{12}}.\end{aligned} \tag{A16}$$

Now define

$$\theta_i \overset{defn}{=} -\frac{\sigma_{ii}^c}{U_{ii}} \geq 0, i \in \{1,2\}; \quad \rho \overset{defn}{=} \frac{\sigma_{12}^c}{\sqrt{\sigma_{11}^c\sigma_{22}^c}} \in [-1,1]; \quad \hat{U} \overset{defn}{=} \left|\sqrt{\frac{U_{11}U_{22}}{U_{12}^2}}\right| \geq 1; \tag{A17}$$

where $\overset{defn}{=}$ means equality by definition. Then the right-most expression set in (A16) may be re-written as

$$\frac{\sigma_{11}^c}{U_{11}} + \frac{\sigma_{22}^c}{U_{22}} - 2\frac{\sigma_{12}^c}{U_{12}} = -\theta_1 - \theta_2 - 2\rho\hat{U}\sqrt{\theta_1\theta_2} \times \text{sign}(U_{12}). \tag{A18}$$

If $\text{sign}(U_{12}) = +$, then (A18) is negative whenever $\rho\hat{U} \geq -1$. This is because

$$-\theta_1 - \theta_2 - 2\rho\hat{U}\sqrt{\theta_1\theta_2} = -\overbrace{\left(\sqrt{\theta_1} - \sqrt{\theta_2}\right)^2}^{+} - 2\overbrace{(\rho\hat{U} + 1)}^{+}\overbrace{\sqrt{\theta_1\theta_2}}^{+}. \tag{A19}$$

If $\text{sign}(U_{12}) = -$, then (A19) is negative whenever $\rho\widehat{U} \le 1$. This is because

$$-\theta_1 - \theta_2 + 2\rho\widehat{U}\sqrt{\theta_1\theta_2} = -\overbrace{\left(\sqrt{\theta_1} - \sqrt{\theta_2}\right)^2}^{+} + 2\overbrace{(\rho\widehat{U} - 1)}^{+}\overbrace{\sqrt{\theta_1\theta_2}}^{+}. \quad \text{(A20)}$$

So the expression is negative whenever either of the following sufficient condition sets apply:

$$\begin{aligned} &\text{sign}(U_{12}) = + \ \text{ and } \ \frac{\sigma_{12}^c}{\sqrt{\sigma_{11}^c\sigma_{22}^c}} \ge -\left|\sqrt{\frac{U_{12}^2}{U_{11}U_{22}}}\right|; \\ &\text{sign}(U_{12}) = - \ \text{ and } \ \frac{\sigma_{12}^c}{\sqrt{\sigma_{11}^c\sigma_{22}^c}} \le \left|\sqrt{\frac{U_{12}^2}{U_{11}U_{22}}}\right|. \end{aligned} \quad \text{(A21)}$$

This is as stated in the Proposition. Q

Demonstration of system (13). For jointly producing firms, aggregation of (11) supports

$$\begin{aligned} &N^J\left[(1-\tau_1)U_1 - c_1^{av,J} - t_1\right] + (1-\tau_1)U_{11}\hat{X}_1^J + (1-\tau_2)U_{12}\hat{X}_2^J = 0; \\ &N^J\left[(1-\tau_2)U_2 - c_2^{av,J} - t_2\right] + (1-\tau_1)U_{12}\hat{X}_1^J + (1-\tau_2)U_{22}\hat{X}_2^J = 0. \end{aligned} \quad \text{(A22)}$$

where $c_m^{av,J} = (1/N^J)\sum_{n\in\Psi_J} c_m^n$ and $\hat{X}_m^J = \sum_{n\in\Psi_J} \hat{x}_m^n$. Now invert (A22) to obtain

$$\begin{aligned} \hat{X}_1^J &= \frac{N^J B_{11}\left[c_1^{av,J} + t_1 - (1-\tau_1)U_1\right]}{1-\tau_1} + \frac{N^J B_{12}\left[c_2^{av,J} + t_2 - (1-\tau_2)U_2\right]}{1-\tau_1}; \\ \hat{X}_2^J &= \frac{N^J B_{12}\left[c_1^{av,J} + t_1 - (1-\tau_1)U_1\right]}{1-\tau_2} + \frac{N^J B_{12}\left[c_2^{av,J} + t_2 - (1-\tau_2)U_2\right]}{1-\tau_2}. \end{aligned} \quad \text{(A23)}$$

Divide these equations through by N^J and subtract from (11), when the relevant equations in that system have also been inverted. Q

Proof of Proposition 5. Write the sum of the terms involving $\sigma_{mm}^{c,S}$ in (17) as

$$N^J \frac{B_{mm}\sigma_{mm}^{c,J}}{1-\tau_m} + N_m^S \frac{\sigma_{mm}^{c,S}}{(1-\tau_m)U_{mm}} \overset{sign}{=} N^J B_{mm}\sigma_{mm}^{c,J} + N_m^S \frac{\sigma_{mm}^{c,S}}{U_{mm}}. \quad \text{(A24)}$$

Now to understand the shift in cost deviations, write

$$c_m^{av} = \frac{\sum_{n\in\Psi_J} c_m^n + \sum_{n\in\Psi_{m,S}} c_m^n}{N^J + N_m^S} = \alpha_m c_m^{av,J} + (1-\alpha_m)c_m^{av,S}; \quad \alpha_m = \frac{N^J}{N^J + N_m^S}. \quad \text{(A25)}$$

Define $\Gamma_m = c_m^{av,J} - c_m^{av,S}$ so that $c_m^{av} = c_m^{av,J} - (1-\alpha_m)\Gamma_m = c_m^{av,S} + \alpha_m \Gamma_m$. Then, with $\sigma_{mm}^{c,Total}$ as *m*th product variance over the entire set of producing firms, we have

$$\begin{aligned}
(N^J + N_m^S)\sigma_{mm}^{c,Total} &= \sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av})^2 + \sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av})^2 \\
&= \sum_{n\in\Psi_J} \left[c_m^{n,J} - c_m^{av,J} + (1-\alpha_m)\Gamma_m \right]^2 + \sum_{n\in\Psi_{m,S}} \left[c_m^{n,S} - c_m^{av,S} - \alpha_m \Gamma_m \right]^2 \\
&= \sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av,J})^2 + \sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av,S})^2 + \left[N^J (1-\alpha_m)^2 + N_m^S \alpha_m^2 \right] (\Gamma_m)^2 .
\end{aligned} \tag{A26}$$

Since $[N^J (1-\alpha_m)^2 + N_m^S \alpha_m^2](\Gamma_m)^2$ is independent of cost deviations, the reallocation $\sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av})^2 \to \sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av})^2 + \zeta$ can be written as reallocation $\sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av,J})^2 \to \sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av,J})^2 + \zeta$ and the reallocation $\sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av})^2 \to \sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av})^2 - \zeta$ can be written as $\sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av,S})^2 \to \sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av,S})^2 - \zeta$. From (A26),

$$\begin{aligned}
N^J B_{mm}\sigma_{mm}^{c,J} + N_m^S \frac{\sigma_{mm}^{c,S}}{U_{mm}} &= B_{mm} \sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av,J})^2 + \frac{\sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av,S})^2}{U_{mm}} \\
&\to B_{mm} \left[\sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av,J})^2 + \zeta \right] + \frac{\sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av,S})^2 - \zeta}{U_{mm}} \\
&= B_{mm} \sum_{n\in\Psi_J} (c_m^{n,J} - c_m^{av,J})^2 + \frac{\sum_{n\in\Psi_{m,S}} (c_m^{n,S} - c_m^{av,S})^2}{U_{mm}} + \left(B_{mm} - \frac{1}{U_{mm}} \right) \zeta .
\end{aligned} \tag{A27}$$

So cost declines whenever the change $[B_{mm} - 1/U_{mm}]\zeta$ is negative, i.e., $B_{mm} < 1/U_{mm}$. But $B_{11} - 1/U_{11} = U_{12}^2 / [U_{11}(U_{11}U_{22} - U_{12}^2)] < 0$ and also $B_{22} - 1/U_{22} = U_{12}^2 / [U_{22}(U_{11}U_{22} - U_{12}^2)] < 0$. Q

Proof of Proposition 6. Under the stated conditions then (18) becomes

$$\frac{N_m^S \left[(1-\tau)t_m - \tau c_m^{av,J} \right]}{U_{mm}} + N^J \sum_{i\in\{1,2\}} B_{im} \times \left[(1-\tau)t_i - \tau c_i^{av,J} \right] = 0 \quad \forall m \in \{1,2\}. \tag{A28}$$

If this is to be true regardless of the values of the B_{ij}, then $t_1 = \lambda c_1^{av,J}$ and $t_2 = \lambda c_2^{av,J}$ must apply. Furthermore, this relation sets <u>both</u> left-hand terms in (18) to equal zero. That is, it imposes no change in total output by either single-product firms or multi-product firms. Q

Proof of Proposition 7. Substitute (20) into (19) to obtain $\forall m \in \{1,2\}$:

$$\frac{\vartheta(1-\tau_m)\xi_m}{U_{mm}} + \sum_{i\in\{1,2\}} B_{im} \times \left[\frac{[(1-\tau_i)U_i - c_i^{av,J}](\tau_m - \tau_i)}{1-\tau_i} + (1-\tau_m)\xi_i \right] = 0. \tag{A29}$$

If, in addition, $\tau_1 > \tau_2 = 0$ then (A29) resolves to the system

$$\begin{pmatrix} \dfrac{\vartheta}{U_{11}} + B_{11} & B_{12} \\ B_{12} & \dfrac{\vartheta}{U_{22}} + B_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{\tau_1 B_{12}}{1-\tau_1} \begin{pmatrix} c_2^{av,J} - U_2 \\ (1-\tau_1)U_1 - c_1^{av,J} \end{pmatrix}. \tag{A30}$$

Since $B_{11} = U_{22}/\Phi$, $B_{22} = U_{11}/\Phi$ and $B_{12} = -U_{12}/\Phi$ where $\Phi = U_{11}U_{22} - U_{12}^2 > 0$, inversion delivers

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{\tau_1 B_{12}}{(1-\tau_1)\Lambda} \begin{pmatrix} \dfrac{\vartheta}{U_{22}} + B_{22} & -B_{12} \\ -B_{12} & \dfrac{\vartheta}{U_{11}} + B_{11} \end{pmatrix} \begin{pmatrix} c_2^{av,J} - U_2 \\ (1-\tau_1)U_1 - c_1^{av,J} \end{pmatrix}; \qquad \Lambda = \frac{\vartheta^2}{U_{11}U_{22}} + \frac{1+2\vartheta}{\Phi} > 0. \tag{A31}$$

Consider (A31). Part (*b*) of the Proposition is immediate, while part (*a*) follows since

$$\begin{aligned} \xi_1 &\overset{sign}{=} \overbrace{B_{12}}^{-} \overbrace{\left(\frac{\vartheta}{U_{22}} + B_{22} \right)}^{-} \overbrace{(c_2^{av,J} - U_2)}^{-} - \overbrace{B_{12}^2}^{+} \overbrace{\left[(1-\tau_1)U_1 - c_1^{av,J} \right]}^{+} < 0; \\ \xi_2 &\overset{sign}{=} \overbrace{B_{12}}^{-} \overbrace{\left(\frac{\vartheta}{U_{11}} + B_{11} \right)}^{-} \overbrace{\left[(1-\tau_1)U_1 - c_1^{av,J} \right]}^{+} - \overbrace{B_{12}^2}^{+} \overbrace{(c_2^{av,J} - U_2)}^{-} > 0. \end{aligned} \tag{A32}$$

Here, $(1-\tau_1)U_1 - c_1^{av,J} > 0 > c_2^{av,J} - U_2$ for otherwise profit per unit output would be non-positive for some single-product firms and outputs would not be interior.

Part (*c*): We may use (A31) or (A32) to write

$$\left[\xi_1 \geq (\leq)\, 0\right] \Leftrightarrow \left[\frac{U_{11}U_{22} + \vartheta\Phi}{U_{22}U_{12}} \geq (\leq)\, K\right];$$
$$\left[\xi_2 \geq (\leq)\, 0\right] \Leftrightarrow \left[\frac{U_{11}U_{22} + \vartheta\Phi}{U_{11}U_{12}} \leq (\geq)\, \frac{1}{K}\right]; \qquad K = \frac{(1-\tau_1)U_1 - c_1^{av,J}}{U_2 - c_2^{av,J}} > 0. \tag{A33}$$

Suppose, contrary to the assertion in (*c*), that $\xi_2 \geq 0 \geq \xi_1$. Then $[U_{11}U_{22} + \vartheta\Phi]/[U_{22}U_{12}] \leq K$ and $[U_{11}U_{22} + \vartheta\Phi]/[U_{11}U_{12}] \leq \frac{1}{K}$. Since the expressions on both sides of both inequalities are positive, the inequality

$$\left(\frac{U_{11}U_{22} + \vartheta\Phi}{U_{11}U_{12}}\right)\left(\frac{U_{11}U_{22} + \vartheta\Phi}{U_{22}U_{12}}\right) \leq K \times \frac{1}{K} = 1 \tag{A34}$$

must hold. This implies that $[U_{11}U_{22} + \vartheta\Phi]^2 = U_{11}^2U_{22}^2 + 2U_{11}U_{22}\vartheta\Phi + \vartheta^2\Phi^2 \leq U_{11}U_{22}U_{12}^2$. But $2U_{11}U_{22}\vartheta\Phi + \vartheta^2\Phi^2 > 0$ and $U_{11}^2U_{22}^2 - U_{11}U_{22}U_{12}^2 = U_{11}U_{22}\Phi > 0$, so that (A34) is invalid. Q

Proof of Proposition 8. The method is as in Proposition 2, so we will only give guidelines. Total input use in each market is the same under either system whenever taxes satisfy

$$t_m = \tau_m H_m(\hat{Z}) + \sum_{s\in\Omega_M} \tau_s H_{ms}(\hat{Z})\hat{z}_s^{av} \quad \forall m \in \Omega_M; \qquad \hat{z}_s^{av} = \frac{1}{N}\sum_{n\in\Omega_N} \hat{z}_s^n. \tag{A35}$$

Total tax revenue under the unit tax system ($T^u = \sum_{m\in\Omega_M} t_m \hat{Z}_m$) exceeds (weakly) that under the proportional tax system ($T^{adv} = \sum_{m\in\Omega_M} \tau_m H_m(\hat{Z})\hat{Z}_m$) if $\sum_{m\in\Omega_M}\left[t_m - \tau_m H_m(\hat{Z})\right]\hat{Z}_m \geq 0$. Using (A35), this may be written as

$$\sum_{m\in\Omega_M}\left[\sum_{s\in\Omega_M} \tau_s H_{ms}(\hat{Z})\hat{Z}_s\right]\hat{Z}_s \geq 0. \tag{A36}$$

The result follows from convexity of $H(Z)$ and use of the method in Proposition 2, part (*a*). Q