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Cost Arrangement and Welfare in a Multi-Product Cournot Oligopoly

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Abstract

Welfare in a two-product Cournot oligopoly is shown to increase (decrease) with an increase in correlation between unit costs when the outputs complement (substitute) in demand. A more qualified correlation structure is required for the result to apply in a three-product Cournot oligopoly when products complement in demand.

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It has long been known that social welfare in a homogeneous good Cournot oligopoly will increase when firms have more heterogeneous costs (Bergstrom and Varian, 1985; Février and Linnemer). Recent research has demonstrated the relevance of cost dispersion in this market structure when the firms cooperate at an earlier stage (Salant and Shaffer, 1999; Van Long and Soubeyran, 2001). Initially symmetric firms have incentives to cooperate at an earlier stage in order to create later-stage cost asymmetries so that total profit to oligopolists is maximized. In this class of models, early cooperation will also increase social welfare because aggregate output is invariant to cost dispersion. This letter extends the analysis on social welfare to oligopoly in plural markets. In a two-product Cournot oligopoly, we show how correlation between unit costs matters when the outputs are complements or substitutes in demand. The same essential insights apply for a three-product oligopoly under complementarity in demand, but the rearrangement in cost structure needs to be more qualified in order to avoid interaction effects.

1. Two products

A fixed set of N firms, denoted as $n \in \{0, 1, 2, \ldots, N-1\} = \Omega_N$, each produce strictly positive amounts of goods x and y in a two-market Cournot oligopoly. The nth firm produces quantity x_n of good x at constant unit cost c_x^n, and firm outputs in this market are perfect substitutes. The nth firm produces y_n of good y at unit cost c_y^n and firm outputs in this market are also perfect substitutes. Aggregate outputs are $X = \sum_{n \in \Omega_N} x_n$ and $Y = \sum_{n \in \Omega_N} y_n$.

With linear-in-income (I) and concave preferences for a *representative consumer*, the utility function may be specified as $u(X, Y, I) = I + U(X, Y)$. Inverse demands for the goods are given as continuously differentiable functions

$$P^x = U_x(X, Y); \qquad P^y = U_y(X, Y); \tag{1}$$

where the law of demand requires $U_{xx}(\cdot) \le 0$ and $U_{yy}(\cdot) \le 0$. The solutions to (1) are demand

functions $X^D(P^x, P^y)$ and $Y^D(P^x, P^y)$.

The nth firm's profit is

$$\pi^n = \left[U_x(\cdot) - c_x^n\right]x_n + \left[U_y(\cdot) - c_y^n\right]y_n, \tag{2}$$

with private optimality derivatives

$$U_x(\cdot) + U_{xx}(\cdot)x_n + U_{xy}(\cdot)y_n = c_x^n; \qquad U_y(\cdot) + U_{xy}(\cdot)x_n + U_{yy}(\cdot)y_n = c_y^n. \tag{3}$$

We assume unique interior solutions throughout, with firm values $(\hat{x}_n, \hat{y}_n), n \in \Omega_N$. Write

$C^x = \sum_{n \in \Omega_N} c_x^n$ and $C^y = \sum_{n \in \Omega_N} c_y^n$, then sum (3) across firms to observe that C^x and C^y are

sufficient statistics for identifying how aggregate outputs $\hat{X} = \sum_{n \in \Omega_N} \hat{x}_n$ and $\hat{Y} = \sum_{n \in \Omega_N} \hat{y}_n$

relate to costs. Consumer surplus will not change so long as all firms produce while unit cost

sums are C^x and C^y.

With $\Delta = U_{xx}U_{yy} - [U_{xy}]^2 > 0$ from the preference structure, write (3) at Nash solutions in

matrix form as

$$\begin{pmatrix} U_{xx} & U_{xy} \\ U_{xy} & U_{yy} \end{pmatrix}\begin{pmatrix} \hat{x}_n \\ \hat{y}_n \end{pmatrix} = \begin{pmatrix} c_x^n - U_x \\ c_y^n - U_y \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{x}_n \\ \hat{y}_n \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} U_{yy} & -U_{xy} \\ -U_{xy} & U_{xx} \end{pmatrix}\begin{pmatrix} c_x^n - U_x \\ c_y^n - U_y \end{pmatrix}. \tag{4}$$

With c representing the ordered set of costs $(c_x^0, c_x^1, \ \dots \ , c_x^{N-1}, c_y^0, c_y^1, \ \dots \ , c_y^{N-1})$, aggregate profit is

$$\begin{aligned}
V(c) = \sum_{n \in \Omega_N} \pi^n &= \frac{U_x}{\Delta} \times \left[U_{yy}C^x - U_{xy}C^y\right] + \frac{[NU_x - C^x]}{\Delta}\left[U_{xy}U_y - U_{yy}U_x\right] \\
&+ \frac{U_y}{\Delta}\left[U_{xx}C^y - U_{xy}C^x\right] + \frac{[NU_y - C^y]}{\Delta}\left[U_{xy}U_x - U_{xx}U_y\right] \\
&- \frac{U_{yy}}{\Delta}\sum_{n \in \Omega_N}\left(c_x^n\right)^2 + 2\frac{U_{xy}}{\Delta}\sum_{n \in \Omega_N} c_x^n c_y^n - \frac{U_{xx}}{\Delta}\sum_{n \in \Omega_N}\left(c_y^n\right)^2.
\end{aligned} \tag{5}$$

Only the last three summation terms depend on cost statistics other than C^x and C^y.

Following Lapan and Hennessy (2002), assert

Definition 1. Consider the function $f(x_0, x_1, \ \dots \ , x_{N-1}; y_0, y_1, \ \dots \ , y_{N-1}): \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$. Let

operation $(0, 1, \ \dots \ , N-1) \circ \tau$ map $(0, 1, \ \dots \ , N-1)$ onto $(\tau(0), \tau(1), \ \dots \ , \tau(N-1))$, a permutation

on Ω_N. Then $f(\cdot)$ is said to be permutation invariant if $f(x_0, \dots, x_{N-1}; y_0, \dots, y_{N-1}) \equiv$

$f(x_{\tau(0)}, \dots, x_{\tau(N-1)}; y_{\tau(0)}, \dots, y_{\tau(N-1)})$ for all permutations on Ω_N.

Definition 2. The function $f(\cdot)$ in definition 1 is said to exhibit permutation order if

$f(x_0, \dots, x_{N-1}; y_{\tau(0)}, \dots, y_{\tau(N-1)}) \geq f(x_0, \dots, x_{N-1}; y_0, \dots, y_{N-1})$ for all permutations τ on Ω_N

that a) interchange two indices j and k for which $(x_j - x_k)(y_j - y_k) \leq 0$, and b) leave all other

indices unchanged.

Definition 3. The function $f(\cdot)$ in definition 1 is said to be arrangement increasing (AI) if it

exhibits permutation invariance and permutation order. It is said to be arrangement decreasing

(AD) if $-f(\cdot)$ is AI.

The simplest AI function is $f(\cdot) = \sum_{n \in \Omega_N} x_n y_n$, where $\sum_{n \in \Omega_N} x_n y_n = \sum_{n \in \Omega_N} x_{\tau(n)} y_{\tau(n)}$

demonstrates permutation invariance and $x_j y_k + x_k y_j \geq x_j y_j + x_k y_k$ under $(x_j - x_k)(y_j - y_k) \leq 0$

demonstrates permutation order. Turning to demand-side interactions, we say that x and y

complement (substitute) in demand if $\partial X^D(P^x, P^y)/\partial P^y \equiv \partial Y^D(P^x, P^y)/\partial P^x \leq (\geq) 0$. It follows

from (1) that the goods complement (substitute) in demand whenever $U_{xy} \geq (\leq) 0$.

Proposition 1. *If x and y are complements (substitutes) in demand, then social welfare is*

arrangement increasing (decreasing) in unit costs.

Proof. A sequence of interchanges satisfying definition 2 leaves the values of $\sum_{n \in \Omega_N} \left(c_x^n\right)^2$ and

$\sum_{n \in \Omega_N} \left(c_y^n\right)^2$ invariant. But the sequence increases the value of $\sum_{n \in \Omega_N} c_x^n c_y^n$ as it is an AI

function. Given complementarity in demand, $2\Delta^{-1} U_{xy} \sum_{n \in \Omega_N} c_x^n c_y^n$ is AI and increases along the

sequence. All other terms on the right-hand side in (5) do not change in value because C^x and

3

C^y do not change. So the value of $V(c)$ at Nash equilibrium increases under the sequence of interchanges. The values of $\hat{X}$ and $\hat{Y}$ do not change and neither does consumer surplus. Social welfare, the sum of industry profits and consumer surplus, must increase. Given substitution in demand, then $2\Delta^{-1}U_{xy}\sum_{n\in\Omega_N}c_x^n c_y^n$ is AD and similar logic applies. □

The cost rearrangement makes the firms that are most competitive (low cost) in one market more competitive in the other too. Viewing (3), when there is complementarity in demand then a strengthening of the cost correlation helps market incentives to guide production toward lower cost firms. Market power across products and firms will be reassigned, but the overall effect of market power in the marketplace will be unaffected in the sense that market outputs do not change under the AI cost rearrangement. So industry cost declines while outputs do not change.

When there is substitution in demand, then firms with low unit costs in one product have a revenue-side incentive to be low output in the second product. A firm with low unit costs in both outputs will have strong incentives to under-produce relative to the firm's comparative cost advantages, so market power incentives tend to distort production toward higher-cost firms.

We conclude this section with an alternative perspective on increasing correlation. When unit costs correlate positively, correlation should increase if we increase the magnitude of a product's high unit cost and compensate by decreasing the magnitude of a low unit cost for that product. From (5) we know that social welfare increases in the two product model if the constant sums cost structure change increases the value of $2U_{xy}\sum_{n\in\Omega_N}c_x^n c_y^n - U_{yy}\sum_{n\in\Omega_N}\left(c_x^n\right)^2$

$-U_{xx}\sum_{n\in\Omega_N}\left(c_y^n\right)^2$. Without loss of generality, let product x unit costs be ordered by the firm index, i.e., $c_x^0 \le c_x^1 \le ... \le c_x^{N-1}$. Additional structure is required on how unit costs are arranged.

Definition 4. Unit costs c_y^k and c_y^m, $k < m$, are said to be positively arranged if $c_y^k \le c_y^m$ and negatively arranged if $c_y^k > c_y^m$.

Fixing the values of the c_x^n, consider the following rearrangement of the c_y^n: 1) $c_y^j \to \tilde{c}_y^j \equiv$

$c_y^j \ \forall j \in \Omega_N, j \neq k, j \neq m$; 2) $c_y^k \to \tilde{c}_y^k \equiv c_y^k - \varepsilon$ and $c_y^m \to \tilde{c}_y^m \equiv c_y^m + \varepsilon$ where $\varepsilon > 0$ and $k < m$.[1]

Sums of unit costs do not change after this change in unit costs for product y. Under 1) and 2),

the difference in oligopoly profits satisfies

$$V(\tilde{c}) - V(c) \overset{sign}{=} (c_x^m - c_x^k)U_{xy} - (\varepsilon - c_y^k + c_y^m)U_{yy}. \tag{6}$$

Under 2), if products complement in demand, then $(c_x^m - c_x^k)U_{xy} \geq 0$. If unit costs are positively

arranged, then $(\varepsilon - c_y^k + c_y^m)U_{yy} \leq 0$ while $(c_x^m - c_x^k)U_{xy} \leq 0$ when products substitute in demand.

Finally, $(\varepsilon - c_y^k + c_y^m)U_{yy} \geq 0$ when unit costs are negatively arranged and $\varepsilon < c_y^k - c_y^m$. Thus, a

small increase in dispersion of unit costs for either product increases social welfare when a) the

goods complement in demand and unit costs are positively arranged, or b) the goods substitute in

demand and unit costs are negatively arranged.

2. Three product model

In this case the firms produce a third good, z, with quantity z_n at unit cost c_z^n for the nth

firm. Aggregate output is $Z = \sum_{n \in \Omega_N} z_n$, and the product's sum of unit costs is $C^z = \sum_{n \in \Omega_N} c_z^n$.

With a representative consumer, quasi-linear and concave preferences $u(X,Y,Z,I) = I +$

$U(X,Y,Z)$, then inverse demands are

$$P^x = U_x(X,Y,Z); \qquad P^y = U_y(X,Y,Z); \qquad P^z = U_z(X,Y,Z). \tag{7}$$

The nth firm's profit is

$$\pi^n = \left[U_x - c_x^n\right]x_n + \left[U_y - c_y^n\right]y_n + \left[U_z - c_z^n\right]z_n. \tag{8}$$

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[1] This transfer approach can be shown to be equivalent to an increase in variance of the c_y^j, and also to dominance in the majorization pre-ordering. See pp. 3-12 in Marshall and Olkin (1979).

Write the private optimality derivatives in matrix form as

$$
\begin{pmatrix} \hat{x}_n \\ \hat{y}_n \\ \hat{z}_n \end{pmatrix} = \frac{1}{\Gamma} \begin{pmatrix} U_{yy}U_{zz} - [U_{yz}]^2 & U_{yz}U_{xz} - U_{xy}U_{zz} & U_{xy}U_{yz} - U_{yy}U_{xz} \\ U_{yz}U_{xz} - U_{xy}U_{zz} & U_{xx}U_{zz} - [U_{xz}]^2 & U_{xy}U_{xz} - U_{xx}U_{yz} \\ U_{xy}U_{yz} - U_{yy}U_{xz} & U_{xy}U_{xz} - U_{xx}U_{yz} & U_{xx}U_{yy} - [U_{xy}]^2 \end{pmatrix} \begin{pmatrix} c_x^n - U_x \\ c_y^n - U_y \\ c_z^n - U_z \end{pmatrix}, \tag{9}
$$

where $\Gamma < 0$ due to concavity. Insert solutions into (8) to see that aggregate profit depends directly on

$$
\begin{aligned}
& \left\{ U_{yy}U_{zz} - [U_{yz}]^2 \right\} \sum\nolimits_{n\in\Omega_N} \left(c_x^n \right)^2 + \left\{ U_{xx}U_{zz} - [U_{xz}]^2 \right\} \sum\nolimits_{n\in\Omega_N} \left(c_y^n \right)^2 \\
& + \left\{ U_{xx}U_{yy} - [U_{xy}]^2 \right\} \sum\nolimits_{n\in\Omega_N} \left(c_z^n \right)^2 + 2\left\{ U_{xz}U_{yz} - U_{xy}U_{zz} \right\} \sum\nolimits_{n\in\Omega_N} c_x^n c_y^n \\
& + 2\left\{ U_{xy}U_{yz} - U_{xz}U_{yy} \right\} \sum\nolimits_{n\in\Omega_N} c_x^n c_z^n + 2\left\{ U_{xy}U_{xz} - U_{xx}U_{yz} \right\} \sum\nolimits_{n\in\Omega_N} c_y^n c_z^n.
\end{aligned} \tag{10}
$$

From differentiating (7), assert that x, y, and z are *mutual complements* if $U_{xz}U_{yz} \geq U_{xy}U_{zz}$, $U_{xy}U_{yz} \geq U_{xz}U_{yy}$, and $U_{xy}U_{xz} \geq U_{xx}U_{yz}$. Then all three coefficients on the interaction sums in (10) are positive. Under mutual complementarity, therefore, the effect of some cost rearrangement on social welfare is determined by the effects on the covariability summations $\sum_{n\in\Omega_N} c_x^n c_y^n$, $\sum_{n\in\Omega_N} c_x^n c_z^n$, and $\sum_{n\in\Omega_N} c_y^n c_z^n$.

Without loss of generality, suppose that firms are ordered in ascending order of marginal costs for production of x, i.e., $c_x^0 \leq c_x^1 \leq \ ... \ \leq c_x^{N-1}$. Consider the following change in the specification of marginal costs.[2]

Case *i*) If $c_y^k \leq c_y^j$ and $c_z^k \leq c_z^j$ for some $j \in \{0,1, \ ... \ , k-1\}$, then map $c_y^k \to c_y^j = \tilde{c}_y^k$, $c_y^j \to c_y^k = \tilde{c}_y^j$, $c_z^k \to c_z^j = \tilde{c}_z^k$, $c_z^j \to c_z^k = \tilde{c}_z^j$, and leave other costs unchanged.

Case *ii*) If $c_y^k \leq c_y^j$ and $c_z^k > c_z^j$ for some $j \in \{0,1, \ ... \ , k-1\}$, then map $c_y^k \to c_y^j = \tilde{c}_y^k$, $c_y^j \to c_y^k = \tilde{c}_y^j$, and leave other costs unchanged.

Case *iii*) If $c_z^k \leq c_z^j$ and $c_y^k > c_y^j$ for some $j \in \{0,1, \ ... \ , k-1\}$, then map $c_z^k \to c_z^j = \tilde{c}_z^k$, $c_z^j \to$

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[2] A generalization of the vector rearrangement we are about to explain was developed in Boland

$c_z^k = \tilde{c}_z^j$, and leave other costs unchanged.

If costs change in this manner, then we say that costs are better ordered in the trivariate arrangement increasing (TAI) sense.

Example 1. A four-firm three-market oligopoly has cost structure

$$\mathcal{C}^0 = \left\{ (c_x^1, c_x^2, c_x^3, c_x^4), (c_y^1, c_y^2, c_y^3, c_y^4), (c_z^1, c_z^2, c_z^3, c_z^4) \right\} = \{(1,3,4,6), (7,6,5,4), (4,6,2,3)\}. \tag{11}$$

Now $c_y^1 \geq c_y^4$ and $c_z^1 \geq c_z^4$, so that case *i*) applies. We must interchange both if we are to interchange either. Write the rearranged cost structure as $\mathcal{C}^1 = \{(1,3,4,6), (4,6,5,7), (3,6,2,4)\}$. But the 6 and 5 in the second vector and the 6 and 2 in the third vector are not properly aligned with the 3 and 4 in the first vector, another example of case *i*). A further rearrangement consistent with the idea of TAI is to interchange both at the same time. Doing so, we obtain $\mathcal{C}^2 = \{(1,3,4,6), (4,5,6,7), (3,2,6,4)\}$.

The x and y unit product costs are aligned. But z product costs are not aligned with those of the other products, an example of case *iii*). Switch the 6 with the 4 in that third vector; $\mathcal{C}^3 = \{(1,3,4,6), (4,5,6,7), (3,2,4,6)\}$. One additional case *iii*) rearrangement brings costs in line across all products, i.e., write $\mathcal{C}^4 = \{(1,3,4,6), (4,5,6,7), (2,3,4,6)\}$. Each among this sequence of rearrangements weakly increases the values of $\sum_{n \in \Omega_N} c_x^n c_y^n$, $\sum_{n \in \Omega_N} c_x^n c_z^n$, and $\sum_{n \in \Omega_N} c_y^n c_z^n$.

Regardless of the number of firms in the industry, a TAI cost rearrangement is designed so that $\sum_{n \in \Omega_N} c_x^n c_y^n$, $\sum_{n \in \Omega_N} c_x^n c_z^n$, and $\sum_{n \in \Omega_N} c_y^n c_z^n$ increase. Inspection of (10) then establishes

Proposition 2. *Let x, y, and z be mutual complements in demand. Then social welfare is larger after a TAI in unit costs occurs.*

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and Proschan (1988). See also Hennessy and Lapan (2003).

When one or more of weak inequalities $U_{xz}U_{yz} \geq U_{xy}U_{zz}$, $U_{xy}U_{yz} \geq U_{xz}U_{yy}$, and $U_{xy}U_{xz} \geq U_{xx}U_{yz}$ fail then the result does not necessarily apply because a firm's output expansion incentives are not well-aligned across products. Market power considerations mean that lower cost firms might not have strong incentives to produce larger quantities of all three commodities. The proposition 1 finding on two-product oligopoly under substitution does not extend to multi-product oligopoly. There is a unique way to rearrange unit costs in duopoly so that that they are less well aligned, and this allows for a substitution result. There is not a unique way to rearrange unit costs across three or more products so that unit costs are less well aligned.

3. Conclusion

When there is complementarity in demand, intuition might suggest that more strongly correlated unit costs would encourage low cost firms to expand and high cost firms to contract. As low cost firms produce more, a cost rearrangement that favors these firms should reduce industry costs and increase social welfare. In controlling for market power effects, our model provides formal confirmation for this intuition.

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