Statistical Moments Analysis of Production and Welfare in Multi-Product Cournot Oligopoly

April 2007

Harvey E. Lapan
University Professor
Department of Economics
283 Heady Hall
Iowa State University
Ames, IA 50011-1070

David A. Hennessy
Professor
Department of Economics & CARD
578C Heady Hall
Iowa State University
Ames, IA 50011-1070

Electronic information for correspondence with contact author: David A. Hennessy, Ph: US 515-294-6171, Fax: US 515-294-6336, e-mail: hennessy@iastate.edu.

Statistical Moments Analysis of Production and Welfare in Multi-Product Cournot Oligopoly

Abstract

Our context involves N Cournot oligopolists producing M products at constant marginal costs when preferences are quasi-linear. We identify relationships between second moments of unit costs and second moments of firm-level production. For example, a larger variance in unit costs of a product increases own output variance and the variance of any other output. We also investigate how second moments of unit costs affect industry cost efficiency. Industry costs can rise if the wrong firm secures a cost reduction. For quadratic preferences, it is shown that Zhao's (2001) share criteria for an increase in unit costs to increase welfare extend to the multi-product setting.

JEL Classification Numbers: C6; D43

Keywords: Complementarity; Covariance matrix; Heterogeneous costs; Innovation

1. Introduction

Curiosities abound concerning the comparative statics of cost for the standard Cournot model in which a single homogeneous good is produced at a constant firm-specific unit cost. Seade (1985), for example, has shown that Cournot oligopolists can gain from an excise tax whereas competitive firms and monopolists cannot. Among these curiosities, our interest concerns two. One is that, as observed by Bergstrom and Varian (1985), a sum-preserving increase in the variance of these unit costs decreases industry costs if the set of active firms does not change. This observation has generated a growing body of research regarding what are referred to as cost manipulation games, see Salant and Shaffer (1999) and Van Long and Soubeyran (1997, 2001, 2005). The second curiosity, initially identified by Lahiri and Ono (1988), is that assistance to a small firm can reduce social welfare. This is because smaller active firms are socially inefficient since the largest firm (if it does indeed have unlimited capacity to produce at its given unit cost) should produce all that is consumed under social efficiency. Février and Linnemer (2004) have generalized the observation to quite arbitrary cost shocks while Zhao (2001) and Wang and Zhao (2007) have provided explicit conditions in particular cases.

These two peculiarities are, of course, related. When the unit cost of a high cost firm increases, then mean unit cost increases and the variance of unit costs increases. The effect on mean unit cost should reduce welfare, but that on variance should increase welfare. If the firm is small enough, then the latter dominates since more efficient firms pick up some of the smaller firm's decline in output and the efficiency gap is large enough. With two exceptions, those of Lapan and Hennessy (2006) and Wang and Zhao (2007), the literature to date has not extended the analysis to the case of multi-product oligopoly. Lapan and Hennessy study the implications of cost correlation structures for welfare in two- and three-product Cournot oligopoly. Wang and

Zhao (2007) consider the case of product differentiation in the (linear) Bertrand-Shubik model.

The general intent of the present work is to look at how the moments of unit costs affect equilibrium in multi-product Cournot oligopoly, subject to the standard assumption that income effects and other consumer-side heterogeneities do not matter. Our first set of results, given in Section 2, establishes relationships between unit cost moments and output moments. A sample inference is what one might call a law of own variance equilibrium response: the variance of any output across firms must increase with the unit cost variance for that output. This is true regardless of how the output interacts in demand with other goods. A less intuitive result is that the variance in output for good *A* must increase with the variance of unit costs for good *B*. Similarly, the covariance between two outputs must increase with the unit cost covariance between those two outputs.

The effects of cost moments on social welfare are considered in sections 3 and 4. It is found that an increase in a cost covariance increases (decreases) the sum of firm profits while leaving consumer surplus unaffected whenever goods complement (substitute). Concerning how an increase in mean unit costs might change welfare, the case of quadratic preferences is studied. Then the introduction of plural markets does not affects Zhao's (2001) criteria for when an increase in the unit cost of a small firm increases industry profits and when it increases social welfare. Social welfare increases with an increase in some firm's unit cost of production in a given market whenever the firm's output share in that market is smaller than $2/(N+1)$, where N is the number of multi-market firms. The paper concludes with a brief review.

2. Model

The model involves M markets and N firms, where each firm is active in all markets.[1] On

[1] The results we will establish need to be modified when not all firms are active in all markets. Further analyses to accommodate firms inactive in some markets are available from the authors

the demand side there are H price-taking consumers, $h \in \{1, 2, \ldots, H\} = \Omega_H$. Availability of goods is denoted by $X^{tr} = (X_1, \ldots, X_M)$ where X_m is the aggregate amount of the *m*th good available and the superscripted *tr* identifies the transpose operation. With prices $P_m(X), m \in \Omega_M$, with I as aggregate income, and with $z = I - \sum_{m\in\Omega_M} P_m(X)X_m$ as the numeraire good, we assume the existence of a representative consumer where the utility function is[2]

$$V = z + U(X) = I + U(X) - \sum_{m\in\Omega_M} P_m(X)X_m, \tag{2.1}$$

and $U(X)$ is an increasing, strictly concave, twice continuously differentiable function.

Upon optimizing in (2.1), inverse demand functions are identified as $P_m(X) = \partial U(X)/\partial X_m \equiv U_m(X)$.[3] Letting $A_{ij} \equiv \partial^2 U(X)/\partial X_i \partial X_j \equiv U_{ij}(X)$ be entries in the aggregate utility function's Hessian matrix, we may describe demand system comparative statics as[4]

$$\begin{pmatrix} dX_1 \\ dX_2 \\ \vdots \\ dX_M \end{pmatrix} = B \begin{pmatrix} dP_1 \\ dP_2 \\ \vdots \\ dP_M \end{pmatrix} = B \begin{pmatrix} dU_1 \\ dU_2 \\ \vdots \\ dU_M \end{pmatrix}; \qquad B = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1M} \\ A_{12} & A_{22} & \cdots & A_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1M} & A_{2M} & \cdots & A_{MM} \end{pmatrix}^{-1}; \tag{2.2}$$

---

upon request.

[2] See Chapter 3 in Vives (1999) for a comprehensive discussion of this preference structure. It is assumed in the standard one-market oligopoly model, so that income effects do not enter as a consideration when specifying aggregate demand.

[3] Rather than intersperse usage of both $P_m(X)$ and $U_m(X)$, we intend to use $U_m(X)$ in the main. We will use $P_m(X)$ when price needs to be emphasized in conveying a point.

[4] Since complementarity and substitution interactions on the demand side are important when seeking to understand equilibrium in our model, some words of caution are warranted. Function $U_{ij} = U_{ij}(X)$ conveys how the inverse demand for the *i*th good changes as X_j increases, *holding all other goods fixed*. Function $B_{ij} = B_{ij}(P)$ conveys how the demand for the *i*th good changes as P_j changes, *holding all other prices fixed*. Thus, a complementarity (resp., substitution) interaction in the sense of demand functions may or may not support that respective interaction in the sense of inverse demand functions. Both notions of interaction are consistent for the two-product system, but are not necessarily consistent when three or more products enter $U(\cdot)$. Throughout the manuscript, we will seek to clarify what we mean when we assert the nature of a demand side interaction.

or $dX = BdP$ where B has entries labeled B_{ms}. The linear-in-income preference structure ensures that there are no income effects; Hicksian and Marshallian demand functions are the same.[5]

In N-firm oligopoly, the nth firm with constant unit production costs $c_m^n, m \in \Omega_M$, chooses outputs x_m^n, to maximize profit

$$\pi^n = \sum_{m\in\Omega_M} \left[U_m(X) - c_m^n\right] x_m^n. \tag{2.3}$$

Nash first-order optimality conditions are[6]

$$U_m(X) - c_m^n + \sum_{s\in\Omega_M} U_{ms}(X) x_s^n = 0 \quad \forall m \in \Omega_M, \forall n \in \Omega_N. \tag{2.4}$$

Throughout we make some standard assumptions:

Assumption 1. *A unique pure-strategy interior solution exists to the* $N \times M$ *system in (2.4).*

Production values in this solution are indicated as $x_m^{n,*}$.[7] The reader's appendix shows that

$$\delta_i^{n,*} \equiv \sum_{m\in\Omega_M} B_{im}\varepsilon_m^n; \qquad \delta_i^{n,*} \equiv x_i^{n,*} - \bar{x}_i^*; \qquad \varepsilon_m^n \equiv c_m^n - \bar{c}_m; \tag{2.5}$$

where $\bar{x}_m^* \equiv (1/N)\sum_{n\in\Omega_N} x_m^{n,*}$ and $\bar{c}_m \equiv (1/N)\sum_{n\in\Omega_N} c_m^n$. The product of a firm's production deviations is

$$\delta_i^{n,*}\delta_j^{n,*} = \left(\sum_{m\in\Omega_M} B_{im}\varepsilon_m^n\right)\left(\sum_{s\in\Omega_M} B_{js}\varepsilon_s^n\right) = \sum_{m\in\Omega_M}\sum_{s\in\Omega_M} B_{im}B_{js}\varepsilon_m^n\varepsilon_s^n. \tag{2.6}$$

If we define the covariance of unit costs for goods *m* and *s* across firms as $\mathrm{Cov}_{ms}^c =$

---

[5] See footnote 2 above. With income effects, then B is not symmetric. This would weaken somewhat some of our results.

[6] We note in passing that a firm may be viewed as producing at under marginal cost if $\sum_{s\in\Omega_M} U_{ms}(X)x_s^n \geq 0$ when evaluated at an equilibrium. This is more likely when goods complement in the sense of $U_{ms} \geq 0 \ \forall m,s \in \Omega_M, m \neq s$, and c_m^n is high. Of course, price cannot be less than marginal cost for all goods since this would mean negative profits in our model.

[7] Chapter 3 in Okuguchi and Szidarovszky (1990) provides conditions under which a unique pure-strategy solution exists. Uniqueness is the more problematic of the two concerns. Szidarovszky and Li (2000) identify conditions under which local stability is guaranteed when

$(1/N)\sum_{n\in\Omega_N}\varepsilon_m^n\varepsilon_s^n$, and the covariance of outputs for goods *i* and *j* across firms as $\text{Cov}_{ij}^x =$ $(1/N)\sum_{n\in\Omega_N}\delta_i^{n,*}\delta_j^{n,*}$, then an interchange of summation signs confirms

$$\text{Cov}_{ij}^x = \frac{1}{N}\sum_{n\in\Omega_N}\sum_{m\in\Omega_M}\sum_{s\in\Omega_M} B_{im}B_{js}\varepsilon_m^n\varepsilon_s^n = \sum_{m\in\Omega_M}\sum_{s\in\Omega_M} B_{im}B_{js}\text{Cov}_{ms}^c. \tag{2.7}$$

This implies

$$\frac{\partial\text{Cov}_{ij}^x}{\partial\text{Cov}_{ms}^c} = B_{im}B_{js} + B_{is}B_{jm}, \tag{2.8}$$

or

Proposition 1. *Assume multi-product Cournot oligopoly with constant marginal costs, representative consumer utility of form (2.1), and a unique interior solution. Then the response of output covariances to unit cost covariances can be written as*

$$\frac{P_m P_s}{X_i X_j}\frac{\partial\text{Cov}_{ij}^x}{\partial\text{Cov}_{ms}^c} = \eta_{im}\eta_{js} + \eta_{is}\eta_{jm}; \qquad \eta_{kt} = B_{kt}\frac{P_t}{X_k}; \tag{2.9}$$

where η_{kt} *is the Hicksian (and Marshallian) elasticity of demand.*

The proposition's contents are best illustrated through identifying some implications.

Corollary 1.1. *Under the assumptions in Proposition 1, (a)* $\partial\text{Cov}_{ii}^x/\partial\text{Cov}_{mm}^c \ge 0$; *(b)* $\partial\text{Cov}_{ij}^x/\partial\text{Cov}_{ij}^c \ge 0$; *(c)* $\partial\text{Cov}_{ii}^x/\partial\text{Cov}_{is}^c \overset{sign}{=} -\eta_{is}$ *; (d)* $\partial\text{Cov}_{im}^x/\partial\text{Cov}_{mm}^c \overset{sign}{=} -\eta_{im}$ *; (e)* $\partial\text{Cov}_{ij}^x/\partial\text{Cov}_{ms}^c$ $\equiv B_{im}B_{js} + B_{is}B_{jm} \equiv \partial\text{Cov}_{ms}^x/\partial\text{Cov}_{ij}^c$ *(as* $B_{im} \equiv B_{mi}$*).*

Part (*a*) shows that variance of output $x_i^{n,*}$ across firms must increase with an increase in the variance of unit costs for <u>any</u> output. Consider the consequences for concentration in the *m*th market, as measured by the Herfindahl-Hirschman index, $\sum_{n\in\Omega_N}(x_m^{n,*}/X_m^*)^2$ with $X_m^* =$

---

adjustments occur according to adaptive expectations.

$\sum_{n\in\Omega_N} x_m^{n,*}$. Since aggregate output does not change when unit cost sums are held fixed, the concentration index for <u>each</u> market must increase whenever the variance of unit costs increases in <u>any</u> market. In part (*b*), and bearing in mind that we have controlled for unit cost variances, a stronger correlation between unit costs leads to a stronger correlation between firm outputs regardless of how the markets interact in demand.

To rationalize part (*a*), suppose $\eta_{im} < 0$ so that goods complement in demand in the direct (as distinct from inverse) demand sense. Then an increase in the variance of c_m^n should induce more dispersion in the $x_i^{n,*}$ to the extent that it induces more dispersion in the $x_m^{n,*}$. The large firms should tend to expand in both products while the small firms should tend to contract in both products. Suppose instead that $\eta_{im} > 0$ so that firms with low (high) values of c_m^n tend to have low (high) values of $x_i^{n,*}$. While the alignment is reversed, it is still true that an increase in the variance of c_m^n should induce more dispersion in the $x_i^{n,*}$. In this case a low value of c_m^n that falls further will tend to be associated with low values of $x_i^{n,*}$ and to induce a further decline in these values. So while the sign of η_{im} will determine how the individual $x_i^{n,*}$ values change in response to a change in Cov_{mm}^c, it is irrelevant in determining the sign of $\partial\mathrm{Cov}_{ii}^x / \partial\mathrm{Cov}_{mm}^c$. Part (*b*) can be motivated by a similar argument.

Regarding parts (*c*) and (*d*), consider $\partial\mathrm{Cov}_{im}^x / \partial\mathrm{Cov}_{mm}^c$ and suppose $\eta_{im} < 0$ so that there is a complementary relation in the direct demand sense. Then firms with low (high) values of c_m^n tend to have high (low) values of $x_i^{n,*}$. An increase in the variance of c_m^n should induce a stronger covariance in outputs because it induces more dispersion in the $x_m^{n,*}$, with more dispersed marginal revenues for the other market. A similar argument applies for part (*c*). Part (*e*) has an analog in dual demand theory in that it identifies some of the behavioral symmetries that demand system integrability requires.

In general, little can be said when no coordinates among the index quintuple (i, j, m, s) in (2.9) are the same. For two situations, however, we show that some more structure identifies strong implications. For one situation, suppose the dual expenditure function for $U(X)$ is of weakly separable form $C[c^A(P_i, P_j, \dots), c^B(P_m, P_s, \dots), \dots]$. Thus, prices P_i and P_j are in group *A* and prices P_m and P_s are in group *B*. Then, upon using monotonicity of costs in prices and also Shepherd's lemma to develop expressions for demand elasticities,

$$\eta_{im}\eta_{js} + \eta_{is}\eta_{jm} \overset{sign}{=} c^A_{P_i}(P_i, P_j, \dots) c^A_{P_j}(P_i, P_j, \dots) c^B_{P_m}(P_m, P_s, \dots) c^B_{P_s}(P_m, P_s, \dots) \geq 0. \tag{2.10}$$

Stated formally,

Corollary 1.2. *Under the assumptions in Proposition 1, suppose that the expenditure function for* $U(X)$ *is of weakly separable form* $C[c^A(P_i, P_j, \dots), c^B(P_m, P_s, \dots), \dots]$. *Then an increase in the unit cost covariance across the mth and sth goods elicits an increase in the output covariance across the ith and jth goods, i.e.,* $\partial \mathrm{Cov}^x_{ij} / \partial \mathrm{Cov}^c_{ms} \geq 0$.

The inference is somewhat surprising; no conditions are placed on the sign of $\partial^2 C / \partial c^A \partial c^B$. One way of viewing the corollary is with reference to part (*a*) of Corollary 1.1. Good pair (m, s), being in the same partitioned set, could be viewed as constituents of the group B composite good. An increase in correlation between unit costs for the *m*th and *s*th goods may be viewed as an increase in variance of the unit cost of composite good B. Upon providing a similar interpretation for good pair (i, j), the lemma may be viewed as asserting that an increase in the variance of unit costs for composite good B increases the variance of firm outputs for composite good A.

The other situation directly places restrictions on demand responses. Suppose that goods are all substitutes in demand in the sense of $B_{ms} \geq 0\ \forall m, s \in \Omega_M, m \neq s$.[8] This would, by itself, be supportive of an industry complex in which most firms tend toward specialization in a particular product. Then equation (2.8) reveals that $\partial \mathrm{Cov}_{ij}^{x} / \partial \mathrm{Cov}_{ms}^{c} \geq 0$ whenever $i, j \notin \{m, s\}$. This is because the outputs in question differ from the unit costs in question. Firms with high (low) *m*th and *s*th unit costs tend to have high (low) *i*th and *j*th outputs. Each of the output responses will be in the same direction, but the response we are considering is the covariation between the *i*th and *j*th outputs. An increase in covariance between the *m*th and *s*th unit costs will act to better align the responses of the *i*th and *j*th outputs across firms and so will strengthen the covariation between the *i*th and *j*th outputs.

Suppose, on the other hand, that all goods complement ($\eta_{im} < 0$) in demand. Then equation (2.8) reveals that $\partial \mathrm{Cov}_{ij}^{x} / \partial \mathrm{Cov}_{ms}^{c} \geq 0$ for <u>every</u> four-index combination. In this situation, each of the output responses will be in the opposite direction, i.e., firms with high (low) *m*th and *s*th unit costs tend to have low (high) *i*th and *j*th outputs. Again, an increase in covariance between the *m*th and *s*th unit costs will act to better align the responses of the *i*th and *j*th outputs across firms. So while the signs of output responses to unit costs differ when comparing substitutes and complements, for second moment comparative statics what matters is how cost innovations affect the alignment between outputs. That has more to do with whether the outputs under consideration are consistent in their responses than the nature of those responses.

A final point to note concerns when unit costs are associated so that $\mathrm{Cov}_{ms}^{c} \geq 0\ \forall m \in \Omega_M$, $\forall s \in \Omega_M$.[9] Then $\mathrm{Cov}_{ij}^{x} = \sum_{m \in \Omega_M} \sum_{s \in \Omega_M} B_{im} B_{js} \mathrm{Cov}_{ms}^{c} \geq 0\ \forall i \in \Omega_M$, $\forall j \in \Omega_M$, when goods

[8] An example of an industry where firms with large market shares are involved in plural markets such that substitution in demand is almost certain is meat packing in the United States (Moschini, Moro, and Green, 1994) and elsewhere.

[9] A vector z of random variables is said to have the association property if, for any pair of increasing functions $a(z)$ and $b(z)$, $\mathrm{Cov}(a(z), b(z)) \geq 0$ holds. Clearly, this implies uniformly positive pair-wise correlation, which is all we require. Affiliation, a standard assumption in auction theory, is implied by association. See Shaked and Shanthikumar (2007) on association.

complement in demand. So a system-wide positive covariance among unit costs implies a system-wide positive output covariance whenever demand also exhibits system-wide complementarity. But cost association is insufficient to sign Cov_{ij}^{x} when goods substitute in demand because own-price effects confound matters.

3. Welfare

Turning to industry profit effects, mean firm profit is

$$\frac{\pi^{ind}}{N} \equiv \bar{\pi} = \frac{1}{N}\sum_{m\in\Omega_M} U_m(X^*)X_m^* - \frac{1}{N}\sum_{n\in\Omega_N}\sum_{m\in\Omega_M} c_m^n x_m^{n,*}; \qquad X^* = (X_1^*, \ldots, X_M^*)^{tr}. \quad (3.1)$$

In deviations form, use covariance relation $E[z_1 z_2] = E[z_1]E[z_2] + \mathrm{Cov}(z_1, z_2)$ to write (3.1) as

$$\bar{\pi} = \pi^0 - \frac{1}{N}\sum_{n\in\Omega_N}\sum_{m\in\Omega_M} \varepsilon_m^n \delta_m^{n,*}; \qquad \pi^0 = \frac{1}{N}\sum_{m\in\Omega_M} U_m(X^*)X_m^* - \sum_{m\in\Omega_M} \bar{c}_m \bar{x}_m^*; \qquad (3.2)$$

where $\bar{c}_m$ and $\bar{x}_m^*$ are the *m*th entries for the relevant average value vectors as defined under (2.5) above. Insertion of (2.5) into (3.2) leads to

$$\begin{aligned} \bar{\pi} &= \pi^0 - \frac{1}{N}\sum_{n\in\Omega_N}\sum_{m\in\Omega_M} \varepsilon_m^n \sum_{s\in\Omega_M} B_{ms}\varepsilon_s^n \\ &= \pi^0 - \frac{1}{N}\sum_{m\in\Omega_M}\sum_{s\in\Omega_M} B_{ms}\sum_{n\in\Omega_N} \varepsilon_m^n \varepsilon_s^n = \pi^0 - \sum_{m\in\Omega_M}\sum_{s\in\Omega_M} \frac{X_m}{P_s}\eta_{ms}\mathrm{Cov}_{ms}^{c} \geq \pi^0, \end{aligned} \qquad (3.3)$$

where concavity of $U(X)$ ensures $\sum_{m\in\Omega_M} \varepsilon_m^n \sum_{s\in\Omega_M} B_{ms}\varepsilon_s^n \leq 0$. Inspection of (3.3) supports

Proposition 2. *Under the assumptions of Proposition 1, an increase in the value of unit cost covariance* Cov_{ms}^{c} *has the following effect on average firm profit and so on social welfare:*

(a) increases it whenever the mth and sth goods are direct complements ($\eta_{ms} < 0$*);*

(b) decreases it whenever the mth and sth goods are direct substitutes ($\eta_{ms} > 0$*).*

From (*a*) it is seen that an increase in any unit cost variance increases profits regardless of the number of goods. So the standard result in Bergstrom and Varian (1985) and Salant and Shaffer

(1999) continues to apply in our multi-product setting. As for covariance effects, when the sums of unit costs remain fixed and outputs remain interior then consumer surplus is invariant to the change in covariance. Lower industry costs, therefore, imply that overall welfare increases (decreases) with an increase in Cov_{ms}^{c} when $\eta_{ms} < (>) \, 0$. A closely related result was developed in Lapan and Hennessy (2006) for two- and three-product Cournot oligopolies. There it was assumed that a change in covariance was arrived at through inter-firm rearrangements of unit costs that were tailored to leave the marginal distributions of unit costs unaffected.

In order to better understand the implications of the preceding results, consider an innovation (or set of innovations) that can <u>reduce</u> the unit cost sums of producing good one by μ. The adoption of this innovation shifts the unit cost vector for firm *n* in industry 1 from c_1^n to $c_1^n - s_1^n \mu$, where $\sum_{n\in\Omega_N} s_1^n = 1$. For the case in which this innovation benefits only one firm, say firm *k*, then $s_1^n = 0 \; \forall n \neq k$. The question is how equilibrium is affected by which firm adopts (or purchases) the innovation.

Since summed unit costs will be the same regardless of which firms' costs are reduced, provided all firms remain active, the equilibrium price and aggregate output will not be affected by which firms adopt the innovation. In the case of a single product it was shown in earlier work (e.g., Bergstrom and Varian, 1985) that, given unit cost sums, any increase in the variance of the cost vector will result in (*i*) lower industry production costs and (*ii*) higher variance of firm outputs. Thus, the industry cost reduction will be maximized if the most efficient firm experiences the unit cost reduction, and this will also lead to the largest variance in firm outputs. We inquire into whether similar results hold in the multi-product model.

Consider two cases concerning how the benefits of the innovation are distributed;

Case I: the *n*th firm unit cost of good 1 is $c_1^n - \tilde{s}_1^n \mu$, and

Case II: the *n*th firm unit cost of good 1 is $c_1^n - \hat{s}_1^n \mu$.

Let ε_m^n, with $\varepsilon_m^n \equiv c_m^n - \overline{c}_m$ as in (2.5), denote the deviation of firm *n's* unit cost in industry *m*

prior to adoption of the innovation. Let $\tilde{\varepsilon}_m^n$ and $\hat{\varepsilon}_m^n$ represent the deviation after the innovation, for cases I and II respectively. By construction, $\tilde{\varepsilon}_m^n = \hat{\varepsilon}_m^n = \varepsilon_m^n$ for all $m \neq 1$, while $\tilde{\varepsilon}_1^n = \varepsilon_1^n + \tilde{\omega}_1^n$ and $\hat{\varepsilon}_1^n = \varepsilon_1^n + \hat{\omega}_1^n$, where $\tilde{\omega}_1^n \equiv (\mu / N) - \mu \tilde{s}_1^n$ and $\hat{\omega}_1^n \equiv (\mu / N) - \mu \hat{s}_1^n$. Hence, $\hat{\varepsilon}_1^n - \tilde{\varepsilon}_1^n =$ $(\tilde{s}_1^n - \hat{s}_1^n)\mu$ and $\hat{\varepsilon}_m^n - \tilde{\varepsilon}_m^n = 0$ for all $m \neq 1$.

Using (2.5), a firm's output deviation for each good under each scenario is given by:

$$\begin{aligned} &\tilde{\delta}_i^n \equiv \tilde{x}_i^n - \overline{x}_i^\mu = \sum\nolimits_{m\in\Omega_M} B_{im}^\mu \tilde{\varepsilon}_m^n; \qquad \hat{\delta}_i^n \equiv \hat{x}_i^n - \overline{x}_i^\mu = \sum\nolimits_{m\in\Omega_M} B_{im}^\mu \hat{\varepsilon}_m^n; \\ &\hat{\delta}_i^n - \tilde{\delta}_i^n = \hat{x}_i^n - \tilde{x}_i^n = \sum\nolimits_{m\in\Omega_M} B_{im}^\mu \left(\hat{\varepsilon}_m^n - \tilde{\varepsilon}_m^n\right) = \mu B_{i1}^\mu \left(\tilde{s}_1^n - \hat{s}_1^n\right). \end{aligned} \tag{3.4}$$

In (3.4), $\overline{x}_i^\mu$ denotes the average output of each good *after* the cost shift. Since the unit cost sum has changed, the values of the B_{im} change relative to when the industry 1 mean unit cost was $\overline{c}_1$. However, the new B_{ms} are the same under Case I as under Case II because summed unit costs are the same. We label the new values of the B_{im} as B_{ms}^μ so as to recognize that these values will, in general, depend upon μ. Since aggregate industry outputs, and hence prices, are the same under cases I and II, the only source of welfare difference between the two scenarios will be due to the difference in industry costs.

To illustrate cases I and II, assume the entire cost reduction accrues to one firm, as when the innovation is excludable and protected by intellectual property rights. In Case I, assume firm *a* gets the cost reduction so that $\tilde{s}_1^a = 1$ with $\tilde{s}_1^n = 0$ for $n \neq a$. In Case II, firm *b* gets the cost reduction so that $\hat{s}_1^b = 1$ with $\hat{s}_1^n = 0$ for $n \neq b$ and $b \neq a$. So $\tilde{s}_1^n - \hat{s}_1^n = 1$ for $n = a$, $\tilde{s}_1^n - \hat{s}_1^n = -1$ for $n = b$, and $\tilde{s}_1^n - \hat{s}_1^n = 0$ otherwise. Let $\Delta^{II-I}TC$ denote total costs under Case II less total costs under Case I, and let $\Delta^{II-I}\mathrm{Cov}_{ij}^x$ represent the difference in the covariance of outputs across the two cases. Work in the appendix shows that

$$\begin{aligned} &\Delta^{II-I}TC = 2\mu \sum\nolimits_{t\in\Omega_M,\, t\neq 1} B_{1t}^\mu \left(\varepsilon_t^a - \varepsilon_t^b\right); \\ &\Delta^{II-I}\mathrm{Cov}_{ij}^x = \mu \sum\nolimits_{t\in\Omega_M,\, t\neq 1} \left[B_{it}^\mu B_{j1}^\mu + B_{i1}^\mu B_{jt}^\mu\right]\left(\varepsilon_t^a - \varepsilon_t^b\right). \end{aligned} \tag{3.5}$$

These equations confirm:

Lemma 1. *Let firm b be the low cost producer of good 1, or* $\varepsilon_1^a - \varepsilon_1^b > 0$. *Assigning the cost reduction to firm b may not result in a greater reduction in total costs if either (i) goods are substitutes (* $B_{1t}^{\mu} > 0 \; \forall t \in \Omega_M, t \neq 1$ *) and firm b is also the low cost producer of the other goods, or (ii) goods are complements and firm a is the low cost producer of the other goods.*

To understand what happens in either of situations (*i*) and (*ii*), consider Case II. Even though more of good 1 is produced by the lower cost producer, the changes in other outputs across firms cause more of those goods to be produced by the higher cost producer.

Turning to the covariance difference in (3.5), when comparing output variances we can write:

$$\Delta^{II-I}\mathrm{Var}_{ii}^{x} = 2\mu B_{i1}^{\mu}\sum_{\substack{t\in\Omega_M,\\ t\neq 1}} B_{it}^{\mu}\left(\varepsilon_t^a - \varepsilon_t^b\right) = B_{i1}^{\mu}\left(\Delta^{II-I}TC\right). \tag{3.6}$$

Thus, for the special rearrangement of a cost reduction that was considered here, the difference in output variance can be related to the difference in output costs. When $i = 1$ then this is the efficiency result in Bergstrom and Varian (1985), albeit for multi-product oligopoly.

4. Welfare under higher costs and quadratic preference structure

Even within the single market setting, special demand functions, such as linear demand, are widely used when seeking to understand how cost shocks might affect welfare (Vives, 1999; Grossman, 2007; Wang and Zhao, 2007). In this section we will look at what can be said about welfare if one is willing to accept additional structure. We have not to this point imposed specific structure on $U(X)$. We didn't have to because interior solutions in the presence of constant unit costs ensured constant aggregate outputs. Now we seek to understand consequences of shifts in mean unit costs so that, inevitably, aggregate outputs will change.

In order to obtain some further insights when firms have heterogeneous costs, we will henceforth restrict attention to the quadratic preference structure:

$$U(X) = A_0 + \sum_{m\in\Omega_M} A_m X_m + 0.5\sum_{m\in\Omega_M}\sum_{s\in\Omega_M} A_{ms} X_m X_s, \tag{4.1}$$

where the double summation is a negative definite quadratic form, i.e., has A_{ms} values such that the form's Hessian is a negative definite matrix. The inverse demand functions are $P_m(X) = A_m + \sum_{s\in\Omega_M} A_{ms} X_s$ for all $m \in \Omega_M$. We have then:

Proposition 3. *Under the assumptions of Proposition 1 and, in addition, quadratic $U(X)$ then (a) a firm-invariant increase in the unit cost of good 1 reduces industry profits and consumer welfare.*

In addition, an increase in some firms' (say firm 1) unit cost of good 1 (b) decreases consumer surplus; (c) increases (decreases) industry profits if $x_1^{n,*}/X_1^* < (>)\, 1/(N+1)$; and (d) increases (decreases) welfare if $x_1^{n,*}/X_1^* < (>)\, 2/(N+1)$.

This proposition extends findings in Zhao (2001) to the multi-product setting. Interestingly, the share bounds he identified for a single good oligopoly are as in parts (*c*) and (*d*) above; i.e., interactions in demand do not affect these share bounds. The key insight is that an adverse cost shock to a large share firm is worse for industry profits than an adverse cost shock to a small share firm if the set of active firms remains the same after the shock. The general intuition has been developed extensively in earlier work for a single output market, see Février and Linnemer (2004) and papers referenced therein. We have shown that, for quadratic preferences at least, Zhao's specific bounds extend to multi-product oligopoly.[10] Other sector costs do matter, of course, because they determine the values of $x_1^{n,*}$ and X_1^*. But $x_1^{n,*}/X_1^*$ is a sufficient summary statistic.

[10] We leave it to the interested reader to consider unit cost shocks of form γ that map $c_m^n \to c_m^n + \gamma k^m(c_m^n)$. Correlation analogs of Proposition 3 in Février and Linnemer (2004) can then be

5. Conclusion

We have extended the literature on relationships between the distributions of unit costs, unit cost innovations, equilibrium actions and welfare measures in multi-product Cournot oligopoly. It is not surprising that the nature of interactions in preferences between consumed goods is prominent in these relationships. Sometimes though, such as when a unit cost increase improves welfare given quadratic utility, the nature of interactions is of little consequence.

---

arrived at under linear demand, or $\Theta^{*} = 0$ in their notation.

References

Bergstrom, T.C., Varian, H.R. Two remarks on Cournot equilibria. Economics Letters 1985;19; (1), 5–8.

Février, P., Linnemer, L. Idiosyncratic shocks in an asymmetric Cournot oligopoly. International Journal of Industrial Organization 2004;22; (6, June), 835–848.

Grossmann, V. Firm size and diversification: Multiproduct firms in asymmetric oligopoly. International Journal of Industrial Organization, 2007;25; (1, February), 51–67.

Lahiri, S., Ono, Y. Helping minor firms reduces welfare. Economic Journal 1988;98; (393, December), 1199–1202.

Lapan, H.E., Hennessy, D.A. A note on cost arrangement and market performance in a multi-product Cournot oligopoly. International Journal of Industrial Organization 2006;24; (3, May), 583–591.

Moschini, G., Moro, D., Green, R.D. Maintaining and testing separability in demand systems. American Journal of Agricultural Economics 1994;76; (1, February), 61–73.

Okuguchi, K., Szidarovszky, F. The theory of oligopoly with multi-product firms. *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag: Berlin; 1990.

Salant, S.W., Shaffer, G. Unequal treatment of identical agents in Cournot equilibrium. American Economic Review 1999;89; (3, June), 585–604.

Seade, J. Profitable cost increases and the shifting of taxation: Equilibrium responses of markets in oligopoly. Dept. of Economics Working Paper #260, University of Warwick; April 1985.

Shaked, M., Shanthikumar, J.G. Stochastic orders. Springer: New York, NY; 2007.

Szidarovszky, F., Li, W. A note on the stability of a Cournot–Nash equilibrium: The multiproduct case with adaptive expectations. Journal of Mathematical Economics 2000;33; (1, February), 101–107.

Van Long, N., Soubeyran, A. Cost heterogeneity, industry concentration and strategic trade policy. Journal of International Economics 1997;43; (1-2, August), 207–220.

Van Long, N., Soubeyran, A. Cost manipulation games in oligopoly, with costs of manipulating. International Economic Review 2001;42; (2, May), 505–533.

Van Long, N., Soubeyran, A. Selective penalization of polluters: an inf-convolution approach. Economic Theory 2005;25; (2, February), 421–454.

Vives, X. Oligopoly pricing: Old ideas and new tools. MIT Press: Cambridge, MA; 1999.

Wang, X.H., Zhao, J. Welfare reductions from small cost reductions in differentiated oligopoly. International Journal of Industrial Organization, 2007;25; (1, February), 173–185.

Zhao, J. A characterization for the negative welfare effects of cost reduction in Cournot oligopoly. International Journal of Industrial Organization 2001;19; (3-4, March), 455–469.

<u>Reader's Appendix</u>

<u>Establishing Equation (2.5)</u>:

Using the A matrix inverted in (2.2), producer optimality conditions (2.4), and $U_m = P_m$, the *n*th firm's optimality conditions may be written as

$$Ax^{n,*} = c^n - P; \qquad x^{n,*} = \begin{pmatrix} x_1^{n,*} \\ x_2^{n,*} \\ \vdots \\ x_M^{n,*} \end{pmatrix}; \qquad c^n = \begin{pmatrix} c_1^n \\ c_2^n \\ \vdots \\ c_M^n \end{pmatrix}; \qquad P = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_M \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{pmatrix}. \tag{A1}$$

Upon aggregation across firms, using (A1), market equilibrium must satisfy

$$A\bar{x}^* = \bar{c} - P; \qquad \bar{x}^* = \frac{1}{N}\begin{pmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_M^* \end{pmatrix}; \qquad \bar{c} = \frac{1}{N}\begin{pmatrix} C_1^{sum} \\ C_2^{sum} \\ \vdots \\ C_M^{sum} \end{pmatrix}; \tag{A2}$$

where $X_m^* = \sum_{n\in\Omega_N} x_m^{n,*}$ and $C_m^{sum} = \sum_{n\in\Omega_N} c_m^n$. We see from (A2) that if both the number of active firms and the average unit cost for each product are held fixed, then the per-household mean consumption vector $N\bar{x}^*/H$ does not change and each household's utility does not change. Subtract (A2) from (A1). Then invert matrix A, as in (2.2), to obtain (2.5).

Proof of Proposition 3. First note that equilibrium consumer surplus is

$$CS = A_0 - 0.5\sum_{m\in\Omega_M}\sum_{s\in\Omega_M} A_{ms} X_m^* X_s^*; \tag{B1}$$

equilibrium industry profits is

$$\begin{aligned} \pi^{ind} &= \sum_{m\in\Omega_M} (A_m - \bar{c}_m) X_m^* + \sum_{m\in\Omega_M}\sum_{s\in\Omega_M} A_{ms} X_m^* X_s^* \\ &\quad - \sum_{m\in\Omega_M}\sum_{s\in\Omega_M} B_{ms} \sum_{n\in\Omega_N} \varepsilon_m^n \varepsilon_s^n; \\ &= \sum_{m\in\Omega_M} (A_m - \bar{c}_m) X_m^* + 2A_0 - 2\times(CS) - \sum_{m\in\Omega_M}\sum_{s\in\Omega_M} B_{ms} \sum_{n\in\Omega_N} \varepsilon_m^n \varepsilon_s^n; \end{aligned} \tag{B2}$$

and equilibrium welfare is

$$W = A_0 + \sum_{m\in\Omega_M}(A_m - \overline{c}_m)X_m^* + 0.5\sum_{m\in\Omega_M}\sum_{s\in\Omega_M} A_{ms}X_m^* X_s^* \\ -\sum_{m\in\Omega_M}\sum_{s\in\Omega_M} B_{ms}\sum_{n\in\Omega_N}\varepsilon_m^n\varepsilon_s^n. \tag{B3}$$

Part (*a*): Work out equilibrium surpluses by using Nash private optimality conditions:

$$A_m + \sum_{s\in\Omega_M} A_{ms}X_s^* + \sum_{s\in\Omega_M} A_{ms}x_s^{n,*} - c_m^n = 0, \quad \forall m\in\Omega_M, \quad \forall n\in\Omega_N. \tag{B4}$$

Aggregate to establish

$$NA_m + (N+1)\sum_{s\in\Omega_M} A_{ms}X_s^* - \sum_{n\in\Omega_N} c_m^n = 0, \quad \forall m\in\Omega_M. \tag{B5}$$

Invert to obtain

$$X_m^* = \frac{N}{N+1}\sum_{s\in\Omega_M} B_{ms}(\overline{c}_s - A_s) \quad \forall m\in\Omega_M. \tag{B6}$$

From (B1) and (B6),

$$\frac{dCS}{d\overline{c}_1} = -\sum_{m\in\Omega_M} X_m^* \sum_{s\in\Omega_M} A_{ms}\frac{dX_s^*}{d\overline{c}_1} = -\frac{N}{N+1}\sum_{m\in\Omega_M} X_m^*\sum_{s\in\Omega_M} A_{ms}B_{1s} \\ = -\frac{N}{N+1}\left[X_1^*(1) + X_2^*(0) + \ldots + \quad\right] = -\frac{N}{N+1}X_1^* < 0, \tag{B7}$$

where use has been made of the fact that $B = A^{-1}$.

For the effect on equilibrium industry profits, differentiate (B2) while using (B6) and (B7):

$$\frac{d\pi^{ind}}{d\overline{c}_1} = -X_1^* + \sum_{m\in\Omega_M}(A_m - \overline{c}_m)\frac{dX_m^*}{d\overline{c}_1} - 2\frac{dCS}{d\overline{c}_1} \\ = -X_1^* + \frac{N}{N+1}\sum_{m\in\Omega_M}(A_m - \overline{c}_m)B_{m1} + 2\frac{N}{N+1}X_1^* = -\frac{2}{N+1}X_1^* < 0. \tag{B8}$$

Part (*b*): This follows immediately from (B1) and (B7) since only average costs enter consumer surplus.

Part (*c*): Note first that $d\overline{c}_1 / dc_1^1 = 1/N$. From (B2) and (B8),

$$\frac{d\pi^{ind}}{dc_1^1} = \frac{\partial\pi^{ind}}{\partial\overline{c}_1}\frac{\partial\overline{c}_1}{\partial c_1^1} - \frac{\partial}{\partial c_1^1}\left[\sum_{m\in\Omega_M}\sum_{s\in\Omega_M} B_{ms}\sum_{n\in\Omega_N}(c_m^n - \overline{c}_m)(c_s^n - \overline{c}_s)\right] \\ = -2\frac{X_1^*}{N(N+1)} - 2\sum_{s\in\Omega_M} B_{1s}(c_s^1 - \overline{c}_s). \tag{B9}$$

Use (B4) and (B5) to obtain $x_m^{n,*} - X_m^*/N = \sum_{s\in\Omega_M} B_{ms}(c_s^n - \overline{c}_s)$ so that

$$\frac{d\pi^{ind}}{dc_1^1} = -2\frac{X_1^*}{N(N+1)} - 2\left(x_1^{n,*} - \frac{X_1^*}{N}\right) = 2\frac{X_1^*}{N+1} - 2x_1^{n,*} = 2\left(\frac{1}{N+1} - \frac{x_1^{n,*}}{X_1^*}\right)X_1^*. \qquad \text{(B10)}$$

Finally, using (B7) and (B10), find

$$\frac{dW}{dc_1^1} = \frac{\partial CS}{\partial \overline{c}_1}\frac{\partial \overline{c}_1}{\partial c_1^1} + \frac{d\pi^{ind}}{dc_1^1} = -\frac{X_1^*}{N+1} + 2\left(\frac{1}{N+1} - \frac{x_1^{n,*}}{X_1^*}\right)X_1^* = 2\left(\frac{1}{2(N+1)} - \frac{x_1^{n,*}}{X_1^*}\right)X_1^* \qquad \text{(B11)}$$

to demonstrate part (*d*). Q

<u>Establishing Relations (3.5)</u>:

Use of (3.2) and (3.4) provides

$$\Delta^{II-I}TC = \sum_{n\in\Omega_N}\sum_{i\in\Omega_M}\left(\hat{\varepsilon}_i^n\hat{\delta}_i^n - \tilde{\varepsilon}_i^n\tilde{\delta}_i^n\right) = \sum_{n\in\Omega_N}\sum_{s\in\Omega_M}\sum_{m\in\Omega_M}\left[\hat{\varepsilon}_s^n B_{sm}^\mu \hat{\varepsilon}_m^n - \tilde{\varepsilon}_s^n B_{sm}^\mu \tilde{\varepsilon}_m^n\right]. \quad \text{(C1)}$$

In light of $\hat{\varepsilon}_m^n = \tilde{\varepsilon}_m^n \ \forall m \neq 1$, (C1) simplifies to:

$$\begin{aligned}\Delta^{II-I}TC &= B_{11}^\mu \sum_{n\in\Omega_N}\left[\left(\hat{\varepsilon}_1^n\right)^2 - \left(\tilde{\varepsilon}_1^n\right)^2\right] + 2\sum_{n\in\Omega_N}\sum_{\substack{t\in\Omega_M,\\ t\neq 1}}\left[\tilde{\varepsilon}_t^n B_{1t}^\mu \hat{\varepsilon}_1^n - \tilde{\varepsilon}_t^n B_{1t}^\mu \tilde{\varepsilon}_1^n\right]\\
&= B_{11}^\mu \sum_{n\in\Omega_N}\left[\left(\hat{\varepsilon}_1^n\right)^2 - \left(\tilde{\varepsilon}_1^n\right)^2\right] + 2\sum_{n\in\Omega_N}\left\{\sum_{\substack{t\in\Omega_M,\\ t\neq 1}} B_{1t}^\mu \tilde{\varepsilon}_t^n\left(\hat{\varepsilon}_1^n - \tilde{\varepsilon}_1^n\right)\right\} \qquad \text{(C2)}\\
&= B_{11}^\mu \sum_{n\in\Omega_N}\left[\left(\hat{\varepsilon}_1^n\right)^2 - \left(\tilde{\varepsilon}_1^n\right)^2\right] + 2\mu\sum_{n\in\Omega_N}\left\{\sum_{\substack{t\in\Omega_M,\\ t\neq 1}} B_{1t}^\mu \tilde{\varepsilon}_t^n\left(\tilde{s}_1^n - \hat{s}_1^n\right)\right\}.\end{aligned}$$

Similarly, upon remembering that $\tilde{\varepsilon}_m^n = \hat{\varepsilon}_m^n = \varepsilon_m^n$ for all $m \neq 1$,

$$\begin{aligned}\Delta^{II-I}\mathrm{Cov}_{ij}^x &= \frac{1}{N}\sum_{n\in\Omega_N}\left(\hat{\delta}_i^n\hat{\delta}_j^n - \tilde{\delta}_i^n\tilde{\delta}_j^n\right) = \frac{1}{N}\sum_{n\in\Omega_N}\sum_{t\in\Omega_M}\sum_{k\in\Omega_M} B_{it}^\mu B_{jk}^\mu\left(\hat{\varepsilon}_t^n\hat{\varepsilon}_k^n - \tilde{\varepsilon}_t^n\tilde{\varepsilon}_k^n\right)\\
&= \frac{1}{N}B_{i1}^\mu B_{j1}^\mu \sum_{n\in\Omega_N}\left[\left(\hat{\varepsilon}_1^n\right)^2 - \left(\tilde{\varepsilon}_1^n\right)^2\right] + \frac{1}{N}\sum_{n\in\Omega_N}\sum_{\substack{t\in\Omega_M,\\ t\neq 1}}\left[B_{it}^\mu B_{j1}^\mu + B_{i1}^\mu B_{jt}^\mu\right]\tilde{\varepsilon}_t^n\left(\hat{\varepsilon}_1^n - \tilde{\varepsilon}_1^n\right) \qquad \text{(C3)}\\
&= \frac{1}{N}B_{i1}^\mu B_{j1}^\mu \sum_{n\in\Omega_N}\left[\left(\hat{\varepsilon}_1^n\right)^2 - \left(\tilde{\varepsilon}_1^n\right)^2\right] + \frac{\mu}{N}\sum_{n\in\Omega_N}\sum_{\substack{t\in\Omega_M,\\ t\neq 1}}\left[B_{it}^\mu B_{j1}^\mu + B_{i1}^\mu B_{jt}^\mu\right]\tilde{\varepsilon}_t^n\left(\tilde{s}_1^n - \hat{s}_1^n\right).\end{aligned}$$

In the case of only one good, or where there are no market interactions ($B_{ij}^\mu = 0 \ \forall i \neq j$), (C2) and (C3) show that $\mathrm{sign}\{\Delta^{II-I}TC\} = -\mathrm{sign}\{\Delta^{II-I}\mathrm{Cov}_{11}^x\}$, so an increase in the variance of costs must increase the variance of outputs and decrease total costs. Hence, in this case, the largest cost reductions are achieved when the most efficient firm in industry 1 adopts the innovation. But when $B_{ij}^{\mu} \neq 0,$ then no such conclusion is possible.

When $\tilde{s}_1^n - \hat{s}_1^n = 1$ for $n = a$, $\tilde{s}_1^n - \hat{s}_1^n = -1$ for $n = b$, and $\tilde{s}_1^n - \hat{s}_1^n = 0$ otherwise, then (C2) and (C3) simplify to (3.5). Bear in mind when doing the algebra that the difference in squares cancels since the sum of squares is the same. Also, the vector difference $\hat{\varepsilon}_1 - \tilde{\varepsilon}_1$ has zero entries apart from when $n = a$ and when $n = b$.